Distance Magic Index One Graphs

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Abstract

Let S be a finite set of positive integers. A graph $G = (V(G), E(G))$ is said to be S-magic if there exists a bijection $f : V(G) \to S$ such that for any vertex u of G, $\sum_{v \in N_G(u)} f(v)$ is a constant, where $N_G(u)$ is the set of all vertices adjacent to u. Let $\alpha(S) = \max_{x \in S} x$. Define $i(G) = \min_{S} \alpha(S)$, where the minimum runs over all S for which the graph G is S-magic. Then $i(G) - |V(G)|$ is called the distance magic index of a graph G. In this paper, we compute the distance magic index of graphs $G[\bar{K_n}]$, where G is any arbitrary regular graph, disjoint union of m copies of complete multi-partite graph and disjoint union of m copies of graph $C_p[\bar{K_n}]$, with $m \geq 1$. In addition to that, we also prove some necessary conditions for an regular graph to be of distance magic index one.

Keywords: Distance magic, S-magic graph, distance magic index, complete multi-partite graphs, lexicographic product.
2010 MSC: 05C78, 05C76.

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1. Introduction

In this paper, we consider only simple and finite graphs. We use $V(G)$ for the vertex set and $E(G)$ for the edge set of a graph G. The neighborhood, $N_G(v)$ or shortly $N(v)$ of a vertex v of G is the set of all vertices adjacent to v in G. For further graph theoretic terminology and notation, we refer Bondy and Murty [1] and Hammack *et al.*[2].

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A distance magic labeling of a graph G is a bijection $f : V(G) \to \{1, ..., |V(G)|\}$, such that for any u of G, the weight of u, $w_G(u) = \sum_{v \in N_G(u)} f(v)$ is a constant, say c. A graph G that admits such a labeling is called a distance magic graph.

The motivation for distance magic labeling came from the concept of magic squares and tournament scheduling. An equalized incomplete tournament, denoted by $EIT(n, r)$, is a tournament, with n teams and r rounds, which satisfies the following conditions:

(i) every team plays against exactly r opponents.

(ii) the total strength of the opponents, against which each team plays is a constant.

Therefore, finding a solution for an equalized incomplete tournament $EIT(n, r)$ is equivalent to establish a distance magic labeling of an r-regular graph of order n. For more details, one can refer [3, 4].

The following results provide some necessary condition for distance magicness of regular graphs.

Theorem 1. [5, 6, 7, 8] *No r-regular graph with r-odd can be a distance magic graph.*

Theorem 2. [4] *Let $EIT(n, r)$ be an equalized tournament with an even number n of teams and $r \equiv 2 \mod 4$. Then $n \equiv 0 \mod 4$.*

In [6], Miller *et al.* discussed the distance magic labeling of the graph $H_{n,p}$, the complete multi-partite graph with p partitions in which each partition has exactly n vertices, $n \geq 1$ and $p \geq 1$. It is clear that $H_{n,1}$ is a distance magic graph. From [6] it is observed that K_n is distance magic if and only if $n = 1$ and hence, $H_{1,p} \cong K_p$ is not distance magic for all $p \neq 1$. The next result gives a characterization for the distance magicness of $H_{n,p}$.

Theorem 3. [6] *Let $n > 1$ and $p > 1$. $H_{n,p}$ has a labeling if and only if either n is even or both n and p are odd.*

Recall a standard graph product (see [2]). Let G and H be two graphs. Then, the lexicographic product $G \circ H$ or $G[H]$ is a graph with the vertex set $V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent in $G[H]$ if and only if g is adjacent to g' in G, or $g = g'$ and h is adjacent to h' in H.

Miller *et al.* [6] proved the following.

Theorem 4. [6] *Let G be an arbitrary regular graph. Then $G[\bar{K_n}]$ is distance magic for any even n.*

Later, Froncek *et al*. [4, 9] proved the following results.

