

Non-integer valued winding numbers and a generalized Residue Theorem

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Abstract

We define a generalization of the winding number of a piecewise C^1 cycle in the complex plane which has a geometric meaning also for points which lie *on* the cycle. The computation of this winding number relies on the Cauchy principal value, but is also possible in a real version via an integral with bounded integrand. The new winding number allows to establish a generalized residue theorem which covers also the situation where singularities lie on the cycle. This residue theorem can be used to calculate the value of improper integrals for which the standard technique with the classical residue theorem does not apply.

Key words: Cauchy principal value, winding number, residue theorem

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1 Introduction

One of the most prominent tools in complex analysis is Cauchy's Residue Theorem. To state the classical version of this theorem (see, e.g., [1] or [8, Theorem 1, p. 75])

we briefly recall the following notions: A *chain* is a finite formal linear combination

$$\Gamma = \sum_{\ell=1}^k m_{\ell} \gamma_{\ell}, \quad m_{\ell} \in \mathbb{Z},$$

of continuous curves $\gamma_{\ell} : [0, 1] \rightarrow \mathbb{C}$. A *cycle* C is a chain, where every point $a \in \mathbb{C}$ is, counted with the corresponding multiplicity m_{ℓ} , as often a starting point of a curve γ_{ℓ} as it is an endpoint. A cycle C is *null-homologous* in $D \subset \mathbb{C}$, if its winding number for all points in $\mathbb{C} \setminus D$ vanishes. Equivalently, C is null-homologous in D , if it can be written as a linear combination of closed curves which are contractible in D . Then the residue theorem can be expressed as follows:

Theorem (Classical Residue Theorem). *Let $U \subset \mathbb{C}$ be an open set and $S \subset U$ be a set without accumulation points in U such that $f : U \setminus S \rightarrow \mathbb{C}$ is holomorphic. Furthermore, let C be a null-homologous cycle in $U \setminus S$. Then there holds*

$$\frac{1}{2\pi i} \oint_C f(z) dz = \sum_{s \in S} n_s(C) \operatorname{res}_s f(z), \quad (1.1)$$

where $n_s(C)$ denotes the winding number of C with respect to s .

Henrici considers in [4, Theorem 4.8f] a version of the residue theorem where C is the boundary of a semicircle in the upper half-plane with diameter $[-R, R]$, and where f is allowed to have poles on $(-R, R)$ which involve odd powers only. The result is basically a version of the classical formula (1.1), but with winding number $\frac{1}{2}$ for the singularities on the real axis, and where the integral on the left hand side of (1.1) is interpreted as a Cauchy Principal Value. Another very recent version of the residue theorem, where poles of order 1 on the piecewise C^1 boundary curve γ of an open set are allowed, is discussed in [5, Theorem 1]. There, if a pole is sitting on a corner of γ , the winding number is replaced by the angle formed by γ in this point, divided by 2π . In [6] a version of the residue theorem for functions f with finitely many poles is presented where singularities in points z_0 on a closed curve Γ are allowed provided $|f(z)| = O(|z - z_0|^{\mu})$ near z_0 , with $-1 < \mu \leq 0$. Further versions of generalizations of the residue theorem are described in [10] and [11]: there, versions for unbounded multiply connected regions of the second class are applied to higher-order singular integrals and transcendental singular integrals.

In the present article, we introduce a generalized, non-integer winding number for piecewise C^1 cycles C , and a general version of the residue theorem which covers all

cases of singularities on C . We will assume throughout the article that all curves are continuous. In particular, a piecewise C^1 curve is a continuous curve which is piecewise C^1 . Recall that a closed piecewise C^1 immersion $\Lambda : [a, b] \rightarrow \mathbb{C}$ is a closed curve such that there is a partition $a = a_0 < a_1 < \dots < a_n = b$ such that $\Lambda|_{[a_k, a_{k+1}]}$ is of class C^1 and such that $\dot{\Lambda}|_{[a_k, a_{k+1}]} \neq 0$ for all $k = 0, \dots, n-1$. If $\dot{\Lambda}|_{[a_k, a_{k+1}]}$ is furthermore a Lipschitz function for all $k = 0, \dots, n-1$, then Λ is called a closed piecewise $C^{1,1}$ immersion.

2 A Generalized Winding Number

The aim of this section is to generalize the winding number to piecewise C^1 cycles with respect to points sitting on the cycle itself.

The usual standard situation is the following: The winding number of a closed piecewise C^1 curve $\gamma : [a, b] \rightarrow \mathbb{C} \setminus \{0\}$ around $z = 0$ is given by

$$n_0(\gamma) = \frac{1}{2\pi i} \oint_{\gamma} \frac{dz}{z} \in \mathbb{Z},$$

see for example [8, p. 70] or [2, p. 75]. More generally, one can replace the curve γ by a piecewise C^1 cycle $C = \sum_{\ell=1}^k m_{\ell} \gamma_{\ell}$. An integral over the cycle is then

$$\oint_C f(z) dz := \sum_{\ell=1}^k m_{\ell} \oint_{\gamma_{\ell}} f(z) dz.$$

In order to make sense of the winding number also for points *on* the curve, we use the Cauchy principal value:

Definition 2.1. The winding number of a piecewise C^1 cycle $C : [a, b] \rightarrow \mathbb{C}$ with respect to $z_0 \in C$ is

$$n_{z_0}(C) := \text{PV} \frac{1}{2\pi i} \oint_C \frac{dz}{z - z_0} = \lim_{\varepsilon \searrow 0} \frac{1}{2\pi i} \int_{|C - z_0| > \varepsilon} \frac{dz}{z - z_0}. \quad (2.1)$$

