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STACKY HAMILTONIAN ACTIONS AND SYMPLECTIC REDUCTION

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Abstract. We introduce the notion of a Hamiltonian action of an étale Lie group stack on an étale symplectic stack and establish versions of the Kirwan convexity theorem, the Meyer-Marsden-Weinstein symplectic reduction theorem, and the Duistermaat-Heckman theorem in this context.

Contents

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1. Introduction

The leaf space of a foliation on a smooth manifold can be interpreted as a differentiable stack. Many invariants of the foliation, such as its K-theory and cyclic homology, depend only on this stack. In this paper we are concerned with transversely symplectic foliations and with some properties of their associated stacks.

For instance, work of He [20], Ishida [23], Ratiu and Zung [43], and Lin and Sjamaar [30] shows that a version of Kirwan's convexity theorem [24] for the moment map in symplectic geometry holds for certain transversely symplectic foliations. Our first main result, Theorem 7.4.2, upgrades their results to a convexity theorem for Hamiltonian actions of étale Lie group stacks on étale symplectic stacks. To avoid technicalities, let us here state our theorem in the case where the Lie group stack is a stacky torus **T**. Our assertion is that if **T** acts on a symplectic stack **X** and if the action admits a moment map, then under a certain "cleanness" hypothesis the image of the moment map is a convex polytope. This statement extends the Atiyah-Guillemin-Sternberg convexity theorem, which holds in the case where **X** is a symplectic manifold and **T** an ordinary torus, but the stacky situation has an interesting new feature, namely that the normal fan of the moment polytope is not necessarily rational. More precisely, the normal vectors to the polytope are cocharacters of **T**, and the cocharacter group of a stacky torus is not a lattice, but a quasi-lattice in the Lie algebra. We call a pair consisting of a stacky torus and a convex polytope in the dual of its Lie algebra a *stacky polytope*.

The second main result of this paper is Theorem 9.1.1, which generalizes the Meyer-Marsden-Weinstein symplectic reduction theorem to the setting of group stack actions on symplectic stacks. We give a necessary and sufficient condition for the reduction of a symplectic stack by a Hamiltonian action of an étale Lie group stack to be again a symplectic stack. The theorem holds under a regularity hypothesis on the moment map, but we make no assumption on the compactness of the group stack or the properness of the action. This generalizes a theorem of Lerman and Malkin [27, Theorem 3.13], who considered the case of a Hamiltonian action of a compact Lie group on a separated symplectic stack. In view of work of Calaque [10], Pecharich [40], and Safronov [45] we expect that in the absence of any regularity assumptions the reduction of a symplectic stack by the action of a Lie group stack is a derived symplectic stack.

Our third main result is Theorem 10.3, which is an extension of the Duistermaat-Heckman theorem to Hamiltonian actions of stacky tori. The Duistermaat-Heckman theorem has two parts: (1) the variation of the reduced symplectic form is linear, and (2) the moment map image of the Liouville measure is piecewise polynomial. It is only the first part that we generalize here, leaving the second part for later.

These results require a basic theory of Hamiltonian actions of Lie group stacks, which we outline in Sections 6–8 and in Appendices A-C. We include some elementary, but apparently new, material concerning étale stacks and stacky Lie groups, such as the fact that the Lie 2-algebra of vector fields of a differentiable stack is equivalent to a Lie algebra if the stack is étale (Proposition 5.7.5), a structure theorem for Lie 2-groups of compact type (Proposition 6.4.7), and a strictification theorem for stacky actions (Theorem 6.8.1). Our starting point is the theory developed by Lerman and Malkin [27], which we extend in two respects: the stacks that we deal with are étale, but usually not separated, and the groups that act on them are themselves étale stacks.

What we call a symplectic stack is a 0-*shifted symplectic* 1-*stack* in the terminology of Pantev et al. [39], except that our stacks are defined over the category of differentiable manifolds instead of the category of algebraic schemes. See Getzler's lecture notes [18] for an introduction to higher symplectic stacks over manifolds and for an explanation of how Weinstein's symplectic groupoids [49] and Xu's quasi-symplectic groupoids [50] are presentations of 1-shifted symplectic 1-stacks. We expect that some of our results, especially the reduction theorem, can be extended to higher symplectic stacks.

An illuminating example of stacky symplectic reduction is Prato's construction of toric quasifolds [41], which predates many of these developments. We present (a slight extension of) her construction as a running example in order to show that every simple stacky polytope is the moment polytope of what we call a *toric* symplectic stack. Toric symplectic stacks are a C^∞ counterpart of the toric stacks of algebraic geometry, a comprehensive treatment of which was given by Gerashchenko and Satriano [17]. However, the two theories are very different. The correspondence between toric symplectic manifolds and nonsingular complex projective toric varieties established by Delzant [15] breaks down in the world of stacks, because C^∞ stacky tori, which include such objects as the quotient of a two-dimensional torus by a dense line, are seldom algebraic. Toric symplectic stacks are classified by the first author in [22].

Two further instances of stacky symplectic reduction are the space of geodesics of a Riemannian manifold (see Example 9.2.3) and the quotient of a contact manifold by the Reeb flow (see Example 9.2.4).

We are grateful to Chenchang Zhu for her contribution to this project, which appears in Appendix C. We also thank the referees for their careful reading and many useful suggestions.

2. Notation and conventions

All manifolds are required to be C^∞ and second countable, but not necessarily Hausdorff. Manifolds and smooth maps form the category **Diff**. Lie groups are group objects in **Diff**, in other words are required to have countably many components. The functor taking a Lie group to its Lie algebra is denoted Lie. The space of smooth global sections of a vector bundle E is denoted by $\Gamma(E)$. The Lie algebra of vector fields on a manifold X is denoted by $\mathrm{Vect}(X) = \Gamma(TX)$. Given a smooth map $f\colon X \to Y$, two vector fields $v \in \mathrm{Vect}(X)$ and $w \in \mathrm{Vect}(Y)$ are *f-related* (notation: $v \sim_f w$) if $T_x f(v_x) = w_{f(x)}$ for all $x \in X$. A Lie groupoid with object manifold X_0 and arrow manifold X_1 is denoted by $X_1 \rightrightarrows X_0$ or $X_\bullet$. Morphisms of Lie groupoids $\phi_\bullet\colon X_\bullet \to Y_\bullet$ are also decorated with a subscript "$\bullet$", as are basic differential forms $\zeta_\bullet \in \Omega^\bullet_{\mathrm{bas}}(X_\bullet)$ and basic vector fields $v_\bullet \in \mathrm{Vect}_{\mathrm{bas}}(X_\bullet)$. We write weak (Morita) equivalence of groupoids as $X_\bullet \simeq Y_\bullet$. Stacks over **Diff** are written in boldface, $\mathbf{X}$, $\mathbf{Y}$, and so are their 1-morphisms $\boldsymbol{\phi}\colon \mathbf{X} \to \mathbf{Y}$, 2-morphisms $\boldsymbol{\alpha}\colon \boldsymbol{\phi} \Rightarrow \boldsymbol{\psi}$, and differential forms $\boldsymbol{\zeta} \in \Omega^\bullet(\mathbf{X})$, etc. The classifying stack of a Lie groupoid $X_\bullet$ is denoted by $\mathbf{B}X_\bullet$. We write equivalences of stacks as $\mathbf{X} \simeq \mathbf{Y}$. We denote by $\star$ a terminal object in the 2-category of stacks.

See also the notation index at the end.

3. Hamiltonian actions on presymplectic manifolds

This section is a brief exposition of the presymplectic convexity theorem [30, Theorem 2.2], which is the prototype of our stacky convexity theorem, Theorem 7.4.2.

A *presymplectic manifold* is a Hausdorff manifold X equipped with a closed 2-form ω of constant rank. The kernel $\ker(\omega)$ defines an involutive distribution on X. The corresponding foliation of X is called the *null foliation*, which we denote by $\mathscr{F}$. For $x \in X$, we write $\mathscr{F}(x)$ for the leaf of $\mathscr{F}$ containing x.

Consider a left action of a connected Lie group G on a presymplectic manifold (X, ω). For $\xi \in \mathfrak{g} = \mathrm{Lie}(G)$, denote by ξ_X the fundamental vector field of ξ on X; then the assignment $\xi \mapsto \xi_X$ is a Lie algebra anti-homomorphism. Let

$$\mathfrak{n}(\mathscr{F}) = \{\, \xi \in \mathfrak{g} \mid (\xi_X)_x \in T_x\mathscr{F} \text{ for all } x \in X \,\}.$$

The subspace $\mathfrak{n}(\mathscr{F}) \subseteq \mathfrak{g}$ is an ideal in $\mathfrak{g}$, which we call the *null ideal* of $\mathscr{F}$, following [30]. Let $N(\mathscr{F}) \subseteq G$ be the connected immersed Lie subgroup of G with Lie algebra $\mathfrak{n}(\mathscr{F})$, which we will call the *null subgroup*. The action of G on (X, ω) is *clean* if

$$T_x(N(\mathscr{F}) \cdot x) = T_x(G \cdot x) \cap T_x\mathscr{F}$$

for all $x \in X$. For a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, let

$$\mathrm{ann}(\mathfrak{h}) = \{\, \eta \in \mathfrak{g}^* \mid \langle \eta, \mathfrak{h} \rangle = 0 \,\} \cong (\mathfrak{g}/\mathfrak{h})^*$$

be the annihilator of $\mathfrak{h}$, with $\langle \cdot, \cdot \rangle\colon \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ being the natural pairing. The action of G on (X, ω) is *Hamiltonian* if there is a *moment map*, i.e. a map $\mu\colon X \to \mathfrak{g}^*$ that satisfies the following conditions:

(i) $d\mu^\xi = \iota_{\xi_X}\omega$ for all $\xi \in \mathfrak{g}$, where $\mu^\xi(x) = \langle \mu(x), \xi \rangle$ denotes the component of μ along ξ;
(ii) μ intertwines the G action on X and the coadjoint action of G on $\mathfrak{g}^*$;
(iii) $\mu(X) \subseteq \mathrm{ann}(\mathfrak{n}(\mathscr{F}))$.

(Since G is connected, these assumptions imply that ω is G-invariant.) The tuple (X, ω, G, μ) is then called a *presymplectic Hamiltonian G-manifold*. An *isomorphism* between two Hamiltonian G-manifolds (X, ω, G, μ) and (X', ω', G, μ') is a G-equivariant diffeomorphism $\phi\colon X \to X'$ which preserves the presymplectic structure and the moment map.

The conditions (i)–(iii) on the moment map are not independent. by [30, proposition 2.9.1], if G is compact and if $\mu\colon X \to \mathfrak{g}^*$ satisfies (i) and (ii), there exists $\lambda \in \mathfrak{g}^*$ fixed by the coadjoint action of G such that $\mu + \lambda$ satisfies (i)–(iii).

3.1. **Example.** This example is drawn from [15] and [41]. Let $\mathbb{T}$ be the circle $\mathbb{R}/\mathbb{Z}$, G the n-dimensional torus $\mathbb{T}^n$, and $N \subseteq G$ an immersed Lie subgroup. Consider the Hamiltonian G-manifold (X, ω, G, μ), where

$$X = \mathbb{C}^n, \qquad \omega = \frac{1}{2\pi i}\sum_j dz_j \wedge d\bar{z}_j, \qquad \mu(z) = \sum_j |z_j|^2 e_j^* + \lambda,$$

and G acts on $\mathbb{C}^n$ in the standard way. Here e_j^* is the dual of the standard basis of $\mathbb{R}^n = \mathfrak{g}$, and $\lambda \in \mathfrak{g}^*$ is in the open negative orthant. Let $\iota\colon \mathfrak{n} \to \mathfrak{g}$ be the inclusion of Lie algebras and $\iota^*\colon \mathfrak{g}^* \to \mathfrak{n}^*$ the dual projection. Then $(X_0, \omega_0, G, \mu_0)$ is a presymplectic Hamiltonian G-manifold, where

$$X_0 = (\iota^* \circ \mu)^{-1}(0), \quad \omega_0 = \omega|_{X_0}, \quad G = \mathbb{T}^n, \quad \mu_0 = \mu|_{X_0}.$$

Note that μ_0 takes values in $\operatorname{ann}(\mathfrak{n})$. We will return to this example throughout the text.

The cleanness condition is essential for the following to be true.

3.2. **Theorem** (Lin and Sjamaar [30]). *Let (X, ω, G, μ) be a Hamiltonian presymplectic G-manifold, where X is connected, and G is compact and connected. Assume that the G-action is clean, and the moment map $\mu\colon X \to \mathfrak{g}^*$ is proper. Choose a maximal torus T of G and a closed Weyl chamber C in $\mathfrak{t}^*$, where $\mathfrak{t} = \operatorname{Lie}(T)$, and define $\Delta(X) = \mu(X) \cap C$.*

(i) *The fibres of μ are connected and $\mu\colon X \to \mu(X)$ is an open map.*
(ii) *$\Delta(X)$ is a closed convex polyhedral set.*
(iii) *$\Delta(X)$ is rational if and only if the null subgroup $N(\mathscr{F})$ of G is closed.*

4. Lie groupoids and differentiable stacks

This section is a summary of definitions, conventions, and well-known facts. For more about Lie groupoids see e.g. Moerdijk and Mrčun [36] or Crainic and Moerdijk [14]. For the relationship between Lie groupoids and differentiable stacks see e.g. Behrend and Xu [3], Blohmann [5], Carchedi [11], [12], Lerman [25], Metzler [34], Noohi [37], or Villatoro [47]. See also the notation index at the end.

4.1. **Lie groupoids.** A Lie groupoid $X_\bullet = (X_1 \rightrightarrows X_0)$ has structure maps s, t, m, $(\cdot)^{-1}$, and u which are called *source, target, multiplication, inversion*, and the *identity bisection*, respectively. When two arrows f, $g \in X_1$ have $s(f) = t(g)$, they are *composable*. We typically write $f \circ g$ for the multiplication $m(f, g)$ of two composable arrows. The *object manifold* X_0 and the *arrow manifold* X_1 are (not necessarily Hausdorff) manifolds, and the maps s and t are required to be surjective submersions. If all source fibres $s^{-1}(x)$ are connected (resp. simply connected), then $X_\bullet$ is *source-connected* (resp. *source-simply connected*). For $x \in X_0$, the *orbit* of x is $X_\bullet \cdot x = t(s^{-1}(x))$, and the *isotropy group* of x is

$$\operatorname{Iso}(x) = \operatorname{Iso}_{X_\bullet}(x) = s^{-1}(x) \cap t^{-1}(x).$$

It is known (see e.g. [36, Theorem 5.4]) that $X_\bullet \cdot x$ is an immersed submanifold of X_0, that $\operatorname{Iso}(x)$ is a closed submanifold of X_1, and that $\operatorname{Iso}(x)$ is a Lie group. The set of orbits equipped with the quotient topology is the *orbit space* or *coarse quotient space* X_0/X_1 of the Lie groupoid.

If a Lie group G acts on a manifold X, we denote the action groupoid by $G \ltimes X \rightrightarrows X$. We sometimes denote a Lie groupoid by its space of arrows; for instance in this notation the action groupoid is $G \ltimes X$. If X is a smooth manifold, we consider it as the identity Lie groupoid $X \rightrightarrows X$. For any Lie groupoid $X_\bullet$ we have a natural inclusion $X_0 \to X_\bullet$ of (the identity groupoid of) X_0 into $X_\bullet$.

4.1.1. **Definition.** The *Lie algebroid* of a Lie groupoid $X_\bullet$ is the vector bundle $\operatorname{Alg}(X_\bullet)$ over X_0 given by

$$\operatorname{Alg}(X_\bullet) = \{\, w \in TX_1|_{u(X_0)} \mid Ts(w) = 0 \,\}.$$

The *anchor map* is the vector bundle morphism $\rho\colon \operatorname{Alg}(X_\bullet) \to TX_0$ given by $\rho = Tt|_{\operatorname{Alg}(X_\bullet)}$.

Sections of the Lie algebroid extend uniquely to right-invariant vector fields on X_1, and so the space of sections of $\operatorname{Alg}(X_\bullet)$ carries a natural Lie bracket.

4.1.2. **Definition.** A *morphism of Lie groupoids* $\phi_\bullet\colon X_\bullet \to Y_\bullet$ is a smooth functor, i.e. a morphism of groupoids which is smooth on the manifolds of objects $\phi_0\colon X_0 \to Y_0$ and on the manifolds of arrows $\phi_1\colon X_1 \to Y_1$. For two morphisms $\phi_\bullet, \psi_\bullet\colon X_\bullet \to Y_\bullet$ of Lie groupoids, a 2-*morphism* or *natural transformation* $\gamma\colon \phi_\bullet \Rightarrow \psi_\bullet$ is a smooth map $\gamma\colon X_0 \to Y_1$ with the property that for each $x \in X_0$, $\gamma(x)$ is an arrow from $\phi_0(x)$ to $\psi_0(x)$ in $Y_\bullet$, and for every arrow $f\colon x_1 \to x_2$ in $X_\bullet$, the following diagram commutes in $Y_\bullet$:

$$\begin{array}{ccc} \phi_0(x_1) & \xrightarrow{\gamma(x_1)} & \psi_0(x_1) \\ {\scriptstyle \phi_1(f)}\downarrow & & \downarrow{\scriptstyle \psi_1(f)} \\ \phi_0(x_2) & \xrightarrow{\gamma(x_2)} & \psi_0(x_2) \end{array}$$

The 2-category of Lie groupoids is denoted **LieGpd**.

A natural transformation of Lie groupoids is automatically a natural isomorphism.

4.1.3. **Definition.** Let $\phi_\bullet\colon X_\bullet \to Y_\bullet$ be a morphism of Lie groupoids. Then $\phi_\bullet$ is *essentially surjective* if the map $t \circ \operatorname{pr}_1\colon Y_1 \times_{Y_0} X_0 \to Y_0$ which sends (g, x) to $t(g)$ is a surjective submersion. Here $Y_1 \times_{Y_0} X_0 = Y_1 \times_{s,Y_0,\phi_0} X_0$ means as usual the fibred product of Y_1 and X_0, which consists of all $(g, x) \in Y_1 \times X_0$ satisfying $s(g) = \phi_0(x)$. The morphism $\phi_\bullet$ is *fully faithful* if the square

$$\begin{array}{ccc} X_1 & \xrightarrow{\phi_1} & Y_1 \\ {\scriptstyle (s,t)}\downarrow & & \downarrow{\scriptstyle (s,t)} \\ X_0 \times X_0 & \xrightarrow{\phi_0\times\phi_0} & Y_0 \times Y_0 \end{array}$$

is a fibred product of manifolds. If $\phi_\bullet$ is both essentially surjective and fully faithful, we say that $\phi_\bullet$ is a *Morita morphism* or *weak equivalence*. If there is a zigzag of Morita morphisms $X_\bullet \to Y_\bullet$ and $X_\bullet \to Z_\bullet$, then $Y_\bullet$ and $Z_\bullet$ are *Morita equivalent*.

4.1.4. **Definition.** Given a Lie groupoid $Y_\bullet$ and a smooth map $\phi_0\colon X \to Y_0$, the *pullback groupoid* $X_\bullet = \phi_0^*(Y_\bullet)$ is defined by $X_0 = X$ and

$$\begin{aligned} X_1 &= X \times_{\phi_0,Y_0,s} Y_1 \times_{t,Y_0,\phi_0} X \\ &= \{\, (x, g, y) \in X \times Y_1 \times X \mid \phi_0(x) = s(g) \text{ and } t(g) = \phi_0(y) \,\}. \end{aligned}$$

The pullback groupoid is a Lie groupoid whenever $(\phi_0, \phi_0)\colon X \times X \to Y_0 \times Y_0$ is transverse to $(s, t)\colon Y_1 \to Y_0 \times Y_0$. The map ϕ_0 then lifts to a Lie groupoid homomorphism $\phi_\bullet\colon \phi_0^*(Y_\bullet) \to Y_\bullet$, which is fully faithful, and which is essentially surjective if and only if $t \circ \operatorname{pr}_1\colon Y_1 \times_{Y_0} X \to Y_0$ is surjective.

4.1.5. **Lemma.** (i) *For every Lie groupoid $Y_\bullet$ there is a Morita morphism $X_\bullet \to Y_\bullet$ from a Lie groupoid $X_\bullet$ with Hausdorff object manifold X_0.*

(ii) *Let $X_\bullet$ be a Lie groupoid with Hausdorff object manifold X_0. Then for every $x \in X_0$ the source fibre $s^{-1}(x)$ is Hausdorff.*

Proof. (i) Choose any surjective étale map $\phi_0 \colon X_0 \to Y_0$ from a Hausdorff manifold X_0 to Y_0 (e.g. X_0 is the disjoint union of charts in a countable atlas of Y_0); then $X_\bullet = \phi_0^* Y_\bullet$ is Morita equivalent to $Y_\bullet$ and has object manifold X_0.

(ii) Let $x \in X_0$ and let t_x be the restriction of the target map to the source fibre $s^{-1}(x)$. According to [36, Theorem 5.4(iv)], $t_x \colon s^{-1}(x) \to X_\bullet \cdot x$ is a locally trivial principal bundle with structure group $\mathrm{Iso}(x)$. The orbit $X_\bullet \cdot x \subseteq X_0$ is Hausdorff and so is the Lie group $\mathrm{Iso}(x)$, and therefore the total space $s^{-1}(x)$ is Hausdorff. QED

4.1.6. **Definition.** Let $\phi_\bullet \colon X_\bullet \to Z_\bullet$ and $\psi_\bullet \colon Y_\bullet \to Z_\bullet$ be Lie groupoid morphisms. The *weak fibred product* is the topological groupoid $X_\bullet \times^{(w)}_{Z_\bullet} Y_\bullet$ whose space of objects is

$$X_0 \times_{Z_0} Z_1 \times_{Z_0} Y_0 = \{ (x, k, y) \in X_0 \times Z_1 \times Y_0 \mid \phi_0(x) = s(k),\ \psi_0(y) = t(k) \},$$

whose space of arrows is

$$X_1 \times_{Z_0} Z_1 \times_{Z_0} Y_1 = \{ (f, k, g) \in X_1 \times Z_1 \times Y_1 \mid \phi_0(s(f)) = s(k),\ \psi_0(s(g)) = t(k) \},$$

and whose groupoid structure maps are as in [36, § 5.3].

If either ϕ_0 or ψ_0 is a submersion, the weak fibred product is a Lie groupoid, and it is a weak pullback in **LieGpd**.

4.2. **Differentiable stacks.** We consider stacks over the site **Diff** of (C^∞ and second countable, but not necessarily Hausdorff) manifolds, equipped with the open cover Grothendieck topology.

Stacks and their 1-morphisms and 2-morphisms form a strict (2, 1)-category **Stack**. An *atlas* of a stack $\mathbf{X}$ is a representable epimorphism $X \to \mathbf{X}$ from a manifold X to $\mathbf{X}$. A *differentiable stack* is a stack that admits an atlas. Every differentiable stack $\mathbf{X}$ admits an atlas $X \to \mathbf{X}$ with X Hausdorff (namely starting from an arbitrary atlas $Y \to \mathbf{X}$, let X be the disjoint union of all charts in a countable atlas of Y). We denote the full subcategory of **Stack** formed by all differentiable stacks by **DiffStack**.

The *classifying stack* $\mathbf{B}X_\bullet$ of a Lie groupoid $X_\bullet$ is the stack of right torsors (right principal bundles) of $X_\bullet$. The *classifying functor* is the 2-functor $\mathbf{B} \colon$ **LieGpd** $\to$ **DiffStack** that takes $X_\bullet$ to $\mathbf{B}X_\bullet$. (See [5, § 2], [34, § 3], [11, § I.2], or [12, § 2].) A *presentation* of a stack $\mathbf{X}$ is an equivalence $\mathbf{B}X_\bullet \simeq \mathbf{X}$, where $X_\bullet$ is a Lie groupoid.

For a Lie groupoid $X_\bullet$, the natural inclusion of Lie groupoids $X_0 \to X_\bullet$ induces a morphism of stacks $X_0 \to \mathbf{B}X_\bullet$, which is an atlas for $\mathbf{B}X_\bullet$. Conversely, if $\mathbf{X}$ is a stack and $\boldsymbol{\phi} \colon X_0 \to \mathbf{X}$ is an atlas, then there exists an equivalence $\mathbf{B}X_\bullet \simeq \mathbf{X}$ such that $\boldsymbol{\phi}$ is 2-isomorphic to the composition $X_0 \longrightarrow \mathbf{B}X_\bullet \xrightarrow{\simeq} \mathbf{X}$, where $X_\bullet$ is the Lie groupoid $X_0 \times_{\mathbf{X}} X_0 \rightrightarrows X_0$. (See e.g. [11, § I.2.7] or [37, § 3.2].) Thus a stack is differentiable if and only if it admits a presentation.

The functor $\mathbf{B}$ takes essentially surjective Lie groupoid morphisms to stack epimorphisms, and it takes fully faithful Lie groupoid morphisms to stack monomorphisms. Thus $\mathbf{B}$ takes Morita morphisms to equivalences of stacks. By the universal property of 2-localization (see [42, § 2]), $\mathbf{B}$ extends to a 2-functor

$$\mathbf{B}^{\mathrm{loc}} \colon \mathbf{LieGpd}\big[\mathscr{M}^{-1}\big] \xrightarrow{\simeq} \mathbf{DiffStack}, \tag{4.2.1}$$

from the bicategory of Lie groupoids, localized at the class $\mathscr{M}$ of all Morita morphisms, to the 2-category **DiffStack**. The functor $\mathbf{B}^{\mathrm{loc}}$ is an equivalence of bicategories; a result of this type in a topological context was proved in [42] and a proof in our specific context is given in [47, Theorem 1.3.27]. The equivalence (4.2.1) implies that for all Lie groupoids $X_\bullet$ and $Y_\bullet$ we have an equivalence of categories $\mathbf{B}^{\mathrm{loc}}\colon \operatorname{Hom}(X_\bullet, Y_\bullet) \to \operatorname{Hom}(\mathbf{B}X_\bullet, \mathbf{B}Y_\bullet)$, where the first Hom is taken in $\mathbf{LieGpd}[\mathscr{M}^{-1}]$.

4.2.2. **Lemma** (Behrend-Xu [3, Theorem 2.2])**.** *Two Lie groupoids are Morita equivalent if and only if their classifying stacks are equivalent.*

A convenient model for the bicategory $\mathbf{LieGpd}\left[\mathscr{M}^{-1}\right]$ is the bicategory of Lie groupoids where the 1-morphisms are bibundles and the 2-morphisms are isomorphisms of bibundles as in [5] or [25].

The classifying functor has the following useful property.

4.2.3. **Lemma.** *The functor* $\mathbf{B}$ *preserves weak pullbacks.*

Proof. This is shown in [37, § 9] for topological stacks, but the result carries over directly to differentiable stacks. See also [37, Theorem 12.6] for a comparison of their quotient stack functor to our classifying stack functor. QED

5. Foliation groupoids and étale stacks

In this section we review basic facts about foliation groupoids and their classifying stacks, and draw some elementary consequences. The sources include Crainic and Moerdijk [14], Hepworth [21], and Berwick-Evans and Lerman [4]. Our notion of basic vector fields is adapted from Lerman and Malkin [27]. We show that the notions of basic differential forms and basic vector fields are Morita invariant (Proposition 5.3.12) and that the Lie 2-algebra of vector fields of a stack is equivalent to a Lie algebra if the stack is étale (Proposition 5.7.5).

5.1. **Foliation groupoids.** A *foliation groupoid* is a Lie groupoid $X_\bullet = (X_1 \rightrightarrows X_0)$ whose object manifold X_0 is Hausdorff and whose isotropy groups $\operatorname{Iso}(x)$ are discrete for all $x \in X_0$. An *étale groupoid* is a Lie groupoid $X_\bullet$ whose object manifold X_0 is Hausdorff and whose source map s is étale (i.e. a local diffeomorphism).

Clearly, étale groupoids are foliation groupoids. If $X_\bullet$ is a source-connected foliation groupoid, then the orbits form a (constant rank) foliation $\mathscr{F} = \mathscr{F}_{X_\bullet}$ of X_0, the anchor map $\rho\colon \operatorname{Alg}(X_\bullet) \to TX_0$ is injective, and the image of ρ is the tangent bundle $T\mathscr{F}$ of the foliation. There is no loss of generality in the assumption that X_0 is Hausdorff; see Lemma 4.1.5. If $X_\bullet$ is a foliation groupoid, every Lie groupoid with Hausdorff object manifold which is Morita equivalent to $X_\bullet$ is also a foliation groupoid.

Let X be a Hausdorff manifold equipped with a (regular) foliation $\mathscr{F}$ and let $T\mathscr{F}$ be the tangent bundle of the foliation. A Lie groupoid $X_\bullet$ over $X_0 = X$ with the property that the anchor map $\rho\colon \operatorname{Alg}(X_\bullet) \to TX$ is injective and has image equal to $T\mathscr{F}$ is said to *integrate* the foliation $\mathscr{F}$.

The integrations of a foliation $\mathscr{F}$ form a category, the objects of which are pairs $(X_\bullet, \psi)$, where $X_\bullet$ is a Lie groupoid integrating $\mathscr{F}$ and $\psi\colon \operatorname{Alg}(X_\bullet) \to T\mathscr{F}$ is an isomorphism of Lie algebroids, and the arrows $\phi_\bullet\colon (X_\bullet, \psi) \to (X'_\bullet, \psi')$ of which are morphisms $\phi_\bullet\colon X_\bullet \to X'_\bullet$ that respect the maps ψ and ψ'.

Given a foliated Hausdorff manifold $(X, \mathscr{F})$, there are two important source-connected foliation groupoids called the *monodromy groupoid* $\operatorname{Mon}(X, \mathscr{F})$, and the *holonomy groupoid* $\operatorname{Hol}(X, \mathscr{F})$, both of which integrate $\mathscr{F}$. (See e.g. [36, § 5.2].) There is a Lie groupoid morphism $\operatorname{hol}\colon \operatorname{Mon}(X, \mathscr{F}) \to \operatorname{Hol}(X, \mathscr{F})$ which is the identity map on the object manifold X and sends an arrow in $\operatorname{Mon}(X, \mathscr{F})$ to its holonomy action. The following theorem says that the category of source-connected integrations of $(X, \mathscr{F})$ is a preorder with the monodromy groupoid as a greatest element and the holonomy groupoid as a least element.

5.1.1. **Theorem** (Crainic and Moerdijk [14, Proposition 1])**.** *Let $(X, \mathscr{F})$ be a foliated Hausdorff manifold. For every source-connected Lie groupoid $X_\bullet = (X_1 \rightrightarrows X_0)$ over $X_0 = X$ integrating $\mathscr{F}$, there is a natural factorization of the holonomy morphism* $\mathrm{Mon}(X, \mathscr{F}) \to \mathrm{Hol}(X, \mathscr{F})$ *into morphisms of Lie groupoids over X,*

$$\mathrm{Mon}(X, \mathscr{F}) \xrightarrow{\psi_{X_\bullet}} X_\bullet \xrightarrow{\mathrm{hol}_{X_\bullet}} \mathrm{Hol}(X, \mathscr{F}).$$

The maps $\psi_{X_\bullet}$ and $\mathrm{hol}_{X_\bullet}$ are étale and surjective on the manifolds of arrows, and $X_\bullet$ is source-simply connected if and only if $\psi_{X_\bullet}$ is an isomorphism.

5.1.2. **Definition.** A *Lie group bundle* is a Lie groupoid where every arrow f has $s(f) = t(f)$. Let $X_\bullet$ and $X'_\bullet$ be Lie groupoids with the same object manifold $X_0 = X'_0$, and let $\psi_\bullet \colon X_\bullet \to X'_\bullet$ be a Lie groupoid morphism which is the identity on X_0. The *kernel* of $\psi_\bullet$ is

$$\ker(\psi_\bullet) = \{\, f \in X_1 \mid \psi_1(f) = u(s(f)) = u(t(f)) \,\}.$$

If ψ_1 is transverse to the identity bisection of $X'_\bullet$, then $\ker(\psi_\bullet)$ is a Lie group bundle over X_0. For instance, the kernels of $\psi_{X_\bullet}$ and $\mathrm{hol}_{X_\bullet}$ in Theorem 5.1.1 are Lie group bundles.

5.1.3. **Definition.** Let $(X, \mathscr{F})$ be a foliated Hausdorff manifold. A smooth map $\phi \colon Y \to X$ is *transverse to* $\mathscr{F}$ if it is transverse to each leaf of $\mathscr{F}$. A transverse map $\phi \colon Y \to X$ is *complete* if $\phi(Y)$ intersects each leaf of $\mathscr{F}$ at least once.

This extends the usual notion of a complete transversal, where ϕ is an injective immersion and $\dim Y = \operatorname{codim} \mathscr{F}$.

5.1.4. **Lemma.** *Let $X_\bullet$ be a foliation groupoid integrating a foliation $\mathscr{F}$ on X_0 and let $\phi_0 \colon Y_0 \to X_0$ be transverse to $\mathscr{F}$. Then the pullback groupoid $Y_\bullet = \phi_0^*(X_\bullet)$ is a foliation groupoid which integrates the foliation $\phi_0^*\mathscr{F}$, and the induced morphism $\phi_\bullet \colon Y_\bullet \to X_\bullet$ is fully faithful. If ϕ_0 is complete, then $\phi_\bullet$ is a Morita morphism.*

Sketch of proof. This is well-known when ϕ_0 is a transversal in the usual sense, in which case $\phi_0^*\mathscr{F}$ is zero-dimensional and $Y_\bullet$ is étale; see e.g. Crainic and Moerdijk [14, Lemma 2]. The general case is proved in a similar way: since ϕ_0 is transverse to $\mathscr{F}$, (ϕ_0, ϕ_0) is transverse to (s, t), so $Y_\bullet$ is a Lie groupoid, $\phi_1 \colon Y_1 \to X_1$ given by $\phi_1(x, f, y) = f$ is smooth, $\phi_\bullet$ is fully faithful and, if ϕ_0 is complete, essentially surjective. QED

5.2. **The Bott connection.** Let $(X, \mathscr{F})$ be a foliated Hausdorff manifold and let $N\mathscr{F} = TX/T\mathscr{F} \to X$ be the normal bundle of the foliation. The vector fields tangent to $\mathscr{F}$ form a Lie subalgebra $\Gamma(T\mathscr{F})$ of $\mathrm{Vect}(X)$. Since X is Hausdorff, we have $\Gamma(N\mathscr{F}) \cong \mathrm{Vect}(X)/\Gamma(T\mathscr{F})$, so $N\mathscr{F}$ is a $\Gamma(T\mathscr{F})$-module. Let us write $\nabla_w v$ for the action of a section $w \in \Gamma(T\mathscr{F})$ on a section $v \in \Gamma(N\mathscr{F})$. The operation

$$\nabla \colon \Gamma(T\mathscr{F}) \times \Gamma(N\mathscr{F}) \longrightarrow \Gamma(N\mathscr{F}) \tag{5.2.1}$$

is the *Bott connection* or *partial connection* of $\mathscr{F}$. A section $v \in \Gamma(N\mathscr{F})$ is $\Gamma(T\mathscr{F})$*-invariant* if $\nabla_w v = 0$ for all $w \in \Gamma(T\mathscr{F})$. We denote by

$$\mathrm{Vect}_0(X, \mathscr{F}) = \Gamma(N\mathscr{F})^{\Gamma(T\mathscr{F})} \tag{5.2.2}$$

the space of all $\Gamma(T\mathscr{F})$-invariant sections of $N\mathscr{F}$. Let $\mathfrak{N}$ be the normalizer of the Lie subalgebra $\Gamma(T\mathscr{F})$ of $\mathrm{Vect}(X)$. Elements of $\mathfrak{N}$ are vector fields v satisfying $[v, w] \in \Gamma(T\mathscr{F})$ for all $w \in \Gamma(T\mathscr{F})$, in other words, whose flow maps integral manifolds of $\mathscr{F}$ to integral manifolds of $\mathscr{F}$. The natural map $\mathfrak{N} \to \mathrm{Vect}(X) \to \Gamma(N\mathscr{F})$ has image equal to $\mathrm{Vect}_0(X, \mathscr{F})$ and kernel equal to $\Gamma(T\mathscr{F})$, so $\mathrm{Vect}_0(X, \mathscr{F}) \cong \mathfrak{N}/\Gamma(T\mathscr{F})$ is naturally a Lie algebra. If the space of leaves $X/\mathscr{F}$ has a manifold structure making the quotient map $X \to X/\mathscr{F}$ a submersion, then $\mathrm{Vect}_0(X, \mathscr{F}) \cong \mathrm{Vect}(X/\mathscr{F})$.

5.2.3. **Remark.** If X is a non-Hausdorff manifold equipped with a foliation $\mathscr{F}$, we may not have $\Gamma(N\mathscr{F}) \cong \mathrm{Vect}(X)/\Gamma(T\mathscr{F})$, but we can still define a Bott connection in the following manner. Let $\mathscr{C}^\infty(E)$ denote the sheaf of smooth sections of a vector bundle E over X. Then $\mathscr{C}^\infty(N\mathscr{F})$ is the sheaf associated to the presheaf $U \mapsto \mathrm{Vect}(U)/\Gamma(T\mathscr{F}|_U)$, and $\Gamma(N\mathscr{F})$ is the space of global sections of $\mathscr{C}^\infty(N\mathscr{F})$. For each open $U \subseteq X$ we have an operation

$$\nabla_U : \Gamma(T\mathscr{F}|_U) \times \mathrm{Vect}(U)/\Gamma(T\mathscr{F}|_U) \longrightarrow \mathrm{Vect}(U)/\Gamma(T\mathscr{F}|_U),$$

which is a morphism of presheaves. This presheaf morphism extends to a morphism of sheaves $\nabla\colon \mathscr{C}^\infty(T\mathscr{F}) \times \mathscr{C}^\infty(N\mathscr{F}) \to \mathscr{C}^\infty(N\mathscr{F})$. On global sections this gives the Bott connection (5.2.1). We define the space $\mathrm{Vect}_0(X, \mathscr{F})$ as in (5.2.2); it carries a natural Lie bracket just as in the Hausdorff case.

5.2.4. **Definition.** Let $(X, \mathscr{F})$ and $(X', \mathscr{F}')$ be foliated manifolds. A smooth map $\phi\colon X \to X'$ is *foliate* if the image of each leaf $\mathscr{F}(x)$ of $\mathscr{F}$ is contained in the leaf $\mathscr{F}'(\phi(x))$ of $\mathscr{F}'$.

The tangent map $T\phi\colon TX \to TX'$ of a foliate map $\phi\colon (X, \mathscr{F}) \to (X', \mathscr{F}')$ descends to a map $\phi_*\colon N\mathscr{F} \to N\mathscr{F}'$. We say that sections $v \in \Gamma(N\mathscr{F})$ and $v' \in \Gamma(N\mathscr{F}')$ are *ϕ-related*, and we write

$$v \sim_\phi v', \tag{5.2.5}$$

if $\phi_*(v(x)) = v'(\phi(x))$ for all $x \in X$. The Bott connection has the following naturality property with respect to foliate maps:

$$\nabla_w v \sim_\phi \nabla_{w'} v' \tag{5.2.6}$$

for all $v \in \Gamma(N\mathscr{F})$, $w \in \Gamma(T\mathscr{F})$, $v' \in \Gamma(N\mathscr{F}')$, and $w' \in \Gamma(T\mathscr{F}')$ satisfying $v \sim_\phi v'$ and $w \sim_\phi w'$.

