

MINIMALITY AND GLUING ORBIT PROPERTY

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ABSTRACT. We show that a dynamical system with gluing orbit property is either minimal or have positive topological entropy. Moreover, for equicontinuous systems, we show that topological transitivity, minimality and orbit gluing property are equivalent. These facts reflect the similarity and dissimilarity of gluing orbit property with specification like properties.

1. INTRODUCTION

The notion of gluing orbit property was introduced in [12], [5] and [2]. As a weaker form of the well-studied specification properties, it turns out to be a more general property which still captures crucial topological features of the systems, especially the non-hyperbolic ones. A number of results have been obtained based on this property. See also [1], [4], [13] and [14]. For classical results with specification property and specification like properties, the readers are referred to [6] and [10].

There is a remarkable difference between gluing orbit property and specification property, as well as most weaker forms of the latter. As illustrated in [1] and [2], certain examples far from specification, such as irrational rotations, have gluing orbit property. We can see that gluing orbit property only requires topological transitivity, and is compatible with zero topological entropy, while specification property implies topological mixing and positive topological entropy. In general, topological mixing should not be expected for a system that only has gluing orbit property. For example, the direct product of the irrational rotation and any system with specification property has gluing orbit property and is not topologically mixing. In this article, we consider the entropy and find that there is a dichotomy: a system with gluing orbit property is either minimal or of positive topological entropy.

Theorem 1.1. *Assume that (X, f) is not minimal and has the gluing orbit property, then it has positive topological entropy.*

We remark that Theorem 1.1 is not trivial as there are plenty of systems with zero topological entropy that are not minimal. Besides those simple examples, there are also complicated ones (cf. Example 6.5). We are not clear whether there exists a system with gluing orbit property that is both minimal and of positive topological entropy. We suspect that the answer is positive and Herman's example [9] (or something like it) may be a possible candidate.

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As a direct corollary of Theorem 1.1, periodic gluing orbit property implies positive topological entropy, just as the specification properties do. The only exception that should be ruled out is the trivial case that X consists of a single periodic orbit.

Corollary 1.2. *A non-trivial system with the periodic gluing orbit property must have positive topological entropy and exponential growth of periodic orbits.*

By Theorem 1.1, for a system with zero topological entropy, gluing orbit property implies minimality. Example 6.6 shows that the converse is not true. We can show that the converse holds if the system is equicontinuous, in which case gluing orbit property is also equivalent to topological transitivity. This extends the examples in [1]. We doubt if there are systems that are not equicontinuous, of zero topological entropy and have gluing orbit property (hence minimal).

Theorem 1.3. *Assume that (X, f) is equicontinuous. Then the followings are equivalent:*

- (1) (X, f) is topologically transitive.
- (2) (X, f) is minimal.
- (3) (X, f) has gluing orbit property.

Our results hold for both invertible and non-invertible cases, and both discrete-time and continuous-time cases as well. In this article we mainly work with homeomorphisms. There are some extra technical difficulties in the proof of the non-invertible and continuous-time cases. We give a proof of Theorem 1.1 in the semiflow case in Section 7 to illustrate the difference.

Some preliminaries are introduced in Section 2, including definitions and notations we shall use. We prove Theorem 1.1 in Section 3 and discuss some corollaries in Section 4. Theorem 1.3 is proved in Section 5. Some examples are investigated in Section 6.

2. PRELIMINARIES

Let (X, d) be a compact metric space. Let $f : X \rightarrow X$ be a homeomorphism on X . Conventionally, (X, f) is called a *topological dynamical system* or just a system.

Definition 2.1. (X, f) is said to be *equicontinuous* if for every $\varepsilon > 0$, there is $\delta > 0$ such that for any $x, y \in X$ with $d(x, y) < \delta$, we have

$$d(f^n(x), f^n(y)) < \varepsilon \text{ for every } n \geq 0.$$

Definition 2.2. (X, f) is said to be *topologically transitive* if for any open sets U, V in X , there is $n \in \mathbb{Z}$ such that

$$U \cap f^{-n}(V) \neq \emptyset.$$

Definition 2.3. (X, f) is said to be *minimal* if every orbit is dense, i.e. for every $x \in X$,

$$\overline{\{f^n(x) : n \in \mathbb{Z}\}} = X.$$

Definition 2.4. For $n \in \mathbb{Z}^+$ and $\varepsilon > 0$, a subset $E \subset X$ is called an (n, ε) -*separated* if for any distinct points x, y in E , there is $k \in \{0, \dots, n-1\}$ such that

$$d(f^k(x), f^k(y)) > \varepsilon.$$

Denote by $s(n, \varepsilon)$ the maximal cardinality of (n, ε) -separated subsets of X . Then the *topological entropy* of f is defined as

$$h(f) := \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{\ln s(n, \varepsilon)}{n}.$$

Definition 2.5. We call the finite sequence of ordered pairs

$$\mathcal{C} = \{(x_j, m_j) \in X \times \mathbb{Z}^+ : j = 1, \dots, k\}$$

an *orbit sequence of rank k* . A *gap* for an orbit sequence of rank k is a $(k-1)$ -tuple

$$\mathcal{G} = \{t_j \in \mathbb{Z}^+ : j = 1, \dots, k-1\}.$$

For $\varepsilon > 0$, we say that $(\mathcal{C}, \mathcal{G})$ can be ε -shadowed by $z \in X$ if for every $j = 1, \dots, k$,

$$(f^{s_j+l}(z), f^l(x_j)) < \varepsilon \text{ for every } l = 0, 1, \dots, m_j - 1,$$

where

$$s_1 = 0 \text{ and } s_j = \sum_{i=1}^{j-1} (m_i + t_i - 1) \text{ for } j = 2, \dots, k.$$

Definition 2.6. (X, f) is said to have *specification property* if for every $\varepsilon > 0$, there is $M(\varepsilon) > 0$ such that any $(\mathcal{C}, \mathcal{G})$ with $\min \mathcal{G} \geq M(\varepsilon)$ can be ε -shadowed.