It is not a priori clear whether this limit exists and what its geometric meaning is. So, we start by looking at the following model case: Using the Cauchy principal

value we can easily compute the winding number with respect to $z = 0$ of the *model sector-curve* $\gamma = \gamma_1 + \gamma_2 + \gamma_3$, where

$$\begin{aligned}\gamma_1 &: [0, r] \rightarrow \mathbb{C}, \quad t \mapsto t, \quad \gamma_1'(t) = 1 \\ \gamma_2 &: [0, \alpha] \rightarrow \mathbb{C}, \quad t \mapsto re^{it}, \quad \gamma_2'(t) = rie^{it} \\ \gamma_3 &: [0, r] \rightarrow \mathbb{C}, \quad t \mapsto (r-t)e^{i\alpha}, \quad \gamma_3'(t) = -e^{i\alpha}\end{aligned}$$

for $\alpha \in [0, 2\pi]$ (see Figure 1). Since

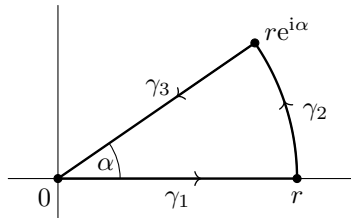


Figure 1: The model sector-curve $\gamma = \gamma_1 + \gamma_2 + \gamma_3$.

$$\begin{aligned}\text{PV} \oint_{\gamma} \frac{dz}{z} &= \lim_{\varepsilon \searrow 0} \left(\int_{\varepsilon}^r \frac{dt}{t} + \int_0^{\alpha} \frac{rie^{it}}{re^{it}} dt + \int_0^{r-\varepsilon} \frac{-e^{i\alpha}}{(r-t)e^{i\alpha}} dt \right) = \\ &= \lim_{\varepsilon \searrow 0} (\ln r - \ln \varepsilon + i\alpha + \ln \varepsilon - \ln r) = i\alpha,\end{aligned}$$

we obtain

$$n_0(\gamma) = \text{PV} \frac{1}{2\pi i} \oint_{\gamma} \frac{dz}{z} = \frac{\alpha}{2\pi}$$

with a meaningful geometrical interpretation.

Consider now a closed piecewise C^1 immersion $\Gamma : [a, b] \rightarrow \mathbb{C}$ starting and ending in 0 but such that $\Gamma(t) \neq 0$ for all $t \neq a, b$ and such that the (positively oriented) angle between $\lim_{t \searrow a} \dot{\Gamma}(t)$ and $-\lim_{t \nearrow b} \dot{\Gamma}(t)$ equals $\alpha \in [0, 2\pi]$. By a suitable rotation we may assume, without loss of generality, that $\lim_{t \searrow a} \dot{\Gamma}(t)$ is a positive real number (see Figure 2). We assume that Γ is homotopic to a model sector-curve γ with the same angle α in the following sense: There is a continuous function $H : [a, b] \times [0, 1] \rightarrow \mathbb{C}$ such that

$$\begin{aligned}H(t, 0) &= \Gamma(t) && \text{for all } t \in [a, b], \\ H(t, 1) &= \gamma(t) && \text{for all } t \in [a, b], \\ 0 = H(a, s) &= H(b, s) && \text{for all } s \in [0, 1], \\ H(t, s) &\neq 0 && \text{for all } t \in (a, b) \text{ and all } s \in [0, 1].\end{aligned}\tag{2.2}$$

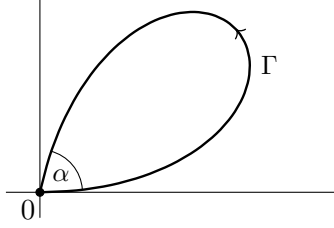


Figure 2: Winding number with respect to the origin sitting on the curve Γ .

Then we claim that

$$\lim_{\varepsilon \searrow 0} \int_{|\Gamma| > \varepsilon} \frac{dz}{z} = \lim_{\varepsilon \searrow 0} \int_{|\gamma| > \varepsilon} \frac{dz}{z}. \quad (2.3)$$

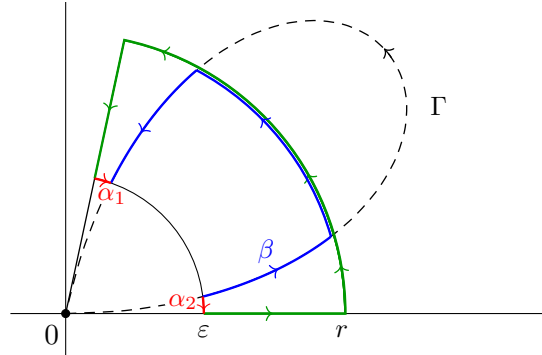


Figure 3: The curves Γ and γ .

For $0 < \varepsilon < r$ small enough we have (see Figure 3 for the definition of β)

$$\int_{|\Gamma| > \varepsilon} \frac{dz}{z} = \int_{\beta} \frac{dz}{z}$$

by Cauchy's integral theorem, and hence

$$\begin{aligned} \int_{|\Gamma| > \varepsilon} \frac{dz}{z} - \int_{|\gamma| > \varepsilon} \frac{dz}{z} &= \int_{\beta} \frac{dz}{z} - \int_{|\gamma| > \varepsilon} \frac{dz}{z} = \\ &= \int_{\beta} \frac{dz}{z} - \int_{|\gamma| > \varepsilon} \frac{dz}{z} + \int_{(|\gamma| > \varepsilon) + \alpha_1 - \beta + \alpha_2} \frac{dz}{z} = \\ &= \int_{\alpha_1 + \alpha_2} \frac{dz}{z}. \end{aligned}$$

Since

$$\left| \int_{\alpha_1 + \alpha_2} \frac{dz}{z} \right| \leq \frac{1}{\varepsilon} \underbrace{\text{Length}(\alpha_1 + \alpha_2)}_{=o(\varepsilon)} \rightarrow 0 \text{ for } \varepsilon \searrow 0,$$

the claim (2.3) follows. Thus we get the geometrically reasonable result that the winding number of the curve Γ with respect to $z = 0$ is, as we have just seen, the angle α divided by 2π :

$$n_0(\Gamma) = \text{PV} \frac{1}{2\pi i} \oint_{\Gamma} \frac{dz}{z} = \frac{\alpha}{2\pi}. \quad (2.4)$$