5.3. **Basic vector fields and forms.** Let $X_\bullet = (X_1 \rightrightarrows X_0)$ be a foliation groupoid integrating a foliation $\mathscr{F}_0 = \mathscr{F}_0(X_\bullet)$ of X_0. Let $N_0 = N_0(X_\bullet) = TX_0/T\mathscr{F}_0$ be the normal bundle of the foliation. The leaves of $\mathscr{F}_0$ are the connected components of the orbits $X_\bullet \cdot x$, and for all $f \in X_1$ with $s(f) = x$ we have

$$T_x\mathscr{F}_0 \cong \ker(T_f s)/(\ker(T_f s) \cap \ker(T_f t)).$$

There is a left action of the product groupoid $X_\bullet \times X_\bullet$ on the arrow manifold X_1

$$(X_1 \times X_1) \times_{(s,s), X_0 \times X_0, (t,s)} X_1 \to X_1$$

given by the formula $(g, h) \cdot f = g \circ f \circ h^{-1}$. The tangent space to the orbit $(X_\bullet \times X_\bullet) \cdot f$ is $\ker(T_f s) + \ker(T_f t)$, which is of constant dimension. So the components of the $X_\bullet \times X_\bullet$-orbits define a foliation $\mathscr{F}_1 = \mathscr{F}_1(X_\bullet)$ of X_1 with normal bundle equal to

$$N_1 = N_1(X_\bullet) = TX_1/T\mathscr{F}_1 = TX_1/(\ker(Ts) + \ker(Tt)).$$

The source map $s\colon X_1 \to X_0$ is a foliate map and therefore descends to a vector bundle map

$$s_*\colon N_1 \longrightarrow N_0. \tag{5.3.1}$$

5.3.2. **Lemma.** *The map (5.3.1) induces an isomorphism $N_1 \cong s^*N_0$. Therefore we have a well-defined pullback map*

$$s^*\colon \Gamma(N_0) \longrightarrow \Gamma(N_1),$$

which restricts to a Lie algebra homomorphism $s^*\colon \mathrm{Vect}_0(X_0, \mathscr{F}_0) \to \mathrm{Vect}_0(X_1, \mathscr{F}_1)$. Similarly, the target map induces an isomorphism $N_1 \cong t^*N_0$ and pullback maps

$$t^*\colon \Gamma(N_0) \longrightarrow \Gamma(N_1), \qquad t^*\colon \mathrm{Vect}_0(X_0, \mathscr{F}_0) \longrightarrow \mathrm{Vect}_0(X_1, \mathscr{F}_1).$$

Proof. For $f \in X_1$ and $x = s(f)$ we have

$$\begin{aligned}(N_1)_f &= T_f X_1/(\ker(T_f s) + \ker(T_f t)) \\ &\cong \big(T_f X_1/\ker(T_f s)\big)\big/\big((\ker(T_f s) + \ker(T_f t))/\ker(T_f s)\big) \\ &\cong \big(T_f X_1/\ker(T_f s)\big)\big/\big(\ker(T_f s)/(\ker(T_f s) \cap \ker(T_f t))\big) \\ &\cong T_x X_0/T_x\mathscr{F}_0 \\ &= (N_0)_x,\end{aligned}$$

which proves the first statement. The existence of the pullback map follows immediately from this, and the naturality property (5.2.6) of the Bott connection implies that the induced map $\mathrm{Vect}_0(X_0, \mathscr{F}_0) \to \mathrm{Vect}_0(X_1, \mathscr{F}_1)$ preserves the Lie bracket. The proof of the last statement is analogous. QED

Because of Lemma 5.3.2 the following definition makes sense.

5.3.3. **Definition.** A *basic vector field* on a foliation groupoid $X_\bullet$ is an element of

$$\begin{aligned}\mathrm{Vect}_{\mathrm{bas}}(X_\bullet) &= \{\, v_\bullet = (v_0, v_1) \in \Gamma(N_0) \times \Gamma(N_1) \mid s^*v_0 = v_1 = t^*v_0 \,\} \\ &\cong \{\, v \in \Gamma(N_0) \mid s^*v = t^*v \,\}.\end{aligned}$$

Thus a basic vector field is not a vector field, but a pair of equivalence classes of vector fields. Although a basic vector field $v_\bullet = (v_0, v_1)$ is determined by its first component v_0, we prefer, mainly for notational consistency, to think of it as a pair of sections.

5.3.4. **Definition.** A *basic differential form* on a foliation groupoid $X_\bullet$ is a pair of differential forms $\zeta_\bullet = (\zeta_0, \zeta_1) \in \Omega^\bullet(X_0) \times \Omega^\bullet(X_1)$ satisfying $s^*\zeta_0 = \zeta_1 = t^*\zeta_0$. The set of basic differential forms on $X_\bullet$ is the *basic de Rham complex*

$$\begin{aligned}\Omega^\bullet_{\mathrm{bas}}(X_\bullet) &= \{\, \zeta_\bullet = (\zeta_0, \zeta_1) \in \Omega^\bullet(X_0) \times \Omega^\bullet(X_1) \mid s^*\zeta_0 = \zeta_1 = t^*\zeta_0 \,\} \\ &\cong \{\, \zeta \in \Omega^\bullet(X_0) \mid s^*\zeta = t^*\zeta \,\}.\end{aligned}$$

Again, although a basic form $\zeta_\bullet = (\zeta_0, \zeta_1)$ is determined by its first component ζ_0, we prefer to think of it as a pair of forms. Clearly $\Omega^\bullet_{\mathrm{bas}}(X_\bullet)$ is a subcomplex of $\Omega^\bullet(X_0) \times \Omega^\bullet(X_1)$. On the basic de Rham complex of a foliation groupoid we have the contraction operators and Lie derivatives

$$\begin{aligned}\iota\colon\ &\mathrm{Vect}_{\mathrm{bas}}(X_\bullet) \times \Omega^\bullet_{\mathrm{bas}}(X_\bullet) \longrightarrow \Omega^{\bullet-1}_{\mathrm{bas}}(X_\bullet), \\ \mathscr{L}\colon\ &\mathrm{Vect}_{\mathrm{bas}}(X_\bullet) \times \Omega^\bullet_{\mathrm{bas}}(X_\bullet) \longrightarrow \Omega^\bullet_{\mathrm{bas}}(X_\bullet)\end{aligned} \tag{5.3.5}$$

defined by

$$\iota_{v_\bullet}\zeta_\bullet = (\iota_{\tilde v_0}\zeta_0, \iota_{\tilde v_1}\zeta_1), \qquad \mathscr{L}_{v_\bullet}\zeta_\bullet = (\mathscr{L}_{\tilde v_0}\zeta_0, \mathscr{L}_{\tilde v_1}\zeta_1) \tag{5.3.6}$$

for basic forms $\zeta_\bullet = (\zeta_0, \zeta_1)$ and basic vector fields $v_\bullet = (v_0, v_1)$, where $(\tilde v_0, \tilde v_1) \in \mathrm{Vect}(X_0) \times \mathrm{Vect}(X_1)$ is a representative of v. The contraction operators $\iota_{v_\bullet}$ are well-defined, i.e. independent of the choice of representatives $(\tilde v_0, \tilde v_1)$. To see this, we must show that for all $\zeta \in \Omega^\bullet(X_0)$ satisfying $s^*\zeta = t^*\zeta$ and for all tangent vectors $v \in T_x\mathscr{F}$ we have $\iota_v\zeta = 0$. There exist $f \in s^{-1}(x)$ and $w \in \ker(T_f s)$ such that $v = T_f t(w)$, so

$$t^*\iota_v\zeta = \iota_w t^*\zeta = \iota_w s^*\zeta = s^*\iota_{T_f s(w)}\zeta = 0,$$

and therefore $\iota_v\zeta = 0$. Similarly, the operators $\mathscr{L}_{v_\bullet}$ are well-defined.

5.3.7. **Definition.** Let $X_\bullet$ be a foliation groupoid. A differential form $\zeta \in \Omega^k(X_0)$ is *horizontal* if $\iota_{\rho(\sigma)}\zeta = 0$ for all sections σ of the Lie algebroid $\mathrm{Alg}(X_\bullet)$, and *infinitesimally invariant* if $\mathscr{L}_{\rho(\sigma)}\zeta = 0$ for all sections σ of $\mathrm{Alg}(X_\bullet)$. We denote by $\Omega^k_0(X_0, \mathscr{F}_0)$ the set of all k-forms on X_0 which are horizontal and infinitesimally invariant.

The notions in Definition 5.3.7 depend only on the Lie algebroid of $X_\bullet$, i.e. on the foliation $\mathscr{F}_0$. The notions of horizontal, basic, and invariant forms are well-known in the context of Lie group actions, for which the second part of the next result is standard.

5.3.8. **Lemma.** *Let $X_\bullet$ be a foliation groupoid.*

(i) *Under the identification* $\mathrm{Vect}_{\mathrm{bas}}(X_\bullet) \cong \{\, v \in \Gamma(N_0) \mid s^*v = t^*v \,\}$, *the set of basic vector fields* $\mathrm{Vect}_{\mathrm{bas}}(X_\bullet)$ *is a Lie subalgebra of the Lie algebra* $\mathrm{Vect}_0(X_0, \mathscr{F}_0)$ *defined in* (5.2.2).

(ii) *Under the identification* $\Omega^\bullet_{\mathrm{bas}}(X_\bullet) \cong \{\, \zeta \in \Omega^\bullet(X_0) \mid s^*\zeta = t^*\zeta \,\}$, *the basic complex* $\Omega^\bullet_{\mathrm{bas}}(X_\bullet)$ *is a subcomplex of the complex* $\Omega^\bullet_0(X_0, \mathscr{F}_0)$ *of Definition* 5.3.7.

Both inclusions are equalities if $X_\bullet$ is source-connected.

Proof. (i) Let $v_\bullet = (v_0, v_1) \in \mathrm{Vect}_{\mathrm{bas}}(X_\bullet)$. Let σ be a section of the Lie algebroid $\mathrm{Alg}(X_\bullet)$, which has anchor map $\rho = Tt|_{\mathrm{Alg}(X_\bullet)}$. Let σ_R denote the right-invariant vector field on X_1 induced by σ. Then σ_R is s-related to the zero vector field on X_0 because σ_R is tangent to the source fibres, and σ_R is t-related to the vector field $\rho(\sigma) \in \Gamma(T\mathscr{F})$ on X_0. In other words,

$$\sigma_R \sim_s 0, \qquad \sigma_R \sim_t \rho(\sigma). \tag{5.3.9}$$

Since $v_1 = s^*v_0 = t^*v_0$, the naturality of the Bott connection (5.2.6) yields

$$t^*\nabla_{\rho(\sigma)}v_0 = \nabla_{\sigma_R}t^*v_0 = \nabla_{\sigma_R}v_1 = \nabla_{\sigma_R}s^*v_0 = 0,$$

and hence $\nabla_{\rho(\sigma)}v_0 = 0$ for all σ, which is to say that v_0 is $\Gamma(T\mathscr{F})$-invariant. This shows that $\mathrm{Vect}_{\mathrm{bas}}(X_\bullet)$ is contained in $\mathrm{Vect}_0(X_0, \mathscr{F}_0)$. That $\mathrm{Vect}_{\mathrm{bas}}(X_\bullet)$ is a Lie subalgebra follows from the fact that s^* and t^* are Lie algebra homomorphisms. Conversely, suppose $\nabla_{\rho(\sigma)}v_0 = 0$ for all $\sigma \in \Gamma(\mathrm{Alg}(X_\bullet))$ and that $X_\bullet$ is source-connected. From (5.3.9) we obtain $\nabla_{\sigma_R}s^*v_0 = 0$ and $\nabla_{\sigma_R}t^*v_0 = t^*\nabla_{\rho(\sigma)}v_0 = 0$ for all σ. For $x \in X_0$ we have $(s^*v_0)_{u(x)} = (t^*v_0)_{u(x)}$. The vector fields σ_R span the subbundle $\ker(Ts)$ of TX_1. The source fibre $s^{-1}(x)$ is Hausdorff (by Lemma 4.1.5, since X_0 is Hausdorff by definition) and connected, so we have $(s^*v_0)_f = (t^*v_0)_f$ for every $f \in s^{-1}(x)$. This shows that $\mathrm{Vect}_0(X_0, \mathscr{F}_0)$ is contained in $\mathrm{Vect}_{\mathrm{bas}}(X_\bullet)$.

(ii) Let $\zeta \in \Omega^k(X_0)$ be a differential form. It follows from (5.3.9) that

$$\mathscr{L}_{\sigma_R}s^*\zeta = 0, \qquad \mathscr{L}_{\sigma_R}t^*\zeta = t^*\mathscr{L}_{\rho(\sigma)}\zeta \tag{5.3.10}$$

for all sections σ of the Lie algebroid $\mathrm{Alg}(X_\bullet)$. At the identity bisection $u(X_0) \subseteq X_1$ the tangent bundle of X_1 is a direct sum $TX_1|_{u(X_0)} = \mathrm{Alg}(X_\bullet) \oplus u^*TX_0$. Let $x \in X_0$. For σ_1, $\sigma_2, \dots, \sigma_k \in \mathrm{Alg}(X_\bullet)_{u(x)}$ and for $w_1, w_2, \dots, w_k \in T_xX_0$ we have

$$\begin{aligned}
&(s^*\zeta)_{u(x)}(\sigma_1 + u_*w_1, \sigma_2 + u_*w_2, \dots, \sigma_k + u_*w_k)\\
&\qquad = \zeta_x(s_*\sigma_1 + w_1, s_*\sigma_2 + w_2, \dots, s_*\sigma_k + w_k)\\
&\qquad = \zeta_x(w_1, w_2, \dots, w_k),\\
&(t^*\zeta)_{u(x)}(\sigma_1 + u_*w_1, \sigma_2 + u_*w_2, \dots, \sigma_k + u_*w_k)\\
&\qquad = \zeta_x(t_*\sigma_1 + w_1, t_*\sigma_2 + w_2, \dots, t_*\sigma_k + w_k)\\
&\qquad = \zeta_x\big(\rho(\sigma_1) + w_1, \rho(\sigma_2) + w_2, \dots, \rho(\sigma_k) + w_k\big).
\end{aligned} \tag{5.3.11}$$

Now assume $s^*\zeta = t^*\zeta$. Then $\mathscr{L}_{\rho(\sigma)}\zeta = 0$ by (5.3.10) and $\iota_{\rho(\sigma)}\zeta = 0$ by (5.3.11), so $\zeta \in \Omega^\bullet_0(X_0, \mathscr{F}_0)$. Finally, suppose that $\zeta \in \Omega^\bullet_0(X_0, \mathscr{F}_0)$ and that $X_\bullet$ is source-connected. From horizontality and from (5.3.11) we obtain that $(s^*\zeta)_{u(x)} = (t^*\zeta)_{u(x)}$ for every $x \in X_0$. From invariance and from (5.3.10) we obtain that $\mathscr{L}_{\sigma_R}s^*\zeta = \mathscr{L}_{\sigma_R}t^*\zeta = 0$. The vector fields σ_R span the subbundle $\ker(Ts)$ of TX_1. The source fibre $s^{-1}(x)$ is Hausdorff and connected, so we have $(s^*\zeta)_f = (t^*\zeta)_f$ for every $f \in s^{-1}(x)$. QED

The next result states that the notions of basic vector fields and basic differential forms are Morita invariant.

5.3.12. **Proposition.** *Let $X_\bullet$ and $Y_\bullet$ be foliation groupoids. A Morita morphism $\phi_\bullet \colon X_\bullet \to Y_\bullet$ induces*

(i) *an isomorphism $\phi_\bullet^* \colon \mathrm{Vect}_{\mathrm{bas}}(Y_\bullet) \xrightarrow{\cong} \mathrm{Vect}_{\mathrm{bas}}(X_\bullet)$ of Lie algebras, and*

(ii) *an isomorphism $\phi_\bullet^* \colon \Omega^\bullet_{\mathrm{bas}}(Y_\bullet) \xrightarrow{\cong} \Omega^\bullet_{\mathrm{bas}}(X_\bullet)$ of complexes.*

If $\phi_\bullet$ and $\chi_\bullet$ are naturally isomorphic Morita morphisms, then $\phi_\bullet^* = \chi_\bullet^*$.

Proof. We will prove (i); the proof of (ii) is similar. Let $\mathscr{F}_0(X_\bullet)$ be the foliation of X_0 and $N_0(X_\bullet) = TX_0/T\mathscr{F}_0(X_\bullet)$ its normal bundle. Let $\mathscr{F}_1(X_\bullet)$ be the foliation of X_1 and $N_1(X_\bullet) = TX_1/(\ker(Ts) + \ker(Tt))$ its normal bundle. Similarly we have foliations $\mathscr{F}_0(Y_\bullet)$ of Y_0, $\mathscr{F}_1(Y_\bullet)$ of Y_1, and normal bundles $N_0(Y_\bullet)$ over Y_0 and $N_1(Y_\bullet)$ over Y_1. The tangent maps $T\phi_0 \colon TX_0 \to TY_0$ and $T\phi_1 \colon TX_1 \to TY_1$ descend to vector bundle maps

$$(\phi_0)_* \colon N_0(X_\bullet) \longrightarrow N_0(Y_\bullet), \qquad (\phi_1)_* \colon N_1(X_\bullet) \longrightarrow N_1(Y_\bullet).$$

Let $\chi_\bullet \colon X_\bullet \to Y_\bullet$ be another Morita morphism and $\gamma \colon \phi \Rightarrow \chi$ a 2-morphism. Then $s \circ \gamma = \phi_0$ and $t \circ \gamma = \chi_0$, and for all arrows $f \in X_1$ we have

$$\gamma(x') = \chi(f) \circ \gamma(x) \circ \phi(f)^{-1},$$

where $x = s(f)$ and $x' = t(f)$. This shows that γ maps the orbit $X_\bullet \cdot x$ to the orbit $(Y_\bullet \times Y_\bullet) \cdot \gamma(x)$, so $\gamma \colon (X_0, \mathscr{F}_0(X_\bullet)) \to (Y_1, \mathscr{F}_1(Y_\bullet))$ is a foliate map. Therefore γ induces a vector bundle map $\gamma_* \colon N_0(X_\bullet) \to N_1(Y_\bullet)$, which makes the diagram

(5.3.13)
$$\begin{array}{ccc} N_0(X_\bullet)_x & \xrightarrow{(\phi_0)_*} & N_0(Y_\bullet)_{\phi_0(x)} \\ {\scriptstyle (\chi_0)_*}\downarrow & \searrow^{\gamma_*} & \uparrow{\scriptstyle s_*}\,{\scriptstyle\cong} \\ N_0(Y_\bullet)_{\chi_0(x)} & \xleftarrow[\cong]{t_*} & N_1(Y_\bullet)_{\gamma(x)} \end{array}$$

commute for all $x \in X_0$. We will use this to establish the following four facts:

(a) $(\phi_0)_* \colon N_0(X_\bullet)_x \to N_0(Y_\bullet)_{\phi_0(x)}$ is an isomorphism for all $x \in X_0$;

(b) the squares

$$\begin{array}{ccc} N_0(X_\bullet) & \xrightarrow{(\phi_0)_*} & N_0(Y_\bullet) \\ \downarrow & & \downarrow \\ X_0 & \xrightarrow{\phi_0} & Y_0 \end{array} \qquad\qquad \begin{array}{ccc} N_1(X_\bullet) & \xrightarrow{(\phi_1)_*} & N_1(Y_\bullet) \\ \downarrow & & \downarrow \\ X_1 & \xrightarrow{\phi_1} & Y_1 \end{array}$$

are cartesian, $\phi_\bullet$ induces pullback maps

$$\phi_0^* \colon \Gamma(N_0(Y_\bullet)) \longrightarrow \Gamma(N_0(X_\bullet)), \qquad \phi_1^* \colon \Gamma(N_1(Y_\bullet)) \longrightarrow \Gamma(N_1(X_\bullet)),$$

and a Lie algebra homomorphism $\phi_\bullet^* \colon \mathrm{Vect}_{\mathrm{bas}}(Y_\bullet) \longrightarrow \mathrm{Vect}_{\mathrm{bas}}(X_\bullet)$;

(c) $\phi_\bullet^* = \chi_\bullet^*$;

(d) $\phi_\bullet^*$ is an isomorphism.

We assert that fact (a) implies fact (b). Indeed, it follows from fact (a) that the square on the left in (b) is cartesian. The map $(\phi_1)_*$ is obtained by lifting $(\phi_0)_*$ via s_*, and therefore the right square is cartesian as well. This means we have isomorphisms $N_0(X_\bullet) \cong \phi_0^* N_0(Y_\bullet)$ and $N_1(X_\bullet) \cong \phi_1^* N_1(Y_\bullet)$, and the existence of the pullback maps ϕ_0^*, ϕ_1^*, $\phi_\bullet^*$ is a formal consequence of this. To check that $\phi_\bullet^*$ is a Lie algebra homomorphism, it is enough to show that for all basic vector fields $u_\bullet, v_\bullet \in \mathrm{Vect}_{\mathrm{bas}}(Y_\bullet)$ we have

$$[\phi_0^* u_0, \phi_0^* v_0] \sim_{\phi_0} [u_0, v_0] \tag{5.3.14}$$

in the sense of (5.2.5). To prove this, we may work in foliation charts and assume that the leaf spaces $\bar{X} = X_0/\mathscr{F}_0(X_\bullet)$ and $\bar{Y} = Y_0/\mathscr{F}_0(Y_\bullet)$ are both smooth manifolds, and that ϕ_0 descends to a smooth map $\bar{\phi}\colon \bar{X} \to \bar{Y}$. By Lemma 5.3.8 we have an inclusion

$$\mathrm{Vect}_{\mathrm{bas}}(X_\bullet) \subseteq \mathrm{Vect}_0(X_0, \mathscr{F}_0(X_\bullet)) \cong \mathrm{Vect}(\bar{X})$$

and a similar inclusion for $Y_\bullet$. The identity (5.3.14) now follows from the naturality of the Lie bracket on $\mathrm{Vect}(\bar{X})$ and $\mathrm{Vect}(\bar{Y})$ with respect to $\bar{\phi}$. Thus fact (b) follows from fact (a). Fact (c) is proved as follows: if $v \in \Gamma(N_0(Y_\bullet))$ satisfies $s^*v = t^*v$, then

$$\phi_0^* v = \gamma^* s^* v = \gamma^* t^* v = \chi_0^* v,$$

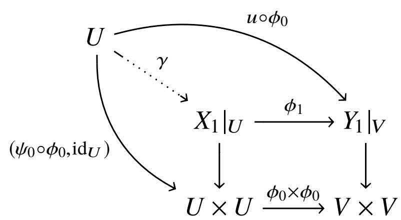
so $\phi_\bullet^* = \chi_\bullet^*$. It remains to prove (a) and (d). We consider three cases.

Case 1. Suppose $\phi_\bullet$ has a weak inverse, i.e. a triple $(\psi_\bullet, \delta, \varepsilon)$ consisting of a Morita morphism $\psi_\bullet\colon Y_\bullet \to X_\bullet$ and 2-morphisms $\delta\colon \psi_\bullet \circ \phi_\bullet \Rightarrow \mathrm{id}_{X_\bullet}$ and $\varepsilon\colon \phi_\bullet \circ \psi_\bullet \Rightarrow \mathrm{id}_{Y_\bullet}$. Applying (5.3.13) to the morphisms $\psi_\bullet \circ \phi_\bullet$ and $\mathrm{id}_{X_\bullet}$ shows that

$$(\psi_0)_* \circ (\phi_0)_*\colon (N_0)_x \to (N_0)_{\psi(\phi(x))}$$

is an isomorphism for all $x \in X_0$. Similarly, $(\phi_0)_* \circ (\psi_0)_*\colon (N_0)_y \to (N_0)_{\phi(\psi(y))}$ is an isomorphism for all $y \in Y_0$. Hence $(\phi_0)_*\colon (N_0)_x \to (N_0)_{\phi(x)}$ is an isomorphism for all x, which establishes (a). It follows from fact (c) that $\psi_\bullet^*\phi_\bullet^* = \mathrm{id}$ and $\phi_\bullet^*\psi_\bullet^* = \mathrm{id}$, so $\phi_\bullet^*$ is an isomorphism, which proves (d).

Case 2. Suppose $\phi_0\colon X_0 \to Y_0$ is a surjective submersion. Consider an open subset V of Y_0 over which there exists a smooth right inverse $\psi_0\colon V \to X_0$ of ϕ_0. Let $Y_\bullet|_V$ be the restriction of $Y_\bullet$ to V, which has object manifold V and arrow manifold $Y_1|_V = s^{-1}(V) \cap t^{-1}(V)$. Let $U = \phi_0^{-1}(V)$ and let $X_\bullet|_U$ be the restriction of $X_\bullet$ to U. Then $\phi_\bullet$ restricts to a Morita morphism $X_\bullet|_U \to Y_\bullet|_V$, which we also denote by $\phi_\bullet$. By full faithfulness of $\phi_\bullet$, the map ψ_0 lifts uniquely to a morphism $\psi_\bullet = (\psi_0, \psi_1)\colon Y_\bullet|_V \to X_\bullet|_U$ satisfying $\phi_\bullet \circ \psi_\bullet = \mathrm{id}_{Y_\bullet|_V}$. Full faithfulness also gives us a map γ completing the following commutative diagram:



This means that for each $x \in U$, $\phi_1(\gamma(x))$ is the identity arrow at $\phi_0(x)$. Again by full faithfulness, this implies that $\gamma(x)$ is an arrow from $\psi_0(\phi_0(x))$ to x satisfying $f \circ \gamma(x) = \gamma(x') \circ \psi_1(\phi_1(f))$ for all arrows $f \in X_1$ with $s(f) = x \in U$, $t(f) = x' \in U$. In other words, γ is a 2-morphism $\psi \circ \phi \Rightarrow \mathrm{id}_{X_\bullet|_U}$. By case 1, $\phi_\bullet$ induces isomorphisms $N_0(X_\bullet)_x \cong N_0(Y_\bullet)_{\phi_0(x)}$ for all $x \in U$, and

$$\phi_\bullet^*\colon \mathrm{Vect}_{\mathrm{bas}}(Y_\bullet|_V) \xrightarrow{\cong} \mathrm{Vect}_{\mathrm{bas}}(X_\bullet|_U), \tag{5.3.15}$$

which are independent of the section ψ_0. Now choose a covering $\mathfrak{V}$ of Y_0 consisting of open sets V over which ϕ_0 admits a right inverse. The sets $U = \phi^{-1}(V)$ cover X_0, so we get $N_0(X_\bullet)_x \cong N_0(Y_\bullet)_{\phi_0(x)}$ for all $x \in X_0$, which proves fact (a). Since ϕ_0 is surjective, the pullback map $\phi_0^*\colon \Gamma(N_0(Y_\bullet)) \to \Gamma(N_0(X_\bullet))$ is injective, so in particular $\phi_\bullet^*\colon \mathrm{Vect}_{\mathrm{bas}}(Y_\bullet) \to \mathrm{Vect}_{\mathrm{bas}}(X_\bullet)$ is injective. To show surjectivity of $\phi_\bullet^*$, let $v \in \Gamma(N_0(X_\bullet))$ be a section satisfying $s^*v = t^*v$. By (5.3.15) there is for each $V \in \mathfrak{V}$ a unique section w_V of $N_0(X_\bullet)$ over V satisfying $s_V^* w_V = t_V^* w_V$ and $\phi_0^* w_V = v|_U$. By injectivity of ϕ_0^* we have $w_V = w_{V'}$ on $V \cap V'$ for all $V, V' \in \mathfrak{V}$. Therefore the w_V glue together to a global section w of $N_0(Y_\bullet)$ with $\phi_0^* w = v$. Since

$$\phi_0^* s^* w = s^* \phi_0^* w = s^* v = t^* v = t^* \phi_0^* w = \phi_0^* t^* w,$$

we have $s^*w = t^*w$, so $\phi_\bullet^*$ is surjective, which establishes fact (d).

Case 3. Given a general Morita morphism $\phi_\bullet\colon X_\bullet \to Y_\bullet$, essential surjectivity gives us a surjective submersion

$$\tau_0 = t \circ \mathrm{pr}_1\colon Z_0 = Y_1 \times_{Y_0} X_0 \longrightarrow Y_0.$$

Let $Z_\bullet = \tau_0^* Y_\bullet$; then the canonical morphism $\tau_\bullet\colon Z_\bullet \to Y_\bullet$ is Morita. The projection onto the second factor $\pi_0 = \mathrm{pr}_2\colon Z_0 \to X_0$ is also a surjective submersion, and lifts to a Morita morphism $\pi_\bullet\colon Z_\bullet \to X_\bullet$. We do *not* have $\phi_\bullet \circ \pi_\bullet = \tau_\bullet$, but the map $\gamma = \mathrm{pr}_1\colon Z_0 \to Y_1$ defines a 2-morphism $\gamma\colon \phi_\bullet \circ \pi_\bullet \Rightarrow \tau_\bullet$. By case 2, the morphisms $\pi_\bullet$ and $\tau_\bullet$ induce isomorphisms

$$(\pi_0)_*\colon N_0(Z_\bullet)_z \longrightarrow N_0(X_\bullet)_{\pi_0(z)} \quad \text{and} \quad (\tau_0)_\bullet\colon N_0(Z_\bullet)_z \longrightarrow N_0(Y_\bullet)_{\tau_0(z)}$$

for all $z \in Z_0$. Applying (5.3.13) to the morphisms $\phi_\bullet \circ \pi_\bullet$ and $\tau_\bullet$ shows that $(\phi_0)_* \circ (\pi_0)_*\colon N_0(Z_\bullet)_z \to N_0(Y_\bullet)_{\phi_0(\pi_0(z))}$ is an isomorphism for all $z \in Z_0$. Hence $(\phi_0)_*\colon N_0(X_\bullet)_x \to N_0(Y_\bullet)_{\phi_0(x)}$ is an isomorphism for all $x \in X_0$, which proves (a). Fact (c) shows that $\pi_\bullet^* \circ \phi_\bullet^* = \tau_\bullet^*$. The maps $\pi_\bullet^*$ and $\tau_\bullet^*$ are isomorphisms by case 2, so $\phi_\bullet^*$ is an isomorphism as well, which proves (d). QED

5.3.16. **Remark.** If we extend the definitions of $\Omega^\bullet_{\mathrm{bas}}(X_\bullet)$ and $\Omega^\bullet_0(X_\bullet)$ verbatim to arbitrary Lie groupoids $X_\bullet$, Lemma 5.3.8(ii) and Proposition 5.3.12(ii) still hold. We omit the proof of this fact, as we have no use for it in this paper.

5.4. **Basic versus multiplicative vector fields.** Let $X_\bullet$ be a Lie groupoid. Applying the tangent functor T to all the structure maps of $X_\bullet$ gives a Lie groupoid $TX_\bullet$ called the *tangent groupoid*, which is equipped with an obvious morphism $p_\bullet\colon TX_\bullet \to X_\bullet$.

Mackenzie and Xu [31] defined a *multiplicative vector field* on $X_\bullet$ to be a Lie groupoid morphism $v_\bullet\colon X_\bullet \to TX_\bullet$ satisfying $p_\bullet \circ v_\bullet = \mathrm{id}_{X_\bullet}$. We will denote the set of multiplicative vector fields by $\mathfrak{M}(X_\bullet)$.

By [31, Proposition 3.5] a multiplicative vector field is the same as a pair of vector fields $v_0 \in \mathrm{Vect}(X_0)$, $v_1 \in \mathrm{Vect}(X_1)$ whose flows form a (local) one-parameter group of Lie groupoid automorphisms of $X_\bullet$. It follows that $\mathfrak{M}(X_\bullet)$ is a Lie subalgebra of $\mathrm{Vect}(X_0) \times \mathrm{Vect}(X_1)$.

By [31, Example 3.4], for each section σ of the Lie algebroid $\mathfrak{A}(X_\bullet) = \Gamma(\mathrm{Alg}(X_\bullet))$ the pair $\partial(\sigma) = (\rho(\sigma), \sigma_L + \sigma_R)$ is a multiplicative vector field. This defines a Lie algebra homomorphism

$$\partial\colon \mathfrak{A}(X_\bullet) \longrightarrow \mathfrak{M}(X_\bullet).$$

Differentiating the conjugation action of $X_\bullet$ on X_1 gives a Lie algebra action $\mathfrak{M}(X_\bullet) \to \mathrm{Der}(\mathfrak{A}(X_\bullet))$, which makes the pair of Lie algebras $\mathfrak{M}(X_\bullet)$, $\mathfrak{A}(X_\bullet)$ a crossed module of Lie algebras. The associated (strict) Lie 2-algebra

$$\mathrm{Vect}_{\mathrm{mult}}(X_\bullet) = \big(\mathfrak{A}(X_\bullet) \rtimes \mathfrak{M}(X_\bullet) \rightrightarrows \mathfrak{M}(X_\bullet)\big)$$

is the *Lie 2-algebra of multiplicative vector fields* of $X_\bullet$; see Berwick-Evans and Lerman [4, § 2]. We assert that for a foliation groupoid this Lie 2-algebra is equivalent to the Lie algebra of basic vector fields (Definition 5.3.3).

5.4.1. **Proposition.** *Let $X_\bullet$ be a foliation groupoid. The natural Lie algebra homomorphism $v = v_{X_\bullet}\colon \mathfrak{M}(X_\bullet) \to \mathrm{Vect}_{\mathrm{bas}}(X_\bullet)$ induces an isomorphism*

$$\mathfrak{M}(X_\bullet)/\partial\mathfrak{A}(X_\bullet) \cong \mathrm{Vect}_{\mathrm{bas}}(X_\bullet).$$

Thus the Lie 2-algebra of multiplicative vector fields $\mathrm{Vect}_{\mathrm{mult}}(X_\bullet)$ is equivalent to the Lie algebra of basic vector fields $\mathrm{Vect}_{\mathrm{bas}}(X_\bullet)$.

Proof. Let $\mathscr{F}_0$, resp. $\mathscr{F}_1$, be the foliation of X_0, resp. X_1, induced by $X_\bullet$, and let $v_\bullet = (v_0, v_1)$ be a multiplicative vector field. Let $\bar{v}_\bullet = (\bar{v}_0, \bar{v}_1) \in \Gamma(N\mathscr{F}_0)\times\Gamma(N\mathscr{F}_1)$ be the pair of sections of the normal bundles determined by $v_\bullet$. The flow of $v_\bullet$ acts by groupoid automorphisms, so the flow of v_0 is foliate (Definition 5.2.4). Hence v_0 is in the normalizer of $\Gamma(T\mathscr{F}_0)$ in $\mathrm{Vect}(X_0)$, that is to say $\bar{v}_\bullet$ is a basic vector field. This defines the natural homomorphism $v\colon \mathfrak{M}(X_\bullet) \to \mathrm{Vect}_{\mathrm{bas}}(X_\bullet)$. The anchor $\rho\colon\ \mathrm{Alg}(X_\bullet) \to TX_0$ is injective and its image is $T\mathscr{F}_0$. In particular ∂ is injective, so the Lie 2-algebra $\mathrm{Vect}_{\mathrm{mult}}(X_\bullet)$ is equivalent to the quotient Lie algebra $\mathfrak{M}(X_\bullet)/\partial\mathfrak{A}(X_\bullet)$. Moreover v descends to a homomorphism

$$\bar{v}\colon \mathfrak{M}(X_\bullet)/\partial\mathfrak{A}(X_\bullet) \longrightarrow \mathrm{Vect}_{\mathrm{bas}}(X_\bullet).$$

We must show that $\bar{v}$ is an isomorphism. First suppose $X_\bullet$ is étale. Then $\mathrm{Alg}(X_\bullet) = 0$, the foliations $\mathscr{F}_0$ and $\mathscr{F}_1$ are discrete, and v is injective. If $v_0 \in \mathrm{Vect}(X_0)$ satisfies $s^*v_0 = t^*v_0$, then the vector field (v_1, v_1) on $X_1 \times X_1$, where $v_1 = s^*v_0 \in \mathrm{Vect}(X_1)$, is tangent to the submanifold $X_2 = X_1 \times_{s,X_0,t} X_1$ of $X_1 \times X_1$. So v_1 restricts to a vector field v_2 on X_2, and we have $v_2 = m^*v_1$, where $m\colon X_2 \to X_1$ is the multiplication map. In other words, every basic vector field is multiplicative and we have equality

$$\mathfrak{M}(X_\bullet)/\partial\mathfrak{A}(X_\bullet) = \mathfrak{M}(X_\bullet) = \mathrm{Vect}_{\mathrm{bas}}(X_\bullet).$$

For an arbitrary foliation groupoid $X_\bullet$, pick a Morita morphism $\phi_\bullet\colon Y_\bullet \to X_\bullet$ from an étale groupoid $Y_\bullet$. By Proposition 5.3.12(i) $\phi_\bullet$ induces an isomorphism $\phi_\bullet^*\colon\ \mathrm{Vect}_{\mathrm{bas}}(X_\bullet) \to \mathrm{Vect}_{\mathrm{bas}}(Y_\bullet)$. We let $Z = Y_0 \times_{\phi_0,X_0,s} X_1$ and as in [4, Theorem 4.4] we form the linking groupoid $L_\bullet$ with $L_0 = Y_0 \coprod X_0$ and $L_1 = Y_1 \coprod Z \coprod Z^{-1} \coprod X_1$. We have obvious open embeddings $\iota_{X_\bullet}\colon X_\bullet \hookrightarrow L_\bullet$ and $\iota_{Y_\bullet}\colon Y_\bullet \hookrightarrow L_\bullet$, both of which are essentially surjective. The triangle

$$\begin{array}{ccc} Y_\bullet & \xrightarrow{\ \phi_\bullet\ } & X_\bullet \\ & {\scriptstyle \iota_{Y_\bullet}}\searrow \quad \swarrow{\scriptstyle \iota_{X_\bullet}} & \\ & L_\bullet & \end{array}$$

is 2-commutative: the map $\gamma\colon Y_0 \to L_1$ given by $\gamma(y) = (\mathrm{id}_{\phi(y)}, y) \in Z \hookrightarrow L_1$ is a 2-morphism $\gamma\colon \iota_{Y_\bullet} \Rightarrow \iota_{X_\bullet} \circ \phi_\bullet$. It follows from [4, Lemma 4.14] that $\phi_\bullet$ induces an isomorphism

$$\phi_\bullet^* = \iota_{Y_\bullet}^* \circ (\iota_X^*)^{-1}\colon \mathfrak{M}(X_\bullet)/\partial\mathfrak{A}(X_\bullet) \to \mathfrak{M}(Y_\bullet).$$

The square

$$\begin{array}{ccc} \mathfrak{M}(X_\bullet)/\partial\mathfrak{A}(X_\bullet) & \xrightarrow{\ \bar{v}_{X_\bullet}\ } & \mathrm{Vect}_{\mathrm{bas}}(X_\bullet) \\ {\scriptstyle \phi_\bullet^*}\Big\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\Big\downarrow{\scriptstyle \phi_\bullet^*} \\ \mathfrak{M}(Y_\bullet) & \xrightarrow[=]{\ \bar{v}_{Y_\bullet}\ } & \mathrm{Vect}_{\mathrm{bas}}(Y_\bullet) \end{array}$$

commutes, so $\bar{v}_{X_\bullet}$ is an isomorphism. QED

5.5. **0-Symplectic groupoids.** Let $X_\bullet$ be a foliation groupoid with foliation $(X_0, \mathscr{F})$ and let $\omega_\bullet = (\omega_0, \omega_1) \in \Omega^2_{\mathrm{bas}}(X_\bullet)$ be a basic 2-form. We call $\omega_\bullet$ *nondegenerate* if $\ker(\omega_0) = T\mathscr{F}$, or equivalently, $\ker(\omega_1) = \ker(Ts) + \ker(Tt)$. We say $\omega_\bullet$ is 0-*symplectic*, and the pair $(X_\bullet, \omega_\bullet)$ is a 0-*symplectic Lie groupoid*, if $d\omega_\bullet = 0$ and $\omega_\bullet$ is nondegenerate.