Definition 2.7. (X, f) is said to have *periodic specification property* if for every $\varepsilon > 0$, there is $M(\varepsilon) > 0$ such that for any $t \geq M(\varepsilon)$, any $(\mathcal{C}, \mathcal{G})$ with $\min \mathcal{G} \geq M(\varepsilon)$ can be ε -shadowed by a periodic point of the period $s_k + t$.

Definition 2.8. (X, f) is said to have *gluing orbit property* if for every $\varepsilon > 0$ there is $M(\varepsilon) > 0$ such that for any orbit sequence \mathcal{C} , there is a gap \mathcal{G} such that $\max \mathcal{G} \leq M(\varepsilon)$ and $(\mathcal{C}, \mathcal{G})$ can be ε -shadowed.

Definition 2.9. (X, f) is said to have *periodic gluing orbit property* if for every $\varepsilon > 0$, there is $M(\varepsilon) > 0$ such that for any orbit sequence \mathcal{C} , there are $t \leq M(\varepsilon)$ and a gap \mathcal{G} with $\max \mathcal{G} \leq M(\varepsilon)$ such that $(\mathcal{C}, \mathcal{G})$ can be ε -shadowed by a periodic point of the period $s_k + m_k + t$.

The notion of specification property was first introduced by Bowen in [3]. It has a number of variations and their names also vary in different literatures. An overview of these specification like properties can be found in [10]. Gluing orbit property first appeared in [12], where it is called transitive specification. It is called weak specification in [5] in a slightly generalized form. It is in [2] that the name gluing orbit is called to indicate its dissimilarity with specification like properties.

Here we attempt to reformulate the definitions of specification and gluing orbit properties to make our argument more clear and more convenient. We follow the names called in [2], [10] and [13]. Note that in our definitions of periodic specification and periodic gluing orbit properties the gap \mathcal{G} may be \emptyset .

Definition 2.5 naturally extends to infinite orbit sequences. Definition 2.6 and 2.8 are conventional definitions speaking of finite orbit sequences. However, they are equivalent to the definitions speaking of infinite ones. This is clear for specification. For gluing orbit property a little extra work should be done. The flow version of the following lemma is contained in [4]. A similar technique is also part of the proof of Theorem 1.1.

Lemma 2.10. *(X, f) has gluing orbit property if and only if for every $\varepsilon > 0$, there is $L(\varepsilon) > 0$ such that for any infinite orbit sequence $\mathcal{C} = \{(x_j, m_j) : j \in \mathbb{Z}\}$, there is \mathcal{G} with $\max \mathcal{G} \leq L(\varepsilon)$, $(\mathcal{C}, \mathcal{G})$ can be ε -shadowed. Moreover, we can take $L(\varepsilon) \leq M(\varepsilon')$ for any $\varepsilon' < \varepsilon$.*

Proof. The if part is trivial.

Assume that (X, f) has the gluing orbit property as defined in Definition 2.8. Let $\varepsilon' < \varepsilon$, $m = M(\varepsilon')$ and $\mathcal{C} = \{(x_j, m_j) : j \in \mathbb{Z}^+\}$ be any infinite (forward) orbit sequence. Proof for two-sided infinite sequences is analogous. We denote for $k \geq 2$,

$$\mathcal{C}_k = \{(x_j, m_j) : j = 1, \dots, k\}.$$

For each $k \geq 2$, there is $\mathcal{G}_k = \{t_1(k), \dots, t_{k-1}(k)\}$ with $\max \mathcal{G}_k \leq m$ and $z_k \in X$ such that $(\mathcal{C}_k, \mathcal{G}_k)$ is ε' -shadowed by z_k .

There is $t_1 \in \{1, \dots, m\}$ and a subsequence $\{z_{n(1,k)}\}$ of $\{z_k\}$ such that

$$t_1(n(1, k)) = t_1 \text{ for every } k.$$

There is $t_2 \in \{1, \dots, m\}$ and a subsequence $\{z_{n(2,k)}\}$ of $\{z_{n(1,k)}\}$ such that

$$t_2(n(2, k)) = t_2 \text{ for every } k.$$

Apply this procedure inductively, we obtain a sequence $\mathcal{G} = \{t_j\}_{j=1}^\infty$ and subsequence $\{z_{n(j,k)}\}$ for each $j \in \mathbb{Z}^+$ such that

$$t_j(n(j, k)) = t_j \text{ for every } k.$$

Let z be a subsequential limit of $\{z_{n(k,k)}\}$. Then for every $j \in \mathbb{Z}^+$ and $l = 0, 1, \dots, m_j - 1$,

$$d(f^{s_j+l}(z), f^l(x_j)) \leq \limsup_{k \rightarrow \infty} d(f^{s_j+l}(z_{n(k,k)}), f^l(x_j)) \leq \varepsilon' < \varepsilon,$$

where

$$s_1 = 0 \text{ and } s_j = \sum_{i=1}^{j-1} (m_i + t_i - 1) \text{ for } j \geq 2.$$

So $(\mathcal{C}, \mathcal{G})$ is ε -shadowed by z and $\max \mathcal{G} \leq m$. \square

Initial idea of the proof of Theorem 1.1 comes from the following classical result.