Next, we consider an closed piecewise C^1 immersion $\Lambda : [a, b] \rightarrow \mathbb{C}$ with one zero x_1 ,

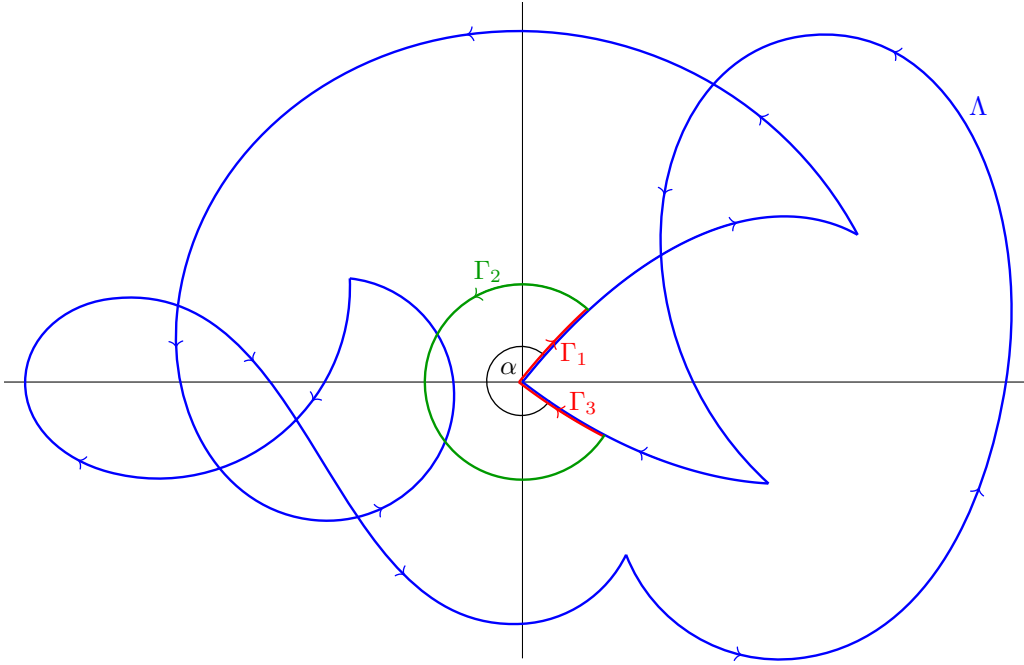


Figure 4: Winding number of Λ .

and a positively oriented angle $\alpha \in [0, 2\pi]$ between $\lim_{t \searrow x_1} \dot{\Lambda}(t)$ and $-\lim_{t \nearrow x_1} \dot{\Lambda}(t)$. Let $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$ be a closed piecewise C^1 curve which coincides with Λ in a neighborhood of x_1 and which is homotopic in the sense of (2.2) to a model sector-curve with the same angle α (see Figure 4). Then, we decompose Λ by $\Lambda = \tilde{\Lambda} + \Gamma$.

By (2.4) we get

$$n_0(\Lambda) = \text{PV} \frac{1}{2\pi i} \oint_{\Lambda} \frac{dz}{z} = n_0(\tilde{\Lambda}) + \frac{\alpha}{2\pi}. \quad (2.5)$$

Finally, if Λ has more than one zero, we obtain in the same way the following proposition:

Proposition 2.2. *Let $\Lambda : [a, b] \rightarrow \mathbb{C}$ be a closed piecewise C^1 immersion and $z_0 \in \mathbb{C}$. Then there exist at most finitely many points $x_1, \dots, x_n \in [a, b]$ such that $\Lambda(x_\ell) = z_0$. Consider a decomposition $\Lambda = \tilde{\Lambda} + \Gamma_1 + \dots + \Gamma_n$, where $\tilde{\Lambda}$ coincides with Λ outside of small neighborhoods of the points x_ℓ and avoids the point z_0 by driving around it on small circular arcs in clockwise direction. The closed curves Γ_ℓ are homotopic in the sense of (2.2) to a model sector-curve with oriented angle α_ℓ between $\lim_{t \searrow x_\ell} \dot{\Lambda}(t)$ and $-\lim_{t \nearrow x_\ell} \dot{\Lambda}(t)$ (see Figure 5). Then, the winding number of Λ with respect to z_0 is*

$$n_{z_0}(\Lambda) = \text{PV} \frac{1}{2\pi i} \oint_{\Lambda} \frac{dz}{z - z_0} = n_{z_0}(\tilde{\Lambda}) + \sum_{\ell=1}^n \frac{\alpha_\ell}{2\pi}.$$

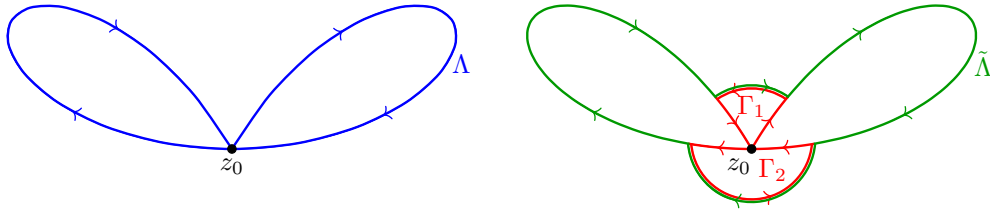


Figure 5: Decomposition of $\Lambda = \tilde{\Lambda} + \Gamma_1 + \Gamma_2$.