We have adopted the term "0-symplectic" to avoid confusion with Weinstein's notion of a symplectic groupoid [49]. The nondegeneracy of $\omega_\bullet$ can be restated as the vertical map

$$\begin{array}{ccccc} T\mathscr{F} & \longrightarrow & TX_0 & \longrightarrow & 0 \\ \downarrow & & \downarrow \omega_0^\flat & & \downarrow \\ 0 & \longrightarrow & T^*X_0 & \longrightarrow & (T\mathscr{F})^* \end{array}$$

from the tangent to the cotangent complex of $X_\bullet$ being a quasi-isomorphism. The notion of a 0-symplectic form is analogous to that of a 0-shifted symplectic structure in the sense of [39, Definition 0.2] (whereas the form on a Weinstein symplectic groupoid is analogous to a 1-shifted symplectic structure). Nondegeneracy implies that contraction with a 0-symplectic form $\omega_\bullet$ induces a linear isomorphism

$$\omega_\bullet^\flat\colon \operatorname{Vect}_{\mathrm{bas}}(X_\bullet) \xrightarrow{\cong} \Omega^1_{\mathrm{bas}}(X_\bullet).$$

5.5.1. **Remark.** Conversely, if a form $\omega_\bullet \in \Omega^2_{\mathrm{bas}}(X_\bullet)$ induces a linear isomorphism $\operatorname{Vect}_{\mathrm{bas}}(X_\bullet) \cong \Omega^1_{\mathrm{bas}}(X_\bullet)$, it is not always the case that $\omega_\bullet$ is nondegenerate. For example, consider the action groupoid $H \ltimes \mathbb{R}^2$, where $H = \mathbb{Z}/2\mathbb{Z} \ltimes \mathbb{Q}^2$ is the semidirect product of the group of half rotations of $\mathbb{R}^2$ around the origin, and translation by $\mathbb{Q}^2$. Then $\operatorname{Vect}_{\mathrm{bas}}(X_\bullet) = \Omega^1_{\mathrm{bas}}(X_\bullet) = 0$, so $\omega_\bullet = 0$ induces an isomorphism $\operatorname{Vect}_{\mathrm{bas}}(X_\bullet) \cong \Omega^1_{\mathrm{bas}}(X_\bullet)$, but is not a 0-symplectic form.

5.5.2. **Proposition.**
(i) *Let $(X_\bullet, \omega_\bullet)$ be a 0-symplectic groupoid with foliation $\mathscr{F}$. Then (X_0, ω_0) is a presymplectic manifold with $\ker(\omega_0) = T\mathscr{F}$.*
(ii) *Let (X, ω) be a presymplectic manifold with null foliation $\mathscr{F}$. Let $X_\bullet$ be a source-connected foliation groupoid with Lie algebroid equal to $T\mathscr{F}$. Then $\omega_\bullet = (\omega, s^*\omega)$ is $X_\bullet$-basic and hence defines a 0-symplectic structure on $X_\bullet$.*

Proof. (i) This follows immediately from the definition of presymplectic (Section 3) and 0-symplectic.

(ii) The form ω is horizontal with respect to $\mathscr{F}$. Since it is closed, it is also infinitesimally invariant. By Lemma 5.3.8(ii) it is basic on $X_\bullet$. QED

5.5.3. **Example.** We revisit our Example 3.1. There are many foliation groupoids $X_\bullet$ that integrate $\ker(\omega_0)$. For instance, we can take $\tilde{N} \to N$ to be any étale Lie group homomorphism, let $\tilde{N}$ act on X_0 through this homomorphism, and take $X_\bullet$ to be the action groupoid $\tilde{N} \ltimes X_0$. This groupoid is not source-connected unless $\tilde{N}$ is connected. Nevertheless, the presymplectic form ω_0 is basic with respect to the $\tilde{N}$-action, so $X_\bullet$ is 0-symplectic with 0-symplectic form $(\omega_0, s^*\omega_0) \in \Omega^2_{\mathrm{bas}}(X_\bullet)$. Possible choices of $\tilde{N}$ are $\tilde{N} = N$, or $\tilde{N} = \operatorname{Lie}(N)$, the universal cover of the identity component of N. Another alternative is the surjective simply connected covering group $\tilde{N} = \pi^{-1}(N)$, where $\pi\colon \mathbb{R}^n \to \mathbb{T}^n$ is the projection.

5.6. **Étale stacks.** An atlas $X \to \mathbf{X}$ of a differentiable stack is *étale* if, for every morphism $M \to \mathbf{X}$ from a smooth manifold M, the pullback $M \times_{\mathbf{X}} X \to M$ is étale, i.e. a local diffeomorphism. An *étale stack* is a stack that admits an étale atlas.

5.6.1. **Lemma.** *Let $\mathbf{X}$ be a differentiable stack. The following are equivalent:*

(i) $\mathbf{X}$ *is an étale stack;*
(ii) $\mathbf{X} \simeq \mathbf{B}X_\bullet$ *for some foliation groupoid* $X_\bullet$*;*
(iii) $\mathbf{X} \simeq \mathbf{B}X_\bullet$ *for some étale groupoid* $X_\bullet$.

Sketch of proof. (i) $\Longleftrightarrow$ (iii): Suppose $\mathbf{X}$ is étale. Let $X \to \mathbf{X}$ be a Hausdorff étale atlas of $\mathbf{X}$ and let $X_\bullet$ be the Lie groupoid $X_\bullet = (X \times_{\mathbf{X}} X \rightrightarrows X)$. Then $X_\bullet$ is an étale groupoid by the definition of étale atlas, and $\mathbf{X} \simeq \mathbf{B}X_\bullet$. Conversely, assume $X_1 \rightrightarrows X_0$ is an étale groupoid, let $f : M \to \mathbf{X}$ be a map from a smooth manifold M, and consider the diagram

$$X_1 \rightrightarrows X_0 \to \mathbf{X}.$$

Forming the pullback over f gives the diagram

$$X_1 \times_{\mathbf{X}} M \rightrightarrows X_0 \times_{\mathbf{X}} M \to M;$$

that is to say a presentation for M by the Lie groupoid $X_1 \times_{\mathbf{X}} M \rightrightarrows X_0 \times_{\mathbf{X}} M$. This Lie groupoid is also étale, and so the map $X_0 \times_{\mathbf{X}} M \to M$ is étale, as desired.

(ii) $\Longleftrightarrow$ (iii): This follows from Lemma 4.2.2 and the fact that every foliation groupoid $X_\bullet$ is Morita equivalent to an étale groupoid $Y_\bullet$ (namely, take a complete transversal $\phi : Y_0 \to X_0$ to the foliation of X_0, and let $Y_\bullet = \phi^* X_\bullet$ be the pullback groupoid). QED

5.7. **Vector fields on étale stacks.** We now recall the definition of vector fields on stacks from [21] and show that equivalence classes of vector fields on an étale stack correspond to basic vector fields on a presenting Lie groupoid.

In [21, § 3], Hepworth constructs the *tangent stack functor* T, which is a lax endofunctor of the 2-category **Stack**, and a transformation $\boldsymbol{p} : T \Rightarrow \mathrm{id}$ from the tangent stack functor to the identity functor, which is unique up to modification. For a differentiable stack $\mathbf{X}$, we will often denote the associated morphism $\boldsymbol{p}_{\mathbf{X}} : T\mathbf{X} \to \mathbf{X}$ simply by $\boldsymbol{p}$. By [21, Theorem 3.11], we have an equivalence

$$\mathbf{B}TX_\bullet \simeq T\mathbf{B}X_\bullet \tag{5.7.1}$$

which is natural in $X_\bullet$, and a 2-isomorphism $\boldsymbol{p} \cong \mathbf{B}p_\bullet$, where $p_\bullet : TX_\bullet \to X_\bullet$ is the tangent groupoid projection.

5.7.2. **Definition** ([21, § 4])**.** A *vector field* on a stack $\mathbf{X}$ is a pair $(\boldsymbol{v}, \boldsymbol{\alpha})$ consisting of a stack morphism $\boldsymbol{v} : \mathbf{X} \to T\mathbf{X}$ and a 2-isomorphism $\boldsymbol{\alpha} : \boldsymbol{p} \circ \boldsymbol{v} \cong \mathrm{id}_{\mathbf{X}}$ to the identity. An *equivalence* of vector fields $(\boldsymbol{v}, \boldsymbol{\alpha})$ and $(\boldsymbol{w}, \boldsymbol{\beta})$ is a 2-arrow $\boldsymbol{\lambda} : \boldsymbol{v} \Rightarrow \boldsymbol{w}$ satisfying $\boldsymbol{\alpha} = \boldsymbol{\beta} \circ (\mathrm{id}_{\boldsymbol{p}} * \boldsymbol{\lambda})$.

Vector fields on $\mathbf{X}$ and their equivalences form a groupoid $\mathbf{Vect}(\mathbf{X})$. We denote by $\mathrm{Vect}(\mathbf{X})$ the set of equivalence classes of the groupoid $\mathbf{Vect}(\mathbf{X})$, and we view $\mathrm{Vect}(\mathbf{X})$ as a groupoid with only identity arrows. An equivalence of stacks $\boldsymbol{\phi} : \mathbf{X} \to \mathbf{Y}$ induces an equivalence of groupoids $\boldsymbol{\phi}^* : \mathbf{Vect}(\mathbf{Y}) \to \mathbf{Vect}(\mathbf{X})$. (See [21, § 4].)

A multiplicative vector field $v_\bullet : X_\bullet \to TX_\bullet$ on a Lie groupoid $X_\bullet$ gives rise to a stack morphism $\mathbf{B}v_\bullet : \mathbf{B}X_\bullet \to \mathbf{B}TX_\bullet \simeq T\mathbf{B}X_\bullet$. Hepworth shows that $\mathbf{B}v_\bullet$ determines a vector field $(\mathbf{B}v_\bullet, \boldsymbol{\alpha}_{v_\bullet})$ on $\mathbf{X}$ and that the assignment $v_\bullet \mapsto (\mathbf{B}v_\bullet, \boldsymbol{\alpha}_{v_\bullet})$ defines a functor

$$\mathrm{Vect}_{\mathrm{mult}}(X_\bullet) \longrightarrow \mathbf{Vect}(\mathbf{B}X_\bullet).$$

If we restrict our attention to Lie groupoids where the object manifold X_0 is Hausdorff, then by [4, Theorem 4.11, Remark 5.4] this functor is an equivalence of categories and is natural with respect to Morita morphisms. ([4, Theorem 4.11] is stated for Lie groupoids with X_0 and X_1 Hausdorff, but its proof only makes use of this assumption on X_0.) If $\mathbf{X}$ is an arbitrary differentiable stack and $\boldsymbol{\phi}\colon \mathbf{B}X_\bullet \simeq \mathbf{X}$ is a presentation by a Lie groupoid with X_0 Hausdorff, we get an equivalence of groupoids

$$\mathrm{Vect}_{\mathrm{mult}}(X_\bullet) \longrightarrow \mathbf{Vect}(\mathbf{B}X_\bullet) \xrightarrow{(\boldsymbol{\phi}^*)^{-1}} \mathbf{Vect}(\mathbf{X}), \tag{5.7.3}$$

which Berwick-Evans and Lerman [4, § 5] call a "Lie 2-algebra atlas" on $\mathbf{Vect}(\mathbf{X})$. A different choice of presentation $\mathbf{B}Y_\bullet \simeq \mathbf{X}$ leads to an equivalent Lie 2-algebra atlas $\mathrm{Vect}_{\mathrm{mult}}(Y_\bullet) \simeq \mathrm{Vect}_{\mathrm{mult}}(X_\bullet)$.

For étale stacks $\mathbf{X}$ the situation is simpler. Let $\mathbf{B}X_\bullet \simeq \mathbf{X}$ be a presentation of $\mathbf{X}$. Then $X_\bullet$ is a foliation groupoid, so Proposition 5.4.1 tells us that the set of equivalence classes of the groupoid $\mathrm{Vect}_{\mathrm{mult}}(X_\bullet)$ is equal to $\mathrm{Vect}_{\mathrm{bas}}(X_\bullet)$. By convention foliation groupoids $X_\bullet$ have Hausdorff X_0 (see § 5.1), so combining this with the equivalence (5.7.3) we see that the quotient map $\mathbf{Vect}(\mathbf{X}) \to \mathrm{Vect}(\mathbf{X})$ is an equivalence, i.e. vector fields on étale stacks have no non-trivial self-equivalences, and that the induced map

$$\mathrm{Vect}_{\mathrm{bas}}(X_\bullet) \xrightarrow{\cong} \mathrm{Vect}(\mathbf{X}) \tag{5.7.4}$$

is a bijection.

5.7.5. **Proposition.** *Let $\mathbf{X}$ be an étale stack. There is a unique Lie algebra structure on the set* $\mathrm{Vect}(\mathbf{X})$ *with the property that for every presentation $\mathbf{B}X_\bullet \to \mathbf{X}$ by a foliation groupoid $X_\bullet$ the bijection (5.7.4) is a Lie algebra isomorphism.*

Proof. By Lemma 4.1.5, there exists a presentation $\mathbf{B}X_\bullet \simeq \mathbf{X}$ by a Lie groupoid with X_0 Hausdorff, and so the bijection (5.7.4) endows $\mathrm{Vect}(\mathbf{X})$ with the structure of a Lie algebra. Uniqueness of this Lie algebra structure now follows immediately from Proposition 5.3.12(i). QED

5.8. **Differential forms on étale stacks.** The functor $\Omega^k\colon \mathbf{Diff} \to \mathbf{Set}$ that takes a manifold M to its set of k-forms $\Omega^k(M)$ is a sheaf on the site $\mathbf{Diff}$. By viewing $\Omega^k(M)$ as a groupoid with only identity arrows, we may think of the sheaf Ω^k as a (non-differentiable) stack.

Let $\mathbf{X}$ be an étale stack. We define a *differential form of degree k* on $\mathbf{X}$ to be a morphism of stacks $\mathbf{X} \to \Omega^k$. The collection of k-forms on $\mathbf{X}$ is a groupoid $\boldsymbol{\Omega}^k(\mathbf{X}) = \mathrm{Hom}(\mathbf{X}, \Omega^k)$. For a morphism of étale stacks $\boldsymbol{\phi}\colon \mathbf{X} \to \mathbf{Y}$ and a k-form $\boldsymbol{\zeta}\colon \mathbf{Y} \to \Omega^k$ on $\mathbf{Y}$ we define the *pullback form* on $\mathbf{X}$ by $\boldsymbol{\phi}^*\boldsymbol{\zeta} = \boldsymbol{\zeta} \circ \boldsymbol{\phi}\colon \mathbf{X} \to \Omega^k$.

Since Ω^k takes values in $\mathbf{Set}$, the groupoid $\mathrm{Hom}(\mathbf{X}, \Omega^k)$ has no nontrivial arrows. So $\boldsymbol{\Omega}^k$ really takes values in $\mathbf{Set}$, and henceforth we write $\Omega^k = \boldsymbol{\Omega}^k$ to emphasize this. By the Yoneda lemma, the set $\Omega^k(M)$ of k-forms on M (considered as a manifold) is naturally isomorphic to the set $\Omega^k(M)$ of k-forms on M (considered as a stack).

5.8.1. **Remark.** This notion of a differential form on a stack corresponds to the notion of a differential form on the diffeological coarse moduli space of the stack (see [48]), or to that of a basic form on a presenting Lie groupoid (see Proposition 5.8.3 below), and is suitable for the purposes of this paper. A different notion (which corresponds to the notion of a simplicial or Bott-Shulman differential form on a presenting Lie groupoid, and which is more adequate for other purposes, especially those involving non-étale stacks) is explained for instance in [3, § 3].

Let $\mathbf{X}$ be an étale stack, and let $\mathbf{B}X_\bullet \to \mathbf{X}$ be a presentation of $\mathbf{X}$. Composing the diagram $X_1 \rightrightarrows X_0 \to \mathbf{X}$ with a k-form $\boldsymbol{\zeta}\colon \mathbf{X} \to \Omega^k$ determines a commutative diagram $X_1 \rightrightarrows X_0 \to \Omega^k$, i.e. a basic form on $X_\bullet$. This defines a map

$$\Omega^k(\mathbf{X}) \longrightarrow \Omega^k_{\mathrm{bas}}(X_\bullet). \tag{5.8.2}$$

5.8.3. **Proposition.** *Let* $\mathbf{X}$ *be an étale stack and let* $\mathbf{B}X_\bullet \to \mathbf{X}$ *be a presentation of* $\mathbf{X}$*. The map* (5.8.2) *is a bijection* $\Omega^k(\mathbf{X}) \cong \Omega^k_{\mathrm{bas}}(X_\bullet)$*.*

Proof. We display an inverse to (5.8.2). Let $D(X_\bullet)$ be the 2-truncated semisimplicial nerve of the Lie groupoid $X_1 \rightrightarrows X_0$, that is to say the diagram

$$D(X_\bullet)\colon \quad X_2 = X_1 \times_{X_0} X_1 \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} X_1 \rightrightarrows X_0,$$

where the arrows from X_2 to X_1 are the two projections and the composition map. It is known (see e.g. [11, Appendix A]) that $\mathbf{X}$ is the weak colimit in **Stack** of the diagram $D(X_\bullet)$. Let $\zeta_\bullet = (\zeta_0, \zeta_1)$ be a basic k-form on $X_\bullet$. Then ζ_0, viewed as a map $\zeta_0\colon X_0 \to \Omega^k$, has the property that the diagram

$$X_1 \rightrightarrows X_0 \xrightarrow{\zeta_0} \Omega^k$$

commutes. Using this, it is straightforward to check that the diagram $D(X_\bullet) \xrightarrow{\zeta_0} \Omega^k$ commutes as well. Hence, by the universal property of weak colimits, ζ_0 descends to a morphism of stacks $\boldsymbol{\zeta}\colon \mathbf{X} \to \Omega^k$. Since Ω^k takes values in the 1-category **Set** the morphism $\boldsymbol{\zeta}$ is unique; and the assignment $\zeta_\bullet \mapsto \boldsymbol{\zeta}$ is inverse to (5.8.2). QED

Abusing notation, we will write

$$\mathbf{B}\zeta_\bullet \in \Omega^k(\mathbf{X})$$

for the differential form on $\mathbf{X} = \mathbf{B}X_\bullet$ determined by the basic k-form $\zeta_\bullet \in \Omega^k(X_\bullet)$ via the bijection (5.8.2). The following is the analogue of Proposition 5.7.5.

5.8.4. **Proposition.** *Let* $\mathbf{X}$ *be an étale stack. There is a unique differential graded algebra structure on the set* $\Omega^\bullet(\mathbf{X})$ *so that for every presentation* $\mathbf{B}X_\bullet \to X_\bullet$ *by a foliation groupoid* $X_\bullet$ *the bijection* (5.8.2) *is an isomorphism of differential graded algebras. There are unique contraction and derivation operations*

$$\iota\colon \operatorname{Vect}(\mathbf{X}) \times \Omega^\bullet(\mathbf{X}) \longrightarrow \Omega^{\bullet-1}(\mathbf{X}),$$
$$\mathscr{L}\colon \operatorname{Vect}(\mathbf{X}) \times \Omega^\bullet(\mathbf{X}) \longrightarrow \Omega^\bullet(\mathbf{X})$$

which correspond to the operations (5.3.5) *via the bijections* (5.7.4) *and* (5.8.2)*.*

Proof. This follows immediately from Proposition 5.3.12(ii) and the naturality of the operations (5.3.5) with respect to Morita morphisms. QED

5.9. **Symplectic stacks.** Let $\mathbf{X}$ be an étale stack. We call a 2-form $\boldsymbol{\omega} \in \Omega^2(\mathbf{X})$ *symplectic* if for some (and hence for every) foliation groupoid $X_\bullet$ presenting $\mathbf{X}$ the basic 2-form $\omega_\bullet \in \Omega_{\mathrm{bas}}(X_\bullet)$ corresponding to $\boldsymbol{\omega}$ via the isomorphism (5.8.2) is 0-symplectic in the sense of § 5.5. A *symplectic stack* is a pair $(\mathbf{X}, \boldsymbol{\omega})$ consisting of an étale stack $\mathbf{X}$ and a symplectic form $\boldsymbol{\omega}$.

This extends the notion of a symplectic stack introduced in [27, § 2.12] in that we allow our stacks to be non-separated. If $(\mathbf{X}, \boldsymbol{\omega})$ is a symplectic stack, the linear map $\operatorname{Vect}(\mathbf{X}) \to \Omega^1(\mathbf{X})$ defined by $\mathbf{v} \mapsto \iota_{\mathbf{v}}\boldsymbol{\omega}$ is an isomorphism. The next statement is an immediate consequence of the definitions.

5.9.1. **Proposition.** *Let* $(X_\bullet, \omega_\bullet)$ *be a* 0*-symplectic groupoid. Then* $(\mathbf{B}X_\bullet, \mathbf{B}\omega_\bullet)$ *is a symplectic stack, where* $\mathbf{B}\omega_\bullet \in \Omega^2(\mathbf{B}X_\bullet)$ *is the form corresponding to* $\omega_\bullet \in \Omega^2_{\mathrm{bas}}(X_\bullet)$ *under the isomorphism* $\Omega^2(\mathbf{B}X_\bullet) \cong \Omega^2_{\mathrm{bas}}(X_\bullet)$ *of Proposition 5.8.3. Conversely, if* $(\mathbf{X}, \boldsymbol{\omega})$ *is a symplectic stack and* $\mathbf{X} \simeq \mathbf{B}X_\bullet$*, then* $X_\bullet$ *is a* 0*-symplectic groupoid.*

From Propositions 5.5.2 and 5.9.1, we see that any presymplectic manifold (X, ω) gives rise to many symplectic stacks, each of which we may interpret as a stacky quotient of X along the null foliation of ω.

6. Lie 2-groups and Lie group stacks

This section starts with a review of Lie 2-groups and Lie group stacks, which we view as group objects internal to the 2-categories **LieGpd** and **DiffStack**, respectively. (Some basic definitions regarding group objects in a 2-category are recalled in Appendix A.) Of main interest to us are connected étale Lie group stacks, which according to [46] are always equivalent to strict Lie group stacks. We show that the action of an étale Lie group stack on an étale stack can also be strictified. We end with a discussion of basic features of étale Lie group stacks, such as the Lie 2-algebra, the adjoint action, the structure of compact étale Lie group stacks, and maximal stacky tori.

6.1. **Lie 2-groups.** A *weak Lie 2-group* is a weak 2-group in the 2-category **LieGpd**, as in Definition A.1. A *strict Lie 2-group* is a strict 2-group in **LieGpd**. We will refer to strict Lie 2-groups as simply *Lie 2-groups*.

More explicitly, a (strict) Lie 2-group is a Lie groupoid $G_\bullet$ so that G_1 and G_0 are both Lie groups, and all the groupoid structure maps are Lie group homomorphisms. We will write $g \cdot h$ for the group product of $g, h \in G_i$ and $g \circ h$ for the groupoid product (composition) of composable $g, h \in G_1$. We denote the group identity of G_1 and G_0 by 1, and $u\colon G_0 \to G_1$ the groupoid identity bisection. We write $m_\bullet\colon G_\bullet \times G_\bullet \to G_\bullet$ for group multiplication in $G_\bullet$, and note that $m_\bullet$ is a Lie groupoid homomorphism. We use $(\cdot)^{-1}$ to denote both inverse with respect to the groupoid structure and the group structure on $G_\bullet$; the meaning should be clear from the context.

A *morphism of Lie 2-groups* is a strict morphism of strict 2-groups in **LieGpd**, as in Definition A.6. Equivalently, it is a map of Lie groupoids which preserves the group structure on both the object and the arrow manifolds. A *Morita morphism* of Lie 2-groups is a morphism of Lie 2-groups which is essentially surjective and fully faithful, i.e. a Morita morphism of the underlying Lie groupoids (Definition 4.1.3). Two Lie 2-groups $G_\bullet$ and $H_\bullet$ are *Morita equivalent* if there exists a Lie 2-group $K_\bullet$ and a zigzag of Morita morphisms of Lie 2-groups $K_\bullet \to G_\bullet$ and $K_\bullet \to H_\bullet$.

The coarse quotient G_0/G_1 of a Lie 2-group $G_\bullet$ is a (not necessarily Hausdorff) topological group, namely $G_0/G_1 = G_0/t(\ker(s))$. The coarse quotient is preserved under Morita equivalence.

A *weak action* of a Lie 2-group $G_\bullet$ on a Lie groupoid $X_\bullet$ is a weak action as in Definition A.12, by considering $G_\bullet$ to be a strict 2-group in **LieGpd**. A *strict action* is also as in Definition A.12. We will usually abbreviate *strict action* to *action*. If $G_\bullet$ is a Lie 2-group acting on $X_\bullet$ and $X'_\bullet$, an *equivariant map* $X_\bullet \to X'_\bullet$ is as in Definition A.16, and is always assumed to be strict.

A (strict) action of a Lie 2-group $G_\bullet$ on $X_\bullet$ is equivalent to a morphism of Lie groupoids $G_\bullet \times X_\bullet \to X_\bullet$ both of whose component maps

$$\begin{aligned} G_0 \times X_0 &\longrightarrow X_0\colon (g, x) \longmapsto L(g)(x) = g \cdot x, \\ G_1 \times X_1 &\longrightarrow X_1\colon (k, f) \longmapsto L(k)(f) = k * f, \end{aligned} \tag{6.1.1}$$

define Lie group actions.

6.1.2. **Example.** Consider the torus $G = \mathbb{T}^n$ and its immersed subgroup N of Examples 3.1 and 5.5.3. Let $\tilde{N} \to N$ be an étale homomorphism as in Example 5.5.3. Form the action groupoid $G_\bullet = (\tilde{N} \ltimes G \rightrightarrows G)$, where $\tilde{N}$ acts on G by left translations via N. Then $G_\bullet$ has the structure of a Lie 2-group. The group structure on $\tilde{N} \ltimes G$ is the product group structure $\tilde{N} \times G$. Taking $X_\bullet = (\tilde{N} \ltimes X_0 \rightrightarrows X_0)$, from the action of G on X_0 we obtain an action $G_\bullet \times X_\bullet \to X_\bullet$, which on arrows is given by

$$(n, g) \cdot (n', x) = (nn', g \cdot x).$$

6.2. **Lie 2-algebras.** A *strict Lie 2-algebra* is a Lie groupoid $\mathfrak{g}_1 \rightrightarrows \mathfrak{g}_0$ so that each $\mathfrak{g}_i$ is a Lie algebra and the groupoid structure maps are Lie algebra homomorphisms. The strict Lie 2-algebra of a strict Lie 2-group $G_1 \rightrightarrows G_0$ is the Lie groupoid $\mathrm{Lie}(G_\bullet) = (\mathfrak{g}_1 \rightrightarrows \mathfrak{g}_0)$ obtained by applying the Lie functor to the structure maps of $G_\bullet$. If $\mathfrak{g}_\bullet = \mathrm{Lie}(G_\bullet)$, we say that $G_\bullet$ *integrates* $\mathfrak{g}_\bullet$.

We will not consider weak Lie 2-algebras, so we refer to strict Lie 2-algebras as simply Lie 2-algebras. Recall (Definition 4.1.1) that the Lie algebroid of $G_\bullet$ is denoted $\mathrm{Alg}(G_\bullet)$, not $\mathrm{Lie}(G_\bullet)$.

A *Morita morphism* of strict Lie 2-algebras is a Morita morphism of the underlying Lie groupoids (Definition 4.1.3) which preserves the Lie algebra structure maps. Two strict Lie 2-algebras $\mathfrak{g}_\bullet$ and $\mathfrak{h}_\bullet$ are *Morita equivalent* if there is a strict Lie 2-algebra $\mathfrak{k}_\bullet$ and a zig-zag of Morita morphisms of Lie 2-algebras $\mathfrak{k}_\bullet \to \mathfrak{g}_\bullet$ and $\mathfrak{k}_\bullet \to \mathfrak{h}_\bullet$.

6.2.1. **Lemma.** (i) *Let $\phi_\bullet \colon G_\bullet \to H_\bullet$ be a Morita morphism of Lie 2-groups. Then* $\mathrm{Lie}(\phi_\bullet) \colon \mathfrak{g}_\bullet \to \mathfrak{h}_\bullet$ *is a Morita morphism of Lie 2-algebras.*

(ii) *Let $\mathfrak{g}$ and $\mathfrak{g}'$ be Lie algebras, considered as Lie 2-algebras $\mathfrak{g} \rightrightarrows \mathfrak{g}$ and $\mathfrak{g}' \rightrightarrows \mathfrak{g}'$. Let $\phi_\bullet \colon \mathfrak{h}_\bullet \to \mathfrak{g}$ be a Morita morphism and $\psi_\bullet \colon \mathfrak{h}_\bullet \to \mathfrak{g}'$ a morphism of Lie 2-algebras. Then there is a unique Lie algebra morphism $\zeta \colon \mathfrak{g} \to \mathfrak{g}'$ such that $\zeta \circ \phi_\bullet = \psi_\bullet$. In particular, if $\mathfrak{g}$ and $\mathfrak{g}'$ are Morita equivalent as Lie 2-algebras, then they are isomorphic as Lie algebras.*

Proof. (i) The Lie functor "Lie" preserves fibred products and takes surjective submersions to surjective submersions. Therefore "Lie" preserves essential surjectivity and full faithfullness.

(ii) Let $\phi \colon \mathfrak{h}_\bullet \to \mathfrak{g}$ be a Morita morphism of Lie 2-algebras. Then $\phi_0 \colon \mathfrak{h}_0 \to \mathfrak{g}$ is surjective, so $\mathfrak{g} \cong \mathfrak{h}_0/\ker(\phi_0)$. Note that $t(\ker(s)) \subseteq \ker(\phi_0)$, since ϕ is a Lie groupoid morphism. Since ϕ is a Morita equivalence, we have

$$\dim \mathfrak{g} = 2 \dim \mathfrak{h}_0 - \dim \mathfrak{h}_1 = \dim \mathfrak{h}_0 - \dim t(\ker(s)),$$

and so $t(\ker(s)) = \ker(\phi_0)$. So $\mathfrak{g} \cong \mathfrak{h}_0/t(\ker(s))$. Similarly, $t(\ker(s)) \subseteq \ker(\psi_0)$, so ψ_0 descends to a unique Lie algebra map $\zeta \colon \mathfrak{g} \to \mathfrak{g}'$, which satisfies $\zeta \circ \phi_\bullet = \psi_\bullet$. The last assertion is an immediate consequence of this. QED

6.2.2. **Example.** The Lie 2-algebra of the 2-group $G_\bullet$ in Example 6.1.2 is $\mathfrak{n} \ltimes \mathfrak{g} \rightrightarrows \mathfrak{g}$, where $\mathfrak{n} \ltimes \mathfrak{g}$ is isomorphic as a Lie algebra to the abelian Lie algebra $\mathfrak{n} \oplus \mathfrak{g}$.

6.3. **Crossed modules.** A *crossed module of Lie groups* is a quadruple (G, H, ∂, α), where G and H are Lie groups, $\partial \colon H \to G$ is a Lie group homomorphism, and $\alpha \colon G \to \mathrm{Aut}(H)$ is an action of G on H subject to the requirements $\partial(\alpha(g)(h)) = g\partial(h)g^{-1}$ and $\alpha(\partial(h))(h') = hh'h^{-1}$ for all $g \in G$ and $h, h' \in H$. We will usually abbreviate $\alpha(g)(h)$ to ${}^g h$. A *morphism (of crossed modules of Lie groups)* $(\psi_G, \psi_H) \colon (G, H, \partial, \alpha) \to (G', H', \partial', \alpha')$ is a pair of Lie group homomorphisms $\psi_G \colon G \to G'$, and $\psi_H \colon H \to H'$ which commute with the structure maps and actions.

There is an equivalence of categories between strict Lie 2-groups and their strict homomorphisms on one hand, and crossed modules of Lie groups and their morphisms on the other; see [8] and [1, Proposition 3.14, § 8.4]. (This can be extended to an equivalence of appropriately defined 2-categories, though we will not need this here). Since many statements involving Lie 2-groups can be more compactly stated and proved in terms of crossed modules, we will use this equivalence throughout what follows.

Let us recall how this equivalence of categories looks on objects. A crossed module of Lie groups (G, H, ∂, α) gives rise to the Lie 2-group $G_\bullet$ with $G_0 = G$ and $G_1 = H \rtimes_\alpha G$. As a Lie groupoid, $G_\bullet$ is the action groupoid for the action of H on G by left translations of $\partial(H)$. As a Lie group, $H \rtimes_\alpha G$ is the semidirect product of H and G with respect to the action $\alpha\colon G \to \mathrm{Aut}(H)$. Conversely, a 2-group $G_\bullet$ determines the crossed module (G, H, ∂, α): we have $G = G_0$, $H = \ker(s)$, $\partial = t|_H\colon H \to G$, and α is the conjugation action of G_1 on H composed with the identity bisection $u\colon G_0 \to G_1$. The Lie 2-algebra $\mathrm{Lie}(H \rtimes_\alpha G)$ is $\mathfrak{h} \rtimes_\alpha \mathfrak{g} \rightrightarrows \mathfrak{g}$, where $\mathfrak{h} \rtimes_\alpha \mathfrak{g}$ is the semidirect product Lie algebra obtained by applying the Lie functor to α.

A crossed module is a nonabelian version of a 2-term chain complex, whose "homology" groups are $\ker(\partial)$ and $\mathrm{coker}(\partial)$. A Morita morphism is a nonabelian version of a quasi-isomorphism.

6.3.1. **Definition.** A *Morita morphism (of crossed modules of Lie groups)* is a morphism

$$(\phi_G, \phi_H)\colon (G, H, \partial, \alpha) \to (G', H', \partial', \alpha')$$

that induces (abstract) group isomorphisms $\ker(\partial) \cong \ker(\partial')$ and $\mathrm{coker}(\partial) \cong \mathrm{coker}(\partial')$.

Thus the kernel and the cokernel of ∂ are Morita invariants of a crossed module (G, H, ∂, α). Note that the cokernel $G/\partial(H)$ is the coarse quotient group of the corresponding 2-group.

The second countability axiom (which we impose on all manifolds; see Section 2) is necessary for the following statement to be true. For instance, if (G, H, ∂, α) is a crossed module of Lie groups and G^d, resp. H^d, denotes G, resp. H, equipped with the discrete topology, then the identity map $(G^d, H^d, \partial, \alpha) \to (G, H, \partial, \alpha)$ is a Morita morphism of crossed modules of (non-second countable) Lie groups, but the corresponding morphism of Lie 2-groups $G^d_\bullet \to G_\bullet$ is not Morita.

6.3.2. **Lemma.** *Let $\phi_\bullet\colon G_\bullet \to G'_\bullet$ be a morphism of Lie 2-groups and let (ϕ_G, ϕ_H) be the associated morphism of crossed modules. Then the following are equivalent:*

(i) *$\phi_\bullet$ is a Morita morphism of Lie 2-groups;*
(ii) *(ϕ_G, ϕ_H) is a Morita morphism of crossed modules;*
(iii) *$G' = \partial'(H')\phi_G(G)$ and the map $\phi_H \times \partial\colon H \to H' \times_{G'} G$ is bijective.*

Sketch of proof. The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) are straightforward. From (iii) one deduces that the morphism $\phi_\bullet\colon G_\bullet \to G'_\bullet$ is essentially surjective and fully faithful considered as a morphism of abstract 2-groups (i.e. strict 2-groups in the 2-category of set-theoretic groupoids). Now use the following two facts to conclude that $\phi_\bullet$ is a Morita morphism of Lie 2-groups: (1) a surjective homomorphism of Lie groups is a submersion (which follows from Sard's theorem); (2) given two Lie group homomorphisms $K_1 \to K$ and $K_2 \to K$, the set-theoretic fibred product $K_1 \times_K K_2$ is a Lie group and is a fibred product in the category of Lie groups (which follows from the closed subgroup theorem). QED

6.3.3. **Remark.** We record three special cases of the lemma. Let (G, H, ∂, α) be a crossed module. First, suppose we are given a closed subgroup G' of G with the property $\partial(H)G' = G$. Then we can form the restricted crossed module $(G', H', \partial', \alpha')$ with $H' = \partial^{-1}(G')$, and the inclusion into (G, H, ∂, α) is a Morita morphism. Second, given a Lie group extension (e.g. a covering homomorphism) $\phi\colon \hat{G} \to G$ we can form the pullback extension $\hat{H} = H \times_G \hat{G}$ of H. Define $\hat{\partial}\colon \hat{H} \to H$ by $\hat{\partial}(h, \hat{g}) = \phi(\hat{g})$ and a $\hat{G}$-action $\hat{\alpha}$ on $\hat{H}$ by ${}^{\hat{g}}(h, \hat{g}') = \left({}^{\phi(\hat{g})}h, \hat{g}\hat{g}'\hat{g}^{-1}\right)$. Then $(\hat{G}, \hat{H}, \hat{\partial}, \hat{\alpha})$ is a crossed module and ϕ induces a Morita morphism $(\hat{G}, \hat{H}, \hat{\partial}, \hat{\alpha}) \to (G, H, \partial, \alpha)$. The third case is the second case run in reverse: given a closed normal subgroup N of G that is contained in $\partial(H)$, we can form the quotient crossed module $(\bar{G}, \bar{H}, \bar{\partial}, \bar{\alpha})$ with $\bar{G} = G/N$ and $\bar{H} = H/\partial^{-1}(N)$, and we have a Morita morphism $(G, H, \partial, \alpha) \to (\bar{G}, \bar{H}, \bar{\partial}, \bar{\alpha})$.

We can also reformulate the notion of a strict Lie 2-group action in terms of the associated crossed module of Lie groups.

6.3.4. **Definition.** Let (G, H, ∂, α) be a crossed module of Lie groups and let $X_\bullet$ be a Lie groupoid. An *action* of (G, H, ∂, α) on $X_\bullet$ consists of three smooth actions

$$G \times X_0 \longrightarrow X_0\colon \qquad (g, x) \longmapsto L(g)(x) = g \cdot x, \tag{6.3.5}$$

$$G \times X_1 \longrightarrow X_1\colon \qquad (g, f) \longmapsto L(g)(f) = g * f, \tag{6.3.6}$$

$$H \times X_1 \longrightarrow X_1\colon \qquad (h, f) \longmapsto L(h)(f) = h * f, \tag{6.3.7}$$

satisfying the compatibility conditions (6.3.8)–(6.3.12) below: for all $g \in G$ the pair of maps $L(g)\colon X_0 \to X_0$ and $L(g)\colon X_1 \to X_1$ is an endofunctor of $X_\bullet$, i.e.

$$g * u(x) = u(g \cdot x), \qquad g \cdot s(f) = s(g * f), \qquad g \cdot t(f) = t(g * f), \tag{6.3.8}$$

$$g * (f_1 \circ f_2) = (g * f_1) \circ (g * f_2), \tag{6.3.9}$$

for all $x \in X_0$ and f, f_1, $f_2 \in X_1$ for which $f_1 \circ f_2$ is defined; for all $h \in H$ the map $L(h)\colon X_1 \to X_1$ is a natural transformation from the identity functor to the functor $L(\partial(h))$, i.e.

$$s(h * f) = s(f), \quad t(h * f) = \partial(h) \cdot t(f), \tag{6.3.10}$$

$$h * f = (h * u(t(f))) \circ f = (\partial(h) * f) \circ (h * u(s(f))), \tag{6.3.11}$$

for all $f \in X_1$; for all $g \in G$, $h \in H$, and $f \in X_1$, one has $L({}^g h) = L(g) \circ L(h) \circ L(g)^{-1}$, i.e.

$$g * (h * f) = {}^g h * (g * f). \tag{6.3.12}$$

We omit the proof of the following straightforward lemma.