Proposition 2.11 (cf. [3]). *Assume that (X, f) has specification property. Assume that $\varepsilon > 0$ and there is a subset E of X that is $(1, 3\varepsilon)$ -separated and $|E| = N \geq 2$. Then*

$$h(f) \geq \frac{\ln N}{M(\varepsilon)}.$$

Proof. Let $m = M(\varepsilon)$. For $n \in \mathbb{Z}^+$ and each $\xi = \{x_1(\xi), \dots, x_n(\xi)\} \in E^n$, let

$$\mathcal{C}_\xi = \{(x_j(\xi), 1) : j = 1, \dots, n\}$$

and $\mathcal{G}_n = \{m, m, \dots, m\}$. There is $z_\xi \in X$ that ε -shadows $(\mathcal{C}_\xi, \mathcal{G}_n)$. If $\xi \neq \xi'$ then there is $j \in \{1, \dots, n\}$ such that

$$d(x_j(\xi), x_j(\xi')) > 3\varepsilon$$

and hence

$$d(f^{s_j}(z_\xi), f^{s_j}(z_{\xi'})) \geq d(x_j(\xi), x_j(\xi')) - d(f^{s_j}(z_\xi), x_j(\xi)) - d(f^{s_j}(z_{\xi'}), x_j(\xi')) > \varepsilon.$$

This implies that

$$A_n = \{z_\xi : \xi \in E^n\}$$

is an (mn, ε) -separated set and hence $s(mn, \varepsilon) \geq |A_n| = N^n$. This yields that

$$h(f) \geq \limsup_{n \rightarrow \infty} \frac{\ln s(mn, \varepsilon)}{mn} \geq \frac{\ln N}{m}.$$

□

Corollary 2.12. *Assume that (X, f) has specification property and X is not a singleton. Then the followings hold.*

- (1) $h(f) > 0$.
- (2) $\lim_{\varepsilon \rightarrow 0} M(\varepsilon) = \infty$.
- (3) Denote the lower box dimension of X by

$$\dim X := \liminf_{\varepsilon \rightarrow 0} -\frac{\ln s(1, \varepsilon)}{\ln \varepsilon}.$$

Then

$$\liminf_{\varepsilon \rightarrow 0} -\frac{M(\varepsilon)}{\ln \varepsilon} \geq \frac{\dim X}{h(f)}.$$

3. POSITIVE ENTROPY

Recall that a point $x \in X$ is called *recurrent* if for every $\varepsilon > 0$ there is $n \in \mathbb{Z} \setminus \{0\}$, such that $d(f^n(x), x) < \varepsilon$. A point is called *non-recurrent* if it is not recurrent. Given a non-minimal system with gluing orbit property, to show that it has positive topological entropy, our idea is based on existence of two non-recurrent points such that the forward orbit of one point stays away from the other point, and vice versa.

Note that without gluing orbit property, a non-minimal system may have no non-recurrent points and a system with non-recurrent points may have zero topological entropy (cf. Example 6.3 and 6.4).

Lemma 3.1. *Assume that (X, f) is not minimal and has gluing orbit property. Then f has a non-recurrent point.*

Proof. As f is not minimal, there is a point whose orbit is not dense. We can find $x, y \in X$ and $\delta > 0$ such that

$$d(f^n(x), y) \geq \delta \text{ for every } n \in \mathbb{Z}.$$

Let $0 < \varepsilon < \frac{1}{3}\delta$. Assume that for every orbit sequence \mathcal{C} there is a gap \mathcal{G} with $\max \mathcal{G} \leq M(\varepsilon)$ such that $(\mathcal{C}, \mathcal{G})$ is ε -shadowed. Let $m = M(\varepsilon)$. For each $n \in \mathbb{Z}^+$, consider

$$\mathcal{C}_n = \{(f^{-(n-1)}(x), n), (y, 1), (x, n)\}.$$

There is $\mathcal{G}_n \in \{1, \dots, m\}^2$ such that $(\mathcal{C}_n, \mathcal{G}_n)$ is ε -shadowed by z'_n . There must be $\mathcal{G} = (t_1, t_2) \in \{1, \dots, m\}^2$ such that

$$\{n \in \mathbb{Z}^+ : \mathcal{G}_n = \mathcal{G}\} \text{ is infinite.}$$

We can find a subsequence $\{z'_{n_k}\}$ with $\mathcal{G}_{n_k} = \mathcal{G}$ for every k . Let $z_{n_k} = f^{n_k+t_1-1}(z'_{n_k})$ and z be a subsequential limit of $\{z_{n_k}\}$. Then

$$d(f^{-t_1-j}(z), f^{-j}(x)) \leq \limsup_{n_k \rightarrow \infty} d(f^{-t_1-j}(z_{n_k}), f^{-j}(x)) \leq \varepsilon \text{ for every } j \geq 0$$

and

$$d(f^{t_2+j}(z), f^j(x)) \leq \limsup_{n_k \rightarrow \infty} d(f^{t_2+j}(z_{n_k}), f^j(x)) \leq \varepsilon \text{ for every } j \geq 0.$$

This implies that

$$d(f^j(z), y) \geq \delta - \varepsilon > 2\varepsilon \text{ for every } j \leq -t_1 \text{ or } j \geq t_2.$$

But $d(z, y) < \varepsilon$. So

$$d(f^j(z), z) \geq d(f^j(z), y) - d(z, y) > \varepsilon \text{ for every } j \leq -t_1 \text{ or } j \geq t_2.$$

This also indicates that z is not periodic. So we have

$$\min\{d(f^j(z), z) : -t_1 < j < t_2, j \neq 0\} = \varepsilon' > 0.$$

Then z is a non-recurrent point as

$$d(f^j(z), z) \geq \min\{\varepsilon, \varepsilon'\} > 0 \text{ for every } j \in \mathbb{Z} \setminus \{0\}.$$