Theorem 5. [4] *For n even an $EIT(n, r)$ exists if and only if $2 \leq r \leq n-2; r \equiv 0 \mod 2$ and either $n \equiv 0 \mod 4$ or $n \equiv r+2 \equiv 2 \mod 4$.*

Theorem 6. [9] *Let n be odd, $p \equiv r \equiv 2 \mod 4$, and G be an r-regular graph with p vertices. Then $G[\bar{K_n}]$ is not distance magic.*

Theorem 7. [9] *Let G be an arbitrary r-regular graph with an odd number of vertices and n be an odd positive integer. Then r is even and the graph $G[\bar{K_n}]$ is distance magic.*

The following results by Shafiq *et al*. [10], discusses the distance magic labeling of disjoint union of m copies of complete multi-partite graphs, $H_{n,p}$, and disjoint union of m copies of product graphs, $C_p[\bar{K_n}]$.

Theorem 8. [10]

(i) *If n is even or mnp is odd, $m \geq 1; n > 1$ and $p > 1$; then $mH_{n,p}$ has a distance magic labeling.*

(ii) *If np is odd, $p \equiv 3 \mod 4$, and m is even, then $mH_{n,p}$ does not have a distance magic labeling.*

Theorem 9. [10] *Let $m \geq 1, n > 1$ and $p \geq 3$. $mC_p[\bar{K_n}]$ has a distance magic labeling if and only if either n is even or mnp is odd or n is odd and $p \equiv 0 \mod 4$.*

In [10], Shafiq *et al*. posted a problem on the graph $mH_{n,p}$.

Problem 1. *For the graph $mH_{n,p}$, where m is even, n is odd, $p \equiv 1 \mod 4$, and $p > 1$, determine if there is a distance magic labeling.*

Later, Froncek *et al*.[9] proved the following necessary condition for $mH_{n,p}$.

Theorem 10. *The graph $mH_{n,p}$, where m is even, n is odd, $p \equiv 1 \mod 4$, and $p > 1$, is not distance magic.*

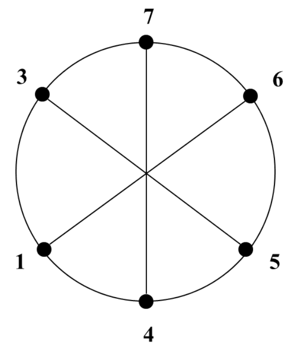

Figure 1: A graph G with $c' = 13$ and $S = \{1, 3, 4, 5, 6, 7\}$.

For more details and results, one can refer Arumugam *et al.* [11].

From Theorem 1, one can observe that any odd-regular graph G of order n is not distance magic. But if we label the graph with respect to a different set S of positive integers with $|S| = n$, then G may admit a magic labeling with a magic constant c'. See Figure 1.

Motivated by this fact Godinho *et al.* [**?**] defined the concept of S-magic labeling of a graph.

Definition 1. [**?**] *Let $G = (V(G), E(G))$ be a graph and let S be a set of positive integers with $|V(G)| = |S|$. Then G is said to be S-magic if there exists a bijection $f : V(G) \to S$ satisfying $\sum_{v \in N(u)} f(v) = c$ (a constant) for every $u \in V(G)$. The constant c is called the S-magic constant.*

Definition 2. [**?**] *Let $\alpha(S) = max\ \{s : s \in S\}$. Let $i(G) = min\ \alpha(S)$, where the minimum is taken over all sets S for which the graph G admits an S-magic labeling. Then $i(G) - |V(G)|$ is called the distance magic index of a graph G and is denoted by $\theta(G)$.*

From above definitions, one can observe that a graph G is distance magic if and only if $\theta(G) = 0$ and if G is not S-magic for any S with $|V(G)| = |S|$, then $\theta(G) = \infty$.

Let G be a graph for which $\theta(G)$ is finite (however so small) and non-zero. Now, a natural question arises that for all such graphs G, does there exist an S-magic labeling with $\theta(G) = 1$?

In the following section, we prove some necessary conditions for an r-regular S-magic graph G to have $\theta(G) = 1$. Further, we compute the distance magic index of disjoint union of m copies of $H_{n,p}$ and disjoint union of m copies of $C_p[\bar{K_n}]$, where $m \geq 1$. Also, for any arbitrary regular graph G, we compute the distance magic index of the graph $G[\bar{K_n}]$. In addition to that, we construct twin sets S and S' for the same graph $H_{n,p}$ with $\theta(G) = 1$, for which $H_{n,p}$ is both S-magic and S'-magic with distinct magic constants. We also discuss the maximum and minimum bounds attained by the magic constant for the graph $H_{n,p}$.