Proof. First we show that for only finitely many points x_ℓ , we have $\Lambda(x_\ell) = z_0$. It suffices to consider a C^1 curve $\Lambda : [a, b] \rightarrow \mathbb{R}^2$ parametrized by arc length, and $z_0 = (0, 0)$. Assume by contradiction that Λ has infinitely many zeros x_ℓ . Then there is a subsequence, again denoted by x_ℓ , which converges to a point $x \in [a, b]$, and we may assume, that x_ℓ is increasing. Then, by Rolle's Theorem, since $\Lambda_1(x_\ell) = \Lambda_2(x_\ell) = 0$, there are points $u_\ell, v_\ell \in (x_\ell, x_{\ell+1})$ such that $\Lambda'_1(u_\ell) = \Lambda'_2(v_\ell) = 0$. But then, $\Lambda'_1(v_\ell) = \Lambda'_2(u_\ell) = 1$. Hence, Λ' cannot be continuous.

For the rest of the proof, observe, that $\tilde{\Lambda}$ avoids the point z_0 . Thus, we have

$$\begin{aligned} n_{z_0}(\Lambda) &= \text{PV} \frac{1}{2\pi i} \oint_{\Lambda} \frac{dz}{z - z_0} = \\ &= \frac{1}{2\pi i} \oint_{\tilde{\Lambda}} \frac{dz}{z - z_0} + \sum_{\ell=1}^n \text{PV} \frac{1}{2\pi i} \oint_{\Gamma_{\ell}} \frac{dz}{z - z_0} = \\ &= n_{z_0}(\tilde{\Lambda}) + \sum_{\ell=1}^n \frac{\alpha_{\ell}}{2\pi}, \end{aligned}$$

where we have used (2.4) in the last step. \square

Proposition 2.2 generalizes immediately from curves to cycles. The Definition 2.1 of a generalized winding number turns out to be useful as it allows to generalize the residue theorem (see Theorem 3.3 below). But before we turn our attention this subject, let us briefly reformulate the formula (2.1) for the generalized winding number as an integral in the real plane. Interestingly, while the winding number as a complex integral requires an interpretation as a principal value, the real counterpart turns out to have a bounded integrand.

If $\Lambda = x + iy : [a, b] \rightarrow \mathbb{C}$ is a closed piecewise C^1 curve, then dz/z decomposes as

$$\frac{dz}{z} = \frac{x\dot{x} + y\dot{y}}{x^2 + y^2} dt + i \frac{x\dot{y} - y\dot{x}}{x^2 + y^2} dt.$$

The considerations above imply that if Λ is a closed piecewise C^1 immersion, then

$$\text{PV} \oint_{\Lambda} \frac{x dx + y dy}{x^2 + y^2} = 0,$$

hence only the imaginary part of dz/z is relevant for computing $n_0(\Lambda)$. We have the following proposition regarding its regularity:

Proposition 2.3. *Let $\Lambda = x + iy : [a, b] \rightarrow \mathbb{C}$ be a closed piecewise $C^{1,1}$ immersion. Then*

$$n_0(\Lambda) = \frac{1}{2\pi} \int_a^b \frac{x(t)\dot{y}(t) - y(t)\dot{x}(t)}{x(t)^2 + y(t)^2} dt$$

and the corresponding integrand is bounded. If Λ is C^2 in a neighbourhood of a point $\tilde{t} \in (a, b)$ with $\Lambda(\tilde{t}) = 0$, then

$$\lim_{t \rightarrow \tilde{t}} \frac{x(t)\dot{y}(t) - y(t)\dot{x}(t)}{x(t)^2 + y(t)^2} = \frac{1}{2} k_{\Lambda}(\tilde{t}) |\dot{\Lambda}(\tilde{t})|,$$

where

$$k_\Lambda(\tilde{t}) = \frac{\dot{x}(\tilde{t})\ddot{y}(\tilde{t}) - \dot{y}(\tilde{t})\ddot{x}(\tilde{t})}{(\dot{x}(\tilde{t})^2 + \dot{y}(\tilde{t})^2)^{3/2}}$$

is the signed curvature of Λ in \tilde{t} .

Proof. Let Λ be a closed piecewise $C^{1,1}$ immersion. If Λ avoids the origin, then the integrand is obviously bounded. On the other hand, as in the proof of Proposition 2.2, it follows that Λ can have at most finitely many zeros, say $a < t_0 < t_1 < \dots < t_n < b$. It suffices to concentrate on one of the zeros t_ℓ of Λ , and to simplify the notation we assume $\tilde{t} = 0$ for the rest of the proof. We first show that the integrand is bounded provided Λ is $C^{1,1}$ in a neighbourhood $U = (-\varepsilon, \varepsilon)$ of 0. In this case, \dot{x} and \dot{y} are Lipschitz functions on U and

$$\begin{aligned} |x(t)\dot{y}(t) - y(t)\dot{x}(t)| &= \left| \int_0^t \dot{x}(s) ds \dot{y}(t) - \int_0^t \dot{y}(s) ds \dot{x}(t) \right| \\ &\leq \int_0^t |(\dot{x}(s) - \dot{x}(t))\dot{y}(t) + \dot{x}(t)(\dot{y}(t) - \dot{y}(s))| ds \\ &\leq C \int_0^t (|s - t|\dot{y}(t) + |\dot{x}(t)||s - t|) ds = O(t^2). \end{aligned} \quad (2.6)$$

On the other hand

$$x(t)^2 + y(t)^2 = (t\dot{x}(0) + o(t))^2 + (t\dot{y}(0) + o(t))^2 = t^2(\dot{x}(0)^2 + \dot{y}(0)^2) + o(t^2). \quad (2.7)$$

Together, (2.6) and (2.7), imply that the integrand in Proposition 2.3 is bounded in U .

If Λ is only $C^{1,1}$ on $U^- = (-\varepsilon, 0]$ and $U^+ = [0, \varepsilon)$, the claim follows in the same way by considering the unilateral intervals left and right of the zero.