6.3.13. **Lemma.** *Let $G_\bullet = (G_1 \rightrightarrows G_0)$ be a Lie 2-group with associated crossed module*

$$(G = G_0, H = \ker(s), \partial = t|_H, \alpha),$$

and let $X_\bullet$ be a Lie groupoid. Let $a_\bullet\colon G_\bullet \times X_\bullet \to X_\bullet$ be a strict $G_\bullet$-action on $X_\bullet$. Then the maps

$$\begin{aligned} a_0\colon G \times X_0 &\longrightarrow X_0 \\ a_1 \circ (u \times \mathrm{id}_{X_1})\colon G \times X_1 &\longrightarrow X_1 \\ a_1|_H\colon H \times X_1 &\longrightarrow X_1 \end{aligned}$$

define an action of (G, H, ∂, α) on $X_\bullet$. Conversely, given a (G, H, ∂, α)-action on $X_\bullet$, the formulas

$$\begin{aligned} G_0 \times X_0 \longrightarrow X_0\colon &\qquad (g, x) \longmapsto g \cdot x, \\ G_1 \times X_1 \longrightarrow X_1\colon &\qquad ((h, g), x) \longmapsto h * (g * x), \end{aligned}$$

define a strict $G_\bullet$-action on $X_\bullet$, where we identify $G_0 = G$ and $G_1 = H \rtimes_\alpha G$. These two constructions are inverse, and so determine a bijection between the set of strict $G_\bullet$-actions on $X_\bullet$ and the set of (G, H, ∂, α)-actions on $X_\bullet$.

The next lemma records a simple consequence of Definition 6.3.4. We use that every element $\xi \in \mathfrak{g}$ gives birth to two vector fields, namely the vector field ξ_{X_1} induced via the G-action on X_1 and the vector field ξ_{X_0} induced via the G-action on X_0. Similarly, every $\eta \in \mathfrak{h}$ gives birth to three vector fields: the vector field η_{X_1} induced via the H-action on X_1, and the two vector fields $\partial(\eta)_{X_1}$ and $\partial(\eta)_{X_0}$ induced by the element $\partial(\eta) \in \mathfrak{g}$ via the G-actions on X_1 and X_0.

6.3.14. **Lemma.** *Let $G_\bullet$ be a strict Lie 2-group acting strictly on a Lie groupoid $X_\bullet$. Let $(G = G_0, H, \partial, \alpha)$ be the crossed module associated with $G_\bullet$. Let $\eta \in \mathfrak{h}$. The vector field η_{X_1} is tangent to the source fibres, right-invariant, and t-related to the vector field $\partial(\eta)_{X_0}$. The vector field $\eta_{X_1} - \partial(\eta)_{X_1}$ is tangent to the target fibres, left-invariant, and s-related to the vector field $\partial(\eta)_{X_0}$.*

Proof. Let $h \in H$. Property (6.3.10) can be restated as $s \circ L(h) = s$ and $t \circ L(h) = L(\partial(h)) \circ t$. In other words, $s : X_1 \to X_0$ is H-invariant and $t : X_1 \to X_0$ is equivariant with respect to the homomorphism $\partial : H \to G$. It follows that η_{X_1} is tangent to the fibres of s and that the vector fields η_{X_1} and $\partial(\eta)_{X_0}$ are t-related. Let $f \in X_1$ have source $s(f) = x$ and target $t(f) = y$. Right composition with f, $R(f)(f') = f' \circ f$, defines a map $R(f) : s^{-1}(y) \to s^{-1}(x)$, and it follows from (6.3.11) that

$$L(h)(f) = R(f)\big(L(h)(u(y))\big).$$

Differentiating this identity with respect to h yields $\eta_{X_1,f} = R(f)_*(\eta_{X_1,u(y)})$, which tells us that η_{X_1} is right-invariant. This proves the first assertion. The second assertion is proved similarly, by considering the element $k = \partial(h)^{-1}h = h\partial(h)^{-1} \in G_1 = H \rtimes G$ and the left composition map $L(f) : t^{-1}(y) \to t^{-1}(x)$, and by noting the properties

$$s \circ L(k) = L(\partial(h)^{-1}) \circ s, \quad t \circ L(k) = t, \quad L(k)(f) = L(f)\big(L(k)(u(x))\big),$$

which follow from (6.3.10) and (6.3.11). QED

6.3.15. **Example.** The crossed module associated to the Lie 2-group $G_\bullet$ of Example 6.1.2 is the homomorphism $\tilde{N} \to G$ obtained by composing the map $\tilde{N} \to N$ with the inclusion $N \to G$. The action of G on $\tilde{N}$ is trivial.

6.4. **Foliation 2-groups.** Let $G_\bullet$ be a Lie 2-group with crossed module $(G = G_0, H, \partial, \alpha)$. The subgroup $\partial(H)$ of G_0 is normal, so

$$\mathrm{Lie}(t)(\ker(\mathrm{Lie}(s))) = \mathrm{Lie}(\partial)(\mathfrak{h})$$

is an ideal of $\mathfrak{g}$, and $\mathfrak{g}/\mathrm{Lie}(\partial)(\mathfrak{h})$ is a Lie algebra.

6.4.1. **Definition.** A Lie 2-group $G_\bullet$ is a *foliation 2-group* if any of the following equivalent conditions hold: (1) $G_\bullet$ is a foliation groupoid; (2) the homomorphism $\partial : H \to G$ has discrete kernel; or (3) the homomorphism $\mathrm{Lie}(\partial) : \mathfrak{h} \to \mathfrak{g}$ is injective. A foliation 2-group $G_\bullet$ is *effective* if ∂ is injective. We say $G_\bullet$ is an *étale 2-group* if either of the following equivalent conditions holds: (1) $G_\bullet$ is an étale groupoid; or (2) H is discrete.

When $G_\bullet$ is a foliation 2-group, we will consider $\mathfrak{h}$ as an ideal of $\mathfrak{g}$. Being a foliation 2-group is a Morita invariant property, and so is being effective.

We denote by $\xi_L \in \mathrm{Vect}(G)$ the left-invariant vector field on a Lie group G induced by a Lie algebra element $\xi \in \mathfrak{g}$. We call a basic vector field $v_\bullet = (v_0, v_1)$ on a foliation 2-group $G_\bullet$ *left-invariant* if for $i = 1, 2$ the vector field v_i is invariant under the left multiplication action of G_i on itself. The left-invariant basic vector fields form a Lie subalgebra of $\mathrm{Vect}_{\mathrm{bas}}(G_\bullet)$ denoted by $\mathrm{Vect}_{\mathrm{bas}}(G_\bullet)_L$.

We require some basic structural results on foliation 2-groups. The first result says that the Lie 2-algebra of a foliation 2-group $G_\bullet$ is equivalent to the quotient Lie algebra $\mathfrak{g}/\mathfrak{h}$, and that $\mathfrak{g}/\mathfrak{h}$ is isomorphic to the Lie algebra of left-invariant basic vector fields on $G_\bullet$. Part (iv) of this result is a special case of Lerman [26, Theorem 1.1], which says that the Lie 2-algebra of left-invariant multiplicative vector fields on a strict Lie 2-group $G_\bullet$ is isomorphic to the Lie 2-algebra $\mathrm{Lie}(G_1) \rightrightarrows \mathrm{Lie}(G_0)$.

6.4.2. **Lemma.** *Let $G_\bullet$ be a foliation 2-group with crossed module (G, H, ∂, α), where $G_0 = G$ and $G_1 = H \rtimes G$. Let $\pi\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ be the quotient map.*

(i) *The map $\pi_\bullet\colon \mathfrak{g}_\bullet \to \mathfrak{g}/\mathfrak{h}$, defined by*

$$\begin{array}{ccc} \mathfrak{g}_1 & \xrightarrow{\pi\circ\mathrm{Lie}(t)} & \mathfrak{g}/\mathfrak{h} \\ \downdownarrows & & \downdownarrows \\ \mathfrak{g}_0 & \xrightarrow{\pi} & \mathfrak{g}/\mathfrak{h} \end{array}$$

is a Morita morphism of Lie 2-algebras.

(ii) *Let $G'_\bullet$ be a second foliation 2-group with crossed module $(G', H', \partial', \alpha')$. A Morita morphism of Lie 2-groups $\phi_\bullet\colon G_\bullet \to G'_\bullet$ induces a Lie algebra isomorphism $\mathfrak{g}/\mathfrak{h} \to \mathfrak{g}'/\mathfrak{h}'$.*

(iii) *The map $f\colon \mathfrak{g} \to \mathrm{Vect}(G_0) \times \mathrm{Vect}(G_1)$ defined by $f(\xi) = (\xi_L, (\mathrm{Lie}(u)\xi)_L)$ descends to a Lie algebra embedding $\mathfrak{g}/\mathfrak{h} \hookrightarrow \mathrm{Vect}_{\mathrm{bas}}(G_\bullet)$, whose image is equal to $\mathrm{Vect}_{\mathrm{bas}}(G_\bullet)_L$.*

(iv) *Let $\phi_\bullet\colon G_\bullet \to G'_\bullet$ be a Morita morphism of Lie 2-groups. Then the diagram*

$$\begin{array}{ccc} \mathfrak{g}/\mathfrak{h} & \longrightarrow & \mathrm{Vect}_{\mathrm{bas}}(G_\bullet) \\ \downarrow\cong & & \uparrow\cong \\ \mathfrak{g}'/\mathfrak{h}' & \longrightarrow & \mathrm{Vect}_{\mathrm{bas}}(G'_\bullet) \end{array}$$

commutes. Here the horizontal arrows are from (iii), the left vertical arrow is from (ii), and the right vertical arrow is from Proposition 5.3.12(i).

Proof. For simplicity, we write $s = \mathrm{Lie}(s)$ for the source map of $\mathfrak{g}_\bullet$, and similarly for the other structure maps.

(i) Since π and t are Lie algebra homomorphisms, to show that $\pi_\bullet$ is a homomorphism of Lie 2-algebras it suffices to show $t(\xi) - s(\xi) \in t(\ker s) = \mathfrak{h}$ for all $\xi \in \mathfrak{g}_1$. Indeed, $\xi - u(s(\xi)) \in \ker s$ and

$$(t - s)(\xi) = (t - s)(\xi - u(s(\xi))) = t(\xi - u(s(\xi))).$$

Next we show that $\pi_\bullet$ is a Morita morphism. Essential surjectivity is automatic since π is a surjective linear map. For full faithfullness it is enough to show that the canonical map $(s, t)\colon \mathfrak{g}_1 \to \mathfrak{g}_0 \times_{\mathfrak{g}_0/\mathfrak{h}} \mathfrak{g}_0$ is a linear isomorphism. To show the map is injective assume $s(\xi) = t(\xi) = 0$. Then $\xi \in \ker(\partial\colon \mathfrak{h} \to \mathfrak{g}_0)$, which is 0 since $G_\bullet$ is assumed to be a foliation groupoid. So $\xi = 0$ and the map is injective. Surjectivity follows from counting dimensions, as in the proof of Lemma 6.2.1.

(ii) By Lemma 6.2.1(i), applying the Lie functor to $\phi_\bullet$ gives a Morita morphism

$$\mathrm{Lie}(\phi_\bullet)\colon \mathfrak{g}_\bullet \to \mathfrak{g}'_\bullet$$

of Lie 2-algebras. Composing with the projection $\pi'_\bullet\colon \mathfrak{g}'_\bullet \to \mathfrak{g}'/\mathfrak{h}'$, by part (i) we have a Morita morphism of Lie 2-algebras $\mathfrak{g}_\bullet \to \mathfrak{g}'/\mathfrak{h}'$. We also have the Morita morphism

$\mathfrak{g}_\bullet \to \mathfrak{g}'/\mathfrak{h}'$. By Lemma 6.2.1(ii), $\pi'_\bullet \circ \mathrm{Lie}(\phi_\bullet)$ descends to a Lie algebra isomorphism $\mathfrak{g}/\mathfrak{h} \cong \mathfrak{g}'/\mathfrak{h}'$.

(iii) For $\xi \in \mathfrak{g}$ we have $s(u(\xi)_L) = t(u(\xi)_L) = \xi_L$, so $u(\xi)_L \sim_s \xi_L$ and $u(\xi)_L \sim_t \xi_L$. The leaves of the foliation $\mathscr{F}$ of G are the connected components of the cosets of the immersed normal subgroup $\partial(H)$. Therefore the flow of ξ_L maps leaves to leaves, and ξ_L is in the normalizer of the subalgebra $\Gamma(T\mathscr{F})$ of $\mathrm{Vect}(G)$. This shows that $f(\xi)$ defines a left-invariant element $\hat{f}(\xi) \in \mathrm{Vect}_{\mathrm{bas}}(G_\bullet)_L$. We have $\hat{f}(\xi) = 0$ if and only if ξ_L is tangent to the leaves of $\mathscr{F}$ if and only if $\xi \in \mathfrak{h}$. So $\hat{f}\colon \mathfrak{g} \to \mathrm{Vect}_{\mathrm{bas}}(G_\bullet)$ induces an isomorphism $\mathfrak{g}/\mathfrak{h} \cong \mathrm{Vect}_{\mathrm{bas}}(G_\bullet)_L$.

(iv) Since the basic vector fields in the image of the embeddings from (iii) are left invariant, they are uniquely determined by their value at the group unit 1. Write $N_0 = N_0(G_\bullet)$ and $N'_0 = N_0(G'_\bullet)$ for the normal bundles of $\mathscr{F}_0(G_\bullet)$ and $\mathscr{F}_0(G'_0)$, respectively. Then $N_0|_1 = \mathfrak{g}/\mathfrak{h}$ and $N'_0|_1 = \mathfrak{g}'/\mathfrak{h}'$, and the pullback of $N'_0|_1$ to $N_0|_1$ is just the inverse of the isomorphism $\mathfrak{g}/\mathfrak{h} \cong \mathfrak{g}'/\mathfrak{h}'$, and the assertion follows. QED

In the remainder of this section we will typically identify the Lie 2-algebra $\mathrm{Lie}(G_\bullet)$ of a foliation 2-group with the quotient Lie algebra $\mathfrak{g}/\mathfrak{h}$. We will without further mention apply Lemma 6.3.2, which allows us to define Morita morphisms of 2-groups in terms of their crossed modules. We will also use the restriction, extension, and quotient Morita morphisms of Remark 6.3.3.

6.4.3. **Lemma.** *Every foliation 2-group $G_\bullet$ is Morita equivalent to an étale 2-group $G'_\bullet$ with the property that the identity component of G'_0 is simply connected.*

Proof. Let (G, H, ∂, α) be the crossed module of $G_\bullet$, then $\ker(\partial)$ is discrete. We must show that (G, H, ∂, α) is Morita equivalent to a crossed module $(G', H', \partial', \alpha')$ where the identity component of G' is simply connected H' is discrete. Let $\phi\colon \hat{G} \to G$ be a surjective étale homomorphism, where $\hat{G}$ is simply connected (this exists even if G is not connected; see [7, Corollary 5.6]). Let $\hat{H} = H \times_G \hat{G}$ be the pullback cover. The extension $(\hat{G}, \hat{H}, \hat{\partial}, \hat{\alpha}) \to (G, H, \partial, \alpha)$ is a Morita morphism. Let N be the identity component of $\hat{\partial}(\hat{H})$. Then N is the connected immersed normal subgroup of $\hat{G}$ with Lie algebra $\mathfrak{h}$. It follows from [6, Proposition III.6.14] that N is closed and the identity component of the quotient $G' = \hat{G}/N$ is simply connected. So we can form the quotient crossed module $(G', H', \partial', \alpha')$ with $H' = \hat{H}/\hat{\partial}^{-1}(N)$, and we have a Morita morphism $(\hat{G}, \hat{H}, \hat{\partial}, \hat{\alpha}) \to (G', H', \partial', \alpha')$. In conclusion we have defined a zigzag of Morita morphisms

$$(G, H, \partial, \alpha) \longleftarrow (\hat{G}, \hat{H}, \hat{\partial}, \hat{\alpha}) \longrightarrow (G', H', \partial', \alpha')$$

with the identity component of G' simply connected and H' discrete. QED

For any foliation 2-group $G_\bullet$, the quotient $G_0/\bar{G}_1 = G/\overline{\partial(H)}$ (i.e. the largest Hausdorff quotient of the coarse quotient group G_0/G_1) is a Lie group, which is a Morita invariant of $G_\bullet$. Compactness properties of this Lie group translate into compactness properties of the Lie algebra of $G_\bullet$.

6.4.4. **Lemma.** *Let $G_\bullet$ be an étale 2-group. If the Lie group $G_0/\bar{G}_1$ is compact, then the Lie algebra of $G_\bullet$ is compact.*

Proof. Let (G, H, ∂, α) be the crossed module of $G_\bullet$. We may assume without loss of generality that G is connected. Let N be the immersed subgroup $\partial(H)$ of G, let $\bar{N}$ be its closure, and let $\mathfrak{n} = \mathrm{Lie}(N)$ and $\bar{\mathfrak{n}} = \mathrm{Lie}(\bar{N})$. Since N is 0-dimensional and normal, it is central in G, so $\bar{N}$ is central in G, so $\bar{\mathfrak{n}}$ is a central subalgebra of $\mathfrak{g}$. Since $K = G/\bar{N}$ is compact

Lie and its Lie algebra is $\mathfrak{g}/\bar{\mathfrak{n}}$, the central extension of Lie algebras $\bar{\mathfrak{n}} \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{k}$ is split. (To produce a splitting, start with an arbitrary linear right inverse s of the projection $\mathfrak{g} \to \mathfrak{k}$. The adjoint action of G on $\mathfrak{g}$ descends to an action of K, and the map $\bar{s} = \int_K \mathrm{Ad}_k \circ s \circ \mathrm{Ad}_k^{-1}\, dk$, where dk is normalized Haar measure on K, is a Lie algebra splitting of p.) This shows that $\mathfrak{g} \cong \bar{\mathfrak{n}} \oplus \mathfrak{k}$ is compact. QED

These lemmas can be made more precise when the coarse quotient G_0/G_1 is connected.

6.4.5. **Definition.** Let $G_\bullet$ be a foliation 2-group. We say $G_\bullet$ is *base-connected* (resp. *base-simply connected*) if G_0 is connected (resp. if G_0 is simply connected). We say $G_\bullet$ is of *compact type* if the Lie algebra $\mathfrak{g}_0$ is a compact Lie algebra and the coarse quotient G_0/G_1 is a compact topological space. We say $G_\bullet$ is *base-compact* if G_0 is compact.

6.4.6. **Lemma.** *Let $G_\bullet$ be a foliation 2-group. The following conditions are equivalent.*

(i) *G_0/G_1 is connected;*
(ii) *$G_0/\bar{G}_1$ is connected;*
(iii) *the inclusion $G'_\bullet \hookrightarrow G_\bullet$ is a Morita morphism, where $G'_\bullet$ is the full subgroupoid obtained by restricting $G_\bullet$ to the identity component G'_0 of G_0;*
(iv) *$G_\bullet$ is Morita equivalent to a base-simply connected étale 2-group.*

Proof. Let (G, H, ∂, α) be the crossed module of $G_\bullet$. The equivalence (i) $\iff$ (ii) is straightforward.

(i) $\implies$ (iii): If $G/\partial(H)$ is connected, then $G = \partial(H)G'$, where G' is the identity component of G. Therefore the inclusion $(G', H', \partial', \alpha') \to (G, H, \partial, \alpha)$ is a Morita morphism, where $(G', H', \partial', \alpha')$ is the restricted crossed module with $H' = \partial^{-1}(H \cap G') = H \times_G G'$.

(iii) $\implies$ (iv): Given the Morita morphism of crossed modules $(G', H', \partial', \alpha') \hookrightarrow (G, H, \partial, \alpha)$, we let $\tilde{G}$ be the universal cover of G', and as in the proof of Lemma 6.4.3 we obtain a base-simply connected crossed module $(\tilde{G}, \tilde{H}, \tilde{\partial}, \tilde{\alpha})$ and a Morita morphism $(\tilde{G}, \tilde{H}, \tilde{\partial}, \tilde{\alpha}) \to (G', H', \partial', \alpha')$. Next we let N the identity component of $\tilde{\partial}(\tilde{H})$, we form the quotient crossed module $(G'', H'', \partial'', \alpha'')$ with $G'' = \tilde{G}/N$, and we obtain a zigzag of Morita morphisms

$$(G, H, \partial, \alpha) \longleftarrow (G', H', \partial', \alpha') \longleftarrow (\tilde{G}, \tilde{H}, \tilde{\partial}, \tilde{\alpha}) \longrightarrow (G'', H'', \partial'', \alpha''),$$

where G'' is simply connected and H'' is discrete.

(iv) $\implies$ (i): If $G_\bullet \simeq G''_\bullet$ with $G''_\bullet$ base-connected, then $G/\partial(H) = G''/\partial''(H'')$ is connected. QED

Under additional compactness assumptions we have the following result, which is a 2-group analogue of the fact that every compact connected Lie group is up to a covering isomorphic to a product of a torus and a simply connected compact Lie group.

6.4.7. **Proposition.** *Let $G_\bullet$ be a foliation 2-group. The following conditions are equivalent.*

(i) *G_0/G_1 is compact and connected;*
(ii) *$G_0/\bar{G}_1$ is compact and connected;*
(iii) *$G_\bullet$ is Morita equivalent to a base-simply connected étale 2-group of compact type;*
(iv) *$G_\bullet$ is Morita equivalent to a base-compact and base-connected étale 2-group.*

Proof. Let (G, H, ∂, α) be the crossed module of $G_\bullet$. The implication (i) $\implies$ (ii) is obvious.

(ii) $\implies$ (iii): Suppose $G/\overline{\partial(H)}$ is compact and connected. By Lemma 6.4.6 we may assume without loss of generality that $G_\bullet$ is base-simply connected and étale. It then follows from Lemma 6.4.4 that the Lie algebra $\mathfrak{g}$ is compact.

(iii) $\implies$ (iv): Suppose that $G_\bullet$ is base-simply connected, étale, and of compact type. Then G is isomorphic to the product $E \times K$ of a vector group E and a simply connected compact Lie group K. Let $\mathrm{pr}_E : G \to E$ be the projection and let $N_E = \mathrm{pr}_E(N)$, where $N = \partial(H)$. Then pr_E induces a surjection $G/N \to E/N_E$, so E/N_E is compact, so N_E contains a basis $e_1, e_2, \ldots, e_k$ of E. Choose $n_i \in N$ with $\mathrm{pr}_E(n_i) = e_i$; then the subgroup L of N generated by the n_i is a discrete cocompact normal subgroup of G isomorphic to $\mathbb{Z}^k$. Then $G' = G/L$ is compact and connected, and we can form the quotient crossed module $(G', H' = H/\partial^{-1}(L), \partial', \alpha')$, which is Morita equivalent to (G, H, ∂, α). The associated 2-group $G'_\bullet$ is Morita equivalent to $G_\bullet$ and is base-compact, base-connected, and étale.

(iv) $\implies$ (i): If $G_\bullet \simeq G'_\bullet$ with $G'_\bullet$ base-compact and base-connected, then $G/\partial(H) = G'/\partial'(H')$ is compact and connected. QED

6.4.8. **Example.** The Lie 2-group $G_\bullet$ of Example 6.1.2 is a foliation 2-group of compact type. Its Lie 2-algebra $\mathfrak{n} \ltimes \mathfrak{g} \rightrightarrows \mathfrak{g}$ (see Example 6.2.2) is Morita equivalent to the abelian Lie algebra $\mathfrak{g}/\mathfrak{n}$.

6.5. **Lie group stacks.** A *weak Lie group stack* is weak 2-group in the 2-category **DiffStack**, as in Definition A.1. A *weak homomorphism* of weak Lie group stacks is a weak homomorphism of 2-groups in **DiffStack**, as in Definition A.6. An *equivalence* of weak Lie group stacks is an equivalence of 2-groups in **DiffStack**, as in Definition A.11. A *strict Lie group stack* is a strict 2-group in **DiffStack**. We will usually abbreviate *strict Lie group stack* to *Lie group stack*.

By Lemma 4.2.3, the 2-functor $\mathbf{B}$: **LieGpd** $\to$ **DiffStack** preserves weak pullbacks. So $\mathbf{B}$ preserves takes weak (resp. strict) Lie 2-groups to weak (resp. strict) Lie group stacks. Note that equivalent weak Lie group stacks are equivalent as differentiable stacks. A weak homomorphism of Lie groups, considered as Lie group stacks, is the same as a homomorphism of Lie groups.

6.5.1. **Definition.** Let $\mathbf{G}$ be a Lie group stack. A *presentation* of $\mathbf{G}$ is an equivalence of Lie group stacks $\mathbf{B}G_\bullet \to \mathbf{G}$ $G_\bullet$, where $G_\bullet$ is a Lie 2-group.

Given a stack $\mathbf{X}$, there is an underlying topological space called the *coarse moduli space*; see for instance [37, § 3] or [48] for a definition. For our purposes, it is enough to know that, given a presentation $\mathbf{B}X_\bullet \simeq \mathbf{X}$ of a differentiable stack, the coarse moduli space is homeomorphic to the coarse quotient X_0/X_1.

6.5.2. **Definition.** A weak Lie group stack $\mathbf{G}$ is *compact*, resp. *connected*, if the coarse moduli space is compact, resp. connected.

Our focus on the strict case is justified by the following strictification theorem, which says that a weak Lie group stack $\mathbf{G}$ that is connected and étale is equivalent to a strict Lie group stack and has a particularly nice atlas.

6.5.3. **Theorem** (Trentinaglia and Zhu [46, Theorem 5.13])**.** *Let* $\mathbf{G}$ *be a weak Lie group stack. Then the following properties are equivalent.*

(i) $\mathbf{G}$ *is connected and étale;*
(ii) $\mathbf{G}$ *is equivalent to a connected étale strict Lie group stack;*
(iii) $\mathbf{G}$ *is equivalent to the classifying stack* $\mathbf{B}G_\bullet$ *of a base-simply connected étale 2-group* $G_\bullet$*;*
(iv) *there exist a simply connected Lie group* G *and an étale atlas* $\boldsymbol{\psi} : G \to \mathbf{G}$ *which is a weak homomorphism.*

We will often use the following case of the strictification theorem. See Definition 6.4.5 for the notions base-connected and base-compact.

6.5.4. **Corollary.** *The following conditions on a weak Lie group stack* $\mathbf{G}$ *are equivalent:* (1) $\mathbf{G}$ *is compact, connected, and étale;* (2) *there exists a presentation* $\mathbf{B}G_\bullet \simeq \mathbf{G}$*, where* $G_\bullet$ *is a base-compact, base-connected, étale* 2*-group;* (3) *there exist a compact connected Lie group* G *and an étale atlas* $\psi\colon G \to \mathbf{G}$ *which is a weak homomorphism.*

Proof. Combine Theorem 6.5.3 with Proposition 6.4.7, using the fact that the coarse moduli space of $\mathbf{G}$ is homeomorphic to the coarse quotient of a presenting 2-group. QED

6.6. **Presentations of equivalent Lie group stacks.** The following result states that the fibred product of two strict Lie group stacks (if it exists as a differentiable stack) is a weak Lie group stack.

6.6.1. **Theorem.** *Let* $\mathbf{G} \to \mathbf{H}$ *and* $\mathbf{G}' \to \mathbf{H}$ *be weak homomorphisms of (strict) Lie group stacks, and assume that the fibred product of stacks* $\mathbf{K} = \mathbf{G} \times_{\mathbf{H}} \mathbf{G}'$ *is a differentiable stack. Then* $\mathbf{K}$ *is naturally a weak Lie group stack, and the projections* $\mathbf{K} \to \mathbf{G}$ *and* $\mathbf{K} \to \mathbf{G}'$ *are strict homomorphisms.*

Proof. See Appendix C. QED

We will deduce from this that if the classifying stacks of two Lie 2-groups are equivalent as Lie group stacks, then the two Lie 2-groups are Morita equivalent as Lie 2-groups.

6.6.2. **Proposition.** *Let* $\mathbf{G}$ *be a (strict) Lie group stack. Assume there are two presentations*

$$\psi\colon \mathbf{B}G_\bullet \xrightarrow{\simeq} \mathbf{G}, \qquad \psi'\colon \mathbf{B}G'_\bullet \xrightarrow{\simeq} \mathbf{G},$$

where $G_\bullet$ *and* $G'_\bullet$ *are Lie* 2*-groups. Let* $K_\bullet$ *be the Lie groupoid defined by*

$$K_0 = G_0 \times_{\mathbf{G}} G'_0 \qquad \text{and} \qquad K_1 = K_0 \times_{\mathbf{G}} K_0 \cong G_1 \times_{G_0} K_0 \times_{G'_0} G'_1.$$

Then

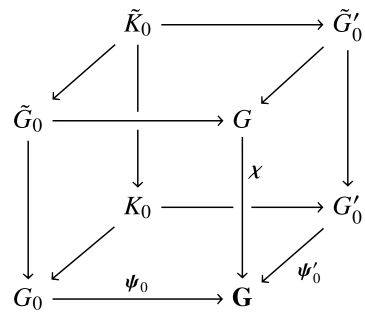
(i) $K_\bullet$ *is naturally a Lie* 2*-group. The maps* ψ *and* ψ' *induce Morita morphisms of Lie* 2*-groups* $K_\bullet \to G_\bullet$ *and* $K_\bullet \to G'_\bullet$*. The maps* $K_0 \to G_0$ *and* $K_0 \to G'_0$ *are surjective submersions.*
(ii) ψ *and* ψ' *induce Morita morphisms of Lie* 2*-algebras* $\mathrm{Lie}(K_\bullet) \to \mathrm{Lie}(G_\bullet)$ *and* $\mathrm{Lie}(K_\bullet) \to \mathrm{Lie}(G'_\bullet)$.
(iii) *If* $\mathbf{G}$ *is compact, connected, and étale, and if* $G_\bullet$ *and* $G'_\bullet$ *are of compact type, then* $K_\bullet$ *is of compact type.*

Proof. (i) Composing ψ and ψ' with the quotient maps $G_0 \to \mathbf{B}G_\bullet$ and $G'_0 \to \mathbf{B}G'_\bullet$ we get two atlases $\psi_0\colon G_0 \to \mathbf{G}$ and $\psi'_0\colon G'_0 \to \mathbf{G}'$, both of which are weak homomorphisms. Because atlases are surjective representable submersions, the fibred product $K_0 = G_0 \times_{\mathbf{G}} G'_0$ is (equivalent to) a manifold, and the projections $K_0 \to G_0$ and $K_0 \to G'_0$ are surjective submersions. On the other hand, by Theorem 6.6.1, K_0 is also a weak Lie group stack. It follows that K_0 is a Lie group. Moreover, the composition $K_0 \to G_0 \to \mathbf{G}$ is an atlas which is a weak homomorphism. Repeating the argument we see that K_1 is likewise a Lie group, and that the Lie groupoid $K_\bullet$ is a Lie 2-group equipped with two Morita morphisms of 2-groups $K_\bullet \to G_\bullet$ and $K_\bullet \to G'_\bullet$.

(ii) This follows from (i) and Lemma 6.2.1.

(iii) It follows from (i) that $K_0/K_1 \cong G_0/G_1$ is compact. Let G be a simply connected Lie group and $\chi\colon G \to \mathbf{G}$ an étale atlas which is a weak homomorphism as in the strictification theorem, Theorem 6.5.3(iv). It follows from (ii) and from Lemma 6.4.2 that ψ and ψ'

induce isomorphisms of Lie algebras $\mathfrak{g}_0/\mathfrak{h} \cong \mathrm{Lie}(G)$ and $\mathfrak{g}_0'/\mathfrak{h}' \cong \mathrm{Lie}(\mathbf{G})$. Therefore the Lie algebra $\mathfrak{k}_0' := \mathfrak{g}_0 \times_{\mathrm{Lie}(G)} \mathfrak{g}_0'$ is compact. We claim that $\mathfrak{k}_0'$ is isomorphic to $\mathfrak{k}_0 = \mathrm{Lie}(K_0)$. Let $\tilde{G}_0 = G_0 \times_{\mathbf{G}} G$ and $\tilde{G}_0' = G_0' \times_{\mathbf{G}} G$. Consider the cube



where $\tilde{K}_0$ is the weak limit of the three squares which contain $\mathbf{G}$. Since these squares are all weak pullbacks, every face of the cube is a weak pullback of stacks and we have $\tilde{K}_0 \cong \tilde{G}_0 \times_G \tilde{G}_0'$. Since $\chi\colon G \to \mathbf{G}$ is étale, the vertical maps are all étale. In particular, $\mathfrak{k}_0 = \mathrm{Lie}(K_0) \cong \mathrm{Lie}(\tilde{K}_0)$. We then have

$$\mathfrak{k}_0 \cong \mathrm{Lie}(\tilde{K}_0) \cong \mathrm{Lie}(\tilde{G}_0 \times_G \tilde{G}_0') \cong \mathrm{Lie}(\tilde{G}_0) \times_{\mathrm{Lie}(G)} \mathrm{Lie}(\tilde{G}_0') \cong \mathfrak{g}_0 \times_{\mathfrak{g}} \mathfrak{g}_0' = \mathfrak{k}_0',$$

as was claimed. QED

6.6.3. **Remark.** Even if the 2-groups $G_\bullet$ and $G_\bullet'$ in part (iii) are base-connected, the fibred product $K_\bullet$ is not necessarily base-connected. However, by Lemma 6.4.6(iii) $K_\bullet$ has a Morita equivalent full subgroupoid $K_\bullet'$ which is a base-connected sub-Lie 2-group, and the maps $K_0' \to G_0$ and $K_0' \to G_0'$ are still surjective submersions.

6.7. **The Lie algebra of an étale Lie group stack.** We define the *Lie algebra* of a connected étale weak Lie group stack $\mathbf{G}$ to be the Lie subalgebra $\mathrm{Lie}(\mathbf{G})$ of $\mathrm{Vect}(\mathbf{G})$ characterized by the next proposition. Here $\mathrm{Vect}(\mathbf{G})$ is the Lie algebra of vector fields of $\mathbf{G}$ described in § 5.7.

6.7.1. **Proposition.** *Let $\mathbf{G}$ be a connected étale weak Lie group stack. There is a unique Lie subalgebra* $\mathrm{Lie}(\mathbf{G})$ *of* $\mathrm{Vect}(\mathbf{G})$ *so that for every presentation $\mathbf{B}G_\bullet \simeq \mathbf{G}$ by a strict Lie 2-group the image of the Lie algebra embedding*

$$\mathfrak{g}/\mathfrak{h} \hookrightarrow \mathrm{Vect}_{\mathrm{bas}}(G_\bullet) \xrightarrow{\cong} \mathrm{Vect}(\mathbf{G}) \tag{6.7.2}$$

is $\mathrm{Lie}(\mathbf{G})$*. The first map of* (6.7.2) *is from Lemma 6.4.2*(iii)*, and the second map is* (5.7.4).

Proof. Let $\mathbf{B}G_\bullet \simeq \mathbf{G}$ and $\mathbf{B}G_\bullet' \simeq \mathbf{G}$ be two presentations of $\mathbf{G}$; such presentations exist by Theorem 6.5.3. By Proposition 6.6.2, the Lie 2-groups $G_\bullet$ and $G_\bullet'$ are Morita equivalent. Without loss of generality we may assume there is a Morita morphism $\phi_\bullet\colon G_\bullet \to G_\bullet'$. Then the result follows from naturality of (5.7.4) with respect to Morita morphisms as well as Lemma 6.4.2(iv). QED

Given a presentation $\mathbf{B}G_\bullet \simeq \mathbf{G}$ by a Lie 2-group, we have canonical isomorphisms

$$\mathrm{Lie}(\mathbf{G}) \cong \mathfrak{g}/\mathfrak{h} \cong \mathrm{Vect}_{\mathrm{bas}}(G_\bullet)_L.$$

If the groupoid $G_\bullet$ is étale, then $\mathrm{Lie}(\mathbf{G}) \cong \mathfrak{g}$.

Let $\boldsymbol{\phi}\colon \mathbf{G} \xrightarrow{\simeq} \mathbf{G}'$ be an equivalence of connected étale weak Lie group stacks. This induces an isomorphism of Lie algebras $\mathrm{Vect}(\mathbf{G}) \cong \mathrm{Vect}(\mathbf{G}')$ by Proposition 5.7.5. By Lemma 6.4.2(iv), this restricts to an isomorphism of Lie algebras, denoted by

$$\mathrm{Lie}(\boldsymbol{\phi})\colon\ \mathrm{Lie}(\mathbf{G}) \xrightarrow{\ \cong\ } \mathrm{Lie}(\mathbf{G}'). \tag{6.7.3}$$

6.8. **Actions of Lie group stacks.** A *weak action* of a (strict) Lie group stack $\mathbf{G}$ on a differentiable stack $\mathbf{X}$ is as in Definition A.12, by considering $\mathbf{G}$ to be a strict 2-group in **DiffStack**. A *strict action* is also as in Definition A.12. A *weakly equivariant morphism* $\mathbf{X} \to \mathbf{X}'$ of stacks with $\mathbf{G}$-actions is as in Definition A.16.

The 2-functor $\mathbf{B}\colon \mathbf{LieGpd} \to \mathbf{DiffStack}$ preserves products (Lemma 4.2.3), and so it takes (strict) actions of Lie 2-groups to (strict) actions of Lie group stacks.

Actions of connected étale Lie group stacks, like the stacks themselves, can be strictified in the sense of the following statement, which generalizes [27, Proposition 3.2].

6.8.1. **Theorem.** *Let $\mathbf{G}$ be a connected (strict) Lie group stack acting weakly on an étale differentiable stack $\mathbf{X}$. Suppose that $\mathbf{G}$ admits a presentation $\mathbf{B}G_\bullet \simeq \mathbf{G}$ by a base-connected Lie 2-group $G_\bullet$. For every such presentation $\mathbf{B}G_\bullet \simeq \mathbf{G}$ there exists a presentation $\mathbf{B}X_\bullet \simeq \mathbf{X}$ of $\mathbf{X}$ by a Lie groupoid $X_\bullet$ so that*

(i) *$G_\bullet$ acts strictly on $X_\bullet$;*
(ii) *identifying $\mathbf{B}G_\bullet = \mathbf{G}$, the equivalence $\mathbf{B}X_\bullet \simeq \mathbf{X}$ is weakly $\mathbf{G}$-equivariant.*

Proof. See Appendix B. QED

Theorem 6.5.3 and Theorem 6.8.1 together justify our focus on strict actions of foliation Lie 2-groups on foliation groupoids.