2. Main results

If G is a graph with $\theta(G) = 1$, then it is clear that G is S-magic for $S = \{1, ..., n+1\} \setminus \{a\}$, for at least one $a \in \{1, ..., n\}$. We call a, the deleted label of S.

The following results are similar to that of Theorem 1 and 2.

Lemma 1. *If G is an odd r-regular S-magic graph with $\theta(G) = 1$, then $a \neq 1$.*

Proof. Assume that G is an r-regular graph with $\theta(G) = 1$, where r is odd. If $S = \{1, ..., n+1\} \setminus \{a\}$ with the S-magic constant c, then,

$$nc = r(1 + ... + n + 1) - ra \tag{1}$$

$$c = \frac{rn + 3r}{2} + \frac{r - ra}{n}. \tag{2}$$

Therefore, if $a = 1$, then c is not an integer, a contradiction. □

Lemma 2. *If G is an r-regular S-magic graph with $\theta(G) = 1$ and $r, n \equiv 2$ mod 4, then a is an even integer, $a \neq 2, n$.*

Proof. Assume that G is an r-regular graph with $\theta(G) = 1$ and $r, n \equiv 2$ mod 4. Let c be the S-magic constant of G, where $S = \{1, 2, ..., n+1\} \setminus \{a\}$ and a is an odd integer belonging to $\{1, 2, .., n\}$. Let $r = 4k + 2$ and $n = 4k' + 2$, with $0 < k < k'$.

Case 1: when $a = 1$, from eq.(2), we have,

$$c = (2k + 1)(4k' + 5).$$

Here c is an odd integer and every vertex is adjacent to odd number of vertices which are labeled with odd integers. Note that, here there are $2k' + 1$ such

vertices. Then the graph induced by the vertices having odd label has every vertex of odd degree, a contradiction.
Case 2: When $a = 2q + 1$, with $q > 0$. Then $rn + 3r \equiv 2 \mod 4$ and $r - ra \equiv 0 \mod 4$ and hence c fails to be an integer.
Case 3: When $a = 2$ or $a = n$, c is not an integer and hence the result follows. □

The following theorem discusses the distance magic index of the graph, $H_{n,p}, n > 1$ and $p > 1$. We define the integer-valued function α given by

$$\alpha(j) = \begin{cases} 0 & \text{for } j \text{ even} \\ 1 & \text{for } j \text{ odd,} \end{cases}$$

and the sets $\Omega_k = \{i : 5 \leq i \leq n-1 \text{ and } i \equiv k \mod 4\}$, where $k \in \{0, 1, 2, 3\}$. Both α and Ω_i's are used in the next theorem.

Theorem 11. *If G is a complete multi-partite graph $H_{n,p}$ with p partitions having n vertices in each partition, then*

$$\theta(G) = \begin{cases} 0 & \text{for } n \text{ even or } n \text{ and } p \text{ both odd} \\ 1 & \text{for } n \text{ odd and } p \text{ even.} \end{cases}$$

Proof. Let $G \cong H_{n,p}$ with $n > 1, p > 1$. From Theorem 3, it is clear that if n is even or when n and p both are odd, then $\theta(G) = 0$.

Now, to construct an $(n \times p)$- rectangular matrix $A = (a_{i,j})$ with distinct entries from a set S having column sum b (a constant) is equivalent to find an S-magic labeling of G with magic constant $(p - 1)b$.

Note that j^{th} column of A can be used to label the vertices of j^{th} partition of G and hence G admits a magic labeling with magic constant $(p - 1)b$. In addition, if the entries of A are all distinct and are from $S = \{1, ..., np + 1\} \setminus \{a\}$, where $a \in \{1, ..., np\}$, then G is S-magic with $\theta(G) = 1$.

Let n be an odd and p be an even integer.

Case 1: If $n = 3$ and $p = 2m, m > 0$, then construct A as,

$$\begin{pmatrix} 1 & 2 & 3 & 4 & ... & 2m-3 & 2m-2 & 2m-1 & 2m \\ 3m & 4m & 3m-1 & 4m-1 & ... & 2m+2 & 3m+2 & 2m+1 & 3m+1 \\ 6m+1 & 5m & 6m & 5m-1 & ... & 5m+3 & 4m+2 & 5m+2 & 4m+1 \end{pmatrix}$$

Note that, the deleted label is $5m + 1$ here. One can observe that each column adds up to a constant $9m + 2$ and thus, $\theta(H_{3,2m}) = 1$.