Now we assume that Λ is C^2 in a neighbourhood $U = (-\varepsilon, \varepsilon)$ of the zero $\tilde{t} = 0$. It remains to show that the limit

$$\lim_{t \rightarrow 0} \frac{x(t)\dot{y}(t) - y(t)\dot{x}(t)}{x(t)^2 + y(t)^2}$$

has the geometrical interpretation stated in the proposition. In fact, if $\Lambda(0) = 0$ we find

$$\begin{aligned} x(t)\dot{y}(t) &= \left(t\dot{x}(0) + \frac{t^2}{2}\ddot{x}(0) + o(t^2) \right) (\dot{y}(0) + t\ddot{y}(0) + o(t)) \\ &= t\dot{x}(0)\dot{y}(0) + t^2 \left(\dot{x}(0)\ddot{y}(0) + \frac{1}{2}\ddot{x}(0)\dot{y}(0) \right) + o(t^2) \end{aligned}$$

and hence

$$\begin{aligned} x(t)\dot{y}(t) - \dot{x}(t)y(t) &= t^2 \left(\dot{x}(0)\ddot{y}(0) - \dot{y}(0)\ddot{x}(0) + \frac{1}{2}(\ddot{x}(0)\dot{y}(0) - \ddot{y}(0)\dot{x}(0)) \right) + o(t^2) \\ &= \frac{t^2}{2} \left(\dot{x}(0)\ddot{y}(0) - \dot{y}(0)\ddot{x}(0) \right) + o(t^2). \end{aligned} \tag{2.8}$$

On the other hand

$$\begin{aligned} x^2(t) + y^2(t) &= \left(t\dot{x}(0) + \frac{t^2}{2}\ddot{x}(0) + o(t^2) \right)^2 + \left(t\dot{y}(0) + \frac{t^2}{2}\ddot{y}(0) + o(t^2) \right)^2 \\ &= t^2(\dot{x}(0)^2 + \dot{y}(0)^2) + t^3(\dot{x}(0)\ddot{x}(0) + \dot{y}(0)\ddot{y}(0)) + o(t^3). \end{aligned} \tag{2.9}$$

From (2.8) and (2.9) we deduce

$$\lim_{t \rightarrow \tilde{t}} \frac{x(t)\dot{y}(t) - y(t)\dot{x}(t)}{x(t)^2 + y(t)^2} = \frac{1}{2}k_\Lambda(\tilde{t})|\dot{\Lambda}(\tilde{t})|.$$

□

It is worth noticing, that Proposition 2.3 is more than just a technical remark. We will see in Section 3.1 an application of the observation that the imaginary part of the integrand is bounded.

The geometrical meaning of the winding number can be used to characterize the topological phases in one-dimensional chiral non-Hermitian systems. Chiral symmetry ensures that the winding number of Hermitian systems are integers, but non-Hermitian systems can take half integer values: see [9] for the corresponding physical interpretation of Proposition 2.3.

Example 2.4. Consider the curve $\Lambda : [0, 2\pi] \rightarrow \mathbb{C}$ given by

$$\Lambda(t) = x(t) + iy(t) := \cos(t) + \cos(2t) + i\sin(2t)$$

which passes through the origin at $t = \pi$ (see Figure 6). According to Proposition 2.3,

$$n_0(\Lambda) = \frac{1}{2\pi} \int_a^b \frac{x(t)\dot{y}(t) - y(t)\dot{x}(t)}{x(t)^2 + y(t)^2} dt = \frac{3}{2}$$

and the corresponding integrand is continuous.

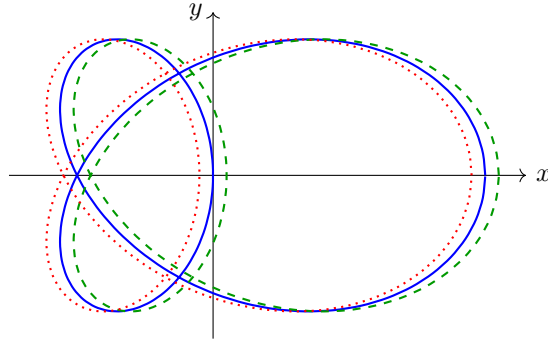


Figure 6: The (solid) curve from Example 2.4 has winding number $\frac{3}{2}$ with respect to the origin, and the dashed and dotted one have winding number 2 and 1 respectively.

3 A Generalized Residue Theorem

Let $U \subset \mathbb{C}$ be an open neighborhood of zero and let f be a holomorphic function on $U \setminus \{0\}$. Then there exists a Laurent series which represents f in a punctured neighborhood $\{z \in \mathbb{C} : 0 < |z| < R\}$ of zero:

$$f(z) = \underbrace{\dots + \frac{a_{-1}}{z}}_{=g(z)} + \underbrace{a_0 + a_1 z + \dots}_{=h(z)}$$

For a closed piecewise C^1 curve γ with $|\gamma| < R$, we have

$$\text{PV} \oint_{\gamma} f(z) dz = \text{PV} \oint_{\gamma} (g(z) + h(z)) dz = \text{PV} \oint_{\gamma} g(z) dz$$

by the Cauchy integral theorem, provided the principal value exists. If f has only a pole of first order in 0, then the discussion in Section 2 shows, that the principal value indeed exists. The general case however is more delicate: let us first consider

a model sector curve γ with angle $\alpha \in [0, 2\pi]$, and let $n > 1$. Then we have

$$\begin{aligned}
\text{PV} \oint_{\gamma} \frac{dz}{z^n} &= \lim_{\varepsilon \searrow 0} \left(\int_{\varepsilon}^r \frac{dt}{t^n} + \int_0^{\alpha} \frac{rie^{it}}{r^n e^{int}} dt + \int_0^{r-\varepsilon} \frac{-e^{i\alpha}}{(r-t)^n e^{i\alpha}} dt \right) \\
&= \lim_{\varepsilon \searrow 0} \left(\frac{r^n \varepsilon - r \varepsilon^n}{(r\varepsilon)^n (n-1)} - \frac{e^{-\alpha(n-1)i} - 1}{r^{n-1} (n-1)} + \frac{r\varepsilon^n - r^n \varepsilon}{(r\varepsilon)^n (n-1)} e^{-\alpha(n-1)i} \right) \\
&= \lim_{\varepsilon \searrow 0} \frac{1 - e^{-i(n-1)\alpha}}{(n-1)\varepsilon^{n-1}} = \begin{cases} 0 & \text{if } \frac{\alpha(n-1)}{2\pi} \in \mathbb{Z}, \\ \text{complex infinity} & \text{otherwise.} \end{cases} \quad (3.1)
\end{aligned}$$