6.9. **Fundamental vector fields.** Let $\mathbf{G}$ be a connected strict Lie group stack, $\mathbf{X}$ an étale stack, and $\boldsymbol{a}\colon \mathbf{G} \times \mathbf{X} \to \mathbf{X}$ a weak action. Recall that $\mathbf{Vect}(\mathbf{X})$ denotes the groupoid of vector fields on $\mathbf{X}$ defined in § 5.7, and that the set $\mathrm{Vect}(\mathbf{X})$ of equivalence classes of $\mathbf{Vect}(\mathbf{X})$ carries a natural Lie bracket (Proposition 5.7.5). Let $\boldsymbol{\xi}$ be an element of the Lie algebra $\mathrm{Lie}(\mathbf{G})$ of $\mathbf{G}$ (as defined in § 6.7), regard $\boldsymbol{\xi}$ as an element of the Lie algebra $\mathrm{Vect}(\mathbf{G})$, and let $\tilde{\boldsymbol{\xi}} \in \mathbf{Vect}(\mathbf{G})$ be a lift of $\boldsymbol{\xi}$ to $\mathbf{Vect}(\mathbf{G})$. The composition of the morphisms

$$\mathbf{X} \simeq \star \times \mathbf{X} \xrightarrow{\mathbf{1}\times\mathrm{id}} \mathbf{G} \times \mathbf{X} \xrightarrow{\tilde{\boldsymbol{\xi}}\times\mathbf{0}} T\mathbf{G} \times T\mathbf{X} \simeq T(\mathbf{G} \times \mathbf{X}) \xrightarrow{T\boldsymbol{a}} T\mathbf{X} \tag{6.9.1}$$

defines an object of the groupoid $\mathbf{Vect}(\mathbf{X})$. Here the equivalence $T\mathbf{G} \times T\mathbf{X} \simeq T(\mathbf{G} \times \mathbf{X})$ follows from [21, Example 4.6], and $T\boldsymbol{a}$ is the tangent morphism of the action $\boldsymbol{a}$ as defined in [21, § 3.1]. The isomorphism class $\boldsymbol{\xi}_{\mathbf{X}} \in \mathrm{Vect}(\mathbf{X})$ of this object does not depend on the choice of lift $\tilde{\boldsymbol{\xi}}$, and we call $\boldsymbol{\xi}_{\mathbf{X}}$ the *fundamental vector field* of $\boldsymbol{\xi}$ associated with the action.

The following describes the fundamental vector field in terms of an atlas.

6.9.2. **Proposition.** *Let a foliation Lie 2-group $G_\bullet$ act strictly on a foliation groupoid $X_\bullet$. The map*

$$\mathfrak{g} \longrightarrow \mathrm{Vect}(X_0) \times \mathrm{Vect}(X_1)$$

defined by $\xi \mapsto (\xi_{X_0}, (\mathrm{Lie}(u)\xi)_{X_1})$ descends to a Lie algebra anti-homomorphism

$$\mathfrak{g}/\mathfrak{h} \cong \mathrm{Vect}_{\mathrm{bas}}(G_\bullet)_L \longrightarrow \mathrm{Vect}_{\mathrm{bas}}(X_\bullet). \tag{6.9.3}$$

The diagram

$$\begin{array}{ccc} \mathfrak{g}/\mathfrak{h} & \longrightarrow & \mathrm{Vect}_{\mathrm{bas}}(X_\bullet) \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ \mathrm{Lie}(\mathbf{B}G_\bullet) & \longrightarrow & \mathrm{Vect}(\mathbf{B}X_\bullet) \end{array}$$

commutes, where the bottom arrow is the assignment $\boldsymbol{\xi} \mapsto \boldsymbol{\xi}_{\mathbf{B}X_\bullet}$ *described in* (6.9.1)*, the left arrow is from Proposition 6.7.1, and the right arrow is* (5.7.4)*.*

Proof. Let us show that the map (6.9.3) is well defined. That it is an anti-homomorphism then follows from the fact that the map $\mathfrak{g} \to \mathrm{Vect}(X_0) \times \mathrm{Vect}(X_1)$ is an anti-homomorphism, and commutivity of the diagram follows from (5.7.1). Let $\xi \in \mathfrak{g}$. Then $\mathrm{Lie}(s)(\mathrm{Lie}(u)\xi) = \xi$. Hence, since the source map is equivariant with respect to the action, the vector field $(\mathrm{Lie}(u)\xi)_{X_1}$ is s-related to ξ_{X_0}. Similarly, $(\mathrm{Lie}(u)\xi)_{X_1}$ is t-related to ξ_{X_0}, so the pair $(\xi_{X_0}, (\mathrm{Lie}(u)\xi)_{X_1})$ descends to an element of $\mathrm{Vect}_{\mathrm{bas}}(X_\bullet)$. Now let $\xi \in \mathfrak{h}$. The vector field $(\xi_L, 0) \in \mathrm{Vect}(G_0) \times \mathrm{Vect}(X_0)$ is tangent to the foliation of $G_0 \times X_0$ by $G_\bullet \times X_\bullet$-orbits, and so its pushforward by the derivative of the action map is tangent to the foliation $\mathscr{F}_0(X_\bullet)$ of X_0. So ξ_{X_0} descends to the zero section of the normal bundle $N_0(X_\bullet)$, which shows that $(\xi_{X_0}, (\mathrm{Lie}(u)\xi)_{X_1})$ is the zero basic vector field. QED

6.9.4. **Definition.** Let a foliation 2-group $G_\bullet$ act strictly on a Lie groupoid $X_\bullet$. For $\xi \in \mathfrak{g}/\mathfrak{h}$, let $\xi_{X_\bullet} \in \mathrm{Vect}_{\mathrm{bas}}(X_\bullet)$ denote the image of ξ under the map (6.9.3). The basic vector field $\xi_{X_\bullet}$ is the *fundamental vector field of* ξ.

6.10. **The (co)adjoint action.** Let $G_\bullet$ be a Lie 2-group. The *adjoint action* of $G_\bullet$ is the Lie groupoid homomorphism

$$\mathrm{Ad}_\bullet \colon G_\bullet \times \mathfrak{g}_\bullet \to \mathfrak{g}_\bullet,$$

where, for $i = 0, 1$, Ad_i is the adjoint action of G_i on its Lie algebra $\mathfrak{g}_i$. The adjoint action is a strict $G_\bullet$-action on $\mathfrak{g}_\bullet$ in the sense of Definition A.12 since each map Ad_i is an action of a Lie group.

Since the adjoint action is natural with respect to Lie group homomorphisms, a homomorphism of Lie 2-groups $G_\bullet \to G'_\bullet$ takes the adjoint action of $G_\bullet$ on $\mathfrak{g}_\bullet$ to the adjoint action of $G'_\bullet$ on $\mathfrak{g}'_\bullet$. In particular, Morita morphisms of Lie 2-groups intertwine the adjoint actions.

6.10.1. **Lemma.** *Let* $G_\bullet$ *be a foliation Lie* 2*-group with crossed module* (G, H, ∂, α)*. Then the adjoint action of* $G_\bullet$ *on* $\mathfrak{g}_\bullet$ *descends to a strict action of* $G_\bullet$ *on* $\mathfrak{g}/\mathfrak{h}$*.*

Proof. Recall Morita morphism of Lie 2-algebras $\pi_\bullet \colon \mathfrak{g}_\bullet \to \mathfrak{g}/\mathfrak{h}$ of Lemma 6.4.2(i): $\pi_0 \colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ is the projection and $\pi_1 \colon \mathfrak{g}_1 \to \mathfrak{g}/\mathfrak{h}$ is π_0 composed with $\mathrm{Lie}(t)$. The kernel of π_0 is $\mathfrak{h}$, and this is preserved by Ad_0 since it is an ideal of $\mathfrak{g}$. The kernel of π_1 is $\ker(\mathrm{Lie}(s)) + \ker(\mathrm{Lie}(t))$, and this is preserved by Ad_1 since it is an ideal of $\mathfrak{g}_1$. Therefore the action $\mathrm{Ad}_\bullet$ descends through the morphism $\pi_\bullet$. QED

By abuse of language we will also call the induced action of $G_\bullet$ on $\mathfrak{g}/\mathfrak{h}$ from Lemma 6.10.1 the *adjoint action* and denote it by

$$\mathrm{Ad}_\bullet \colon G_\bullet \times \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}.$$

Let $(\mathfrak{g}/\mathfrak{h})^*$ be vector space dual to $\mathfrak{g}/\mathfrak{h}$ and let $\langle \cdot, \cdot \rangle \colon (\mathfrak{g}/\mathfrak{h})^* \times \mathfrak{g}/\mathfrak{h} \to \mathbb{R}$ be the dual pairing. The *coadjoint action*

$$\mathrm{Ad}^*_\bullet \colon G_\bullet \times (\mathfrak{g}/\mathfrak{h})^* \to (\mathfrak{g}/\mathfrak{h})^*$$

is the strict action given by the defining property

$$\langle \mathrm{Ad}_i^*(g^{-1})(\lambda), \xi\rangle = \langle g, \mathrm{Ad}_i(g)(\xi)\rangle$$

for $i = 0, 1$ and for all $g \in G_i$, $\lambda \in (\mathfrak{g}/\mathfrak{h})^*$, and $\xi \in \mathfrak{g}/\mathfrak{h}$.

The notions of adjoint and coadjoint action carry over to stacks as follows.

6.10.2. **Proposition.** *Let* $\mathbf{G}$ *be a connected étale weak Lie group stack and let* $\mathrm{Lie}(\mathbf{G})$ *be its Lie algebra, as defined in § 6.7. There is a unique strict action* $\mathbf{Ad}$ *of* $\mathbf{G}$ *on* $\mathrm{Lie}(\mathbf{G})$ *so that for every presentation* $\mathbf{B}G_\bullet \simeq \mathbf{G}$ *by a Lie 2-group the diagram*

$$\begin{array}{ccc} \mathbf{B}G_\bullet \times \mathfrak{g}/\mathfrak{h} & \xrightarrow{\mathbf{B}\,\mathrm{Ad}_\bullet} & \mathfrak{g}/\mathfrak{h} \\ \downarrow & & \downarrow \\ \mathbf{G} \times \mathrm{Lie}(\mathbf{G}) & \xrightarrow{\mathbf{Ad}} & \mathrm{Lie}(\mathbf{G}), \end{array}$$

commutes, where left arrow is the product of the presentation $\mathbf{B}G_\bullet \simeq \mathbf{G}$ *and the map* (6.7.2).

Proof. Given a presentation $\mathbf{B}G_\bullet \simeq \mathbf{G}$, one can define the morphism $\mathbf{Ad}$ so that the diagram commutes. Since $\mathrm{Lie}(\mathbf{G})$ (considered as a differentiable stack) takes values in the 1-category **Set**, the action is automatically strict. Uniqueness of $\mathbf{Ad}$ then follows from naturality of the adjoint action $\mathrm{Ad}_\bullet$ with respect to Morita equivalences. QED

We call the strict action

$$\mathbf{Ad}\colon \mathbf{G} \times \mathrm{Lie}(\mathbf{G}) \to \mathrm{Lie}(\mathbf{G})$$

of Proposition 6.10.2 the *adjoint action* of the Lie group stack $\mathbf{G}$. Let $\mathrm{Lie}(\mathbf{G})^*$ be the vector space dual to $\mathrm{Lie}(\mathbf{G})$ and let $\langle\cdot,\cdot\rangle\colon \mathrm{Lie}(\mathbf{G})^* \times \mathrm{Lie}(\mathbf{G}) \to \mathbb{R}$ be the dual pairing. Then the *coadjoint action* of $\mathbf{G}$ is the strict action

$$\mathbf{Ad}^*\colon \mathbf{G} \times \mathrm{Lie}(\mathbf{G})^* \to \mathrm{Lie}(\mathbf{G})^*$$

given by the defining property

$$\langle \mathbf{Ad}^* \circ ((\cdot)^{-1} \times \mathrm{id}_{\mathrm{Lie}(\mathbf{G})^*}), \mathrm{id}_{\mathrm{Lie}(\mathbf{G})}\rangle = \langle \mathrm{id}_{\mathrm{Lie}(\mathbf{G})^*}, \mathbf{Ad}\rangle.$$

If $\boldsymbol{g}\colon \star \to \mathbf{G}$ is a categorical point of $\mathbf{G}$, the isomorphism

$$\mathrm{Lie}(\mathbf{G}) \xrightarrow{\simeq} \star \times \mathrm{Lie}(\mathbf{G}) \xrightarrow{\boldsymbol{g}\times \mathrm{id}_{\mathrm{Lie}(\mathbf{G})}} \mathbf{G} \times \mathrm{Lie}(\mathbf{G}) \xrightarrow{\mathbf{Ad}} \mathrm{Lie}(\mathbf{G})$$

is denoted by $\mathbf{Ad}_{\boldsymbol{g}}$. The equivalence $\mathbf{Ad}^*_{\boldsymbol{g}}\colon \mathrm{Lie}(\mathbf{G})^* \to \mathrm{Lie}(\mathbf{G})^*$ is similarly defined.

6.11. **Stacky tori.** Stacky tori play an analogous role to that of compact tori in the theory of compact Lie groups. A 2-*torus* is a Lie 2-group which is Morita equivalent to a foliation 2-group $G_\bullet$ with the property that G_0 is a torus. A *stacky torus* is an étale Lie group stack equivalent to $\mathbf{B}G_\bullet$, where $G_\bullet$ is a 2-torus.

6.11.1. **Lemma.** *Suppose that* $G_\bullet$ *is a foliation 2-group and that* G_0 *is connected and abelian. Then* G_1 *is abelian and the action* $\alpha\colon G_0 \to \mathrm{Aut}(\ker(s))$ *is trivial.*

Proof. Let $g \in G$ and $h \in H$. Then $\partial({}^g h) = g\partial(h)g^{-1} = \partial(h)$ because G is abelian. Therefore $\partial({}^g h h^{-1}) = 1$, i.e. ${}^g h h^{-1} \in \ker(\partial)$. In other words, for each $h \in H$ the map $f(g) = {}^g h h^{-1}$ maps G to $\ker(\partial)$. But $G_\bullet$ is a foliation groupoid, so $\ker(\partial)$ is discrete, and G is connected, so f is constant. Thus ${}^g h h^{-1} = {}^1 h h^{-1} = h h^{-1} = 1$, so ${}^g h = h$, i.e. the action $\alpha\colon G \to \mathrm{Aut}(H)$ is trivial. Hence for all $h, h' \in H$ we have $h' = {}^{\partial(h)} h' = h h' h^{-1}$, i.e. $h'h = hh'$. It follows that $G_1 = H \rtimes_\alpha G$ is abelian. QED

In the setting of Lemma 6.11.1 we will often write the crossed module (G, H, ∂, α) simply as $\partial\colon H \to G$ or $H \to G$.

6.11.2. **Definition.** A *quasi-lattice* is a crossed module $\partial\colon A \to E$, where E is (the additive group of) a finite-dimensional real vector space and A is a countable discrete abelian group, and where the image $\partial(A)$ is required to span E as a vector space.

Definition 6.11.2 generalizes a definition of Prato [41], who assumes the map $\partial\colon A \to E$ to be injective. Our next result is a stacky analogue of the familiar fact that a torus is isomorphic to the quotient of its Lie algebra by the exponential lattice. Recall from [46, § 5] that the fundamental group $\pi_1(\mathbf{G})$ of a Lie group stack is the set of equivalence classes of maps $\mathbf{S}^1 \to \mathbf{G}$ based at the identity of $\mathbf{G}$, modulo homotopy.

6.11.3. **Proposition.** *Let* $\mathbf{G}$ *be a stacky torus and let* $G_\bullet$ *be a* 2*-torus with* $\mathbf{G} \simeq \mathbf{B}G_\bullet$*. The crossed module of* $G_\bullet$ *is Morita equivalent to a quasi-lattice* $\partial\colon A \to E$*. This quasi-lattice is isomorphic to* $\pi_1(\mathbf{G}) \to \mathrm{Lie}(\mathbf{G})$*, and hence is uniquely determined up to isomorphism by* $\mathbf{G}$.

Proof. We may assume that $G_\bullet$ is étale and that G_0 is a torus. By Lemma 6.11.1, G_1 is abelian and G_0 acts trivially on G_1. We obtain our Morita equivalence from two weak equivalences as in the diagram

$$\begin{array}{ccccc} H & \longleftarrow & \tilde{H} & \longrightarrow & A \\ {\scriptstyle\partial}\big\downarrow & & {\scriptstyle\tilde{\partial}}\big\downarrow & & {\scriptstyle\partial}\big\downarrow \\ G & \xleftarrow{\ \exp\ } & \mathfrak{g} & \longrightarrow & E. \end{array}$$

Here $\tilde{H}$ is the fibred product

$$H \times_G \mathfrak{g} = \{\, (h, \xi) \mid \partial(h) = \exp(\xi) \,\}$$

and E is the quotient $E = \mathfrak{g}/\mathfrak{h}$. The kernel of the exponential map $\exp\colon \mathfrak{g} \to G$ is isomorphic to $\pi_1(G) = \mathrm{Hom}(\mathbf{U}(1), G)$, the fundamental group of G, and we have a short exact sequence

$$\pi_1(G) \hookrightarrow \tilde{H} \twoheadrightarrow H,$$

which shows that $\tilde{H}$ is an extension of H by $\pi_1(G)$. The group $\tilde{H}$ contains a copy $\tilde{\mathfrak{h}}$ of $\mathfrak{h}$, namely the image of the embedding $\mathfrak{h} \to \tilde{H}$ which sends η to $(\exp(\eta), \mathrm{Lie}(\partial)(\eta))$. We have $\mathrm{Lie}(\tilde{H}) \cong \tilde{\mathfrak{h}}$. We define $A = \tilde{H}/\tilde{\mathfrak{h}}$ to complete the diagram. Then $\mathrm{Lie}(A) = 0$, so A is discrete. The Morita equivalence between $H \to G$ and $A \to E$ gives us a group isomorphism $G/\partial(H) \cong E/\partial(A)$, and hence a surjection $G \to E/\partial(A)$, which implies that $\partial(A)$ generates E as a vector space, because G is compact. This proves the existence of the quasi-lattice $\partial\colon A \to E$.

The uniqueness is proved as follows. Since $H \to G$ is Morita equivalent to $A \to E$, Proposition 6.7.1 guarantees that there is an isomorphism of the abelian Lie algebras

$$\mathrm{Lie}(\mathbf{G}) \cong \mathrm{Lie}(E)/\mathrm{Lie}(A) = \mathrm{Lie}(E) = E$$

Let $E_\bullet$ be the Lie 2-group associated with the crossed module $A \to E$, then the equivalence $\mathbf{G} \simeq \mathbf{B}E_\bullet$ gives a fibration of stacks $A \to E \to \mathbf{G}$ in the sense of Noohi [38, § 5], and hence a long exact homotopy sequence, which yields $A \cong \pi_1(\mathbf{G})$. QED

6.12. **Maximal stacky tori.** We now introduce the appropriate analogue of a maximal torus in the 2-categories of Lie 2-groups and Lie group stacks.

6.12.1. **Definition.** Let $G_\bullet$ be a foliation 2-group of compact type. A *maximal* 2-*torus* of $G_\bullet$ is a full Lie 2-subgroup $T_\bullet$ of $G_\bullet$ such that $T_0 \subseteq G_0$ is connected and $\mathfrak{t}_0 = \mathrm{Lie}(T_0)$ is a maximal abelian subalgebra of $\mathfrak{g}_0$.

A maximal 2-torus is a 2-torus in the sense of § 6.11. If (G, H, ∂, α) is the crossed module of $G_\bullet$, then the crossed module of $T_\bullet$ is $(T, H_T = \partial^{-1}(T), \partial, \alpha)$. By Lemma 6.11.1, the closed subgroup H_T of H is abelian and the action of T on H_T is trivial.

6.12.2. **Definition.** Let $\mathbf{G}$ be a compact connected étale Lie group stack. A *maximal stacky torus* is a strict homomorphism $\mathbf{T} \to \mathbf{G}$ of Lie group stacks with the following property: there exist a foliation 2-group $G_\bullet$ of compact type, a maximal 2-torus $T_\bullet$ of $G_\bullet$, and presentations

$$\mathbf{B}G_\bullet \simeq \mathbf{G} \qquad \text{and} \qquad \mathbf{B}T_\bullet \simeq \mathbf{T},$$

such that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{B}T_\bullet & \longrightarrow & \mathbf{B}G_\bullet \\ {\scriptstyle\simeq}\downarrow & & \downarrow{\scriptstyle\simeq} \\ \mathbf{T} & \longrightarrow & \mathbf{G}. \end{array}$$

It follows from the strictification theorem, Corollary 6.5.4, that every compact connected étale Lie group stack has a maximal stacky torus.

6.12.3. **Lemma.** *Let* $(\psi_G, \psi_H)\colon (G, H, \partial, \alpha) \to (G', H', \partial', \alpha')$ *be Morita morphism of crossed modules. Assume the Lie algebras* $\mathfrak{g}$ *and* $\mathfrak{g}'$ *are compact, and assume* $\psi_G\colon G \to G'$ *is surjective. Let* $\mathfrak{t}$ *and* $\mathfrak{t}'$ *be maximal abelian Lie subalgebras of* $\mathfrak{g}$ *and* $\mathfrak{g}'$, *respectively, chosen so that* $\mathfrak{t}'$ *contains* $\mathrm{Lie}(\psi_G)(\mathfrak{t})$. *Let* $T \subseteq G$ *and* $T' \subseteq G'$ *be the connected subgroups with Lie algebras* $\mathfrak{t}$ *and* $\mathfrak{t}'$, *respectively. Then the restriction*

$$(\psi_T, \psi_{H_T}) := (\psi_G|_T, \psi_H|_{H_T})\colon (T, H_T, \partial, \alpha) \longrightarrow (T', H'_T, \partial', \alpha')$$

is a Morita morphism.

Proof. Let us first show that $\psi_T\colon T \to T'$ is surjective. Since $\mathfrak{g}$ and $\mathfrak{g}'$ are compact, they decompose into $\mathfrak{g} \cong [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{z}(\mathfrak{g})$ and $\mathfrak{g}' \cong [\mathfrak{g}', \mathfrak{g}'] \oplus \mathfrak{z}(\mathfrak{g}')$, where $[\mathfrak{g}, \mathfrak{g}]$ and $[\mathfrak{g}', \mathfrak{g}']$ are the derived subalgebras of $\mathfrak{g}$ and $\mathfrak{g}'$, respectively, and $\mathfrak{z}(\mathfrak{g})$ and $\mathfrak{z}(\mathfrak{g}')$ are the centers of $\mathfrak{g}$ and $\mathfrak{g}'$, respectively. Since ψ_G is surjective, $\mathrm{Lie}(\psi_G)$ maps $\mathfrak{z}(\mathfrak{g})$ onto $\mathfrak{z}(\mathfrak{g}')$. If $[\mathfrak{g}, \mathfrak{g}] = \bigoplus_i \mathfrak{g}_i$ is a decomposition into simple subalgebras, then $[\mathfrak{g}', \mathfrak{g}'] = \bigoplus_i \mathfrak{g}'_i$, where $\mathfrak{g}'_i = \mathrm{Lie}(\psi_G)(\mathfrak{g}_i) \cong \mathfrak{g}_i$ or $\mathfrak{g}'_i = 0$. It follows that $\mathrm{Lie}(\psi_G)(\mathfrak{t})$ is a maximal abelian subalgebra of $\mathfrak{g}'$, and hence $\psi_T(T) = T'$.

Next we show that $H_T = \partial^{-1}(T) \cong T \times_{T'} (\partial')^{-1}(T')$. Since (ψ_G, ψ_H) is a Morita morphism, we may identify $H = G \times_{G'} H'$ and $H_T = T \times_{G'} H' = T \times_{T'} H'$. Assume that $t \in T$ and $h' \in H'$ with $\psi_T(t) = \partial'(h')$. Then $\partial'(h') \in T'$, so $h' \in (\partial')^{-1}(T')$. Thus $H_T = T \times_{T'} (\partial')^{-1}(T')$. The result now follows from Lemma 6.3.2. QED

The conjugation action $\boldsymbol{C}\colon \mathbf{G} \times \mathbf{G} \to \mathbf{G}$ is a strict action of $\mathbf{G}$ on itself. Composing this action with a categorical point $\boldsymbol{g}\colon \star \to \mathbf{G}$ gives an equivalence $\boldsymbol{C_g}\colon \mathbf{G} \to \mathbf{G}$, called *conjugation by* $\boldsymbol{g}$.

6.12.4. **Corollary.** *Let* $\mathbf{G}$ *be a compact connected étale Lie group stack and let* $\mathbf{T} \to \mathbf{G}$ *and* $\mathbf{T}' \to \mathbf{G}$ *be two maximal stacky tori. Then there exist a categorical point* $\boldsymbol{g}\colon \star \to \mathbf{G}$ *and an equivalence of Lie group stacks* $\mathbf{T} \simeq \mathbf{T}'$ *so that the following diagram commutes:*

$$\begin{array}{ccc} \mathbf{T} & \longrightarrow & \mathbf{G} \\ {\scriptstyle\simeq}\downarrow & & \downarrow{\scriptstyle C_{\boldsymbol{g}}} \\ \mathbf{T}' & \longrightarrow & \mathbf{G}. \end{array}$$

Proof. By Lemma 6.12.3 and Proposition 6.6.2, we can assume that there exists a single base-connected foliation Lie 2-groups of compact type $G_\bullet$ presenting $\mathbf{G}$, so that $\mathbf{T}$ and $\mathbf{T}'$ both come from maximal 2-tori $T_\bullet$ and $T'_\bullet$ of $G_\bullet$. From the theory of Lie groups, the subgroups T_0 and T'_0 of G_0 are related by conjugation in G_0. Passing to the associated Lie group stacks gives the result. QED

By an argument as in Corollary 6.12.4, one can also show that a maximal stacky torus $\mathbf{T}$ is maximal in the sense that, if $\mathbf{T}' \to \mathbf{G}$ is a sub-Lie group stack and $\mathbf{T}'$ is a stacky torus, then there is some $\boldsymbol{g}\colon \star \to \mathbf{G}$ so that $\mathbf{T}' \to \mathbf{G}$ factors through the morphism $\mathbf{T} \longrightarrow \mathbf{G} \xrightarrow{C_{\boldsymbol{g}}} \mathbf{G}$.

6.13. **Principal bundles.** Bursztyn, Noseda, and Zhu [9] have introduced the notion of a principal bundle in the 2-category of differentiable stacks. We briefly introduce the analogous notion in the 2-category of Lie groupoids, and compare the two.

6.13.1. **Definition.** Let $G_\bullet$ be a Lie 2-group, $X_\bullet$ and $Y_\bullet$ Lie groupoids, and $a_\bullet\colon G_\bullet \times X_\bullet \to X_\bullet$ a strict action. A Lie groupoid morphism $\psi_\bullet\colon X_\bullet \to Y_\bullet$ is $G_\bullet$*-invariant* if there is a given 2-isomorphism $\gamma\colon \psi_\bullet \circ \mathrm{pr}_2 \Rightarrow \psi_\bullet \circ a_\bullet$ making the following diagram 2-commute:

$$\begin{array}{ccc} G_\bullet \times X_\bullet & \xrightarrow{a_\bullet} & X_\bullet \\ {\scriptstyle \mathrm{pr}_2}\downarrow & & \downarrow{\scriptstyle\psi_\bullet} \\ X_\bullet & \xrightarrow{\psi_\bullet} & Y_\bullet. \end{array}$$

The 2-isomorphism $\gamma\colon G_0 \times X_0 \to Y_1$ is required to satisfy two coherence conditions, for all $x \in X_0$ and $g, h \in G_0$:

$$\gamma(h, g\cdot x) \circ \gamma(g, x) = \gamma(hg, x) \qquad \text{and} \qquad \gamma(1, x) = u(\psi_0(x)). \tag{6.13.2}$$

A Lie groupoid morphism $\psi_\bullet\colon X_\bullet \to Y_\bullet$ is a *principal* $G_\bullet$*-bundle* if

(i) $\psi_\bullet$ is essentially surjective;
(ii) $\psi_\bullet$ is $G_\bullet$-invariant;
(iii) the canonical morphism $G_\bullet \times X_\bullet \to X_\bullet \times^{(w)}_{Y_\bullet} X_\bullet$ is a Morita morphism.

6.13.3. **Example.** Let G be a Lie group (viewed as the identity 2-group $G \rightrightarrows G$) and X a G-manifold (viewed as the identity groupoid $X \rightrightarrows X$). Then the morphism $\psi_\bullet\colon X \to G \ltimes X$ defined by $\psi_1(x) = (1, x)$ is a principal G-bundle.

6.13.4. **Definition.** Let $\mathbf{G}$ be a Lie group stack, $\mathbf{X}$ and $\mathbf{Y}$ differentiable stacks, and $\boldsymbol{a}\colon \mathbf{G} \times \mathbf{X} \to \mathbf{X}$ a strict action. A morphism $\boldsymbol{\psi}\colon \mathbf{X} \to \mathbf{Y}$ is $\mathbf{G}$*-invariant* if there is a given 2-isomorphism $\boldsymbol{\gamma}\colon \boldsymbol{\psi} \circ \mathrm{pr}_2 \Rightarrow \boldsymbol{\psi} \circ \boldsymbol{a}$ satisfying two coherence conditions analogous to (6.13.2), namely that the following diagrams commute:

$$\begin{array}{ccccc} \boldsymbol{\psi} \circ \mathrm{pr}_2 \circ \mathrm{pr}_{23} & \xRightarrow{\boldsymbol{\gamma}} & \boldsymbol{\psi} \circ \boldsymbol{a} \circ \mathrm{pr}_{23} & = & \boldsymbol{\psi} \circ \mathrm{pr}_2 \circ (\mathrm{id}_{\mathbf{G}} \times \boldsymbol{a}) \\ \| & & & & \Downarrow{\scriptstyle\boldsymbol{\gamma}} \\ \boldsymbol{\psi} \circ \mathrm{pr}_2 \circ (\boldsymbol{m} \times \mathrm{id}_{\mathbf{X}}) & \xRightarrow{\boldsymbol{\gamma}} & \boldsymbol{\psi} \circ \boldsymbol{a} \circ (\boldsymbol{m} \times \mathrm{id}_{\mathbf{X}}) & = & \boldsymbol{\psi} \circ \boldsymbol{a} \circ (\mathrm{id}_{\mathbf{G}} \times \boldsymbol{a}) \end{array}$$

$$\begin{array}{ccc} \boldsymbol{\psi} \circ \mathrm{pr}_2 \circ (1_{\mathbf{G}} \times \mathrm{id}_{\mathbf{X}}) & & \\ \boldsymbol{\psi} \Big\Downarrow & \diagdown\!\!\diagdown & \\ \boldsymbol{\psi} \circ \boldsymbol{a} \circ (1_{\mathbf{G}} \times \mathrm{id}_{\mathbf{X}}) & = \!=\!=\!= & \boldsymbol{\psi} \circ \mathrm{pr}_2 \,. \end{array}$$

Here we omit horizontal composition of 1-morphisms and 2-morphisms, and $\mathrm{pr}_{23} \colon \mathbf{G}\times\mathbf{G}\times \mathbf{X} \to \mathbf{G}\times\mathbf{X}$ is the projection to the second and third factors. A stack morphism $\boldsymbol{\psi} \colon \mathbf{X} \to \mathbf{Y}$ is a *principal* $\mathbf{G}$*-bundle* if

(i) $\boldsymbol{\psi}$ is a stack epimorphism, and there is an atlas $X \to \mathbf{X}$ so the composition $X \to \mathbf{X} \to \mathbf{Y}$ is representable;
(ii) $\boldsymbol{\psi}$ is $\mathbf{G}$-invariant;
(iii) the canonical morphism $\mathbf{G} \times \mathbf{X} \to \mathbf{X} \times_{\mathbf{Y}} \mathbf{X}$ is an equivalence.

6.13.5. **Proposition.** *Let* $(\psi_\bullet \colon X_\bullet \to Y_\bullet, \gamma)$ *be a principal* $G_\bullet$*-bundle. Then* $(\mathbf{B}\psi_\bullet \colon \mathbf{B}X_\bullet \to \mathbf{B}Y_\bullet, \mathbf{B}\gamma)$ *is a principal* $\mathbf{B}G_\bullet$*-bundle.*

Proof. Since $\psi_\bullet$ is essentially surjective, the map $\mathbf{B}\psi_\bullet$ is an epimorphism of stacks. To check the conditions of Definition 6.13.4, it suffices to note that $\mathbf{B}$ is a 2-functor which preserves weak fibred products (Lemma 4.2.3), and to show that there is an atlas $M \to \mathbf{B}X_\bullet$ so that the composition $M \to \mathbf{B}X_\bullet \to \mathbf{B}Y_\bullet$ is representable. Indeed, we can take $M = X_0$. Then $\mathbf{B}Y_\bullet \simeq \mathbf{B}\psi_0^*(Y_\bullet)$, where $\psi_0^*(Y_\bullet)$ is the pullback groupoid, and the composition $X_0 \to \mathbf{B}X_\bullet \to \mathbf{B}(\psi_0^*(Y_\bullet) \simeq \mathbf{B}Y_\bullet$ is representable. QED

6.13.6. **Remark.** Let $\psi_\bullet \colon X_\bullet \to Y_\bullet$ be a principal $G_\bullet$-bundle. The canonical morphism $X_\bullet \to \psi_0^*(Y_\bullet)$ is the identity on the manifold of objects X_0, and therefore induces an injection $\Omega^\bullet_{\mathrm{bas}}(\psi_0^*(Y_\bullet)) \to \Omega^\bullet_{\mathrm{bas}}(X_\bullet)$. The canonical morphism $\psi_0^*(Y_\bullet) \to Y_\bullet$ is a Morita morphism, so $\Omega^\bullet_{\mathrm{bas}}(\psi^*(Y_\bullet))$ is isomorphic to $\Omega^\bullet_{\mathrm{bas}}(Y_\bullet)$. We conclude that the pullback map of basic forms $\psi^* \colon \Omega^\bullet_{\mathrm{bas}}(Y_\bullet) \to \Omega^\bullet_{\mathrm{bas}}(X_\bullet)$ is an injection.

7. Hamiltonian actions on groupoids and stacks

In this section we introduce Hamiltonian actions in the 2-categories of Lie groupoids and of differentiable stacks. Our notion of Hamiltonian actions extends that of Hamiltonian Lie group actions on stacks defined by Lerman and Malkin [27] in two ways: we allow our groups to be étale Lie group stacks, and we allow our stacks to be non-separated. We show that every presymplectic Hamiltonian action can be integrated, in several different ways, to a Hamiltonian action of a foliation groupoid on a 0-symplectic groupoid (Theorem 7.2.1). The classifying functor $\mathbf{B}$ takes any such Hamiltonian groupoid to a Hamiltonian stack. The converse is also true: any Hamiltonian stack arises from a Hamiltonian groupoid (Theorem 7.3.2). We then show that, for compact Lie group stacks, the moment map image is an invariant of a stacky Hamiltonian action and obtain the stacky convexity theorem (Theorem 7.4.2).

7.1. Hamiltonian actions on 0-symplectic Lie groupoids.

Let $(X_\bullet, \omega_\bullet)$ be a 0-symplectic Lie groupoid (§ 5.5), let $G_\bullet$ be a foliation 2-group (Definition 6.4.1), and let (G, H, ∂, α) be the associated crossed module (§ 6.3). We assume the coarse quotient group G_0/G_1 to be connected. A strict action $a_\bullet \colon G_\bullet \times X_\bullet \to X_\bullet$ is *Hamiltonian* if there is a morphism of Lie groupoids called the *moment map*

$$\mu_\bullet = (\mu_0, \mu_1) \colon X_\bullet \longrightarrow (\mathfrak{g}/\mathfrak{h})^* \cong \mathrm{ann}(\mathfrak{h}) \subseteq \mathfrak{g}^*$$

which satisfies the following conditions:

(i) Let $\xi \in \mathfrak{g}/\mathfrak{h}$, and let $\xi_{X_\bullet} \in \mathrm{Vect}_{\mathrm{bas}}(X_\bullet)$ be the fundamental vector field of ξ on $X_\bullet$, as in Definition 6.9.4. Then we require

$$d\mu_0^\xi = \iota_{\xi_{X_0}}\omega_0.$$

(ii) Recall the *coadjoint action* of $G_\bullet$ on $(\mathfrak{g}/\mathfrak{h})^*$ defined in § 6.10. We require that $\mu_\bullet$ is (strictly) equivariant with respect to the $G_\bullet$ action on $X_\bullet$ and the coadjoint action.

A *Hamiltonian $G_\bullet$-groupoid* is a tuple $(X_\bullet, \omega_\bullet, G_\bullet, \mu_\bullet)$ consisting of a 0-symplectic groupoid equipped with a Hamiltonian $G_\bullet$-action with moment map $\mu_\bullet$.

7.1.1. **Example.** The actions of Example 6.1.2 are Hamiltonian, with moment map $(\mu_0, \mu_0 \circ s)$ which was described in Example 3.1.

7.2. **From manifolds to Lie groupoids.** Let (X, ω, G, μ) be a Hamiltonian presymplectic G-manifold with null foliation $\mathscr{F} = \ker(\omega)$ and null ideal $\mathfrak{n} = \mathfrak{n}(\mathscr{F})$. We will show we can integrate these data to a Hamiltonian Lie groupoid $(X_\bullet, \omega_\bullet, G_\bullet, \mu_\bullet)$, where $X_\bullet$ is a foliation groupoid that integrates $(X, \mathscr{F})$, and $G_\bullet$ is a foliation 2-group that integrates the Lie 2-algebra $\mathfrak{g}_\bullet = (\mathfrak{n} \rtimes \mathfrak{g} \rightrightarrows \mathfrak{g})$. As usual we can integrate both $(X, \mathscr{F})$ and $\mathfrak{g}_\bullet$ in a number of different ways, but we have to integrate them in a compatible manner. The following theorem shows that the monodromy groupoid of X and the source-simply connected integration of $\mathfrak{g}_\bullet$ always work. If the action of G on the monodromy groupoid descends to an action on the holonomy groupoid, then we can take $X_\bullet$ to be the holonomy groupoid and $G_\bullet = (N \rtimes G \rightrightarrows G)$, where $N = N(\mathscr{F})$ is the null subgroup, i.e. the immersed subgroup of G generated by $\mathfrak{n}$.