□

Lemma 3.2. *Assume that (X, f) is not minimal and has gluing orbit property. Then there are $x, y \in X$ and $\varepsilon > 0$ such that*

$$\begin{aligned} d(f^n(x), x) &\geq \varepsilon \text{ for any } n \in \mathbb{Z} \setminus \{0\}, \\ d(f^n(x), y) &\geq \varepsilon \text{ for any } n \geq 0, \\ d(f^n(y), x) &\geq \varepsilon \text{ for any } n \geq 0, \text{ and} \\ d(f^n(y), y) &\geq \varepsilon \text{ for any } n > 0. \end{aligned}$$

Proof. By Lemma 3.1, there is a non-recurrent point $x \in X$. Assume that

$$d(f^n(x), x) \geq \delta \text{ for every } n \in \mathbb{Z} \setminus \{0\}.$$

Let $\varepsilon_1 = \frac{1}{3}\delta$ and $m_1 = M(\varepsilon_1)$. For each n , there is $t_n \leq m_1$ such that

$$(\{(x, 1), (x, n)\}, \{t_n\})$$

is ε_1 -shadowed by y'_n . Let $y_n = f^{t_n}(y'_n)$. Then

$$d(f^j(y_n), f^j(x)) < \varepsilon_1 \text{ for } j = 0, 1, \dots, n-1.$$

As $t_n \in \{1, \dots, m_1\}$ for every n , there is t such that

$$\{t_n : t_n = t\} \text{ is infinite.}$$

We can find a subsequence $\{y_{n_k}\}$ such that

$$t_{n_k} = t \text{ for every } k.$$

Let y a subsequential limit of $\{y_{n_k}\}$. Then

$$d(f^j(y), f^j(x)) \leq \limsup_{n_k \rightarrow \infty} d(f^j(y_{n_k}), f^j(x)) \leq \varepsilon_1 \text{ for every } j \geq 0.$$

Moreover

$$d(f^{-t}(y), x) \leq \limsup_{n_k \rightarrow \infty} d(y'_{n_k}, x) \leq \varepsilon_1 < d(f^{-t}(x), x),$$

which guarantees that $y \neq x$. Let

$$\varepsilon := d(x, y) = d(f^0(x), y) = d(f^0(y), x) \leq \varepsilon_1.$$

Then for every $n > 0$,

$$\begin{aligned} d(f^n(x), y) &\geq d(f^n(x), x) - d(x, y) \geq \delta - \varepsilon \geq \varepsilon, \\ d(f^n(y), x) &\geq d(f^n(x), x) - d(f^n(x), f^n(y)) \geq \delta - \varepsilon_1 \geq \varepsilon, \\ d(f^n(y), y) &\geq d(f^n(x), x) - d(f^n(y), f^n(x)) - d(x, y) \geq \delta - \varepsilon_1 - \varepsilon \geq \varepsilon. \end{aligned}$$

□

Now we complete the proof of Theorem 1.1. Let $x, y \in X$ and $\varepsilon > 0$ be as in Lemma 3.2. Let $\varepsilon_2 = \frac{1}{3}\varepsilon$ and $m = M(\varepsilon_2)$. For each $\xi = \{x_k(\xi)\}_{k=1}^n \in \{x, y\}^n$, consider

$$\mathcal{C}_\xi = \{(x_k, m) : k = 1, \dots, n\}.$$

There is

$$\mathcal{G}_\xi = \{t_j(\xi) : j = 1, \dots, n-1\}$$

with $\max \mathcal{G}_\xi \leq m$ such that $(\mathcal{C}, \mathcal{G})$ is ε_2 -shadowed by $z_\xi \in X$. We claim that if $\xi \neq \xi'$ then there is $s < 2mn$ such that

$$d(f^s(z_\xi), f^s(z_{\xi'})) > \varepsilon_2.$$

If $\mathcal{G}_\xi = \mathcal{G}_{\xi'}$, then there is k such that $x_k(\xi) \neq x_k(\xi')$. For

$$s := \sum_{j=1}^{k-1} (m + t_j(\xi) - 1) \leq (2m-1)(k-1) < 2mn$$

we have

$$\begin{aligned} d(f^s(z_\xi), f^s(z_{\xi'})) &\geq d(x_k(\xi), x_k(\xi')) - d(f^s(z_\xi), x_k(\xi)) - d(f^s(z_{\xi'}), x_k(\xi')) \\ &> d(x, y) - 2\varepsilon_2 \geq \varepsilon_2. \end{aligned}$$

If $\mathcal{G}_\xi \neq \mathcal{G}_{\xi'}$, we may assume that there is k such that

$$t_j(\xi) = t_j(\xi') \text{ for } j < k \text{ and } t_k(\xi) > t_k(\xi').$$

Let $l = t_k(\xi) - t_k(\xi')$. Then $1 \leq l \leq m-1$. For

$$s := \sum_{j=1}^{k-1} (m + t_j(\xi) - 1) = \sum_{j=1}^{k-1} (m + t_j(\xi') - 1) + l \leq (2m-1)(k-1) < 2mn$$

we have

$$\begin{aligned} d(f^s(z_\xi), f^s(z_{\xi'})) &\geq d(x_k(\xi), f^l(x_k(\xi'))) - d(f^s(z_\xi), x_k(\xi)) - d(f^s(z_{\xi'}), f^l(x_k(\xi'))) \\ &> \min\{d(f^l(x), x), d(f^l(x), y), d(f^l(y), x), d(f^l(y), y)\} - 2\varepsilon_2 \\ &\geq \varepsilon_2. \end{aligned}$$

Above argument shows that

$$E = \{z_\xi : \xi \in \{x, y\}^n\}$$

is a $(2mn, \varepsilon_2)$ -separated subset of X that contains 2^n points. Hence

$$h(f) \geq \limsup_{n \rightarrow \infty} \frac{\ln s(2mn, \varepsilon_2)}{2mn} \geq \limsup_{n \rightarrow \infty} \frac{n \ln 2}{2mn} = \frac{\ln 2}{2m} > 0.$$