On an intuitive level it is clear that an angle condition decides whether the limit exists or not: Indeed, in order to compensate the purely real values on γ_1 (see Figure 1), the integral along γ_3 cannot have a non-real singular part. Hence, the principal value in (3.1) exists (and is actually 0) if and only if

$$\alpha = \frac{2k\pi}{n-1}$$

for some $k \in \mathbb{Z}$. Stated differently: If $\alpha = \frac{p}{q}\pi$ for some $p, q \in \mathbb{N}$, $q \neq 0$, then

$$\text{PV} \oint_{\gamma} \frac{dz}{z^n} = 0 \quad (3.2)$$

if $n = \frac{2kq}{p} + 1$ for an integer $k \geq 0$, otherwise the principal value (3.2) is infinite. Therefore we obtain:

Lemma 3.1. *Let $\alpha = \frac{p}{q}\pi$ for some $p, q \in \mathbb{N}$, $q \neq 0$. If the Laurent series of f only contains terms a_n/z^n for indices of the form $n = \frac{2kq}{p} + 1$ for integers $k \geq 0$, and if γ is a model sector-curve with angle α and radius smaller than the radius of convergence of the Laurent series, then there holds*

$$\text{PV} \frac{1}{2\pi i} \oint_{\gamma} f(z) dz = n_0(\gamma) \text{res}_0 f(z). \quad (3.3)$$

Proof. If f has a pole in 0, then (3.3) follows directly from (3.2). If 0 is an essential singularity of $f = \sum_{k \in \mathbb{Z}} a_k z^k$, we observe that the Laurent series $f_n(z) = \sum_{k=-n}^{\infty} a_k z^k$ converges locally uniformly to $f(z)$. Then we have for $\varepsilon > 0$

$$\begin{aligned}
\frac{1}{2\pi i} \int_{|\gamma|>\varepsilon} f(z) dz &= \frac{1}{2\pi i} \int_{|\gamma|>\varepsilon} (f(z) - f_n(z)) dz + \\
&+ \left(\frac{1}{2\pi i} \int_{|\gamma|>\varepsilon} f_n(z) dz - n_0(\gamma) \text{res}_0 f(z) \right) + n_0(\gamma) \text{res}_0 f(z) =: I + II + III.
\end{aligned}$$

Now, for $\delta > 0$, we choose $\varepsilon > 0$ small enough, such that the absolute value of the second term II on the right is smaller than δ . Note that by (3.1) this choice does not depend on n . Then we can choose n , depending on ε , large enough such that the absolute value of the first term I is also smaller than δ , and the claim follows. \square

For a more general curve than a model sector curve, we need the following definition:

Definition 3.2. Let $\Gamma : (a, b) \rightarrow \mathbb{C}$ be a piecewise C^1 curve and $\Gamma(x_1) =: z_1$. Let t^+ and t^- be the tangents in z_1 in the direction $\lim_{x \searrow x_1} \dot{\Gamma}(x)$ and $-\lim_{x \nearrow x_1} \dot{\Gamma}(x)$ respectively. We say that Γ is flat of order n in x_1 , if

$$\begin{aligned} |\Gamma(x) - P^+(\Gamma(x))| &= o(|\Gamma(x) - z_1|^n) \text{ for } x \searrow x_1 \text{ and} \\ |\Gamma(x) - P^-(\Gamma(x))| &= o(|\Gamma(x) - z_1|^n) \text{ for } x \nearrow x_1 \end{aligned}$$

where P^+ and P^- denote the orthogonal projection to t^+ and t^- respectively (see Figure 7).

Notice, that a piecewise C^1 curve is always flat of order 1 in all of its points.

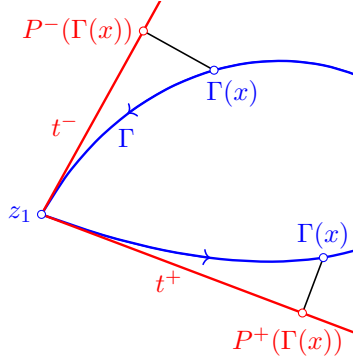


Figure 7: Γ is flat of order n in x_1 .

Now, let us consider a closed piecewise C^1 immersion $\Gamma : [a, b] \rightarrow \mathbb{C}$ starting and ending in 0 but such that $\Gamma(t) \neq 0$ for all $t \neq a, b$ and such that the (positively oriented) angle between $\lim_{t \searrow a} \dot{\Gamma}(t)$ and $-\lim_{t \nearrow b} \dot{\Gamma}(t)$ equals $\alpha \in [0, 2\pi]$. We assume that, after a suitable rotation, $\lim_{t \searrow a} \dot{\Gamma}(t)$ is a positive real number and that Γ is homotopic in the sense of (2.2) to a model sector-curve γ with the same angle α .