7.2.1. **Theorem.** *Let G be a Lie group and (X, ω, G, μ) a presymplectic Hamiltonian G-manifold with null foliation $\mathscr{F}$ and null ideal $\mathfrak{n}$.*

(i) *There exists a source-simply connected Lie 2-group $G_\bullet$ with object group $G_0 = G$ and Lie 2-algebra $\mathrm{Lie}(G_\bullet) = \mathfrak{n} \rtimes \mathfrak{g} \rightrightarrows \mathfrak{g}$. This Lie 2-group is unique up to a unique isomorphism that induces the identity map of G and of $\mathfrak{n} \rtimes \mathfrak{g}$.*

(ii) *Let $X_\bullet$ be a source-connected Lie groupoid over $X_0 = X$ integrating $\mathscr{F}$ and let $\psi_\bullet = \psi_{X_\bullet}\colon \mathrm{Mon}(X, \mathscr{F}) \to X_\bullet$ be the universal morphism as in Theorem 5.1.1. There exists a $G_\bullet$-action on $X_\bullet$ that extends the action of $G_0 = G$ on X if and only if $\ker(\psi_\bullet)$ is preserved by the G-action, where $\ker(\psi_\bullet)$ is the kernel of $\phi_\bullet$ as defined in 5.1.2. This $G_\bullet$-action is unique and it is Hamiltonian with respect to the 0-symplectic structure $\omega_\bullet$ on $X_\bullet$ determined by ω.*

(iii) *Assume that the G-action preserves the kernel of the holonomy homomorphism $\mathrm{hol}\colon \mathrm{Mon}(X, \mathscr{F}) \to \mathrm{Hol}(X, \mathscr{F})$. Then the $G_\bullet$-action on $\mathrm{Hol}(X, \mathscr{F})$ given by (ii) descends to a Hamiltonian action of the Lie 2-group $N \rtimes G \rightrightarrows G$, where $N \subseteq G$ is the null subgroup.*

Proof. (i) Let H be the universal cover of N. We interpret elements of H as homotopy classes relative to endpoints of paths $\nu\colon [0,1] \to N$ starting at $\nu(0) = 1$. Define the homomorphism $\partial\colon H \to G$ by $\partial([\nu]) = \nu(1)$ and the action $\alpha\colon G \to \mathrm{Aut}(H)$ by $\alpha(g)([\nu]) = {}^g[\nu] = [\tau \mapsto g\nu(\tau)g^{-1}]$. This defines a crossed module (G, H, ∂, α) and hence a Lie 2-group $G_\bullet$ with $G_0 = G$, simply connected source fibre $\ker(s) = H$, and $\mathrm{Lie}(G_1) = \mathfrak{n} \rtimes \mathfrak{g}$. The uniqueness of $G_\bullet$ follows from the uniqueness of the simply connected group H.

(ii) We start by showing that the G-action on X extends in at most one way to a $G_\bullet$-action on $X_\bullet$. The G-action on X preserves the foliation $\mathscr{F}$ and therefore induces an action on $\mathrm{Lie}(X_\bullet) = T\mathscr{F}$ by Lie algebroid automorphisms. Since $X_\bullet$ is source-connected, by Lie's theorems for Lie groupoids (see [36, § 6.3]) there can exist no more than one G-action on $X_\bullet$ that is compatible with the action on $\mathrm{Lie}(X_\bullet)$. We show that the H-action on X_1 is unique by showing that the action of its Lie algebra $\mathfrak{h}$ is unique. Let $\eta \in \mathfrak{h}$. By assumption $\partial\colon \mathfrak{h} \to \mathfrak{g}$ is an isomorphism onto $\mathfrak{n}$, so the vector field $\partial(\eta)_X$ is tangent to the foliation, and therefore $\partial(\eta)_X = \rho(\sigma(\eta))$ for a unique section $\sigma(\eta)$ of $\mathrm{Lie}(X_\bullet) = T\mathscr{F}$. Let $\sigma(\eta)_R$ be the right-invariant vector field on X_1 determined by $\sigma(\eta)$. By Lemma 6.3.14 we must have $\sigma(\eta)_R = \eta_{X_1}$. So the $\mathfrak{h}$-action on X_1 is determined by the G-action on X.

Next we show the existence of a $G_\bullet$-action on $X_\bullet$. First consider the case of $M_\bullet = \mathrm{Mon}(X, \mathscr{F})$, the monodromy groupoid of $\mathscr{F}$. An element of M_1 is a leafwise homotopy class relative to endpoints of a path $\gamma\colon [0,1] \to M_0 = X$ in a leaf of $\mathscr{F}$. Let $[\nu] \in H$. The homotopy class of the path $\nu \cdot \gamma\colon \tau \mapsto \nu(\tau) \cdot \gamma(\tau)$ depends only on the homotopy classes $[\nu] \in H$ and $[\gamma] \in M_1$, so we have a well-defined action of H on M_1 given by $[\nu] * [\gamma] = [\nu \cdot \gamma]$. Similarly, the homotopy class of the path $g \cdot \gamma\colon \tau \mapsto g \cdot \gamma(\tau)$ depends only on $g \in G$ and on the homotopy class $[\gamma] \in M_1$, so we have a well-defined action of G on M_1 given by $g * [\gamma] = [g \cdot \gamma]$. The actions of G on M_0 and on M_1 and the action of H on M_1 satisfy the rules (6.3.8)–(6.3.12), and therefore combine to an action of $G_\bullet$ on $M_\bullet$.

Now consider the general case of a source-connected groupoid $X_\bullet$ integrating $(X, \mathscr{F})$ and for which the kernel of $\psi_\bullet\colon M_\bullet \to X_\bullet$ is preserved by G. Then the G-action on M_1 descends to a G-action on $X_1 = M_1/\ker(\psi)$. As we saw in the discussion of uniqueness, the G-action on X_0 determines an action of $\mathfrak{h}$ on X_1. For each $\eta \in \mathfrak{h}$ the vector field η_{X_1} is complete because it lifts to the complete vector field η_{M_1}. Hence, by the Lie-Palais theorem, the $\mathfrak{h}$-action on X_1 integrates to an H-action. (Here we use that the source fibres of $X_\bullet$ are Hausdorff by Lemma 4.1.5 and that the vector fields η_{X_1} are tangent to the source fibres by Lemma 6.3.14.) The conditions (6.3.8)–(6.3.12) hold because they hold on $M_\bullet$.

Finally, the moment map μ is invariant on leaves of the null foliation, so it defines a map of Lie groupoids $\mu\colon X_\bullet \to \mathrm{ann}(\mathfrak{n})$. The map μ_0 is G-equivariant by assumption.

(iii) It suffices to show that the kernel of the homomorphism $\partial\colon H \to G$ acts trivially on the arrows of $\mathrm{Hol}(X, \mathscr{F})$. If $[\nu] \in \ker(\partial)$, then ν is a loop based at the identity of N. Let S be a local transversal to $x \in X$, then $\mathrm{hol}([\nu \cdot u(x)])$ is the germ of the holonomy action of the path $\nu(\tau) \cdot x\colon [0,1] \times \{x\} \to X$ on S. This path extends to $\nu(\tau) \cdot S\colon [0,1] \times S \to X$, and so the holonomy action on S is just $\nu(1) \cdot S = \nu(0) \cdot S$, and therefore $\mathrm{hol}([\nu \cdot u(x)]) = u(x)$. For $\gamma \in \mathrm{Mon}(X, \mathscr{F})$, applying (6.3.11) gives

$$[\nu] * \mathrm{hol}([\gamma]) = \mathrm{hol}([\nu \cdot \gamma]) = \mathrm{hol}([\nu \cdot u(t(g))] \circ [\gamma]) = \mathrm{hol}([\gamma]),$$

which proves the claim. QED

7.3. **Hamiltonian actions on stacks.** Let $(\mathbf{X}, \omega)$ be a symplectic stack (§ 5.9) and let $\mathbf{G}$ be a connected étale Lie group stack (§ 6.1). A weak action $\boldsymbol{a}\colon \mathbf{G} \times \mathbf{X} \to \mathbf{X}$ is *Hamiltonian* if there is a morphism $\boldsymbol{\mu}\colon \mathbf{X} \to (\mathrm{Lie}(\mathbf{G}))^*$ called the *moment map*, where $(\mathrm{Lie}(\mathbf{G}))^*$ is the dual vector space of $\mathrm{Lie}(\mathbf{G})$. We require that

(i) $d\boldsymbol{\mu}^{\boldsymbol{\xi}} = \iota_{\boldsymbol{\xi}_{\mathbf{X}}} \omega$ for all $\boldsymbol{\xi} \in \mathrm{Lie}(\mathbf{G})$, and
(ii) $\boldsymbol{\mu}$ is $\mathbf{G}$-equivariant with respect to the coadjoint action of $\mathbf{G}$ on $\mathrm{Lie}(\mathbf{G})^*$.

A *Hamiltonian* $\mathbf{G}$*-stack* is a tuple $(\mathbf{X}, \omega, \mathbf{G}, \boldsymbol{\mu})$ consisting of a symplectic stack $(\mathbf{X}, \omega)$ and a Hamiltonian $\mathbf{G}$-action with moment map $\boldsymbol{\mu}$.

7.3.1. **Definition.** An *equivalence of Hamiltonian* $\mathbf{G}$*-stacks*

$$\boldsymbol{\phi}\colon (\mathbf{X}, \omega, \mathbf{G}, \boldsymbol{\mu}) \simeq (\mathbf{X}', \omega', \mathbf{G}', \boldsymbol{\mu}')$$

is a pair $(\boldsymbol{\phi}_{\mathbf{X}}, \boldsymbol{\phi}_{\mathbf{G}})$, where $\boldsymbol{\phi}_{\mathbf{X}}\colon (\mathbf{X}, \omega) \simeq (\mathbf{X}', \omega')$ is an equivalence of symplectic stacks, and $\boldsymbol{\phi}_{\mathbf{G}}\colon \mathbf{G} \simeq \mathbf{G}'$ is an equivalence of Lie group stacks, subject to the following conditions. The equivalence $\boldsymbol{\phi}_{\mathbf{G}}$ determines an action of $\mathbf{G}'$ on $\mathbf{X}$, which is

$$\mathbf{G}' \times \mathbf{X} \xrightarrow{\boldsymbol{\phi}_{\mathbf{G}}^{-1} \times \mathrm{id}} \mathbf{G} \times \mathbf{X} \xrightarrow{\boldsymbol{a}} \mathbf{X},$$

where $\boldsymbol{\phi}_{\mathbf{G}}^{-1}$ is a weak inverse of $\boldsymbol{\phi}_{\mathbf{G}}$. The tuple $\left(\mathbf{X}, \omega, \mathbf{G}', \mathrm{Lie}\left(\boldsymbol{\phi}_{\mathbf{G}}^{-1}\right)^* \circ \boldsymbol{\mu}\right)$ is a Hamiltonian $\mathbf{G}'$-stack, where $\mathrm{Lie}(\boldsymbol{\phi}_{\mathbf{G}})$ is as in (6.7.3). We then require that

(i) $\boldsymbol{\phi}_{\mathbf{X}}$ is $\mathbf{G}'$-equivariant, and
(ii) $\boldsymbol{\mu}' \circ \boldsymbol{\phi}_{\mathbf{X}} = \mathrm{Lie}\left(\boldsymbol{\phi}_{\mathbf{G}}^{-1}\right)^* \circ \boldsymbol{\mu}$.

The preceding sections establish the following bijection up to equivalence between Hamiltonian stacks and groupoids. For the following theorem, recall that by the strictification theorem (Theorem 6.5.3), every connected étale Lie group stack $\mathbf{G}$ has a presentation $\mathbf{B}G_\bullet \simeq \mathbf{G}$ by a base-connected foliation 2-group $G_\bullet$.

7.3.2. **Theorem.** (i) *Let $(X_\bullet, \omega_\bullet, G_\bullet, \mu_\bullet)$ be a Hamiltonian $G_\bullet$-groupoid. Then the classifying stack $(\mathbf{B}X_\bullet, \mathbf{B}\omega_\bullet, \mathbf{B}G_\bullet, \mathbf{B}\mu_\bullet)$ is a Hamiltonian $\mathbf{G}$-stack.*
(ii) *Let $(\mathbf{X}, \omega, \mathbf{G}, \boldsymbol{\mu})$ be a Hamiltonian $\mathbf{G}$-stack. For every base-connected Lie 2-group $G_\bullet$ with $\mathbf{B}G_\bullet \simeq \mathbf{G}$, there exists a Hamiltonian $G_\bullet$-groupoid $(X_\bullet, \omega_\bullet, G_\bullet, \mu_\bullet)$ so that $(\mathbf{B}X_\bullet, \mathbf{B}\omega_\bullet, \mathbf{B}G_\bullet, \mathbf{B}\mu_\bullet)$ is equivalent to $(\mathbf{X}, \omega, \mathbf{G}, \boldsymbol{\mu})$ as a Hamiltonian $\mathbf{G}$-stack.*

Proof. (i) This follows from Proposition 5.9.1 and the definitions of Hamiltonian groupoids and stacks.

(ii) Let $X_\bullet$ and $G_\bullet$ be Lie groupoids presenting $\mathbf{X}$ and $\mathbf{G}$ as in Theorem 6.8.1, so that $G_\bullet \times X_\bullet \to X_\bullet$ is a strict action presenting $\boldsymbol{a}$. It follows from Proposition 5.9.1 that $X_\bullet$ is 0-symplectic Lie groupoid. The map $X_0 \longrightarrow \mathbf{X} \xrightarrow{\boldsymbol{\mu}} \mathrm{Lie}(\mathbf{G})^*$ gives a moment map for the $G_\bullet$ action on $X_\bullet$, and $X_\bullet$ is a Hamiltonian $G_\bullet$-groupoid. Finally, $\mathbf{B}X_\bullet$ is equivalent to $\mathbf{X}$ as a Hamiltonian stack by construction. QED

7.4. **The stacky moment body.** In this subsection $\mathbf{G}$ denotes a compact connected étale Lie group stack and $\mathbf{T} \to \mathbf{G}$ a maximal stacky torus. The Lie algebra $\mathrm{Lie}(\mathbf{G})$ is compact and $\mathrm{Lie}(\mathbf{T})$ is a maximal abelian subalgebra. Moreover, $\mathrm{Lie}(\mathbf{T})$ is in a natural way a direct summand of $\mathrm{Lie}(\mathbf{G})$, so we can identify $\mathrm{Lie}(\mathbf{T})^*$ with a subspace of $\mathrm{Lie}(\mathbf{G})^*$. The *Weyl group* of $\mathbf{G}$ relative to $\mathbf{T}$ is by definition $W = W(\mathrm{Lie}(\mathbf{G}), \mathrm{Lie}(\mathbf{T}))$, the Weyl group of $\mathrm{Lie}(\mathbf{G})$ relative to $\mathrm{Lie}(\mathbf{T})$. We choose a closed Weyl chamber C for the W-action on $\mathrm{Lie}(\mathbf{T})^*$.

Let $(\mathbf{X}, \boldsymbol{\omega}, \mathbf{G}, \boldsymbol{\mu})$ be a Hamiltonian $\mathbf{G}$-stack. The *moment map image* is the set $\boldsymbol{\mu}(\mathbf{X}) \subseteq \mathrm{Lie}(\mathbf{G})^*$ defined by $\boldsymbol{\mu}(\mathbf{X}) = \mu_0(X_0) = \mu_1(X_1)$ for any presentation $\mathbf{B}X_\bullet \simeq \mathbf{X}$ of $\mathbf{X}$ and $\mathbf{B}\mu_\bullet \cong \boldsymbol{\mu}$ of $\boldsymbol{\mu}$. Picking different presentations of $\mathbf{X}$ and $\boldsymbol{\mu}$ leaves $\boldsymbol{\mu}(\mathbf{X})$ unchanged. We define the *stacky moment body* of $(\mathbf{X}, \boldsymbol{\omega}, \mathbf{G}, \boldsymbol{\mu})$ to be the pair $(\Delta(\mathbf{X}), \mathbf{T})$, where $\Delta(\mathbf{X}) = \boldsymbol{\mu}(\mathbf{X}) \cap C$. An *equivalence* of stacky moment bodies $(\Delta, \mathbf{T}) \simeq (\Delta', \mathbf{T}')$ is an equivalence $\boldsymbol{\phi}_{\mathbf{T}} : \mathbf{T} \xrightarrow{\simeq} \mathbf{T}'$ so that $\mathrm{Lie}(\boldsymbol{\phi}_{\mathbf{T}})^*(\Delta') = \Delta$.

7.4.1. **Proposition.** *Let $(\mathbf{X}, \omega, \mathbf{G}, \boldsymbol{\mu})$ be a Hamiltonian $\mathbf{G}$-stack.*

(i) *Up to equivalence, the stacky moment body of $\mathbf{X}$ is independent of the choice of the maximal stacky torus $\mathbf{T} \to \mathbf{G}$ and the Weyl chamber C.*
(ii) *If $\boldsymbol{\phi} : (\mathbf{X}, \omega, \mathbf{G}, \boldsymbol{\mu}) \simeq (\mathbf{X}', \omega', \mathbf{G}', \boldsymbol{\mu}')$ is an equivalence of Hamiltonian stacks, then the stacky moment body of $\mathbf{X}$ is equivalent to the stacky moment body of $\mathbf{X}'$.*

Proof. (i) Given two maximal stacky tori $\mathbf{T}$ and $\mathbf{T}'$ of $\mathbf{G}$ and Weyl chambers $C \subseteq \mathrm{Lie}(\mathbf{T})^*$ and $C' \subseteq \mathrm{Lie}(\mathbf{T}')^*$, arguing as in Corollary 6.12.4, there exists a categorical point $\boldsymbol{g} : \star \to \mathbf{G}$ so that $\mathbf{Ad}_{\boldsymbol{g}}^* : \mathrm{Lie}(\mathbf{G})^* \to \mathrm{Lie}(\mathbf{G})^*$ maps $\mathrm{Lie}(\mathbf{T})^*$ to $\mathrm{Lie}(\mathbf{T}')^*$ and C to C'. Therefore, by equivariance of the moment map,

$$\mathbf{Ad}_{\boldsymbol{g}}^*(\boldsymbol{\mu}(\mathbf{X}) \cap C) = \boldsymbol{\mu}(\mathbf{X}) \cap \mathbf{Ad}_{\boldsymbol{g}}^*(C) = \boldsymbol{\mu}(\mathbf{X}) \cap C',$$

as desired.

(ii) Given a pair of equivalences $\boldsymbol{\phi}_{\mathbf{X}}\colon \mathbf{X}\simeq\mathbf{X}'$ and $\boldsymbol{\phi}_{\mathbf{G}}\colon \mathbf{G}\simeq\mathbf{G}'$ as in Definition 7.3.1, we have $\boldsymbol{\mu}'(\mathbf{X}')=\mathrm{Lie}\big(\boldsymbol{\phi}_{\mathbf{G}}^{-1}\big)^*(\boldsymbol{\mu}(\mathbf{X}))$ by Definition 7.3.1(ii). Choose a maximal torus $\mathbf{T}\to\mathbf{G}$ of $\mathbf{G}$ and a chamber $C\subseteq\mathfrak{t}^*$, let C' be the chamber $\mathrm{Lie}\big(\boldsymbol{\phi}_{\mathbf{G}}^{-1}\big)^*(C)$ of the maximal torus

$$\mathbf{T}\longrightarrow\mathbf{G}\xrightarrow{\ \phi_{\mathbf{G}}\ }\mathbf{G}'$$

of $\mathbf{G}'$. Then $\boldsymbol{\phi}_G$ defines an equivalence from $(\Delta(\mathbf{X}),\mathbf{T}\to\mathbf{G})$ to $(\Delta(\mathbf{X}'),\mathbf{T}\to\mathbf{G}')$. QED

We can now rephrase the main result of [30] in the language of stacks.

7.4.2. **Theorem.** *Let $(\mathbf{X},\boldsymbol{\omega},\mathbf{G},\boldsymbol{\mu})$ be a Hamiltonian $\mathbf{G}$-stack. If $(\mathbf{X},\boldsymbol{\omega},\mathbf{G},\boldsymbol{\mu})$ is equivalent to $(\mathbf{B}X_\bullet,\mathbf{B}\omega_\bullet,\mathbf{B}G_\bullet,\mathbf{B}\mu_\bullet)$, where X_0 and G_0 are compact and connected, and the action of G_0 on X_0 is clean, then $\Delta(\mathbf{X})$ is a closed convex polyhedron.*

Proof. This follows from Proposition 7.4.1 and Theorem 3.2. QED

7.4.3. **Example.** Let $(X_\bullet,\omega_\bullet,G_\bullet,\mu_\bullet)$ be the Hamiltonian groupoid of Example 7.1.1. The Lie 2-group $G_\bullet$ presents a stacky torus $\mathbf{G}$. Let $(\mathbf{X},\boldsymbol{\omega},\mathbf{G},\boldsymbol{\mu})$ be the Hamiltonian $\mathbf{G}$-stack presented by $(X_\bullet,\omega_\bullet,G_\bullet,\mu_\bullet)$. Referring back to Example 3.1, we see that the image of the moment map $\mu_0\colon X_0\to\mathrm{ann}(\mathfrak{n})$ is the polyhedron

$$P=\{\,\eta\in\mathrm{ann}(\mathfrak{n})\mid\langle a_i,\eta\rangle\geq\lambda_i\text{ for }1\leq i\leq n\,\}.$$

Here $a_i\in\mathfrak{g}/\mathfrak{n}$ is the image of the standard basis vector $e_i\in\mathfrak{g}=\mathbb{R}^n$ under the projection $\mathfrak{g}\to\mathfrak{g}/\mathfrak{n}$. We conclude that the stacky moment body is $\Delta(\mathbf{X})=(P,\mathbf{G})$. Following Prato [41] we call $\mathbf{X}$ a *toric quasifold* associated with the stacky polyhedron $(P,\mathbf{G})$.

7.4.4. **Remark.** Lerman and Tolman [28] show that compact symplectic toric orbifolds (which can be thought of as certain separated Hamiltonian stacks, as in [27]) are classified by their moment polytopes which have positive integer labels attached to their faces. Let M be a toric $\mathbb{T}^k$-orbifold and let n be the number of faces of the moment polytope Δ of M. The labels of Δ can be thought of as defining a homomorphism $\mathbb{Z}^n\to\mathbb{Z}^k$. More generally, we can consider homomorphisms $\mathbb{Z}^n\to A$ labeling the stacky moment polytope of a compact Hamiltonian $\mathbf{G}$-stack, where $A\to E$ is a quasi-lattice presenting a stacky torus $\mathbf{G}$. See [22] for more on this perspective and for an interpretation of the results of [28], [41], and [43] in a stacky context.

8. Leafwise transitivity

In this section we prove two basic structural results about 2-group actions on groupoids, with an eye toward the symplectic reduction theorem and the Duistermaat-Heckman theorem. We introduce the notion of a leafwise transitive Lie 2-group action. We show that if $(X_\bullet,\omega_\bullet,G_\bullet,\mu_\bullet)$ is a Hamiltonian $G_\bullet$-groupoid, then a regular fibre of $\mu_\bullet$ is Morita equivalent to a Lie groupoid with a locally leafwise transitive $G_\bullet$-action (Proposition 8.2.1). If $G_\bullet$ is a 2-torus acting leafwise transitively on a foliation groupoid $X_\bullet$, we show that $X_\bullet$ is isomorphic to an action groupoid (Proposition 8.3.3).

Throughout Section 8 $X_\bullet$ denotes a foliation groupoid and $G_\bullet$ denotes a foliation 2-group acting strictly on $X_\bullet$. We denote by $\mathscr{F}$ the foliation of X_0 defined by $X_\bullet$, and by (G,H,∂,α) the crossed module associated to $G_\bullet$.

8.1. **Leafwise transitive and regular actions.** Let $x \in X_0$ and $h \in H$. Let $f = h * u(x) \in X_1$. It follows from (6.3.10) that $s(f) = x$ and $t(f) = \partial(h) \cdot x$. This shows that the orbit of x under the $\partial(H)$-action is contained in the $X_\bullet$-orbit of x. We call the $G_\bullet$-action *leafwise transitive* if this inclusion is an equality, in other words if

$$\partial(H) \cdot x = t(s^{-1}(x))$$

for all $x \in X_0$. We call the action *locally leafwise transitive* if for every $x \in X_0$ the image of the map $\mathfrak{h} \to T_x X_0$ defined by $\eta \mapsto (\partial(\eta)_{X_0})_x$ is equal to $T_x \mathscr{F}$.

A leafwise transitive action is locally leafwise transitive. Conversely, if the action is locally leafwise transitive and if additionally $G_\bullet$ and $X_\bullet$ are both source-connected, then $\partial(H) \cdot x = \mathscr{F}(x) = t(s^{-1}(x))$ for all $x \in X_0$, so in particular the action is leafwise transitive.

We call the $G_\bullet$-action *regular* if the following conditions hold: $G_\bullet$-action is locally leafwise transitive; the G-action on X_0 is free; and if $h \in H$ satisfies $h * f = f$ for any $f \in X_1$, then $h \in \ker(\partial)$.

8.2. **Regular form of the zero fibre.** In this subsection $(X_\bullet, \omega_\bullet, G_\bullet, \mu_\bullet)$ denotes a Hamiltonian $G_\bullet$-groupoid (as defined in § 7.1). The *zero fibre* of $\mu_\bullet$ is the subgroupoid $\mu_\bullet^{-1}(0) = (\mu_1^{-1}(0) \rightrightarrows \mu_0^{-1}(0))$ of $X_\bullet$. We say that 0 is a *regular value* of $\mu_\bullet$ if $0 \in (\mathfrak{g}_0/\mathfrak{h})^*$ is a regular value of $\mu_0 \colon X_0 \to (\mathfrak{g}_0/\mathfrak{h})^*$.

By [27, § 3.9], $\mu_\bullet^{-1}(0)$ is a Lie subgroupoid of $X_\bullet$ if 0 is a regular value of $\mu_\bullet$.

Suppose that 0 is a regular value of $\mu_\bullet$. We define a *regular form* of the zero fibre $\mu_\bullet^{-1}(0)$ to be a pair $(R_\bullet, \phi_\bullet)$ with the following properties: $R_\bullet$ is a foliation groupoid equipped with a regular $G_\bullet$-action; $\phi_\bullet \colon R_\bullet \to \mu_\bullet^{-1}(0)$ is a (strictly) $G_\bullet$-equivariant Morita morphism; and the G_0-orbits of R_0 are the leaves of the null foliation of $\phi_0^*\omega_0 \in \Omega^2(R_0)$.

8.2.1. **Proposition.** *Suppose that* 0 *is a regular value of* $\mu_\bullet$. *Then there exists a regular form* $(R_\bullet, \phi_\bullet)$ *of the zero fibre* $Z_\bullet = \mu_\bullet^{-1}(0)$.

Proof. Let $\mathscr{F}_0$ be the restriction of the foliation $\mathscr{F}$ to Z_0. Let D be the subbundle of TZ_0 spanned by $T\mathscr{F}_0$ and the fundamental vector fields of G. Then $\operatorname{rank} D = \operatorname{rank} T\mathscr{F} + \dim \mathfrak{g} - \dim \mathfrak{h}$ is constant because 0 is a regular value of μ. Let $v \in \Gamma(T\mathscr{F})$ and $\xi \in \mathfrak{g}$. The identity

$$\iota_{[\xi_{Z_0}, v]}\omega_0 = \mathscr{L}_{\xi_{Z_0}}\iota_v\omega_0 - \iota_v\mathscr{L}_{\xi_{Z_0}}\omega_0 = 0 - 0 = 0$$

shows that $[\xi_{Z_0}, v] \in \Gamma(T\mathscr{F})$, so D is involutive. Therefore $D = T\mathscr{D}$ for a unique foliation $\mathscr{D}$ of Z_0. This is the foliation denoted by $\mathfrak{g} \ltimes \mathscr{F}_0$ in [29, § 2]; its leaves are the G-orbits $G \cdot \mathscr{F}_0(x)$ of leaves of $\mathscr{F}_0$. Let $S \hookrightarrow Z_0$ be a complete transversal of $\mathscr{D}$ and define $\phi_0 \colon G \times S \to Z_0$ by $\phi_0(g, x) = g \cdot x \in Z_0$. Then ϕ_0 is transverse to $\mathscr{F}_0$ and is complete, so it follows from Lemma 5.1.4 that the pullback groupoid $R_\bullet = \phi_0^* Z_\bullet$ is a Lie groupoid and that the induced morphism $\phi_\bullet \colon R_\bullet \to Z_\bullet$ is a Morita morphism. The object manifold of $R_\bullet$ is $R_0 = G \times S$. Elements of R_1 are tuples $((j, x), f, (j', x')) \in R_0 \times Z_1 \times R_0$ where $s(f) = j \cdot x$ and $t(f) = j' \cdot x'$. Using the action of the crossed module (G, H, ∂, α) on $Z_\bullet$ we define an action on $R_\bullet$ as follows.

$$\begin{aligned} G \times R_0 &\longrightarrow R_0 \colon \quad g \cdot (j, x)) = (gj, x) \\ G \times R_1 &\longrightarrow R_1 \colon \quad g * ((j, x), f, (j', x')) = ((gj, x), g * f, (gj', x')) \\ H \times R_1 &\longrightarrow R_1 \colon \quad h * ((j, x), f, (j', x')) = ((j, x), h * f, (\partial(h)j', x')). \end{aligned}$$

These actions satisfy the conditions (6.3.8)–(6.3.12) and so by Lemma 6.3.13 determine a strict action of $G_\bullet$ on $R_\bullet$. The morphism ϕ is $G_\bullet$-equivariant, the G-action on R_0 is free, and if $h \in H$ fixes any tuple $((j, x), f, (j', x')) \in R_1$, then $\partial(h) = 1$. To check that the action of $G_\bullet$ on $R_\bullet$ is locally leafwise transitive, we count dimensions. Since the G-action on R_0 is free, the H-action $y \mapsto \partial(h)\cdot y$ is locally free. Hence for every $y \in R_0$ the infinitesimal orbit map $\mathfrak{h} \to T_y R_0$ is injective, so the dimension of its image is equal to $\dim \mathfrak{h}$. On the other hand, by Lemma 6.3.14 the image is contained in the fibre at y of the Lie algebroid $\mathrm{Alg}(R_\bullet)$, which has rank equal to $\dim R_1 - \dim R_0 = \dim \mathfrak{h}$. This proves that the $G_\bullet$-action on $R_\bullet$ is regular. Since 0 is a regular value of $\mu_\bullet$, the foliation $\mathscr{D}$ of Z_0 is transversely symplectic, so $(S, \omega_0|_S)$ is a symplectic manifold, and $(R_0, \phi_0^*\omega_0)$ is a presymplectic manifold with null foliation given by the G-orbits in R_0. QED

8.3. **Foliation groupoids versus action groupoids.** Recall (Definition 5.1.2) that a Lie group bundle is a Lie groupoid where every arrow g has $s(g) = t(g)$. For a group K acting on a manifold X, let $K_x \subseteq K$ denote the stabilizer of a point $x \in X$.

8.3.1. **Lemma.** *Let K be a simply connected Lie group acting locally freely on a manifold X and let $\mathscr{F}$ be the foliation of X into K-orbits. Let $L = \ker(K \to \mathrm{Diff}(X))$ be the kernel of the K-action. Assume that the set $X_L = \{\, x \in X \mid K_x = L \,\}$ is dense in X.*

(i) *The monodromy groupoid* $\mathrm{Mon}(X, \mathscr{F})$ *is isomorphic to the action groupoid $K \ltimes X$.*
(ii) *The holonomy groupoid* $\mathrm{Hol}(X, \mathscr{F})$ *is isomorphic to the action groupoid $(K/L) \ltimes X$.*
(iii) *Every source-connected Lie groupoid $X_1 \rightrightarrows X_0 = X$ integrating $\mathscr{F}$ is isomorphic to one of the form $(K \ltimes X)/Z$, where $Z \to X$ is an open Lie group subbundle of the Lie group bundle $L \times X \to X$; and X_1 is Hausdorff if and only if Z is closed.*

Proof. The Lie groupoids $K \ltimes X$ and $(K/L) \ltimes X$ are source-connected, have Lie algebroid isomorphic to $T\mathscr{F}$, and $K \ltimes X$ is source-simply connected. Therefore $K \ltimes X$ is isomorphic to $\mathrm{Mon}(X, \mathscr{F})$, which proves (i).

Let $X_\bullet$ be any source-connected Lie groupoid with base $X_0 = X$ and Lie algebroid $T\mathscr{F}$. By Theorem 5.1.1 the holonomy morphism $\mathrm{Mon}(X, \mathscr{F}) = K \ltimes X \to \mathrm{Hol}(X, \mathscr{F})$ factors as two surjective étale maps:

$$\mathrm{hol}\colon K \ltimes X \xrightarrow{\psi_{X_\bullet}} X_\bullet \xrightarrow{\mathrm{hol}_{X_\bullet}} \mathrm{Hol}(X, \mathscr{F}). \tag{8.3.2}$$

Let us first take $X_\bullet = (K/L) \ltimes X$. To establish (ii) we need to prove that $\mathrm{hol}_{X_\bullet}$ is an isomorphism, which amounts to showing that $\ker(\mathrm{hol}) = \ker(\psi_{X_\bullet})$. The kernel of $\psi_{X_\bullet}$ is the Lie group bundle $L \times X$. Consider any pair $(g, x_0) \in K \ltimes X$ contained in the kernel of hol. The equation $\mathrm{hol}(g, x_0) = u(x_0)$ means that $g \cdot x_0 = x_0$ and, for a sufficiently small section S of the foliation $\mathscr{F}$ at x_0, for every $x \in S$ the points x and $g \cdot x$ are in the same plaque of the foliation near S. The map $\Phi\colon K \times S \to X$ that sends (g, x) to $g \cdot x$ is étale at $(1_K, x_0)$, so, after shrinking S if necessary, Φ restricts to a diffeomorphism $\phi\colon U \times S \to V$, where U is a neighbourhood of $1_K \in K$ and V a neighbourhood of $x_0 \in X$. The inverse ϕ^{-1} is a foliation chart at x_0. The slice $g \cdot S$ of $\mathscr{F}$ is transverse to the orbit $K \cdot x_0$ at x_0 and therefore (if S is small enough) $\phi^{-1}(g \cdot S) \subseteq U \times S$ is the graph of a unique smooth function $f\colon S \to U$. Since x and $g \cdot x$ are in the same plaque we have $\phi^{-1}(g \cdot x) \in U \times \{x\}$, which is equivalent to $g \cdot x = f(x) \cdot x$. Hence $f(x) \in gK_x$ for all $x \in S$. The set X_L, being preserved by the K-action and dense in X, intersects S in a dense set. Thus $f(x) \in g^{-1}L$ for a dense set of $x \in S$. Since L is discrete, the smooth map f is constant on S. It follows that $f(x) = f(x_0) = 1_K$ for $x \in S$, in other words that $g \in L$ and $(g, x_0) \in L \times X$. Thus $\ker(\mathrm{hol}) = L \times X = \ker(\psi_{X_\bullet})$, which proves (ii).

Returning to a general Lie groupoid $X_\bullet$ integrating $\mathscr{F}$ as in the diagram (8.3.2), we see that $X_\bullet \cong (K \ltimes X)/Z$, where $Z = \ker(\psi_{X_\bullet})$ is a group subbundle of the Lie group bundle $\ker(\mathrm{hol}) = L \times X$. The Lie group bundles Z and $L \times X$ are manifolds of dimension equal to that of X, and therefore Z is open in $L \times X$. The equivalence relation on $K \ltimes X$ defined by the action of Z is closed if and only if Z is closed. This proves (iii). QED

Specializing to the abelian case gives the following result.

8.3.3. **Proposition.** *Let $G_\bullet$ be a 2-torus. Assume that $G_\bullet$ and $X_\bullet$ are source-connected. Also assume that X_0 is connected, X_1 is Hausdorff, the action of $G_\bullet$ on $X_\bullet$ is leafwise transitive, and the action $H \xrightarrow{\partial} G \longrightarrow \mathrm{Diff}(X_0)$ of H on X_0 is locally free. Then $X_\bullet$ is isomorphic to the action groupoid $H/Z \ltimes X_0$, where Z is a subgroup of the discrete subgroup $\ker(H \to \mathrm{Diff}(X_0))$ of H. The action of $G_1 \cong H \times G_0$ on $X_1 \cong H/Z \ltimes X_0$ is given by $(h, g) * (kZ, x) = (hkZ, g \cdot x)$.*

Proof. Assume first that H is simply connected. Put $X = X_0$. Let L_G be the kernel of the action $G \to \mathrm{Diff}(X)$ and $L_H = \partial^{-1}(L_G)$ the kernel of the action $H \to \mathrm{Diff}(X)$. Since H acts locally freely, L_H is a discrete subgroup of H. Let $X_{L_G} = \{\, x \in X \mid G_x = L \,\}$ and $X_{L_H} = \{\, x \in X \mid H_x = L \,\}$. Then $X_{L_G} \subseteq X_{L_H}$. Since G is a torus, it follows from the principal orbit type theorem (see e.g. [6, § IX.9, Theorème 2]) that X_{L_G} is dense in X. Hence X_{L_H} is dense in X. Therefore we can apply Lemma 8.3.1(iii) to the H-action on X. An open and closed Lie group subbundle of $\ker(\mathrm{hol}) = L_H \times X$ is a trivial bundle $Z \times X$ for some subgroup Z of L_H, so we see that $X_\bullet \cong H/Z \ltimes X_0$. The identification $\mathrm{Mon}(X, \mathscr{F}) \cong H \ltimes X$ is $G_\bullet$-equivariant with respect to the action of $G_1 = H \rtimes G$ on $H \ltimes X$ given by $(h, g) \cdot (k, x) = (hk, g \cdot x)$. This action descends to the action $(h, g) * (kZ, x) = (hkZ, g \cdot x)$ on $H/Z \ltimes X_0 \cong X_1$. If H is not simply connected, we can apply the previous argument to the crossed module $\tilde{H} \to G$, where $\tilde{H}$ is the universal cover of H, to get that $X_1 \cong \tilde{H}/\tilde{Z} \ltimes X_0$, where $\tilde{Z}$ is a subgroup of $\tilde{H}$. But the action of $\tilde{H}$ on X_1 descends to an action of H, so $N = \ker(\tilde{H} \to H)$ is a subgroup of $\tilde{Z}$. Write $Z = \tilde{Z}/N$; then $H/Z \ltimes X \cong \tilde{H}/\tilde{Z} \ltimes X$. QED

9. Symplectic reduction

The main result of this section is Theorem 9.1.1, which is an extension of the Meyer-Marsden-Weinstein symplectic reduction theorem to the setting of Hamiltonian $\mathbf{G}$-stacks, where $\mathbf{G}$ is an étale Lie group stack. The theorem is valid under the assumption that 0 is a regular value of the moment map $\boldsymbol{\mu}\colon \mathbf{X} \to \mathrm{Lie}(\mathbf{G})^*$, which ensures that the zero fibre $\boldsymbol{\mu}^{-1}(0)$ is a differentiable stack. The group stack $\mathbf{G}$ is not required to be compact or separated, nor is it required to act freely or properly. A quotient stack $\boldsymbol{\mu}^{-1}(0)/\mathbf{G}$ then exists and is symplectic, provided that a "second-order" freeness condition is fulfilled. This second-order condition holds automatically if $\mathbf{G}$ is equivalent to a Lie group. If it fails, the quotient does not exist as a 1-stack, although it might still exist as a higher-order stack. To keep the size of this paper within reasonable limits we have omitted any discussion of reduction at nonzero levels. We draw the reader's attention to the recent preprint [2], which handles a special case of our situation, namely symplectic reduction of toric quasifolds by stacky tori.

9.1. **Symplectic reduction theorem.** Let $\mathbf{G}$ be a connected étale Lie group stack (as defined in § 6.5) and $(\mathbf{X}, \boldsymbol{\omega}, \mathbf{G}, \boldsymbol{\mu})$ a Hamiltonian $\mathbf{G}$-stack (as defined in § 7.3). We denote the zero fibre $\boldsymbol{\mu}^{-1}(0) = \mathbf{X} \times_{\mu, \mathfrak{g}^*, 0} \star$ by $\mathbf{Z}$ and the natural morphism $\mathbf{Z} \to \mathbf{X}$ by $\boldsymbol{i}$. We say that $0 \in \mathrm{Lie}(\mathbf{G})^*$ is a *regular value* of the moment map $\boldsymbol{\mu}$ if 0 is a regular value of the composite map $X_0 \to \mathbf{X} \to \mathrm{Lie}(\mathbf{G})^*$ for every atlas $X_0 \to \mathbf{X}$. If 0 is a regular value of $\boldsymbol{\mu}$, then the zero fibre $\mathbf{Z} = \boldsymbol{\mu}^{-1}(0)$ is an étale stack by [27, § 3.9].