Case 2: If $n = 5$ and $p = 2m, m > 0$, then construct A as,

$$\begin{pmatrix} 1 & 2 & 3 & 4 & ... & 2m-3 & 2m-2 & 2m-1 & 2m \\ 3m & 4m & 3m-1 & 4m-1 & ... & 2m+2 & 3m+2 & 2m+1 & 3m+1 \\ 6m & 5m & 6m-1 & 5m-1 & ... & 5m+2 & 4m+2 & 5m+1 & 4m+1 \\ 7m & 8m & 7m-1 & 8m-1 & ... & 6m+2 & 7m+2 & 6m+1 & 7m+1 \\ 9m+2 & 8m+1 & 9m+3 & 8m+2 & ... & 10m & 9m-1 & 10m+1 & 9m \end{pmatrix}$$

Here, the deleted label is $9m + 1$ and each column adds up to a constant $25m + 3$. Therefore, $\theta(H_{5,2m}) = 1$.

Case 3: If $n > 5$ is odd and $p = 2m, m > 0$, then for each $j \in \{1, ..., p\}$, construct A as follows.

$$a_{i,j} = \begin{cases} j & \text{for } i = 1 \\ (2i-1)m - (\frac{j-1}{2}) + \alpha(j+1)(m + \frac{1}{2}) & \text{for } i = 2, 4 \\ 2mi - (\frac{j-1}{2}) + \alpha(j+1)(-m + \frac{1}{2}) & \text{for } i = 3 \\ 2mi - m + \frac{j}{2} + \alpha(j)(-m + \frac{1}{2}) & \text{for } i \equiv 1 \mod 4, i \in \{5, 6, ..., n-1\} \\ 2mi - m + \frac{j}{2} + \alpha(j)\frac{1}{2} - \alpha(j+1)m & \text{for } i \equiv 2 \mod 4, i \in \{5, 6, ..., n-1\} \\ 2mi - (\frac{j-1}{2}) + \alpha(j+1)(-m + \frac{1}{2}) & \text{for } i \equiv 3 \mod 4, i \in \{5, 6, ..., n-1\} \\ 2mi - (\frac{j-1}{2}) + \alpha(j)(-m) + \alpha(j+1)\frac{1}{2} & \text{for } i \equiv 0 \mod 4, i \in \{5, 6, ..., n-1\} \\ 2mi - m + \frac{j}{2} + \alpha(j)(\frac{3}{2}) - \alpha(j+1)m & \text{for } i = n \equiv 1 \mod 4 \\ 2mi - \frac{j}{2} + \frac{1}{2} - \alpha(j+1)(m + \frac{1}{2}) & \text{for } i = n \equiv 3 \mod 4 \end{cases}$$

Therefore, $m(2n - 1) + 1$ is the deleted label in this case.

Subcase 1: If $n \equiv 1 \mod 4$, then $n - 5 \equiv 0 \mod 4$. Let $n = 4q + 5$, where $q \geq 1$.
Now for any fixed odd j, the j^{th} column sum in A is,

$$\sum_{i=1}^{n} a_{i,j} = \sum_{i=1}^{4} a_{i,j} + \sum_{i\in\Omega_1} a_{i,j} + \sum_{i\in\Omega_2} a_{i,j} + \sum_{i\in\Omega_3} a_{i,j} + \sum_{i\in\Omega_0} a_{i,j} + \left(2mn - m + \frac{j}{2} + \frac{3}{2}\right)$$

$$= j + 3m - (\tfrac{j-1}{2}) + 6m - (\tfrac{j-1}{2}) + 7m - (\tfrac{j-1}{2}) + \textstyle\sum_{k=1}^{q}\left(2m(4k+1) - 2m + \frac{j+1}{2}\right) +$$
$$\textstyle\sum_{k=1}^{q}\left(2m(4k+2) - m + \frac{j+1}{2}\right) + \sum_{k=1}^{q}\left(2m(4k+3) - (\frac{j-1}{2})\right) +$$
$$\textstyle\sum_{k=1}^{q}\left(2m(4k+4) - m - (\frac{j-1}{2})\right) + 2mn - m + 1 + \frac{j+1}{2}$$

$= 15m + 32mq + 16mq^2 + 2mn + 2q + 3 = \frac{n^2p+n+1}{2}$.