Moreover, we assume that Γ is flat of order n in 0 . Then, as in Section 2 (see Figure 3), we have for $n > 1$

$$\begin{aligned} \left| \int_{|\Gamma|>\varepsilon} \frac{dz}{z^n} - \int_{|\gamma|>\varepsilon} \frac{dz}{z^n} \right| &= \left| \int_{\beta} \frac{dz}{z^n} - \int_{|\gamma|>\varepsilon} \frac{dz}{z^n} \right| = \\ &= \left| \int_{\alpha_1+\alpha_2} \frac{dz}{z^n} \right| \\ &\leq \frac{1}{\varepsilon^n} \underbrace{\text{Length}(\alpha_1 + \alpha_2)}_{=o(\varepsilon^n)} \longrightarrow 0 \text{ for } \varepsilon \searrow 0. \end{aligned}$$

Hence, we only have a finite principal value if $\alpha = \frac{p}{q}\pi$ for some $p, q \in \mathbb{N}$, $q \neq 0$, and

$$\text{PV} \oint_{\Gamma} \frac{dz}{z^n} = 0 \quad (3.4)$$

if $n = \frac{2kq}{p} + 1$ for an integer $k \geq 0$, otherwise the principal value is infinite. This leads to the main Theorem:

Theorem 3.3. *Let $U \subset \mathbb{C}$ be an open set, and $S = \{z_1, z_2, \dots\} \subset U$ be a set without accumulation points in U such that $f : U \setminus S \rightarrow \mathbb{C}$ is holomorphic. Moreover, let C be a null-homologous immersed piecewise C^1 cycle in U such that C only contains singularities of f which are poles of order 1. Then*

$$\text{PV} \frac{1}{2\pi i} \oint_C f(z) dz = \sum_{\ell} n_{z_{\ell}}(C) \text{res}_{z_{\ell}} f(z). \quad (3.5)$$

The formula remains correct for poles of higher order on C if the following two conditions hold:

- (A) *If z_0 is a pole on C of order n , then C is flat of order n in z_0 , or, if z_0 is an essential singularity, C coincides near z_0 locally with the tangents t^+ and t^- in z_0 .*
- (B) *If z_0 is a singularity of f on C and α is the angle between the tangents t^+ and t^- in z_0 , then $\alpha = \frac{p}{q}\pi$, $p, q \in \mathbb{N}$, $q \neq 0$, and the Laurent series of f in z_0 contains only terms $a_n/(z-z_0)^n$ with $a_n \neq 0$ for indices of the form $n = \frac{2kq}{p} + 1$, $k \geq 0$ an integer.*

Proof. Let $C = \sum_{\ell=1}^k m_\ell \gamma_\ell$ with $m_\ell \in \mathbb{Z}$ and where $\gamma_\ell : [0, 1] \rightarrow \mathbb{C}$ are closed piecewise C^1 immersions. Then, there are at most finitely many points $x_{\ell 1}, \dots, x_{\ell k_\ell}$ such that $\gamma_\ell(x_{\ell j}) = z_{\ell j} \in S$. For each ℓ consider a decomposition $\gamma_\ell = \tilde{\gamma}_\ell + \Gamma_{\ell 1} + \dots + \Gamma_{\ell k_\ell}$, where $\tilde{\gamma}_\ell$ coincides with γ_ℓ outside of small neighborhoods of the points $x_{\ell j}$ and avoids the singularity at $\gamma_\ell(x_{\ell j})$ by driving around it on small circular arcs in clockwise direction. The closed curves $\Gamma_{\ell j}$ are homotopic in the sense of (2.2) to a model sector-curve with oriented angle $\alpha_{\ell j}$ between the tangents $\lim_{t \searrow x_{\ell j}} \dot{\gamma}_\ell$ and $-\lim_{t \nearrow x_{\ell j}} \dot{\gamma}_\ell$. The circular arcs are chosen small enough such that no singularity lies in the interior of the sectors whose boundary are the curves $\Gamma_{\ell j}$, and such that these sectors are contained in U . Observe that the cycle $\tilde{C} := \sum_{\ell=1}^k m_\ell \tilde{\gamma}_\ell$ avoids the singularities of f and is null-homologous in U . Hence, in the sequel we may apply the classical residue theorem to \tilde{C} .

Now, suppose that the two conditions (A) and (B) hold. This covers in particular the case when only poles of first order lie on C . Then, we have, by the classical residue theorem applied with \tilde{C} , by Lemma 3.1, and (3.4)

$$\begin{aligned} \text{PV} \frac{1}{2\pi i} \oint_C f(z) dz &= \text{PV} \frac{1}{2\pi i} \oint_{\tilde{C}} f(z) dz + \sum_{\ell=1}^k m_\ell \sum_{j=1}^{k_\ell} \text{PV} \frac{1}{2\pi i} \oint_{\Gamma_{\ell j}} f(z) dz = \\ &= \sum_{z \in S} n_z(\tilde{C}) \text{res}_z f(z) + \sum_{\ell=1}^k m_\ell \sum_{j=1}^{k_\ell} n_{z_{\ell j}}(\Gamma_{\ell j}) \text{res}_{z_{\ell j}} f(z). \end{aligned} \quad (3.6)$$

The first sum in (3.6) runs over

- (I) the singularities which are not lying on C , with winding number $\neq 0$.
- (II) the singularities on C .

Thus, the summands in (I) appear exactly also in the sum in (3.5) since for singularities z not on C we have $n_z(C) = n_z(\tilde{C})$. The summands in (II) coming from a singularity $z_{\ell j}$ on C together with the corresponding terms in the double sum in (3.6) give

$$\left(n_{z_{\ell j}}(\tilde{C}) + \sum_{\ell=1}^k m_\ell \sum_{j=1}^{k_\ell} n_{z_{\ell j}}(\Gamma_{\ell j}) \right) \text{res}_{z_{\ell j}} f(z) = n_{z_{\ell j}}(C) \text{res}_{z_{\ell j}} f(z)$$

and we are done. \square

As corollaries of Theorem 3.3 we obtain the residue theorems [5, Theorem 1] and [4, Theorem 4.8f].