4. UNIQUE ERGODICITY AND GROWTH OF PERIODIC ORBITS

Theorem 4.1. *Assume that (X, f) is uniquely ergodic and has gluing orbit property. Then it is minimal.*

Proof. Assume that (X, f) is not minimal and it has gluing orbit property. There are $x, y \in X$ and $\delta > 0$ such that

$$d(f^n(x), y) \geq \delta \text{ for every } n \in \mathbb{Z}.$$

Let $0 < \varepsilon' < \varepsilon < \frac{1}{3}\delta$ and $m = M(\varepsilon')$. Let

$$\mathcal{C} = \{(x_j, 1) : j \in \mathbb{Z}^+ \text{ and } x_j = y \text{ for every } j\}.$$

By Lemma 2.10, there is $y_0 \in X$ that ε -shadows $(\mathcal{C}, \mathcal{G})$ for some \mathcal{G} with $\max \mathcal{G} \leq m$.

Take a continuous function $\varphi : X \rightarrow \mathbb{R}$ such that

$$\begin{aligned} \varphi(z) &= 1 \text{ for every } z \in \overline{B(y, \varepsilon)} \\ \varphi(z) &= 0 \text{ for every } z \notin B(y, 2\varepsilon) \\ 0 &< \varphi(z) < 1 \text{ otherwise.} \end{aligned}$$

We have

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x)) = 0.$$

But

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(y_0)) \geq \frac{1}{m}$$

as the orbit of y_0 enters $\overline{B(y, \varepsilon)}$ at least once in every m iterates. This implies that (X, f) is not uniquely ergodic. \square

Denote by $P_n(f)$ the set of periodic points of f with periods no more than n , and $p_n(f)$ the cardinality of $P_n(f)$. Consider

$$p(f) = \limsup_{n \rightarrow \infty} \frac{\ln p_n(f)}{n}.$$

A flow version of the following theorem is contained in [1].

Theorem 4.2. *If (X, f) has periodic gluing orbit property, then $h(f) \leq p(f)$.*

Proof. Assume that $h < h(f)$. There is $\varepsilon > 0$ and $N > 0$ such that $s(n, \varepsilon) > e^{nh}$ for every $n > N$. Let E be an (n, ε) -separated set with $|E| > e^{nh}$. Denote $m = M(\frac{\varepsilon}{2})$. For every $x \in E$, there is $t < m$ such that $\{(x, n), \emptyset\}$ is $\frac{\varepsilon}{2}$ -shadowed by a periodic point with period $n+t$. Hence every $(n, \frac{\varepsilon}{2})$ -ball around an element of E , which are disjoint with each other, contains an element of $P_{n+m}(f)$. This implies that

$$p_{n+m}(f) \geq |E| > e^{nh}.$$

It follows that

$$p(f) = \limsup_{n \rightarrow \infty} \frac{\ln p_n(f)}{n} \geq h.$$

The result follows as this holds for any $h < h(f)$. \square

Corollary 4.3. *Assume that (X, f) has periodic gluing orbit property and X does not consist of a single periodic orbit. Then*

$$0 < h(f) \leq p(f).$$

5. EQUICONTINUOUS SYSTEMS

Let (X, f) be an equicontinuous system. We shall show that minimality implies gluing orbit property. It is clear that gluing orbit property implies topological transitivity. For completeness, we present a proof that topological transitivity implies minimality. As every equicontinuous system has zero topological entropy, the fact that gluing orbit property implies minimality is also a corollary of Theorem 1.1.

We first prove a lemma that shows that the time needed for the pre-images of ε -balls to cover X is uniform. We remark that this lemma does not require equicontinuity.

Lemma 5.1. *Assume that (X, f) is minimal. Then for every $\varepsilon > 0$, there is $N \in \mathbb{Z}^+$ such that for every $x \in X$,*

$$\bigcup_{n=0}^N f^{-n}(B(x, \varepsilon)) = X.$$

Proof. Let $\varepsilon > 0$ and $x \in X$. As f is minimal, for every $y \in X$, there is $n \in \mathbb{Z}$ such that $f^n(y) \in B(x, \varepsilon)$. Equivalently, $y \in f^{-n}(B(x, \varepsilon))$. This implies that

$$X \subset \bigcup_{n=-\infty}^{\infty} f^{-n}(B(x, \varepsilon)).$$

As X is compact, there is $N_x = 2N'_x$ such that

$$X \subset \bigcup_{n=-N'_x}^{N'_x} f^{-n}(B(x, \varepsilon))$$

and hence

$$X = f^{-N'_x}(X) \subset \bigcup_{n=0}^{N_x} f^{-n}(B(x, \varepsilon)) \quad (1)$$

For every $y \in X$, denote

$$r(y) := \max\{d(f^n(y), x) : 0 \leq n \leq N_x \text{ such that } d(f^n(y), x) < \varepsilon.\}$$

By (1), we have $r(y) < \varepsilon$ for every $y \in X$. We claim that the function $r : X \rightarrow \mathbb{R}$ is upper semi-continuous.