Similarly, for any fixed even j, the j^{th} column sum in A is,

$$\sum_{i=1}^{n} a_{i,j} = \sum_{i=1}^{4} a_{i,j} + \sum_{i\in\Omega_1} a_{i,j} + \sum_{i\in\Omega_2} a_{i,j} + \sum_{i\in\Omega_3} a_{i,j} + \sum_{i\in\Omega_0} a_{i,j} + \left(2mn - 2m + \frac{j}{2}\right)$$

$= j + 4m - (\frac{j-2}{2}) + 5m - (\frac{j-2}{2}) + 8m - (\frac{j-2}{2}) + \sum_{k=1}^{q}\left(2m(4k+1) - m + \frac{j}{2}\right) + \sum_{k=1}^{q}\left(2m(4k+2) - 2m + \frac{j}{2}\right) + \sum_{k=1}^{q}\left(2m(4k+3) - m - (\frac{j-2}{2})\right) + \sum_{k=1}^{q}\left(2m(4k+4) - (\frac{j-2}{2})\right) + 2mn - 2m + \frac{j}{2}$
$= 15m + 32mq + 16mq^2 + 2mn + 2q + 3 = \frac{n^2p+n+1}{2}$.

Subcase 2: if $n \equiv 3 \mod 4$, then $n - 5 \equiv 2 \mod 4$. Let $n = 4q + 3$ where $q \geq 0$.
Now, for any fixed odd j, the j^{th} column sum in A is,

$$\sum_{i=1}^{n} a_{i,j} = \sum_{i=1}^{4} a_{i,j} + \sum_{i\in\Omega_1} a_{i,j} + \sum_{i\in\Omega_2} a_{i,j} + \sum_{i\in\Omega_3} a_{i,j} + \sum_{i\in\Omega_0} a_{i,j} + \left(2mn - \frac{j}{2} + \frac{3}{2}\right)$$

$= j + 3m - (\frac{j-1}{2}) + 6m - (\frac{j-1}{2}) + 7m - (\frac{j-1}{2}) + \sum_{k=1}^{q+1}\left(2m(4k+1) - 2m + \frac{j+1}{2}\right) + \sum_{k=1}^{q+1}\left(2m(4k+2) - m + \frac{j+1}{2}\right) + \sum_{k=1}^{q}\left(2m(4k+3) - (\frac{j-1}{2})\right) + \sum_{k=1}^{q}\left(2m(4k+4) - m - (\frac{j-1}{2})\right) + 2mn + 1 - (\frac{j-1}{2})$
$= 35m + 48mq + 16mq^2 + 2mn + 2q + 4 = \frac{n^2p+n+1}{2}$.

Similarly, for any fixed even j, the j^{th} column sum in A is,

$$\sum_{i=1}^{n} a_{i,j} = \sum_{i=1}^{4} a_{i,j} + \sum_{i\in\Omega_1} a_{i,j} + \sum_{i\in\Omega_2} a_{i,j} + \sum_{i\in\Omega_3} a_{i,j} + \sum_{i\in\Omega_0} + \left(2mn - m - \frac{j}{2} + 1\right)$$

$= j + 4m - (\frac{j-2}{2}) + 5m - (\frac{j-2}{2}) + 8m - (\frac{j-2}{2}) + \sum_{k=1}^{q+1}\left(2m(4k+1) - m + \frac{j}{2}\right) + \sum_{k=1}^{q+1}\left(2m(4k+2) - 2m + \frac{j}{2}\right) + \sum_{k=1}^{q}\left(2m(4k+3) - m - (\frac{j-2}{2})\right) +$

$\sum_{k=1}^{q}\left(2m(4k+4)-(\frac{j-2}{2})\right)+2mn-m-(\frac{j-2}{2})$
$= 35m+48mq+16mq^2+2mn+2q+4=\frac{n^2p+n+1}{2}.$