3.1 Application

In [5], the version of the residue theorem is used to calculate principal values of integrals. At first sight it seems that this is the only advantage of Theorem 3.3 over the classical residue theorem. After all, poles on the curve γ necessarily mean that one is forced to consider principal values. However, Proposition 2.3 shows that it is possible to use Theorem 3.3 to compute integrals with bounded integrand.

Example 3.4. We want to compute the integral

$$\int_0^\infty \frac{\operatorname{sinc}(t) \sinh(t)}{\cos(t) + \cosh(t)} dt.$$

The current computer algebra systems give up on this integral after giving it some thought. To determine the integral we interpret the integrand as follows

$$\frac{\operatorname{sinc}(t) \sinh(t)}{\cos(t) + \cosh(t)} = \operatorname{im} f(\gamma_3(t)) \dot{\gamma}_3(t)$$

for

$$f(z) = -\frac{\cos(z/2)}{z \cosh(z/2)}$$

and $\gamma = \gamma_1 + \gamma_2 - \gamma_3$, where

$$\begin{aligned} \gamma_1 : [0, r] &\rightarrow \mathbb{C}, & t &\mapsto t - it, \\ \gamma_2 : [-\frac{\pi}{4}, \frac{\pi}{4}] &\rightarrow \mathbb{C}, & t &\mapsto \sqrt{2}re^{it}, \\ \gamma_3 : [0, r] &\rightarrow \mathbb{C}, & t &\mapsto t + it. \end{aligned}$$

Note that f has a pole of order 1 in $z = 0$ on γ with residue -1 . The winding number of γ with respect to $z = 0$ is $\frac{1}{4}$. By Theorem 3.3 we get:

$$\operatorname{im} \oint_\gamma f(z) dz = \operatorname{im} \left(\int_{\gamma_1} f(z) dz + \int_{\gamma_2} f(z) dz - \int_{\gamma_3} f(z) dz \right) = \operatorname{im} \left(\frac{1}{4} \cdot (-1) \cdot 2\pi i \right) = -\frac{\pi}{2}.$$

The integral along γ_2 converges to 0 as r tends to infinity, and

$$\operatorname{im} \int_{\gamma_1} f(z) dz = -\operatorname{im} \int_{\gamma_3} f(z) dz = -\int_0^r \frac{\operatorname{sinc}(t) \sinh(t)}{\cos(t) + \cosh(t)} dt.$$

Hence, we find the value of this improper integral:

$$\int_0^\infty \frac{\operatorname{sinc}(t) \sinh(t)}{\cos(t) + \cosh(t)} dt = \frac{\pi}{4}.$$

3.2 Connection to the Sokhotskiĭ-Plemelj Theorem

In this section, we want to briefly show that the above-mentioned version of the residue theorem in [5, Theorem 1] can be obtained as a corollary of the Sokhotskiĭ-Plemelj Theorem.

Let $U \subset \mathbb{C}$ be an open set, let $S = \{z_1, \dots\} \subset U$ be a set without accumulation points and let $f : U \setminus S \rightarrow \mathbb{C}$ be a holomorphic function. Let furthermore D be a bounded domain with piecewise C^1 -boundary C , consisting of finitely many components, such that $\bar{D} \subset U$. As usual, we assume that C is oriented such that D lies always on the left with respect to the direction of the parametrization. If f only has first order poles on C , we may use a decomposition

$$f = g + \sum_{k=1}^m f_k,$$

where f_k is holomorphic away from a single first order pole $z_k \in C$ and g has only singularities in $z_{m+1}, z_{m+2}, \dots \notin C$. Then $\varphi_k(z) := f_k(z)(z - z_k)$ is holomorphic. Let

$$F_k(z) := \frac{1}{2\pi i} \int_C \frac{\varphi_k(\xi)}{\xi - z} d\xi.$$

By Cauchy's integral formula we have $F_k(z) = \varphi_k(z)$ provided $z \in D$. Furthermore

$$\begin{aligned} F_k^+(z_k) &:= \lim_{D \ni z \rightarrow z_k} F_k(z) = \lim_{D \ni z \rightarrow z_k} \varphi_k(z) = \lim_{D \ni z \rightarrow z_k} f_k(z)(z - z_k) \\ &= \operatorname{res}_{z_k} f_k(z) = \operatorname{res}_{z_k} f(z) \end{aligned}$$

and

$$F_k(z_k) = \operatorname{PV} \frac{1}{2\pi i} \int_C \frac{\varphi_k(\xi)}{\xi - z_k} d\xi = \operatorname{PV} \frac{1}{2\pi i} \int_C f_k(\xi) d\xi.$$

According to the Sokhotskiĭ-Plemelj formula (see [3, p. 385, (3)] or [7, Chapter 3]) we find

$$F_k^+(z_k) = F_k(z_k) + \left(1 - \frac{\alpha_k}{2\pi}\right) \varphi_k(z_k),$$

where α_k is the interior angle of C in z_k and hence

$$\operatorname{res}_{z_k} f(z) = \operatorname{PV} \frac{1}{2\pi i} \int_C f_k(\xi) d\xi + \left(1 - \frac{\alpha_k}{2\pi}\right) \operatorname{res}_{z_k} f(z)$$

and after rearranging

$$\operatorname{PV} \frac{1}{2\pi i} \int_C f_k(\xi) d\xi = \frac{\alpha_k}{2\pi} \operatorname{res}_{z_k} f(z).$$

Therefore we find

$$\begin{aligned} \text{PV} \frac{1}{2\pi i} \int_C f(\xi) d\xi &= \text{PV} \frac{1}{2\pi i} \int_C \left(g(\xi) + \sum_{k=1}^m f_k(\xi) \right) d\xi \\ &= \sum_{k \geq m+1} \text{res}_{z_k} f(z) + \sum_{k=1}^m \frac{\alpha_k}{2\pi} \text{res}_{z_k} f(z) \\ &= \sum_{k \geq 1} n_{z_k}(C) \text{res}_{z_k} f(z). \end{aligned}$$

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