Assume that $y \in X$ and $r(y) = d(f^{n_y}(y), x) < \varepsilon$. Then for every $\varepsilon' > 0$, there is $\delta > 0$ such that for every $z \in B(y, \delta)$,

$$d(f^{n_y}(z), f^{n_y}(y)) < \min\{\varepsilon', \varepsilon - r(y)\}.$$

Then

$$d(f^{n_y}(z), x) \leq d(f^{n_y}(y), x) + d(f^{n_y}(z), f^{n_y}(y)) < \varepsilon.$$

This implies that

$$r(z) \geq d(f^{n_y}(z), x) \geq d(f^{n_y}(y), x) - d(f^{n_y}(z), f^{n_y}(y)) > r(y) - \varepsilon'.$$

As r is upper semi-continuous and X is compact, r attains its maximum $R_x < \varepsilon$ on X . Let $\delta_x := \frac{\varepsilon - R_x}{2} > 0$. Then for every $x' \in B(x, \delta_x)$, we have

$$\begin{aligned} & \min\{d(f^n(y), x') : 0 \leq n \leq N_x\} \\ & \leq \min\{d(f^n(y), x) + d(x, x') : 0 \leq n \leq N_x\} \\ & = \min\{d(f^n(y), x) : 0 \leq n \leq N_x\} + d(x, x') \\ & \leq r(y) + \delta_0 \leq R_x + \delta_0 < \varepsilon \end{aligned}$$

for every $y \in X$. This implies that

$$X \subset \bigcup_{n=0}^{N_x} f^{-n}(B(x', \varepsilon)) \text{ for every } x' \in B(x, \delta_x).$$

Note that $\{B(x, \delta_x) : x \in X\}$ is an open cover of X . It has a finite subcover $\{B(x_j, \delta_{x_j}) : j = 1, \dots, k\}$. Let $N = \max\{N_{x_j} : j = 1, \dots, k\}$. Then for every $x \in X$, $x \in B(x_j, \delta_{x_j})$ for some j and hence

$$X \subset \bigcup_{n=0}^{N_{x_j}} f^{-n}(B(x, \varepsilon)) \subset \bigcup_{n=0}^N f^{-n}(B(x, \varepsilon)).$$

□

The proof of Theorem 1.3 is completed by Proposition 5.2 and Proposition 5.3.

Proposition 5.2. *A minimal equicontinuous system has gluing orbit property.*

Proof. Let $\varepsilon > 0$. By equicontinuity, there is $\delta > 0$ such that

$$d(f^n(x), f^n(y)) < \varepsilon \text{ whenever } d(x, y) < \delta. \quad (2)$$

By Lemma 5.1, there is M such that

$$\bigcup_{n=0}^M f^{-n}(B(x, \delta)) = X \text{ for every } x \in X.$$

Let $\mathcal{C} = \{(x_j, m_j) : j = 1, \dots, k\}$ be any orbit chain. We claim that there is a gap \mathcal{G} with $\max \mathcal{G} \leq M + 1$ such that $(\mathcal{C}, \mathcal{G})$ can be ε -shadowed by x_1 .

For each $j = 1, \dots, k - 1$, we have

$$\bigcup_{n=1}^{M+1} f^{-n}(B(x_{j+1}, \delta)) = f^{-1}\left(\bigcup_{n=0}^M f^{-n}(B(x_{j+1}, \delta))\right) = X \ni f^{s_j+m_j-1}(x_1),$$

where

$$s_1 = 0 \text{ and } s_j = \sum_{i=1}^{j-1} (m_i + t_i - 1) \text{ for } j = 2, \dots, k.$$

There is $t_j \in \mathbb{Z}^+$ such that

$$t_j \leq M + 1 \text{ and } f^{t_j}(f^{s_j+m_j-1}(x_1)) \in B(x_{j+1}, \delta).$$

By (2), this implies that

$$(f^{s_{j+1}+l}(x_1), f^l(x_{j+1})) < \varepsilon \text{ for every } l = 0, 1, \dots, m_j - 1.$$

Hence $(\mathcal{C}, \mathcal{G})$ is ε -shadowed by x_1 for $\mathcal{G} = \{t_j : j = 1, \dots, k - 1\}$. □

Proposition 5.3. *A topological transitive equicontinuous system is minimal.*

Proof. Let $x, y \in X$ and $\varepsilon > 0$. As f is equicontinuous, there is $\delta > 0$ such that

$$d(f^n(z), f^n(x)) < \frac{\varepsilon}{2} \text{ for every } z \in B(x, \delta).$$

As f is topologically transitive, there is $n \geq 0$ such that

$$B(x, \delta) \cap f^{-n}(B(y, \frac{\varepsilon}{2})) \neq \emptyset.$$

Take

$$z_0 \in B(x, \delta) \cap f^{-n}(B(y, \frac{\varepsilon}{2})).$$

Then

$$d(f^n(x), y) \leq d(f^n(x), f^n(z_0)) + d(f^n(z_0), y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This implies that the orbit of every $x \in X$ is dense, i.e. f is minimal. □

6. EXAMPLES

Example 6.1. In [2], it is shown that a topologically transitive subshift of finite type has gluing orbit property. Note that such a system has periodic points. As a corollary of Theorem 1.1, it has positive topological entropy if it does not consist of a single periodic orbit.

Example 6.2. As a corollary of Theorem 1.3, adding machines have gluing orbit property.

Example 6.3. A rational rotation is not minimal but every point is recurrent (cf. Lemma 3.1).

Example 6.4. Consider the map

$$f(x) = x^2 \pmod{1}$$

on the unit circle $[0, 1]/\sim$. It is not minimal and has non-recurrent points and zero topological entropy (cf. Theorem 1.1 and Lemma 3.2). It is also uniquely ergodic (cf. Theorem 4.1). It fails to satisfy the theorems as it does not have gluing orbit property.

Example 6.5. According to [11], there are C^∞ interval maps with periodic points of period 2^n for any $n \in \mathbb{Z}^+$ and zero topological entropy that are chaotic in the sense of Li-York. Theorem 1.1 implies that all such maps can not have gluing orbit property.

Example 6.6. The subshift on the closure of the orbit of an almost periodic point, as constructed in [7, 12.28] and [8], does not have gluing orbit property. The gap needed before shadowing an orbit segment of length L may be no less than L and hence neither uniform nor tempered. Such a system is minimal. It can have zero topological entropy and can also have positive topological entropy. So minimality itself does not imply gluing orbit property, no matter how much is the topological entropy (cf. Theorem 1.3).