Assume that $0 \in \mathrm{Lie}(\mathbf{G})^*$ is a regular value of $\boldsymbol{\mu}$. A *symplectic reduction (at 0)* of $\mathbf{X}$ is a triple $(\mathbf{Y}, \boldsymbol{p}, \boldsymbol{\omega}_{\mathbf{Y}})$ consisting of an étale stack $\mathbf{Y}$, a stack morphism $\boldsymbol{p}\colon \mathbf{Z} \to \mathbf{Y}$ which is a principal $\mathbf{G}$-bundle in the sense of Definition 6.13.4, and a symplectic form $\boldsymbol{\omega}_{\mathbf{Y}} \in \Omega^2(\mathbf{Y})$

with the property $\boldsymbol{p}^*\omega_{\mathbf{Y}} = \boldsymbol{i}^*\omega$. As a consequence of [9, Theorem 5.2] and Remark 6.13.6, if a symplectic reduction exists it is unique up to equivalence.

The goal of this section is to prove the following theorem, which provides a necessary and sufficient condition for a symplectic reduction to exist. The theorem is formulated in terms of a presentation of the Hamiltonian stack.

9.1.1. **Theorem.** *Let $\mathbf{G}$ be a connected étale Lie group stack and $(\mathbf{X}, \boldsymbol{\omega}, \mathbf{G}, \boldsymbol{\mu})$ a Hamiltonian $\mathbf{G}$-stack. Assume that $0 \in \mathrm{Lie}(\mathbf{G})^*$ is a regular value of $\boldsymbol{\mu}$. Let $\mathbf{B}G_\bullet \simeq \mathbf{G}$ be a presentation of $\mathbf{G}$ by a base-connected foliation 2-group $G_\bullet$ with crossed module (G, H, ∂, α), and let $(X_\bullet, \omega_\bullet, G_\bullet, \mu_\bullet)$ be a Hamiltonian groupoid presenting $(\mathbf{X}, \boldsymbol{\omega}, \mathbf{G}, \boldsymbol{\mu})$. (Such presentations exists by Corollary 6.5.4 and Theorem 7.3.2(ii)). Let $Z_\bullet = \mu_\bullet^{-1}(0)$ and let $(R_\bullet, \phi_\bullet)$ be a regular form of $Z_\bullet$ (which exists by Proposition 8.2.1). Then a symplectic reduction of $\mathbf{X}$ exists if and only if the Lie group H acts freely on the manifold R_1. If H acts freely on R_1, the 0-symplectic groupoid $(G \times^H R_\bullet, \omega_\bullet^{\mathrm{red}})$ defined in Lemma 9.1.2 below presents a symplectic reduction of $\mathbf{X}$.*

We will give the proof after establishing some preliminary results.

9.1.2. **Lemma.** *In the situation described in Theorem 9.1.1, suppose that H acts freely on R_1. Then the orbit space R_1/H is a (not necessarily Hausdorff) manifold and the projection $p\colon R_1 \to R_1/H$ is a principal H-bundle. The associated bundle $G \times^H R_1$ with fibre G is the arrow manifold of a foliation groupoid $G \times^H R_\bullet = (G \times^H R_1 \rightrightarrows R_0)$. The presymplectic form $\phi_0^*\omega_0 \in \Omega^2(R_0)$ defines a 0-symplectic form $\omega_\bullet^{\mathrm{red}}$ on $G \times^H R_\bullet$.*

Proof. The source map $s\colon R_1 \to R_0$ is H-invariant. Since the action of $G_\bullet$ on $R_\bullet$ is locally leafwise transitive, the kernel of $T_x s$ is precisely the span of the fundamental vector fields at x. The first assertion now follows from Lemma 9.1.3 below. The associated bundle $G \times^H R_1$ is the quotient of $G \times R_1$ by the action $h \cdot (g, f) = (g\partial(h^{-1}), h * f)$. Using the local trivializations of $R_1 \to R_1/H$, one gives $G \times^H R_1$ a smooth manifold structure which makes the projection $G \times R_1 \to G \times^H R_1$ a surjective submersion. Let $[g, f] \in G \times^H R_1$ denote the equivalence class of the pair $(g, f) \in G \times R_1$. We define the groupoid $G \times^H R_\bullet$ as follows. For $[g, f], [g', f'] \in G \times^H R_1$ and $x \in R_0$ put

$$s[g, f] = s(f), \qquad t[g, f] = g \cdot t(f), \qquad u(x) = [1, u(x)],$$
$$[g, f] \circ [g', f'] = [gg', ((g')^{-1} * f) \circ f'],$$
$$[g, f]^{-1} = [g^{-1}, (g * f)^{-1}] = [g^{-1}, g * f^{-1}].$$

It follows from (6.3.8)–(6.3.12) that these structure maps are well defined. Because G is connected, the form $\omega_0^{\mathrm{red}} = \phi^*\omega_0$ is G-invariant. Since the action of H preserves source fibres, the form $\phi^*\omega_1$ is H-invariant, and so descends to a form ω_1^{red} on $G \times^H R_1$. The pair $\omega_\bullet^{\mathrm{red}} = (\omega_0^{\mathrm{red}}, \omega_1^{\mathrm{red}})$ is a basic form on the groupoid $G \times^H R_\bullet$. Since the G-orbits of R_0 are the leaves of the null foliation of ω_0^{red}, the form $\omega_\bullet^{\mathrm{red}}$ is 0-symplectic. QED

Lemma 9.1.2 makes use of the following fact, which is part of [36, Lemma 5.5].

9.1.3. **Lemma.** *Let X be a (possibly non-Hausdorff) manifold and let G be a Lie group with a smooth free action $a\colon G \times X \to X$. There is a (necessarily unique) smooth (possibly non-Hausdorff) manifold structure on orbit space X/G such that the quotient map $X \to X/G$ is a principal G-bundle, if and only if there exist a (possibly non-Hausdorff) manifold Y and a smooth map $f\colon X \to Y$ which is G-invariant and satisfies $\ker(Tf)_x = T_x(G \cdot x)$ for all $x \in X$.*

The next proposition completes one direction of the proof of Theorem 9.1.1.

9.1.4. **Proposition.** *In the situation described in Theorem 9.1.1, suppose that H acts freely on R_1. Let $G \times^H R_\bullet$ be the Lie groupoid described in Lemma 9.1.2. Then the Lie groupoid morphism $\psi_\bullet \colon R_\bullet \to G \times^H R_\bullet$ defined by $\psi_0 = \mathrm{id}_{R_0}$ and $\psi_1(f) = [1_G, f]$ is a principal $G_\bullet$-bundle in the sense of Definition 6.13.1, and satisfies $\psi_\bullet^* \omega_\bullet^{\mathrm{red}} = \phi_\bullet^* \omega_\bullet$.*

Proof. The statement $\psi_\bullet^* \omega_\bullet^{\mathrm{red}} = \phi_\bullet^* \omega_\bullet$ holds because

$$\psi_0^* \omega_0^{\mathrm{red}} = \omega_0^{\mathrm{red}} = \phi_0^* \omega_0 \in \Omega^2(R_0).$$

To show that $\psi_\bullet$ is a principal $G_\bullet$-bundle we will verify conditions (i)–(iii) of Definition 6.13.1. Condition (i) is obvious: $\psi_\bullet$ is essentially surjective because $\psi_0 = \mathrm{id}_{R_0}$. To check condition (ii) let $\mathrm{pr}_2 \colon G_\bullet \times R_\bullet \to R_\bullet$ be the projection and $a_\bullet \colon G_\bullet \times X_\bullet \to X_\bullet$ the action. Recall that $G_0 = G$ and $G_1 = H \rtimes_\alpha G$, and define $\gamma \colon \psi_\bullet \circ \mathrm{pr}_2 \Rightarrow \psi_\bullet \circ a_\bullet$ to be the map

$$\gamma \colon G \times R_0 \longrightarrow G \times^H R_1, \qquad (g, x) \longmapsto [g, u(x)].$$

Let $f \in R_1$ be an arrow in $R_\bullet$ from x to y, and let $((h, g), f) \in G_1 \times R_1$ be an arrow from $s((h, g), f) = (g, x)$ to $t((h, g), f) = (\partial(h)g, y)$. Then the following diagram in $G \times^H R_\bullet$ commutes:

$$\begin{array}{ccc} \psi_0 \circ \mathrm{pr}_2(g, x) = x & \xrightarrow{\gamma(g,x)=[g,u(x)]} & \psi_0 \circ a_0(g, x) = g \cdot x \\ {\scriptstyle \psi_1 \circ \mathrm{pr}_2((h,g),f)=[1,f]} \Big\downarrow & & \Big\downarrow {\scriptstyle \psi_1 \circ a_1((h,g),f)=[1,h*(g*f)]} \\ \psi_0 \circ \mathrm{pr}_2(\partial(h)g, y) = y & \xrightarrow{\gamma(\partial(h)g,y)=[\partial(h)g,u(y)]} & \partial(h)g \cdot y. \end{array}$$

Thus γ is a natural transformation. It is automatically a natural isomorphism, because **LieGpd** is a $(2,1)$-category. The coherence conditions on γ are verified in a similar way. This shows that $\psi_\bullet$ is $G_\bullet$-invariant. To check condition (iii) we first describe the groupoid

$$P_\bullet := R_\bullet \times^{(w)}_{G \times^H R_\bullet} R_\bullet,$$

which is the weak fibre product as described in Definition 4.1.6. The object and arrow manifolds are

$$P_0 = R_0 \times_{R_0} (G \times^H R_1) \times_{R_0} R_0 \cong G \times^H R_1,$$
$$P_1 = R_1 \times_{R_0} (G \times^H R_1) \times_{R_0} R_1.$$

The source and target maps of $P_\bullet$ can be written

$$s(r, [g, f], r') = [g, f], \qquad t(r, [g, f], r') = [g, (g^{-1} * r') \circ f \circ r^{-1}].$$

We must show that the canonical morphism $\tau_\bullet = (\mathrm{pr}_2, a_\bullet) \colon G_\bullet \times R_\bullet \to P_\bullet$ is a Morita morphism. On objects this is the map $\tau_0 \colon G_0 \times R_0 \to P_0$ given by

$$\tau_0(g, x) = \gamma(g, x) = [g, u(x)],$$

and on arrows this is the map $\tau_1 \colon (H \rtimes_\alpha G) \times R_1 \to P_1$ given by

$$\tau_1((h, g), f) = (f, \gamma(s((h, g), f)), h * (g * f)) = (f, [g, u(s(f))], h * (g * f)).$$

First we show that $\tau_\bullet$ is essentially surjective, i.e. the map

$$t \circ \mathrm{pr}_1 \colon P_1 \times_{s, P_0, \tau_0} (G \times R_0) \longrightarrow P_0$$
$$((r, [g, u(s(r))], r'), (g, u(s(r)))) \longmapsto [g, (g^{-1} * r') \circ r^{-1}]$$

is a surjective submersion. Since the action of H on R_1 preserves source fibres and is free, whenever $[g, u(x)] = [g', u(x')] \in G \times^H R_1$ we must have $g = g'$ and $x = x'$. It follows that we have a diffeomorphism $\lambda\colon P_1 \times_{P_0} (G \times R_0) \cong (R_1 \times G) \times_{R_0} R_1$ given by

$$\lambda((r, [g, u(s(r))], r'), (g, u(s(r)))) = (r, g, r'),$$

where

$$(R_1 \times G) \times_{R_0} R_1 = \{\, ((r, g, r') \in R_1 \times G \times R_1 \mid g \cdot s(r) = s(r') \,\}.$$

Under the isomorphism λ the map $t \circ \mathrm{pr}_1$ becomes the map $\kappa\colon (R_1 \times G) \times_{R_0} R_1 \to P_0 = G \times^H R_1$ given by

$$\kappa(r, g, r') = [g, (g^{-1} * r') \circ r^{-1}].$$

For surjectivity, let $[g, f] \in P_0$. Then $(f^{-1}, g, g * u(t(f))) \in (R_1 \times G) \times_{R_0} R_1$ and $\kappa(f^{-1}, g, g * u(t(f))) = [g, f]$. For submersivity, note that there is a G-action on $(R_1 \times G) \times_{R_0} R_1$ given by

$$g \cdot (r, g', r') = (g * r, g' g^{-1}, r')$$

and also a G-action on $P_0 = G \times^H R_1$ given by

$$g \cdot [g', f] = [g' g^{-1}, g * f].$$

The map κ is G-equivariant. So it suffices to show that, for each $[1, f] \in P_0$ and for any $(r, 1, r') \in (R_1 \times G) \times_{R_0} R_1$ with $r' \circ r^{-1} = f$, there is a local section $\sigma\colon P_0 \to (R_1 \times G) \times_{R_0} R_1$ of κ with $\sigma[1, f] = (r, 1, r')$. But this follows from the fact that the multiplication map of $R_\bullet$ is a submersion. Therefore $\tau_\bullet$ is essentially surjective. Next we show $\tau_\bullet$ is fully faithful. Consider the fibred product

$$M = \big((G \times R_0) \times (G \times R_0)\big) \times_{P_0 \times P_0} P_1$$

with respect to the maps $\tau_0 \times \tau_0\colon (G \times R_0) \times (G \times R_0) \to P_0 \times P_0$ and $(s, t)\colon P_1 \to P_0 \times P_0$. A typical element of M is a tuple

$$\big((g, s(r)), (g', x), (r, [g, u(s(r))], r')\big) \in (G \times R_0) \times (G \times R_0) \times P_1$$

satisfying

$$[g', u(x)] = t(r, [g, u(s(r))], r') = \big[g, (g^{-1} * r') \circ r^{-1}\big],$$

where $x = t(r)$ because $h * u(x) = (g^{-1} * r') \circ r^{-1}$ for some $h \in H$, and because the H-action preserves the s-fibres. The universal property of the fibred product M yields a canonical map $\chi\colon (H \rtimes_\alpha G) \times R_1 \to M$ given by

$$\chi(h, g, f) = \big((g, s(f)), (\partial(h) g, t(f)), (f, [g, u(s(f))], h * (g * f))\big).$$

We must show χ is a diffeomorphism. For r, $r' \in R_1$ in the same H-orbit, let $\delta(r, r')$ be the unique element $h \in H$ satisfying $hr = r'$. The map $\delta\colon R_1 \times_{R_1/H} R_1 \to H$ is smooth, because $R_1 \to R_1/H$ is a principal H-bundle. Define $\zeta\colon M \to (H \rtimes_\alpha G) \times R_1$ by

$$\zeta\big((g, s(r)), (g', x), (r, [g, u(s(r))], r')\big) = (\delta(g * r, r'), g, r).$$

We assert that ζ is the inverse of χ. Indeed,

$$\begin{aligned}(\zeta \circ \chi)(h, g, f) &= \zeta\big((g, s(f)), (\partial(h) g, t(f)), (f, [g, u(s(f))], h * (g * f))\big)\\ &= (\delta(g * f, h * (g * f)), g, f)\\ &= (h, g, f),\end{aligned}$$

and

$$\begin{aligned}(\chi \circ \zeta)\big((g, s(r)), (g', t(r)), (r, [g, u(s(r))], r')\big)\\ &= \chi(\delta(g * r, r'), g, r)\\ &= \big((g, s(r)), (\partial(\delta(g * r, r'))g, t(r)), (r, [g, u(s(r))], \delta(g * r, r') * (g * r))\big)\\ &= \big((g, s(r)), (\partial(\delta(g * r, r'))g, t(r)), (r, [g, u(s(r))], r')\big).\end{aligned}$$

It remains to show that $\partial(\delta(g * r, r'))g = g'$. We have $\partial(\delta(g * r, r'))g \cdot t(r) = t(r')$ by (6.3.10). And $g' \cdot t(r) = t(r')$ by the definition of M. Since G acts freely on R_0, we have $\partial(\delta(g * r, r'))g = g'$. So $\zeta = \chi^{-1}$, and $\tau_\bullet$ is fully faithful. QED

Proof of Theorem 9.1.1. If H acts freely on R_1, Propositions 9.1.4 and 6.13.5 show that a symplectic reduction of $\mathbf{X}$ exists. Conversely, suppose that H does not act freely on R_1. It follows from [9, Theorem 5.2] that $\mathbf{B}R_\bullet \to \mathbf{S}$ is a principal $\mathbf{B}G_\bullet$-bundle over some stack $\mathbf{S}$ if and only if the Lie groupoid

$$Y_\bullet := (R_0 \times R_0) \times^{(w)}_{R_\bullet \times R_\bullet} (G_\bullet \times R_\bullet)$$

is Morita equivalent to a manifold. Here the weak fibred product is taken over the canonical map $R_0 \times R_0 \to R_\bullet \times R_\bullet$ and the projection-action map $\mathrm{pr}_2 \times a_\bullet \colon G_\bullet \times R_\bullet \to R_\bullet \times R_\bullet$. We will show that $Y_\bullet$ has non-trivial isotropy groups and so cannot be Morita equivalent to a manifold. Choose $f \in R_1$ and $1 \neq h \in H$ so that $h * f = f$. Let $x = s(f)$ and $y = t(f)$. Consider the point

$$z = ((x, x), (f, f), (1, y)) \in (R_0 \times R_0) \times_{R_0 \times R_0} (R_1 \times R_1) \times_{R_0 \times R_0} (G_0 \times R_0).$$

Let $k \in Y_1$ be the arrow

$$k = ((x, x), (f, f), ((h, 1), u(y))) \in (R_0 \times R_0) \times_{R_0 \times R_0} (R_1 \times R_1) \times_{R_0 \times R_0} (G_1 \times R_1),$$

where we identify $G_1 = H \rtimes_\alpha G$. Then $s(k) = z$ and

$$t(k) = ((x, x), (u(y), h * u(y)) \circ (f, f), (\partial(h), y)) = ((x, x), (f, f), (\partial(h), y)).$$

But since $R_\bullet$ is a regular form and $h * f = f$, we have $\partial(h) = 1$. So $t(k) = z$. Therefore the isotropy group of z is nontrivial. QED

9.2. **Symplectic reduction by a Lie group.** Theorem 9.1.1 gives new information even in the context of symplectic reduction by an ordinary Lie group.

9.2.1. **Corollary.** *Let G be a Lie group and (X, ω) a symplectic manifold on which G acts in a Hamiltonian fashion with moment map $\mu \colon X \to \mathfrak{g}^*$. If 0 is a regular value of μ, then $\mu^{-1}(0)/G$ is a symplectic stack.*

An *orbifold* is a separated étale stack. If in Corollary 9.2.1 we make the extra assumption that G acts properly on $\mu^{-1}(0)$, then the stack $\mu^{-1}(0)/G$ is separated, so we obtain the following familiar Meyer-Marsden-Weinstein reduction theorem [35], [32], [33].

9.2.2. **Corollary.** *Let G be a Lie group and (X, ω) a symplectic manifold on which G acts in a Hamiltonian fashion with moment map $\mu \colon X \to \mathfrak{g}^*$. If 0 is a regular value of μ and if G acts properly on $\mu^{-1}(0)$, then $\mu^{-1}(0)/G$ is a symplectic orbifold. If in addition the action of G on $\mu^{-1}(0)$ is free, then $\mu^{-1}(0)/G$ is a symplectic manifold.*

Here are three instances of Corollary 9.2.1.

9.2.3. **Example** (space of geodesics)**.** Let $X = T^*M$ be the cotangent bundle of a complete Riemannian manifold M. The space of (non-parametrized) geodesics of M is the symplectic quotient of X by the Hamiltonian action of $G = \mathbb{R}$ generated by the kinetic energy (norm square) function. Corollary 9.2.1 says that the space of geodesics can be interpreted as a symplectic stack.

9.2.4. **Example** (space of Reeb orbits)**.** Let M be a manifold equipped with a contact form α and let X be the symplectization of M, i.e. the manifold $X = (0, \infty) \times M$ equipped with the symplectic form $d\alpha + dt \wedge \alpha$. Suppose the Reeb vector field of M is complete. Then the space of Reeb orbits of M is the symplectic quotient of X by the Hamiltonian action of $G = \mathbb{R}$ generated by the function $(t, m) \mapsto t$. Corollary 9.2.1 says that the space of Reeb orbits can be interpreted as a symplectic stack.

9.2.5. **Example** (toric quasi-folds)**.** Consider $\mathbb{C}^n$ with its standard symplectic form ω and the standard action of the torus $G = \mathbb{T}^n$ as in Example 3.1. Let $N \subseteq G$ an immersed Lie subgroup and $\iota^* \circ \mu \colon \mathbb{C}^n \to \mathfrak{n}^*$ the N-moment map with zero fibre X_0. Let $\tilde{N} \to N$ a covering homomorphism as in Example 6.1.2. Let $(X_\bullet, \omega_\bullet)$ be the 0-symplectic groupoid of Example 7.1.1 and let $(\mathbf{X}, \boldsymbol{\omega})$ be the associated symplectic stack. There is an obvious morphism of groupoids $p_\bullet$ from (the identity groupoid of) X_0 to the action groupoid $X_\bullet = \tilde{N} \ltimes X_0$. The associated morphism of stacks $\mathbf{B}p_\bullet \colon X_0 \to \mathbf{X}$ is a principal $\tilde{N}$-bundle in the sense of Definition 6.13.4. By definition the pullback of $\boldsymbol{\omega}$ to X_0 is equal to the presymplectic form ω_0 on X_0. We conclude that the toric quasifold $(\mathbf{X}, \boldsymbol{\omega})$ is the symplectic reduction of $\mathbb{C}^n$ with respect to $\tilde{N}$. The symplectic stack $(\mathbf{X}, \boldsymbol{\omega})$ is a symplectic orbifold if and only if N is a closed subgroup of G and the covering $\tilde{N} \to N$ is finite. The action of $G = \mathbb{T}^n$ on $\mathbb{C}^n$ descends to an action of the quotient Lie group stack $\mathbf{G} = G/\tilde{N}$ on $\mathbf{X}$, which is nothing other than the $\mathbf{G}$-action defined in Example 7.1.1.

10. The Duistermaat-Heckman theorem

In this section we prove an analogue of the Duistermaat-Heckman theorem for Hamiltonian $\mathbf{G}$-stacks, where $\mathbf{G}$ is a stacky torus. The Duistermaat-Heckman theorem has two parts: (1) the variation of the reduced symplectic form is linear, and (2) the moment map image of the Liouville measure is piecewise polynomial. It is only the first part that we generalize here; when the Hamiltonian stack is not proper, it is unclear how to integrate the Liouville measure along fibres of the moment map in a canonical, Morita-invariant fashion. (See [13] for a treatment of measures and densities on differentiable stacks.)

The following version of the Duistermaat-Heckman theorem was obtained by Guillemin and Sternberg by applying the coisotropic embedding theorem to the zero fibre of the moment map. Our approach is to generalize this formulation to Hamiltonian $\mathbf{G}$-stacks. We focus our attention on the situation when we have a presentation of our $\mathbf{G}$-stack $\mathbf{X}$ by a Hamiltonian groupoid with a leafwise transitive action (Theorem 10.3).

10.1. **Theorem** (Duistermaat-Heckman [16], Guillemin-Sternberg [19])**.** *Let G be a torus and let (M, ω, G, μ) be a connected Hamiltonian G-manifold, where μ is a proper map. Let U be an open neighborhood of $0 \in \mathfrak{g}^*$ which consists of regular values of μ, and let $Z = \mu^{-1}(0)$. Let $\theta \in \Omega^1(Z) \otimes \mathfrak{g}$ be a connection form for the locally free action of G on Z, and define the 1-form $\gamma \in \Omega^1(Z \times \mathfrak{g}^*)$ by*

$$\gamma(v|_p, w|_\beta) = \langle \beta, \theta|_p(v|_p) \rangle. \tag{10.2}$$

Then:

(i) *After possibly shrinking U, there is an isomorphism of Hamiltonian G-manifolds*

$$(\mu^{-1}(U), \omega|_{\mu^{-1}(U)}, G, \mu|_{\mu^{-1}(U)}) \cong (Z \times U, \omega|_Z + d\gamma, G, \mathrm{pr}_2),$$

where the G-action on $Z \times U$ is $t \cdot (z, u) = (t \cdot z, u)$.

(ii) *Let $u \in U$. If the action of G on Z is free, there is a symplectic isomorphism of reduced spaces $(\mu^{-1}(u)/G, \omega^{\mathrm{red}(u)}) \cong (\mu^{-1}(0)/G, \omega^{\mathrm{red}(0)} + \langle u, \Gamma\rangle)$, where $\omega^{\mathrm{red}(u)}, \omega^{\mathrm{red}(0)}$ denote the symplectic forms on the reduced spaces at u and 0, and $\Gamma \in \Omega^2(Z/G) \otimes \mathfrak{g}$ is the curvature 2-form for the principal G-bundle $Z \to Z/G$.*

(iii) *The de Rham cohomology class $[\langle u, \Gamma\rangle]$ varies linearly with u and does not depend on the choice of connection θ or the choices involved in constructing the isomorphism in (i).*

We expand on the second part of (iii). In constructing the isomorphism in (i), one views $\mu\colon \mu^{-1}(U) \to U$ as a G-invariant locally trivial fibre bundle. To construct a trivialization $\mu^{-1}(U) \cong Z \times U$, one chooses a G-invariant connection on this fibre bundle and a smooth contraction of U to 0. The horizontal lifts of this contraction gives a G-equivariant diffeomorphism from $\mu^{-1}(0) \times Z$ to $\mu^{-1}(U)$. The isotopy class of this diffeomorphism is independent of the contraction and of the connection on $\mu^{-1}(U)$.

10.3. **Theorem.** *Let $\mathbf{G}$ be a stacky torus, and let $\mathbf{B}G_\bullet \simeq \mathbf{G}$ be a presentation by a Lie 2-group with the property that G_0 is a torus. Let $(\mathbf{X}, \boldsymbol{\omega}, \mathbf{G}, \boldsymbol{\mu})$ be a Hamiltonian $\mathbf{G}$-stack presented by $(X_\bullet, \omega_\bullet, G_\bullet, \mu_\bullet)$. Denote by $\partial\colon H \to G$ the crossed module of $G_\bullet$. Assume the following:*

(a) *X_1 is Hausdorff and $0 \in \mathrm{Lie}(\mathbf{G})^*$ is a regular value of μ;*
(b) *The action of H on X_0 is locally free;*
(c) *The action of $G_\bullet$ on $X_\bullet$ is leafwise transitive;*
(d) *The moment map $\mu_0\colon X_0 \to \mathrm{ann}(\mathfrak{h})$ is proper, and $X_0 = X$ is connected.*

Then:

(i) *There is an open neighborhood U of 0 and an isomorphism of Hamiltonian $\mathbf{G}$-stacks*

$$(\boldsymbol{\mu}^{-1}(U), \boldsymbol{\omega}|_{\mu^{-1}(U)}, \mathbf{G}, \boldsymbol{\mu}) \cong (\boldsymbol{\mu}^{-1}(0) \times U, \boldsymbol{\omega}|_{\boldsymbol{\mu}^{-1}(0)} + d\gamma|_{Z\times U}, \mathbf{G}, \mathrm{pr}_2).$$

where $d\gamma|_{Z\times U}$ is depends on a choice and is described in (10.8) below.

(ii) *If the symplectic reduction of $\mathbf{X}$ exists at 0, then it exists at all points of U. For all $u \in U$, there is an equivalence of symplectic stacks of the reduced spaces*

$$(\boldsymbol{\mu}^{-1}(u)/\mathbf{G}, \boldsymbol{\omega}^{\mathrm{red}(u)}) \cong (\boldsymbol{\mu}^{-1}(0)/\mathbf{G}, \boldsymbol{\omega}^{\mathrm{red}(0)} + \Gamma)$$

where $\boldsymbol{\omega}^{\mathrm{red}(u)}, \boldsymbol{\omega}^{\mathrm{red}(0)}$ denote the symplectic forms on the reduced spaces at u and 0, respectively, and $\Gamma \in \Omega^2(\boldsymbol{\mu}^{-1}(0)/\mathbf{G})$ is described before (10.9) below.

(iii) *The form Γ varies linearly with $u \in \mathrm{Lie}(\mathbf{G})^*$, and, after fixing the presentation $X_\bullet$ of $\mathbf{X}$, its cohomology with respect to the complex $\Omega^\bullet(\boldsymbol{\mu}^{-1}(0)/\mathbf{G})$ does not depend on the choices involved.*

We will prove the theorem after some initial results. For simplicity we will assume that $H = N = N(\mathscr{F})$, the null subgroup of $\mathscr{F}$; the proof carries to other H with minor changes. Then $\mathfrak{h} = \mathfrak{n} = \mathfrak{n}(\mathscr{F})$, the null ideal, and $\mu_0(X_0) \subseteq \mathrm{ann}(\mathfrak{n})$.

10.4. **Lemma.** *Assume we are in the context of Theorem 10.3.*

(i) *There is an isomorphism of Lie groupoids $X_\bullet \cong (N/L \ltimes X) \rightrightarrows X$, where L is a discrete subgroup of N, and where the action of $G_\bullet$ is given as in Proposition 8.3.3.*

Under this isomorphism, the 0*-symplectic form* $\omega = (\omega_0, \omega_1)$ *can be written* $(\omega_0, 0 \oplus \omega_0)$.

(ii) *There is an open set* U *containing* 0 *which consists of regular values of* μ_0.

(iii) *The action of* G *is then locally free on* $V := \mu_0^{-1}(U)$.

Proof. Item (i) follows immediately from Proposition 8.3.3 and the condition $\omega_1 = s^*\omega_0$. Item (ii) holds because μ_0 is proper. Item (iii) follows from (ii) because the action of N is locally free on X_0. QED

Recall the notion of the *symplectization* of a presymplectic manifold; see for instance [30]. In the context of Theorem 10.3, for the presymplectic manifold (X, ω_0), let $T^*\mathscr{F}$ be the vector bundle dual to $T\mathscr{F}$, and let $\mathrm{pr}\colon T^*\mathscr{F} \to X$ be the bundle projection. By choosing a G-invariant metric on X, one can embed $j\colon T^*\mathscr{F} \hookrightarrow T^*X$. Let $\tilde{\omega}$ be the standard symplectic form on the cotangent bundle T^*X, and let $\Omega = \mathrm{pr}^*\,\omega_0 + j^*\tilde{\omega}$ be the 2-form on $T^*\mathscr{F}$. Then Ω is symplectic near the zero section $X \to T^*\mathscr{F}$, which is a coisotropic embedding of X. There is a moment map $\Psi = \mathrm{pr}^*\,\mu_0 + j^*\tilde{\mu}$ for the G-action on $T^*\mathscr{F}$, where $\tilde{\mu}\colon T^*X \to \mathfrak{g}^*$ is the standard moment map for the G-action on T^*X given by

$$\tilde{\mu}^{\xi}(y) = \langle y, \xi_X|_x \rangle, \qquad \text{for } y \in T^*_x X.$$

The germ at X of $T^*\mathscr{F}$ is called a symplectization of X. In [30] it is shown that in the leafwise transitive case, fibres of the moment map $\mu_0\colon X \to \mathrm{ann}(\mathfrak{h}) \subseteq \mathfrak{g}^*$ are fibres of $\Psi\colon T^*\mathscr{F} \to \mathfrak{g}^*$.

10.5. **Lemma.** *Assume we are in the context of Theorem 10.3. Let* $(T^*\mathscr{F}, \Omega, G, \Psi)$ *be the symplectization of* $(X, \omega_0, G_0, \mu_0)$ *just described, and let* U *and* V *be sets as in Lemma 10.4. The restriction of the moment map* $\Psi\colon T^*\mathscr{F}|_V \to \mathfrak{g}^*$ *to* $V = \mu_0^{-1}(U)$ *is proper.*

Proof. Choose a splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{n}$. Because the action of G is locally free on V, the tangent bundle can be split $TV \cong \mathfrak{k} \oplus \mathfrak{n} \oplus (TU/\mathfrak{g})$, and because the action is leafwise transitive, $T\mathscr{F}|_V$ is isomorphic to the trivial bundle with fibre $\mathfrak{n}$. Let us choose the G-invariant metric on X so that $\mathfrak{k}$ is orthogonal to $\mathfrak{n}$, as follows. Fix a metric on TV, by (1) choosing a basis ξ_i of $\mathfrak{k}$ and a basis ν_j of $\mathfrak{n}$ and declaring the corresponding sections of TV to be orthonormal, and by (2) extending to get a metric on TV by choosing an arbitrary smooth metric on $TV/\mathfrak{g}$. Averaging this metric over G gives a G-invariant metric on V where $\mathfrak{k}$ is orthogonal to $\mathfrak{n}$, as desired.

Now consider the moment map Ψ. Under the splitting $\mathfrak{g}^* = \mathfrak{k}^* \oplus \mathfrak{n}^*$, the description of μ_0 and the choice of splitting gives

$$\mathrm{pr}^*\,\mu_0(T\mathscr{F}^*|_V) = U \oplus 0, \qquad j^*\tilde{\mu}(T^*\mathscr{F}|_V) = 0 \oplus \mathfrak{n}^*.$$

Because the action of N is locally free, the restriction of $j^*\tilde{\mu}$ to a fibre $T^*\mathscr{F}|_x$ is a linear isomorphism to $\mathfrak{n}^*$.

To show that the restriction of Ψ is proper, it is enough to show that the preimage of $D \times E \subseteq \mathfrak{k}^* \oplus \mathfrak{n}^*$ in $T^*\mathscr{F}|_U$ is compact, where $D \subseteq \mathfrak{k}^*$ and $E \subseteq \mathfrak{n}^*$ are compact. Indeed, from the description of Ψ above, the preimage of $D \times E$ is homeomorphic to a fibre bundle over $\mu_0^{-1}(D)$ with fibre E. Since μ_0 is proper this space is compact. QED

Proof of Theorem 10.3. Let U and V be as in Lemma 10.4. By Lemma 10.5, we can apply Theorem 10.1 to the symplectization $T^*\mathscr{F}|_V$ of $V \subseteq X$. Let $Z = \mu_0^{-1}(0) = \Psi^{-1}(0)$, and let

$$\theta \in \Omega^1(Z) \otimes \mathfrak{g} \tag{10.6}$$

be a connection form for the action of G on Z, and let $\gamma \in \Omega^1(Z\times\mathfrak{g}^*)$ be as in Theorem 10.1. After possibly shrinking U, there is an open neighborhood U' of 0 in $\mathfrak{g}^*$ so that $U'\cap \mathrm{ann}(\mathfrak{n}) = U$ and an isomorphism of Hamiltonian G-manifolds

$$(\Psi^{-1}(U'), \Omega|_{\Psi^{-1}(U')}, G, \Psi|_{\Psi^{-1}(U')}) \cong (Z\times U', \omega_0|_Z + d\gamma, G, \mathrm{pr}_2).$$

We require that we choose the contraction of U' described after Theorem 10.1 so that it restricts to a contraction of $U = U'\cap \mathrm{ann}(\mathfrak{n})$. Restricting the isomorphism to $\Psi^{-1}(U) = \mu_0^{-1}(U) = V$ gives an isomorphism of presymplectic Hamiltonian G-manifolds

$$F\colon (V, \omega_0|_V, G, \mu_0) \cong (Z\times U, \omega_0|_Z + (d\gamma)|_{Z\times U}, G, \mathrm{pr}_2). \tag{10.7}$$

By Lemma 10.4(i) we can lift this to an isomorphism of Hamiltonian $G_\bullet$-groupoids

$$F_\bullet\colon (X_\bullet|_V, \omega_\bullet, G_\bullet, \mu_\bullet) \cong (X_\bullet|_Z\times U, \omega_\bullet|_Z + (d\gamma)|_{Z\times U}, G_\bullet, \mathrm{pr}_2). \tag{10.8}$$

Note that, by (10.7), the form $(d\gamma)|_{Z\times U}\in\Omega^2(Z\times U)$ determines an element of $\Omega^2_{\mathrm{bas}}(X_\bullet|_Z\times U)\cong\Omega^2(\mathbf{B}X_\bullet|_Z\times U)$ which we have also denoted by $(d\gamma)|_{Z\times U}$ in (10.8). Applying Theorem 7.3.2(i) proves (i).

The first statement of (ii) follows immediately. For the second, since $\omega|_Z$ and $\omega|_Z + (d\gamma)|_{Z\times\{u\}}$ descend to symplectic forms on the stack $\boldsymbol{\mu}^{-1}(0)/\mathbf{G}$, it follows that $(d\gamma)|_{Z\times\{u\}}$ descends to some

$$\Gamma\in\Omega^2(\boldsymbol{\mu}^{-1}(0)/\mathbf{G}). \tag{10.9}$$

This proves (ii). Finally, that Γ varies linearly with $u\in\mathrm{Lie}(\mathbf{G})^*$ is obvious from the definition. It remains to check that its cohomology class does not depend on the choice of connection 1-form $\theta\in\Omega^1(Z)\otimes\mathfrak{g}$ for the action of G on Z or on the choice of isomorphism $X_\bullet|_V\cong X_\bullet|_Z\times U$.

Let $\theta'\in\Omega^1(Z)\otimes\mathfrak{g}$ be another connection form. Then $\theta-\theta'\in\Omega^1(Z)\otimes\mathfrak{g}$ is a G-invariant horizontal 1-form. If γ' is related to θ' as in (10.2) then $\gamma-\gamma'\in\Omega^1(Z\times\mathfrak{g}^*)$ is G-invariant. By the description of $X_\bullet$ in Lemma 10.4 we consider $(\gamma-\gamma')|_{Z\times U}$ as a G-invariant element of $\Omega^1_{\mathrm{bas}}(X_\bullet|_Z\times U)$. So from Lemma 10.10 below, the restriction of $\gamma-\gamma'$ to $X_\bullet|_Z\times\{u\}$ descends to an element of $\Omega^1(\boldsymbol{\mu}^{-1}(0)/\mathbf{G}\times\{u\})$. The form $\Gamma'\in\Omega^2(\boldsymbol{\mu}^{-1}(0)/\mathbf{G}\times\{u\})$ determined by γ' then differs from Γ by an exact form.