Since the sum of the entries in each column of A is $\frac{n^2p+n+1}{2}$ for odd $n>5$, $H_{n,2m}$ is S-magic with magic constant $\frac{n^2p+n+1}{2}(p-1)$ and $\theta(H_{n,2m})=1$. □

Theorem 12. *If $G \cong H_{n,p}$ is an S-magic graph with $\theta(G)=1$ and S-magic constant $\frac{n^2p+n+1}{2}(p-1)$, then there exists a set S' such that G is an S'-magic graph with $\theta(G)=1$ and S'-magic constant $\frac{n^2p+3n-1}{2}(p-1)$.*

Proof. For every S-magic graph $G \cong H_{n,p}$ with $\theta(G)=1$, one can obtain the corresponding rectangular matrix $A=(a_{i,j})$ associated with G by Theorem 11.

Define a new $(n\times p)$- rectangular matrix $A'=(a'_{i,j})$ with entries,

$$a'_{i,j}=(np+2)-a_{i,j} \text{ for all } i \text{ and } j. \quad (3)$$

By Theorem 11, it is clear that the entries in A belong to the set $\{1,...,np+1\}\setminus\{np+1-\frac{p}{2}\}$, which sum up to $\frac{n^2p^2+p(n+1)}{2}$ and is divisible by p. Hence the magic constant is $\frac{n^2p+n+1}{2}(p-1)$.

Now using (3), define the new set $S'=S\cup\{np+1-\frac{p}{2}\}\setminus\{\frac{p}{2}+1\}$ and the sum of all the entries in $A'=np(np+2)-(\frac{n^2p^2+p(n+1)}{2})=\frac{n^2p^2+3np-p}{2}$, which is divisible by p. Therefore, we obtain the magic constant as $\frac{n^2p+3n-1}{2}(p-1)$. □

The rectangular matrices A and A' associated with $H_{5,6}$ are given below,

$$A=\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6\\ 9 & 12 & 8 & 11 & 7 & 10\\ 18 & 15 & 17 & 14 & 16 & 13\\ 21 & 24 & 20 & 23 & 19 & 22\\ 29 & 25 & 30 & 26 & 31 & 27 \end{pmatrix} \qquad A'=\begin{pmatrix} 31 & 30 & 29 & 28 & 27 & 26\\ 23 & 20 & 24 & 21 & 25 & 22\\ 14 & 17 & 15 & 18 & 16 & 19\\ 11 & 8 & 12 & 9 & 13 & 10\\ 3 & 7 & 2 & 6 & 1 & 5 \end{pmatrix}$$

Here, the sum of the entries in each column of A and A' are 78 and 82 respectively. Then, $H_{5,6}$ is S-magic with magic constant 390 and S'-magic

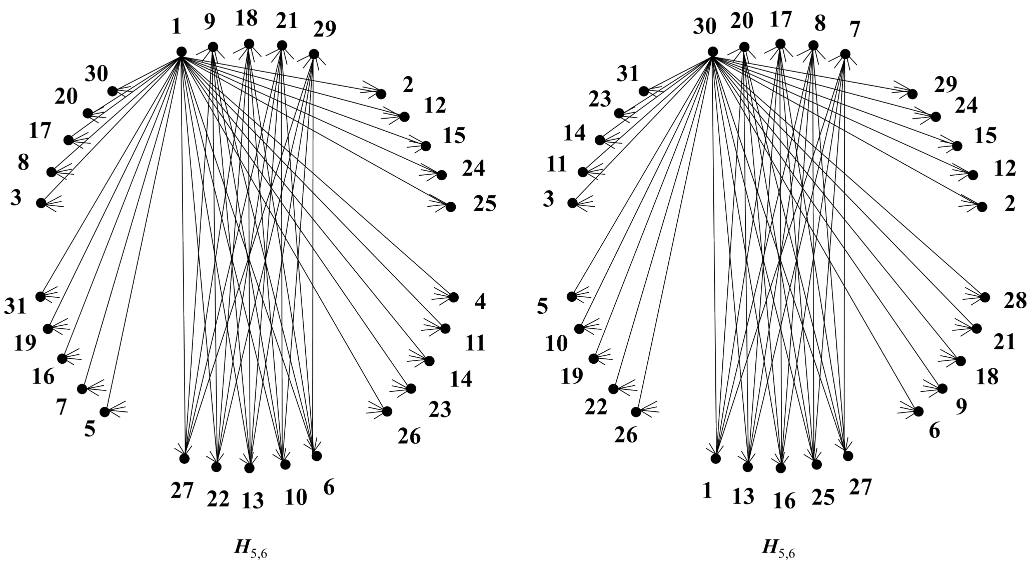

Figure 2: $H_{5,6}$ with S-magic constant 390 and S' magic constant 410.

with magic constant 410.
Now the following result is immediate.