7. THE SEMIFLOW CASE

In this section we give a proof of Theorem 1.1 in the semiflow case. Throughout this section, f^t is assumed to be a semiflow on X that is not minimal and has gluing orbit property. We first state the definition of gluing orbit property in this case and note the difference. Idea of the proof is similar to the homeomorphism case. There are two major technical differences. Non-recurrence is established after a time period and the orbit sequences for finding separated sets are more carefully designed.

Definition 7.1. A semiflow (X, f) is said to have *gluing orbit property* if for every $\varepsilon > 0$ there is $M(\varepsilon) > 0$ such that for any *orbit sequence*

$$\mathcal{C} = \{(x_j, m_j) \in X \times [0, \infty) : j = 1, \dots, k\},$$

there is a *gap*

$$\mathcal{G} = \{t_j \in [0, \infty) : j = 1, \dots, k-1\}$$

such that $\max \mathcal{G} \leq M(\varepsilon)$ and $(\mathcal{C}, \mathcal{G})$ can be ε -shadowed in the following sense: for every $j = 1, \dots, k$,

$$(f^{s_j+t}(z), f^t(x_j)) < \varepsilon \text{ for every } t \in [0, m_j],$$

where

$$s_1 = 0 \text{ and } s_j = \sum_{i=1}^{j-1} (m_i + t_i) \text{ for } j = 2, \dots, k.$$

Lemma 7.2. *There is $x_0 \in X$, $\varepsilon > 0$ and $\tau > 0$ such that*

$$d(f^t(x_0), x_0) > \varepsilon \text{ for any } t \geq \tau.$$

Proof. As f is not minimal, there is a point whose orbit is not dense. We can find $x, y \in X$ and $\delta > 0$ such that

$$d(f^t(x), y) \geq \delta \text{ for every } t \geq 0.$$

Let $0 < \varepsilon < \frac{1}{3}\delta$ and $m = M(\varepsilon)$. For each $n \in \mathbb{Z}^+$, consider

$$\mathcal{C}_n = \{(y, 0), (x, n)\}.$$

There is $\tau_n \in [0, m]$ such that $(\mathcal{C}_n, \{\tau_n\})$ is ε -shadowed by z_n . There must be a subsequence $\{\tau_{n_k}\}$ that converges to $\tau \in [0, m]$. Let x_0 be a subsequential limit of $\{z_{n_k}\}$. Then

$$d(f^{\tau+t}(x_0), f^t(x)) \leq \limsup_{n_k \rightarrow \infty} d(f^{\tau_{n_k}+t}(z_{n_k}), f^t(x)) \leq \varepsilon \text{ for every } t \geq 0.$$

and

$$d(f^{\tau+t}(x_0), y) \geq d(f^t(x), y) - d(f^{\tau+t}(x_0), f^t(x)) > 2\varepsilon \text{ for every } t \geq 0.$$

Note that $d(x_0, y) < \varepsilon$. So

$$d(f^{\tau+t}(x_0), x_0) \geq d(f^{\tau+t}(x_0), y) - d(x_0, y) > \varepsilon \text{ for every } t \geq 0.$$

□

Lemma 7.3. *There are $x, y \in X$, $\varepsilon > 0$ and $T > 0$ such that*

$$\begin{aligned} d(f^t(x), x) &\geq \varepsilon \text{ for any } t \geq T, \\ d(f^t(y), x) &\geq \varepsilon \text{ for any } t \geq T, \\ d(f^t(y), y) &\geq \varepsilon \text{ for any } t \geq T, \text{ and} \\ d(f^t(x), y) &\geq \varepsilon \text{ for any } t \geq 0. \end{aligned}$$

Proof. By Lemma 7.2, there is $x \in X$, $\delta > 0$ and $t_0 > 0$ such that

$$d(f^t(x), x) \geq \delta \text{ for every } t \geq t_0.$$

Let $\varepsilon_1 = \frac{1}{3}\delta$ and $m_1 = M(\varepsilon_1)$. For each n , there is $\tau_n \in [0, m_1]$ such that

$$(\{(x, t_0), (x, n)\}, \{\tau_n\})$$

is ε_1 -shadowed by y_n . There is a subsequence $\{\tau_{n_k}\}$ that converges to $\tau \in [0, m_1]$. Let y be a subsequential limit of $\{y_{n_k}\}$. Then

$$d(f^{t_0+\tau+t}(y), f^t(x)) \leq \varepsilon_1 \text{ for every } t \geq 0. \quad (3)$$

This yields that for every $t \geq 0$, we have

$$\begin{aligned} d(f^{2t_0+\tau+t}(y), x) &\geq d(f^{t_0+t}(x), x) - d(f^{2t_0+\tau+t}(y), f^{t_0+t}(x)) \geq \delta - \varepsilon_1 = 2\varepsilon_1, \\ d(f^{2t_0+\tau+t}(y), y) &\geq d(f^{2t_0+\tau+t}(y), x) - d(x, y) \geq 2\varepsilon_1 - \varepsilon_1 = \varepsilon_1, \text{ and} \\ d(f^{t_0+\tau+t}(x), y) &\geq d(f^{t_0+\tau+t}(x), x) - d(x, y) \geq \delta - \varepsilon_1 = 2\varepsilon_1. \end{aligned} \quad (4)$$

Equation (3) also guarantees that $f^t(x) \neq y$ for any $t \geq 0$, as

$$d(f^{t_0+\tau}(y), x) \leq \varepsilon_1 < \delta \leq d(f^{t_0+\tau+t}(x), x).$$

Let

$$\varepsilon := \min\{d(f^t(x), y) : 0 \leq t \leq t_0 + \tau\}.$$

Then $\varepsilon \in (0, \varepsilon_1]$. Together with (4) we have

$$d(f^t(x), y) \geq \varepsilon \text{ for every } t \geq 0.$$

The lemma holds for x, y, ε and $T = 2t_0 + \tau$. \square

Proposition 7.4. *(X, f) has positive topological entropy.*

Proof. Let $x, y \in X$, $\varepsilon > 0$ and $T > 0$ be as in Lemma 7.3. Let $0 < \varepsilon_2 < \frac{1}{3}\varepsilon$ and $m = M(\varepsilon_2)$. Let

$$Q_1 = \{(y, 2T + 3m)\} \text{ and } Q_2 = \{(x, T + m), (x, T + m)\}.$$

For each $\xi = \{\omega_k(\xi)\}_{k=1}^n \in \{1, 2\}^n$, consider

$$\mathcal{C}_\xi = \{Q_{\omega_k(\xi)} : k = 1, \dots, n\} = \{(x_j(\xi), m_j(\xi)) : j = 1, \dots, n(\xi)\},$$

where

$$n(\xi) = \sum_{k=1}^n \omega_k(\xi).$$

There is $\mathcal{G}_\xi = \{t_j(\xi) : j = 1, \dots, n(\xi) - 1\}$ with $\max \mathcal{G}_\xi \leq m$ such that $(\mathcal{C}, \mathcal{G})$ is ε_2 -shadowed by $z_\xi \in X$. For each ξ , denote

$$s_1(\xi) = 0 \text{ and } s_j(\xi) = \sum_{i=1}^{j-1} (m_i(\xi) + t_i(\xi)) \text{ for } j = 2, \dots, n(\xi).$$

Then

$$s_{n(\xi)}(\xi) \leq (2T + 4m)n \text{ for every } \xi \in \{1, 2\}^n$$

We claim that if $\xi \neq \xi'$ then there is $s \leq (2T + 4m)n$ such that

$$d(f^s(z_\xi), f^s(z_{\xi'})) > \varepsilon_2.$$

Assume that $x_j(\xi) = x_j(\xi')$ for $j = 1, \dots, l-1$, $x_l(\xi) = y$ and $x_l(\xi') = x$. Our discussion can be split into the following cases.

Case 1. $l = 1$. Then

$$d(z_\xi, z_{\xi'}) \geq d(x, y) - d(z_\xi, x) - d(z_{\xi'}, y) > \varepsilon_2.$$

Case 2. $l \geq 2$ and there is $k < l$ with $|s_k(\xi) - s_k(\xi')| \geq T$. Let k be the smallest index satisfying the inequality. Then

$$|s_{k-1}(\xi) - s_{k-1}(\xi')| < T.$$

Then

$$\begin{aligned} r &:= |s_k(\xi) - s_k(\xi')| \\ &= |(s_{k-1}(\xi) + m_{k-1}(\xi) + t_{k-1}(\xi)) - (s_{k-1}(\xi') + m_{k-1}(\xi') + t_{k-1}(\xi'))| \\ &= |(s_{k-1}(\xi) + t_{k-1}(\xi)) - (s_{k-1}(\xi') + t_{k-1}(\xi'))| \\ &\leq |(s_{k-1}(\xi) - s_{k-1}(\xi'))| + |t_{k-1}(\xi) - t_{k-1}(\xi')| \\ &< T + m. \end{aligned}$$

Assume that $s_k(\xi) < s_k(\xi')$. Then

$$\begin{aligned} & d(f^{s_k(\xi')} (z_\xi), f^{s_k(\xi')} (z_{\xi'})) \\ & \geq d(f^r(x_k(\xi)), x_k(\xi')) - d(f^{s_k(\xi)+r}(z_\xi), f^r(x_k(\xi))) - d(f^{s_k(\xi')} (z_{\xi'}), x_k(\xi')) \\ & > \varepsilon_2. \end{aligned}$$

Case 3. $l \geq 2$ and $|s_{l-1}(\xi) - s_{l-1}(\xi')| < T$. A similar argument shows that

$$r := |s_l(\xi) - s_l(\xi')| < T + m.$$

If $s_l(\xi) \geq s_l(\xi')$, then

$$\begin{aligned} & d(f^{s_l(\xi)} (z_\xi), f^{s_l(\xi)} (z_{\xi'})) \\ & \geq d(y, f^r(x)) - d(f^{s_l(\xi)} (z_\xi), y) - d(f^{s_l(\xi')+r}(z_{\xi'}), f^r(x)) \\ & > \varepsilon_2. \end{aligned}$$

If $s_l(\xi) < s_l(\xi')$, then

$$r_1 := s_{l+1}(\xi') - s_l(\xi) = r + (T + m) + t_l(\xi') \in (T + m, 2T + 3m)$$

Note that $x_{l+1}(\xi') = x$. We have

$$\begin{aligned} & d(f^{s_{l+1}(\xi')} (z_\xi), f^{s_{l+1}(\xi')} (z_{\xi'})) \\ & \geq d(f^{r_1}(y), x) - d(f^{s_l(\xi)+r_1}(z_\xi), f^{r_1}(y)) - d(f^{s_{l+1}(\xi')} (z_{\xi'}), x) \\ & > \varepsilon_2. \end{aligned}$$

Above argument shows that

$$E = \{z_\xi : \xi \in \{1, 2\}^n\}$$

is a $((2T + 4m)n, \varepsilon_2)$ -separated subset of X that contains 2^n points. Hence

$$h(f) \geq \limsup_{n \rightarrow \infty} \frac{\ln s((2T + 4m)n, \varepsilon_2)}{(2T + 4m)n} \geq \limsup_{n \rightarrow \infty} \frac{n \ln 2}{(2T + 4m)n} = \frac{\ln 2}{2T + 4m} > 0.$$

□

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