Lemma 3. *If G is an r-regular graph with $\theta(G) = 1$ and with S-magic constant c, then*

$$\frac{nr + r}{2} + \frac{r}{n} \leq c \leq \frac{nr + 3r}{2}.$$

Proof. The proof is obtained from Lemma 1 by substituting $a = 1$ and $a = n$ for c. □

Observation 1. *If $G \cong H_{n,p}$ is a graph with $\theta(G) = 1$ and S-magic constant c, then*

$$\frac{n^2p + n + 1}{2}(p - 1) \leq c \leq \frac{n^2p + 3n - 1}{2}(p - 1)$$

□

The lower and upper bounds in Observation 1 are tight when one compares with Lemma 3. It is noticed that if $S = \{1, ..., np + 1\} \setminus \{a\}$, which confirms that $H_{n,p}$ is S-magic, then the sum of all the entries in S is divisible by p. Therefore, the highest a that can be removed to get a multiple of p is

$np + 1 - \frac{p}{2}$ and the lowest a that can be removed to get a multiple of p is $\frac{p}{2} + 1$. Hence the result follows.

Lemma 4. *Let B be an $(n \times p)$-rectangular matrix with distinct entries from the set $\{1, 2, .., np+1\} \setminus \{a\}$, where $a \in \{1, 2, ..., np\}$ having column sum s. If there exists an integer $m \geq 1$, $m|p$, then there exixts m, $(n \times t)$-rectangular matrices, $B_m, (1 \leq m \leq t)$, having column sum s.*

Proof. Consider the $(n \times mt)$-rectangular matrix B with distinct entries from the set $\{1, 2, .., np+1\} \setminus \{a\}$, where $a \in \{1, 2, ..., np\}$ and having column sum s.

Construct an $(n \times t)$-rectangular matrix, B_1 by choosing any t distinct columns of B and update the B matrix by replacing all the entries in the newly chosen t columns with $0's$. Now the updated B matrix will have exactly $(m - 1)t$ nonzero columns and t columns having all zero entries.

Now, repeat the process to obtain the next matrix B_2 by choosing any t non-zero columns from the remaining $(m - 1)t$ columns and update the B matrix in the same manner as in first step. Now repeatedly apply the above technique to obtain the remaining $m - 2$ matrices, $B_i, (3 \leq i \leq m)$, until the matrix B becomes an zero matrix. □

From Theorem 8, it is observed that in both the cases when n is odd, p is even and when np is odd, $p \equiv 3 \mod 4$ and m is even, $\theta(mH_{n,p}) \neq 0$. The following theorem computes the distance magic index of $mH_{n,p}$ for above cases.

Theorem 13. *If $n > 1, p > 1, m \geq 1$, then*

$$\theta(mH_{n,p}) = \begin{cases} 0 & \text{for } n \text{ even or } mnp \text{ is odd,} \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Using Theorem 8, it is clear that $\theta(mH_{n,p}) = 0$, when either n is even or mnp is odd and $\theta(mH_{n,p}) \neq 0$, when either np is odd, $p \equiv 3 \mod 4$, and m is even. On the other hand, by Theorem 10, one can conclude that $\theta(mH_{n,p}) \neq 0$, when m is even, n is odd, $p \equiv 1 \mod 4$, and $p > 1$.
For all the remaining cases, use Theorem 11 to construct the rectangular matrix A associated with the graph $H_{n,mp}$. Now using Lemma 4, construct the $(n \times p)$-matrices B_k, for $k \in \{1, ..., m\}$ Here, each B_k forms the matrix associated with the k^{th} copy of $H_{n,p}$ and hence we obtain an S-magic labeling of $mH_{n,p}$ with $c = \frac{n^2mp+n+1}{2}(p - 1)$. Therefore, $\theta(mH_{n,p}) = 1$. □

Theorem 9 confirms that if n is even or mnp is odd or n is odd and $p \equiv 0 \mod 4$, then $\theta(mC_p[\bar{K_n}]) = 0$. Now the remaining cases are given below.
Case 1: n is odd, m is even, $p \equiv 2 \mod 4$.
Case 2: n is odd, m is odd, $p \equiv 2 \mod 4$.
Case 3: n is odd, m is even, p is odd.

The following theorem determines the distance magic index of the graph $mC_p[\bar{K_n}]$ for all the above mentioned three cases.

Theorem 14. *Let $m \geq 1, n > 1$ and $p \geq 3$, then*

$$\theta(mC_p[\bar{K_n}]) = \begin{cases} 0 & \text{if } n \text{ is even or } mnp \text{ is odd or } n \text{ is odd, } p \equiv 0 \bmod 4, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Let $G \cong mC_p[\bar{K_n}]$. From Theorem 9, it is clear that $\theta(G) = 0$, when n even or mnp is odd or n is odd and $p \equiv 0 \mod 4$.

Now, for all the remaining cases, using Theorem 11 construct the matrix A associated with the graph $H_{n,mp}$ and use A in Lemma 4 to construct the m rectangular matrices associated with m copies of graph $C_p[\bar{K_n}]$. Hence, we obtain a S-magic labeling of G with $c = n^2mp+n+1$ and hence $\theta(G) = 1$. □

Let G be an r-regular graph on p vertices. From Theorem 5, for the graph $G[\bar{K_n}]$, if n is odd, r is even and p is even except when $p \equiv r \equiv 2 \mod 4$, then $\theta(G[\bar{K_n}] = 0$. The following theorem computes the distance magic index of the graph $G[\bar{K_n}]$.

Theorem 15. *Let G be an r-regular graph on p vertices. Then,*

$$\theta(G[\bar{K_n}]) = \begin{cases} 0 & \text{if } n \text{ is even or } n,p \text{ are odd, } r \text{ is even,} \\ 1 & \text{if } n, r \text{ are odd or } n \text{ is odd, } r \equiv p \equiv 2 \text{ mod } 4 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let G be a graph on p vertices $v_1, ..., v_p$ and let $V_i = \{v_i^1, ..., v_i^n\}$ be set the vertices of $G[\bar{K_n}]$ that replace the vertex v_i of G for all $i = 1, ..., p$. Note that here $V(G[\bar{K_n}]) = \bigcup_{i=1}^{p} V_i$.

When n is even, by Theorem 4, $\theta(G[\bar{K_n}]) = 0$ and when n is odd, p is odd and r is even, by Theorem 7, $\theta(G[\bar{K_n}]) = 0$. Further, when n is odd and $p \equiv r \equiv 2 \mod 4$, then by Theorem 6, $\theta(G[\bar{K_n}]) \neq 0$. Also when n is odd and r is odd, by Theorem 1, $\theta(G[\bar{K_n}]) \neq 0$. Further for all the other cases $\theta(G[\bar{K_n}] = 0$ by Theorem 5. Now for both the cases when $\theta(G[\bar{K_n}]) \neq 0$, use Theorem 11, to construct the rectangular matrix A associated with the graph $H_{n,p}$ and use the i^{th} column of A to label the set of vertices, V_i, for all $i = 1, 2, .., p$. Hence, we obtain a S-magic labeling of $G[\bar{K_n}]$, with $c = r\left(\frac{n^2p+n+1}{2}\right)$. Therefore we obtain that $\theta(C_p[\bar{K_n}]) = 1$. □

3. Conclusion

In this paper, the distance magic index of disjoint union of m copies of $H_{n,p}$ and disjoint union of m copies of $C_p[\bar{K_n}]$ are computed and few necessary conditions are derived for a regular graph G for which $\theta(G)$ is 1. The paper establishes a technique to construct a new set of labels from an existing one in such a way that both magic constants are distinct. Further, the lower and upper bounds of magic constant of a regular graph G with $\theta(G) = 1$, are also determined.

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