Let us now assume we choose a different isomorphism $F'\colon V\cong Z\times U$ from (10.7). Then $F'\circ F^{-1}\colon Z\times U\to Z\times U$ is G-equivariant and isotopic the identity. So $(F'\circ F^{-1})^*(\omega_0|_Z + (d\gamma)|_{Z\times U}) - (\omega_0|_Z + (d\gamma)|_{Z\times U}) = d\eta$, where $\eta\in\Omega^1(Z\times U)$ is G-invariant. We view η as a G-invariant element of $\Omega^1_{\mathrm{bas}}(X_\bullet|_Z\times U)$ as before and apply Lemma 10.10. This proves (iii). QED

10.10. **Lemma.** *Assume we are in the context of Theorem 10.3, and let U and V be as in Lemma 10.4. Let $(R_\bullet,\phi_\bullet)$ be a regular form of the zero fibre $Z_\bullet=\mu_\bullet^{-1}(0)$ as in Proposition 8.2.1 and let $G\times^H R_\bullet$ be the 0-symplectic groupoid of Lemma 9.1.2. Identify $X_\bullet|_V\cong Z_\bullet\times U$ as in (10.8). If $\eta_\bullet\in\Omega^k_{\mathrm{bas}}(Z_\bullet\times U)$ is G-invariant, then $\phi_\bullet^*\eta_\bullet$ descends to an element of $\Omega^k_{\mathrm{bas}}(G\times^H R_\bullet)\cong\Omega^k(\boldsymbol{\mu}^{-1}(0)/\mathbf{G})$.*

Proof. Consider the form $\phi_\bullet^*\eta_\bullet = (\phi_0^*\eta_0, \phi_1^*\eta_1)\in\Omega^k_{\mathrm{bas}}(R_\bullet)$. We will show that $(\phi_0^*\eta_0, 0\oplus\phi_1^*\eta_1)\in\Omega^k_{\mathrm{bas}}(G\times^H R_1\rightrightarrows R_0)$. Consider the maps

$$\mathrm{pr}_2\circ(\mathrm{id}\times s)\colon G\times R_1\to R_0,\qquad a_0\circ(\mathrm{id}\times t)\colon G\times R_1\to R_0,$$

which under the projection $G\times R_1\to G\times^H R_1$ descend to the source and target maps of the groupoid $G\times^H R_1$. It suffices to show that

$$(\mathrm{pr}_2\circ(\mathrm{id}\times s))^*(\phi_0^*\eta_0) = 0\oplus\phi_1^*\eta_1 = (a\circ(\mathrm{id}\times t))^*(\phi_0^*\eta_0).$$

Since η is basic, the first equality is obvious, and the second follows immediately from the fact that η_0 is G-invariant and that ϕ is $G_\bullet$-equivariant. QED

10.11. **Remark.** The cohomology class $[\Gamma]$ of Theorem 10.3(iii) is independent of the choice of presentation of $\mathbf{X}$. We omit the verification of this fact.

10.12. **Example.** Consider the previous theorem in the context of Examples 7.4.3 and 9.2.5. We have $X_0 = (\iota^* \circ \mu)^{-1}(0)$, $X_1 = \tilde{N} \ltimes X_0$, and $G_\bullet = \tilde{N} \ltimes X_0$, with moment map $\mu_\bullet \colon X_\bullet \to \operatorname{ann}(\mathfrak{n})$. Let $U \subseteq \operatorname{ann}(\mathfrak{n})$ be a small neighborhood of 0. Note that the action of $G_\bullet$ on $X_\bullet$ satisfies the assumptions of Theorem 10.3. The reduced spaces over points of U are then equivalent as smooth stacks to the reduced space $\mathbf{B}\mu^{-1}(0)/\mathbf{B}G_\bullet$, which here is just equivalent to a point.

Appendix A. Groups and actions in 2-categories

This is a review of group objects in 2-categories and their actions. The notion of a 2-group internal to a 2-category was developed by Baez and Lauda [1]. We modify their terminology in two ways: our "weak 2-groups" (resp. "weak homomorphisms") are called "coherent 2-groups" (resp. "homomorphisms") in [1].

Throughout this appendix we denote by $\mathscr{C}$ a 2-category with finite products and with terminal object $\star$.

A.1. **Definition.** A *weak* 2*-group in* $\mathscr{C}$ consists of

(i) an object G of $\mathscr{C}$;
(ii) three 1-morphisms: *multiplication* $m \colon G \times G \to G$, the *group unit* $1 \colon \star \to G$, and the *inverse* $(\cdot)^{-1} \colon G \to G$;
(iii) five 2-isomorphisms:
  (a) the *associator* $\alpha \colon m \circ (m \times \mathrm{id}_G) \Rightarrow m \circ (\mathrm{id}_G \times m)$,
  (b) the *left unit law* $\lambda \colon m \circ (1 \times \mathrm{id}_G) \Rightarrow \mathrm{id}_G$,
  (c) the *right unit law* $\rho \colon m \circ (\mathrm{id}_G \times 1) \Rightarrow \mathrm{id}_G$,
  (d) the *unit adjunction law* $d \colon 1 \circ \tau_G \Rightarrow m \circ (\mathrm{id}_G \times (\cdot)^{-1}) \circ \Delta_G$,
  (e) the *counit adjunction law* $e \colon m \circ ((\cdot)^{-1} \times \mathrm{id}_G) \circ \Delta_G \Rightarrow 1 \circ \tau_G$.
  Here τ_G is the unique 1-morphism $G \to \star$ and $\Delta_G \colon G \to G \times G$ is the diagonal.

The 2-morphisms are subject to coherence conditions, which are that the diagrams (A.2)–(A.5) below commute. We write $\mathrm{id} = \mathrm{id}_G$ and $T \colon G \to G \times G \times G$ for the three-way diagonal.

(A.2)
$$\begin{array}{ccccc} & & m \circ (m \times m) & & \\ & \overset{\alpha \circ (m \times \mathrm{id} \times \mathrm{id})}{\nearrow} & & \overset{\alpha \circ (\mathrm{id} \times \mathrm{id} \times m)}{\searrow} & \\ m \circ (m \times \mathrm{id}) \circ (m \times \mathrm{id} \times \mathrm{id}) & & & & m \circ (\mathrm{id} \times m) \circ (\mathrm{id} \times \mathrm{id} \times m) \\ \Downarrow {\scriptstyle m \circ (\alpha \times \mathrm{id})} & & & & \Uparrow {\scriptstyle m \circ (\mathrm{id} \times \alpha)} \\ m \circ (m \times \mathrm{id}) \circ (\mathrm{id} \times m \times \mathrm{id}) & & \xRightarrow{\alpha \circ (\mathrm{id} \times m \times \mathrm{id})} & & m \circ (\mathrm{id} \times m) \circ (\mathrm{id} \times m \times \mathrm{id}) \end{array}$$

(A.3)
$$\begin{array}{ccc} m \circ (m \times \mathrm{id}) \circ (\mathrm{id} \times 1 \times \mathrm{id}) & \xRightarrow{\alpha \circ (\mathrm{id} \times 1 \times \mathrm{id})} & m \circ (\mathrm{id} \times m) \circ (\mathrm{id} \times 1 \times \mathrm{id}) \\ & {\scriptstyle m \circ (\rho \times \mathrm{id})} \searrow \quad \swarrow {\scriptstyle m \circ (\mathrm{id} \times \lambda)} & \\ & m & \end{array}$$

(A.4)

$$\begin{array}{ccc}
m\circ(m\times\mathrm{id})\circ(\mathrm{id}\times(\cdot)^{-1}\times\mathrm{id})\circ T & \xRightarrow{\alpha\circ(\mathrm{id}\times(\cdot)^{-1}\times\mathrm{id})\circ T} & m\circ(\mathrm{id}\times m)\circ(\mathrm{id}\times(\cdot)^{-1}\times\mathrm{id})\circ T \\
\Uparrow{\scriptstyle m\circ(d\times\mathrm{id})} & & \Downarrow{\scriptstyle m\circ(\mathrm{id}\times e)} \\
m\circ(1\times\mathrm{id}) & \xRightarrow{\lambda}\ \mathrm{id}\ \xRightarrow{\rho^{-1}} & m\circ(\mathrm{id}\times 1)
\end{array}$$

(A.5)

$$\begin{array}{ccc}
m\circ(\mathrm{id}\times m)\circ((\cdot)^{-1}\times\mathrm{id}\times(\cdot)^{-1})\circ T & \xRightarrow{\alpha^{-1}\circ((\cdot)^{-1}\times\mathrm{id}\times(\cdot)^{-1})\circ T} & m\circ(m\times\mathrm{id})\circ((\cdot)^{-1}\times\mathrm{id}\times(\cdot)^{-1})\circ T \\
\Uparrow{\scriptstyle m\circ((\cdot)^{-1}\times d)} & & \Downarrow{\scriptstyle m\circ(e\times(\cdot)^{-1})} \\
m\circ((\cdot)^{-1}\times 1) & \xRightarrow{\rho}\ (\cdot)^{-1}\ \xRightarrow{\lambda^{-1}} & m\circ(1\times(\cdot)^{-1})
\end{array}$$

A *strict* 2*-group in* $\mathscr{C}$ is a weak 2-group in $\mathscr{C}$ for which the 2-morphisms α, λ, ρ, d, e are all identity 2-morphisms. In other words, a strict 2-group is a group object in the underlying 1-category of $\mathscr{C}$.

A.6. **Definition.** Let G and G' be weak 2-groups in $\mathscr{C}$. A *weak homomorphism* consists of a 1-morphism $\phi_{(1)} = \phi\colon G \to G'$ and two 2-isomorphisms

$$\phi_{(0)}\colon \phi\circ 1 \Longrightarrow 1' \qquad \text{and} \qquad \phi_{(2)}\colon m'\circ\phi\times\phi \Longrightarrow \phi\circ m.$$

These are required to satisfy certain coherence conditions, namely that the diagrams (A.7)–(A.9) below commute.

(A.7)

$$\begin{array}{ccc}
m'\circ(m'\times\mathrm{id}_{G'})\circ(\phi\times\phi\times\phi) & \xRightarrow{m'\circ(\phi_{(2)}\times\phi)} & m'\circ(\phi\times\phi)\circ(m\times\mathrm{id}_G) \\
{\scriptstyle \alpha\circ(\phi\times\phi\times\phi)}\Downarrow & & \Downarrow{\scriptstyle \phi_{(2)}\circ(m\times\mathrm{id}_G)} \\
m'\circ(\mathrm{id}_{G'}\times m')\circ(\phi\times\phi\times\phi) & & \phi\circ m\circ(m\times\mathrm{id}_G) \\
{\scriptstyle m'\circ(\phi\times\phi_{(2)})}\Downarrow & & \Downarrow{\scriptstyle \phi\circ\alpha} \\
m'\circ(\phi\times\phi)\circ(\mathrm{id}_G\times m) & \xRightarrow{\phi_{(2)}\circ(\mathrm{id}_G\times m)} & \phi\circ m\circ(\mathrm{id}_G\times m)
\end{array}$$

(A.8)

$$\begin{array}{ccc}
m'\circ(1'\times\phi) & \xRightarrow{\lambda'\circ\phi} & \phi \\
{\scriptstyle m'\circ(\phi_{(0)}\times\phi)}\Downarrow & & \Uparrow{\scriptstyle \phi\circ\lambda} \\
m'\circ(\phi\times\phi)\circ(1\times\mathrm{id}_G) & \xRightarrow{\phi_{(2)}\circ(1\times\mathrm{id}_G)} & \phi\circ m\circ(1\times\mathrm{id}_G)
\end{array}$$

(A.9)

$$\begin{array}{ccc}
m'\circ(\phi\times 1') & \xRightarrow{\rho'\circ\phi} & \phi \\
{\scriptstyle m'\circ(\phi\times\phi_{(0)})}\Downarrow & & \Uparrow{\scriptstyle \phi\circ\rho} \\
m'\circ(\phi\times\phi)\circ(\mathrm{id}_G\times 1) & \xRightarrow{\phi_{(2)}\circ(\mathrm{id}_G\times 1)} & \phi\circ m\circ(\mathrm{id}_G\times 1)
\end{array}$$

A *strict homomorphism* is a weak homomorphism ϕ for which $\phi_{(0)}$ and $\phi_{(2)}$ are both identities.

A.10. **Definition.** Let $\phi, \psi \colon G \to G'$ be weak homomorphisms of weak 2-groups in $\mathscr{C}$. A 2-*homomorphism* is a 2-morphism $\theta \colon \phi \Rightarrow \psi$ so that the following diagrams commute:

$$\begin{array}{ccc} m' \circ (\phi \times \phi) & \xRightarrow{m' \circ (\theta \times \theta)} & m' \circ (\psi \times \psi) \\ {\scriptstyle \phi_{(2)}} \Big\Downarrow & & \Big\Downarrow {\scriptstyle \psi_{(2)}} \\ \phi \circ m & \xRightarrow{\theta \circ m} & \psi \circ m \end{array} \qquad \begin{array}{ccc} 1' & \xLeftarrow{\psi_{(0)}} & \psi \circ 1 \\ {\scriptstyle \phi_{(0)}} \Big\Uparrow & \nearrow\!\!\!\nearrow {\scriptstyle \theta \circ 1} & \\ \phi \circ 1 & & \end{array}$$

A.11. **Definition.** A weak homomorphism $\phi \colon G \to G'$ of weak 2-groups is an *equivalence* if there exists a weak homomorphism $\psi \colon G' \to G$, together with 2-homomorphisms $\psi \circ \phi \Rightarrow \mathrm{id}_G$ and $\phi \circ \psi \Rightarrow \mathrm{id}_{G'}$. In this case ψ is called a *weak inverse* to ϕ and we write $\psi = \phi^{-1}$.

The following definition is modeled on [44, Part A]; see also [9, § 3.2].

A.12. **Definition.** Let G be a strict 2-group in $\mathscr{C}$ and let X be an object of $\mathscr{C}$. A *weak action of G on X* consists of the following data: an *action morphism* $a \colon G \times X \to X$, and two 2-isomorphisms

$$\beta \colon a \circ (m \times \mathrm{id}_X) \Rightarrow a \circ (\mathrm{id}_G \times a), \qquad \epsilon \colon a \circ (1 \times \mathrm{id}_X) \Rightarrow \mathrm{id}_X\,.$$

The 2-morphisms β and ϵ are subject to the condition that the diagrams (A.13)–(A.13) below commute. For simplicity, we omit notation for horizontal composition of 1-morphisms and 2-morphisms, and display only the 2-morphism being used.

$$\begin{array}{ccc} a \circ \big((m \circ (\mathrm{id}_G \times 1)) \times \mathrm{id}_X\big) & \xRightarrow{\beta} & a \circ \big(\mathrm{id}_G \times (a \circ (1 \times \mathrm{id}_X))\big) \\ & \diagdown\!\!\diagdown & \Big\Downarrow {\scriptstyle \epsilon} \\ & & a \circ (\mathrm{pr}_G \times \mathrm{id}_X) \end{array} \tag{A.13}$$

$$\begin{array}{ccc} a \circ \big((m \circ (1 \times \mathrm{id}_G)) \times \mathrm{id}_X\big) & \xRightarrow{\beta} & a \circ (1 \times a) \\ & \diagdown\!\!\diagdown & \Big\Downarrow {\scriptstyle \epsilon} \\ & & a \circ (\mathrm{pr}_G \times \mathrm{id}_X) \end{array} \tag{A.14}$$

$$\begin{array}{ccc} a \circ \big((m \circ (m \times \mathrm{id}_G)) \times \mathrm{id}_X\big) \overset{\beta}{\Longrightarrow} a \circ (m \times a) & \overset{\beta}{\Longrightarrow} & a \circ \big(\mathrm{id}_G \times (a \circ (\mathrm{id}_G \times a))\big) \\ \Big\Vert & & \Big\Uparrow {\scriptstyle \beta} \\ a \circ \big((m \circ (\mathrm{id}_G \times m)) \times \mathrm{id}_X\big) & \xRightarrow{\qquad \beta \qquad} & a \circ \big(\mathrm{id}_G \times (a \circ (m \times \mathrm{id}_X))\big). \end{array} \tag{A.15}$$

A *strict action* is a weak action a for which the 2-morphisms β and ϵ are both identities.

A.16. **Definition.** Let G be a strict 2-group in $\mathscr{C}$. Given weak actions

$$a \colon G \times X \to X, \qquad a' \colon G \times X' \to X'$$

on objects X and X' of $\mathscr{C}$, a *weakly G-equivariant map* consists of a 1-morphism $F \colon X \to X'$ and a 2-isomorphism

$$\delta \colon a' \circ (\mathrm{id}_G \times F) \Rightarrow F \circ a,$$

which are subject to the condition that the diagrams (A.17)–(A.18) commute. We omit horizontal composition of 1-morphisms, as above.

(A.17)

$$\begin{array}{ccccc} a' \circ (m \times F) & \overset{\beta'}{\Longrightarrow} & a' \circ (\mathrm{id}_G \times a') \circ (\mathrm{id}_G \times \mathrm{id}_G \times F) & \overset{\delta}{\Longrightarrow} & a' \circ (\mathrm{id}_G \times F) \circ (\mathrm{id}_G \times a) \\ \delta \Big\Downarrow & & & & \Big\Downarrow \delta \\ F \circ a \circ (m \times \mathrm{id}_X) & & \overset{\beta}{\Longrightarrow} & & F \circ a \circ (\mathrm{id}_G \times a) \end{array}$$

(A.18)

$$\begin{array}{ccc} a' \circ (1 \times F) & \overset{\epsilon'}{\Longrightarrow} & F \\ \delta \Big\Downarrow & \overset{\epsilon}{\nearrow} & \\ F \circ a \circ (1 \times \mathrm{id}_X) & & \end{array}$$

If δ is the identity 2-morphism, then F is *strictly equivariant*.

Appendix B. Strictification of stacky actions

This appendix contains the proof of the following result, which is Theorem 6.8.1 in the main text.

B.1. **Theorem.** *Let $\mathbf{G}$ be a connected (strict) Lie group stack acting weakly on an étale differentiable stack $\mathbf{X}$. Suppose that $\mathbf{G}$ admits a presentation $\mathbf{B}G_\bullet \simeq \mathbf{G}$ by a base-connected Lie 2-group $G_\bullet$. For every such presentation $\mathbf{B}G_\bullet \simeq \mathbf{G}$ there exists a presentation $\mathbf{B}X_\bullet \simeq \mathbf{X}$ of $\mathbf{X}$ by a Lie groupoid $X_\bullet$ so that*

(i) *$G_\bullet$ acts strictly on $X_\bullet$;*
(ii) *identifying $\mathbf{B}G_\bullet = \mathbf{G}$, the equivalence $\mathbf{B}X_\bullet \simeq \mathbf{X}$ is weakly $\mathbf{G}$-equivariant.*

We first prove the following.

B.2. **Proposition.** *Let $G_\bullet$ be a base-connected Lie 2-group, let $\mathbf{G} = \mathbf{B}G_\bullet$, and let*

$$\boldsymbol{a}\colon \mathbf{G} \times \mathbf{X} \to \mathbf{X}$$

be a weak action of $\mathbf{G}$ on a differentiable stack $\mathbf{X}$. Let $\boldsymbol{p}\colon Y_0 \to \mathbf{X}$ be an atlas, let $X_0 = G_0 \times Y_0$, and consider the map

$$\boldsymbol{b}\colon X_0 = G_0 \times Y_0 \longrightarrow \mathbf{G} \times \mathbf{X} \xrightarrow{\ \boldsymbol{a}\ } \mathbf{X}.$$

The map $\boldsymbol{b}$ is a representable epimorphism and a submersion. So $\boldsymbol{b}\colon X_0 \to \mathbf{X}$ is an atlas and hence determines a canonical equivalence $\mathbf{B}X_\bullet \simeq \mathbf{X}$, where $X_\bullet$ is the Lie groupoid $X_0 \times_{\mathbf{X}} X_0 \rightrightarrows X_0$.

Proof. Consider the 2-cartesian square

$$\begin{array}{ccc} (G_0 \times Y_0) \times_{\mathbf{X}} Y_0 & \xrightarrow{\ q\ } & Y_0 \\ \mathrm{pr}_1 \Big\downarrow & & \Big\downarrow \boldsymbol{p} \\ X_0 = G_0 \times Y_0 & \xrightarrow{\ \boldsymbol{b}\ } & \mathbf{X} \end{array}$$

where pr_1 and q are the canonical (up to 2-isomorphism) maps out of the fibred product. Then $(G_0 \times Y_0) \times_{\mathbf{X}} Y_0$ is (equivalent to) a manifold, since $\boldsymbol{p}$ is representable. By Lemmas 2.2 and 2.3 of [3], to show that $\boldsymbol{b}$ is a representable submersive epimorphism it is enough to show that q is a surjective submersion. To see this, consider any $g \in G_0$ and form the following diagram composed of two 2-cartesian squares:

$$\begin{array}{ccccc} (\{g\} \times Y_0) \times_{\mathbf{X}} Y_0 & \hookrightarrow & (G_0 \times Y_0) \times_{\mathbf{X}} Y_0 & \xrightarrow{q} & Y_0 \\ \downarrow & & \downarrow & & \downarrow{\boldsymbol{p}} \\ \{g\} \times Y_0 & \hookrightarrow & G_0 \times Y_0 & \xrightarrow{\boldsymbol{b}} & \mathbf{X} \end{array} \tag{B.3}$$

We assert that the bottom row

$$\{g\} \times Y_0 \hookrightarrow G_0 \times Y_0 \longrightarrow \mathbf{X} \tag{B.4}$$

is an atlas for $\mathbf{X}$. Indeed, applying the functor $\mathbf{B}$ gives us a categorical point $\boldsymbol{g} \colon \star \to \mathbf{G}$, which determines a morphism of stacks

$$\boldsymbol{L}_g \colon \mathbf{X} \simeq \star \times \mathbf{X} \xrightarrow{\boldsymbol{g} \times \mathrm{id}} \mathbf{G} \times \mathbf{X} \xrightarrow{\boldsymbol{a}} \mathbf{X}.$$

By the axioms for the weak $\mathbf{G}$-action on $\mathbf{X}$, the morphism $\boldsymbol{L}_g$ is an equivalence with weak inverse $\boldsymbol{L}_{g^{-1}}$. The composition (B.4) is naturally isomorphic to $\boldsymbol{L}_g \circ \boldsymbol{p} \colon X_0 \to \mathbf{X}$. Since $\boldsymbol{L}_g$ is an equivalence and $\boldsymbol{p}$ is an atlas, we have that (B.4) is an atlas. Since being an epimorphism and being a submersion are stable under pullbacks, this implies that the top row of (B.3)

$$(\{g\} \times Y_0) \times_{\mathbf{X}} Y_0 \hookrightarrow (G_0 \times Y_0) \times_{\mathbf{X}} Y_0 \longrightarrow Y_0$$

is a surjective submersion. It follows immediately that q is a surjective submersion. QED

Proof of Theorem B.1. Let $Y_0 \to \mathbf{X}$ be any atlas and let $X_\bullet$ be as in Proposition B.2. Each square of the following diagram 2-commutes:

$$\begin{array}{ccc} G_0 \times G_0 \times Y_0 & \xrightarrow{m_0 \times \mathrm{id}} & G_0 \times Y_0 \\ \downarrow & & \downarrow \\ \mathbf{G} \times \mathbf{G} \times \mathbf{X} & \xrightarrow{\boldsymbol{m} \times \mathrm{id}} & \mathbf{G} \times \mathbf{X} \\ {\scriptstyle \mathrm{id} \times \boldsymbol{a}}\downarrow & & \downarrow{\boldsymbol{a}} \\ \mathbf{G} \times \mathbf{X} & \xrightarrow{\boldsymbol{a}} & \mathbf{X}. \end{array} \tag{B.5}$$

There is a Lie groupoid morphism $A_\bullet \colon G_\bullet \times X_\bullet \to X_\bullet$, where $A_0 = m_0 \times \mathrm{id}$ and $\mathbf{B}A_\bullet \cong \boldsymbol{a}$. Then A_0 determines an action of G_0 on X_0. We write

$$A_0(g, x) = g \cdot x, \qquad A_1(h, y) = h \cdot y$$

for $g \in G_0$, $x \in X_0$, $h \in G_1$, and $y \in X_1$. We will show that $A_\bullet$ is naturally isomorphic to a strict action $\tilde{A}_\bullet \colon G_\bullet \times X_\bullet \to X_\bullet$ with $\tilde{A}_0 = A_0$. Note that despite our notation, A_1 is not in general an action.

There are 2-isomorphisms which come with the action $\boldsymbol{a}$ as in Definition A.12, and we must write these in terms of $A_\bullet$. To do this, we make use of the following fact: let $F_\bullet$, $F'_\bullet$ be morphisms of Lie groupoids and let $\boldsymbol{\alpha} \colon \mathbf{B}F \Rightarrow \mathbf{B}F'$ be a 2-arrow in the category of stacks. Then there is a unique natural isomorphism $\alpha \colon F \Rightarrow F'$ in the category of Lie groupoids so that $\boldsymbol{\alpha} = \mathbf{B}\alpha$. This assertion follows from the equivalence of **DiffStack** and **LieGpd**$[\mathscr{M}^{-1}]$ and the description of 2-cells in [42, § 2.3]. It was also shown in [4, Theorem 2.5].

The 2-isomorphisms which come with $\boldsymbol{a}$ then translate as follows. There are 2-arrows

$$\begin{aligned}
&\beta\colon A_\bullet \circ (m_\bullet \times \mathrm{id}) \Longrightarrow A_\bullet \circ (\mathrm{id} \times A_\bullet);\\
&\beta\colon G_0 \times G_0 \times X_0 \longrightarrow X_1;\\
&\epsilon\colon A_\bullet \circ (1 \times \mathrm{id}) \Longrightarrow \mathrm{id};\\
&\epsilon\colon X_0 \longrightarrow X_1,
\end{aligned}$$

where $1\colon \star \to G_\bullet$ is the 2-group unit, and we write

$$1 \times \mathrm{id}\colon X_\bullet \cong \star \times X_\bullet \longrightarrow G_\bullet \times X_\bullet.$$

We also write the coherence conditions on ϵ and β. For $k, h, g \in G_0$ and $x \in X_0$, they are as follows.

$$(u(g) \cdot \epsilon(x)) \circ \beta(g, 1, x) = u(g \cdot x); \tag{B.6}$$

$$\epsilon(g \cdot x) \circ \beta(1, g, x) = u(g \cdot x); \tag{B.7}$$

$$\beta(k, h, g \cdot x) \circ \beta(kh, g, x) = (u(k) \cdot \beta(h, g, x)) \circ \beta(k, hg, x). \tag{B.8}$$

Since A_0 is an action, we have that $\epsilon(x)$ is an arrow from x to x and $\beta(h, g, x)$ is an arrow from $(hg) \cdot x$ to $(hg) \cdot x$.

Now fix $x \in X_0$. Define smooth maps $\gamma_1, \gamma_2\colon G_0 \times G_0 \to s^{-1}(x) \cap t^{-1}(x)$ as follows.

$$\begin{aligned}
\gamma_1(k, j) &= u((kj)^{-1}) \cdot \beta(k, 1, j \cdot x)\\
\gamma_2(k, j) &= u((kj)^{-1}) \cdot \beta(k, j, x).
\end{aligned}$$

Then, since G_0 is connected and $X_\bullet$ has discrete isotropy groups, the maps γ_1 and γ_2 are constant. Moreover, $\gamma_1(1, 1) = 1 \cdot \beta(1, 1, x) = \gamma_2(1, 1)$, so $\gamma_1 = \gamma_2$ and thus

$$u((kj)^{-1}) \cdot \beta(k, 1, j \cdot x) = u((kj)^{-1}) \cdot \beta(k, j, x)$$

for all $k, j \in G_0$. Therefore,

$$u(kj) \cdot (u((kj)^{-1}) \cdot \beta(k, 1, j \cdot x)) = u(kj) \cdot (u((kj)^{-1}) \cdot \beta(k, j, x))$$

Applying the natural transformation β to both sides, one finds

$$\begin{aligned}
&\beta(kj, (kj)^{-1}, kj \cdot x) \circ (1 \cdot \beta(k, 1, j \cdot x)) \circ \beta(kj, (kj)^{-1}, kj \cdot x)^{-1}\\
&\qquad = \beta(kj, (kj)^{-1}, kj \cdot x) \circ (1 \cdot \beta(k, j, x)) \circ \beta(kj, (kj)^{-1}, kj \cdot x)^{-1},
\end{aligned}$$

and so

$$1 \cdot \beta(k, 1, j \cdot x) = 1 \cdot \beta(k, j, x).$$

Applying the natural transformation ϵ to both sides, one has

$$\epsilon(kj \cdot x)^{-1} \circ \beta(k, 1, j \cdot x) \circ \epsilon(kj \cdot x) = \epsilon(kj \cdot x)^{-1} \circ \beta(k, j, \cdot x) \circ \epsilon(kj \cdot x),$$

and therefore

$$\beta(k, 1, j \cdot x) = \beta(k, j, x). \tag{B.9}$$

As a consequence, by applying the coherence condition (B.6) we have

$$u(k) \cdot \epsilon(j \cdot x) = \beta(k, 1, j \cdot x)^{-1} = \beta(k, j, x)^{-1}. \tag{B.10}$$

Now define $\tilde{A}_\bullet\colon G_\bullet \times X_\bullet \to X_\bullet$ by

$$\begin{aligned}
\tilde{A}_0(g, x) &= A_0(g, x) = g \cdot x\\
\tilde{A}_1(k, f) &= \epsilon(t(k) \cdot t(f)) \circ (k \cdot f) \circ \epsilon(s(k) \cdot s(f))^{-1}.
\end{aligned}$$

We write $\tilde{A}_1(k, f) = k \odot f$. That $\tilde{A}_\bullet$ is a Lie groupoid morphism is easy to check. The smooth map

$$\epsilon \circ A_0 \colon G_0 \times X_0 \longrightarrow X_1$$

is a natural isomorphism from $A_\bullet$ to $\tilde{A}_\bullet$. It remains to check that $\tilde{A}_1 \colon G_1 \times X_1 \longrightarrow X_1$ is an action. First,

$$1 \odot f = \epsilon(t(f)) \circ (1 \cdot f) \circ \epsilon(s(f))^{-1} = f,$$

since ϵ is a natural isomorphism $A_\bullet \circ (1 \times \mathrm{id}) \Rightarrow \mathrm{id}$. Next,

$$\begin{aligned} k \odot (j \odot f) &= k \odot \Big(\epsilon(t(j) \cdot t(f)) \circ (j \cdot f) \circ \epsilon(s(j) \cdot s(f))^{-1}\Big) \\ &= \epsilon(t(kj) \cdot t(f)) \circ F \circ \epsilon(s(kj) \cdot s(f))^{-1}, \end{aligned}$$

where $F = k \cdot \big(\epsilon(t(j) \cdot t(f)) \circ (j \cdot f) \circ \epsilon(s(j) \cdot s(f))^{-1}\big)$. On the other hand,

$$kj \odot f = \epsilon(t(kj) \cdot t(f)) \circ [kj \cdot f] \circ \epsilon(s(kj) \cdot s(f))^{-1}.$$

We must then show that

$$F = kj \cdot f. \tag{B.11}$$

We compute

$$\begin{aligned} F &= [u(t(k)) \circ k \circ u(s(k))] \cdot \big[\epsilon(t(j) \cdot t(f)) \circ (j \cdot f) \circ \epsilon(s(j) \cdot s(f))^{-1}\big] \\ &= [u(t(k)) \cdot \epsilon(t(j) \cdot t(f))] \circ [k \cdot (j \cdot f)] \circ \big[u(s(k)) \cdot \epsilon(s(j) \cdot s(f))^{-1}\big], \end{aligned} \tag{B.12}$$

since $A_\bullet$ is a groupoid morphism. From (B.10), one has that (B.12) is equal to

$$\beta(t(k), t(j), t(f))^{-1} \circ [k \cdot (j \cdot f)] \circ \beta(s(k), s(j), s(f)).$$

But since β is a natural transformation $A_\bullet \circ (m_\bullet \times \mathrm{id}) \Rightarrow A_\bullet \circ (\mathrm{id} \times A_\bullet)$ this is just $kj \cdot f$. So we have found (B.11), and proved (i).

To prove (ii), assume $\mathbf{G} = \mathbf{B}G_\bullet$. It suffices to show that the identity 1-morphism $X_\bullet \to X_\bullet$ together with the 2-isomorphism $\delta = \epsilon \circ A_0 \colon G_0 \times X_0 \longrightarrow X_1$ are the data of a weakly $G_\bullet$-equivariant map intertwining the strict action $\tilde{A}_\bullet$ with the weak action $A_\bullet$. Indeed, the coherence condition (A.17) becomes the requirement that for any $g, h \in G_0$ and $x \in X_0$,

$$\epsilon(gh \cdot x) \circ \big(u(g) \cdot \epsilon(h \cdot x)\big) \circ \beta(g, h, x) = \epsilon(gh \cdot x).$$

This follows from the coherence condition (B.6) above together with (B.9). The coherence condition (A.18) is automatic from the definition of δ. QED

Appendix C. Weak fibred products of Lie group stacks (by C. Zhu)

This appendix is devoted to the proof of the following result, which is Theorem 6.6.1 in the main text.

C.1. **Theorem.** *Let* $\mathbf{G} \to \mathbf{H}$ *and* $\mathbf{G}' \to \mathbf{H}$ *be weak homomorphisms of strict Lie group stacks, and assume that the fibred product of stacks* $\mathbf{K} = \mathbf{G} \times_{\mathbf{H}} \mathbf{G}'$ *is a differentiable stack. Then* $\mathbf{K}$ *is naturally a weak Lie group stack, and the projections* $\mathbf{K} \to \mathbf{G}$ *and* $\mathbf{K} \to \mathbf{G}'$ *are strict homomorphisms.*

Proof. It suffices to prove the statement object-wise. That is, for an object $U \in \mathbf{Diff}$, the groupoid $\mathbf{K}(U) = \mathbf{G}(U) \times^{(w)}_{\mathbf{H}(U)} \mathbf{G}'(U)$ is a weak 2-group. Because $\mathbf{K}$ is assumed to be a differentiable stack, it will then automatically be a Lie group stack.

Let us denote

$$\begin{aligned} G &= (G_1 \rightrightarrows G_0) := \mathbf{G}(U) \\ G' &= (G'_1 \rightrightarrows G'_0) := \mathbf{G}'(U) \\ H &= (H_1 \rightrightarrows H_0) := \mathbf{H}(U) \end{aligned}$$

and the maps between them $\phi\colon G \to H$ and $\phi'\colon G' \to H$. We now define the group structure maps on $G \times_H^{(w)} G'$.

Multiplication. The multiplication

$$\tilde{m}\colon (G \times_H^{(w)} G') \times (G \times_H^{(w)} G') = G^{\times 2} \times_{H^{\times 2}}^{(w)} G'^{\times 2} \longrightarrow (G \times_H^{(w)} G')$$

is essentially given by $m \times m'$, where m and m' are the group multiplication maps for G and G' respectively. We denote by $\cdot$ the multiplications m, m', $\tilde{m}$, etc. Then on the level of objects of $G \times_H^{(w)} G'$, define:

$$(g_0^1, h_1^1, {g'}_0^1) \cdot (g_0^2, h_1^2, {g'}_0^2) := (g_0^1 \cdot g_0^2, \phi'_{(2)} \circ (h_1^1 \cdot h_1^2) \circ \phi_{(2)}^{-1}, {g'}_0^1 \cdot {g'}_0^2),$$

where $\phi_{(2)}\colon \phi_0(g_0^1) \cdot \phi_0(g_0^2) \to \phi_0(g_0^1 \cdot g_0^2)$ comes from the 2-isomorphism data of the weak homomorphism ϕ; and similarly for $\phi'_{(2)}$. The middle element in H_1 is then obtained by the following composition

$$\phi_0(g_0^1 \cdot g_0^2) \xrightarrow{\phi_{(2)}^{-1}} \phi_0(g_0^1) \cdot \phi_0(g_0^2) \xrightarrow{h_1^1 \cdot h_1^2} \phi_0({g'}_0^1) \cdot \phi_0({g'}_0^2) \xrightarrow{\phi'_{(2)}} \phi_0({g'}_0^1 \cdot {g'}_0^2).$$

On the level of morphisms, define

$$(g_1^1, h_1^1, {g'}_1^1) \cdot (g_1^2, h_1^2, {g'}_1^2) := (g_1^1 \cdot g_1^2, \phi'_{(2)} \circ (h_1^1 \cdot h_1^2) \circ \phi_{(2)}^{-1}, {g'}_1^1 \cdot {g'}_1^2)$$

where $s(g_1^1, h_1^1, {g'}_1^1) = (g_0^1, h_1^1, {g'}_0^1)$ and $s(g_1^2, h_1^2, {g'}_1^2) = (g_0^2, h_1^2, {g'}_0^2)$.

Associator. The associator $\tilde{\alpha}$ for $\tilde{m}$ is a natural transformation, whose value at the point

$$\big((g_0^1, g_0^2, g_0^3), (h_1^1, h_1^2, h_1^3), ({g'}_0^1, {g'}_0^2, {g'}_0^3)\big) \in ((G \times_H^{(w)} G')_0)^3$$

is given by the following element in $\big(G \times_H^{(w)} G'\big)_1$,

$$\begin{aligned} &\big((g_0^1 \cdot g_0^2) \cdot g_0^3, \overline{\phi}'_{(2)} \circ [(h_1^1 \cdot h_1^2) \cdot h_1^3] \circ \overline{\phi}_{(2)}^{-1}, ({g'}_0^1 \cdot {g'}_0^2) \cdot {g'}_0^3\big) \\ &\qquad \xrightarrow{\left(\alpha(g_0^1, g_0^2, g_0^3), \overline{\phi}'_{(2)} \circ [(h_1^1 \cdot h_1^2) \cdot h_1^3] \circ \overline{\phi}_{(2)}^{-1}, \alpha'({g'}_0^1, {g'}_0^2, {g'}_0^3)\right)} \\ &\qquad\qquad \big(g_0^1 \cdot (g_0^2 \cdot g_0^3), (\hat{\phi}'_{(2)}) \circ [h_1^1 \cdot (h_1^2 \cdot h_1^3)] \circ \hat{\phi}_{(2)}^{-1}, {g'}_0^1 \cdot ({g'}_0^2 \cdot {g'}_0^3)\big), \end{aligned}$$

where the 1-morphisms

$$\begin{aligned} \overline{\phi}_{(2)}&\colon (\phi_0(g_0^1) \cdot \phi_0(g_0^2)) \cdot \phi_0(g_0^3) \to \phi_0((g_0^1 \cdot g_0^2) \cdot g_0^3) \\ \hat{\phi}_{(2)}&\colon \phi_0(g_0^1) \cdot (\phi_0(g_0^2) \cdot \phi_0(g_0^3)) \to \phi_0(g_0^1 \cdot (g_0^2 \cdot g_0^3)) \end{aligned}$$

are defined using the 2-isomorphism data $\phi_{(2)}$, and similarly for $\overline{\phi}'_{(2)}$ and $\hat{\phi}'_{(2)}$. It is an arrow between these two objects thanks to the compatibility of $\phi_{(2)}$ and the associator α, and that of $\phi'_{(2)}$ and α'.

Naturality of $\tilde{\alpha}$ comes from that of α and α'. The pentagon identity for $\tilde{\alpha}$ is implied by that of α and α'.

Identity. Similarly, we define the identity map $\tilde{1}\colon * \to G\times_H^{(w)} G'$ to be essentially $(1, 1')$. More precisely, $\tilde{1}_0 := (1_0, \phi'^{-1}_{(0)} \circ \phi_{(0)}, 1'_0) \in (G \times_H^{(w)} G')_0$, where we have

$$\phi_0(1_0) \xrightarrow{\phi_{(0)}} 1_0^H \xrightarrow{\phi'^{-1}_{(0)}} \phi_0(1'_0).$$

Here $\phi_{(0)}$ and $\phi'_{(0)}$ come from the 2-isomorphism data in the definition of ϕ and ϕ'. Then $\tilde{1}_1 := (1_1, \phi'^{-1}_{(0)} \circ \phi_{(0)}, 1'_1)$. The 2-morphisms involving $\tilde{1}$ for $\mathbf{G} \times_{\mathbf{H}} \mathbf{G}'$ are similarly defined by those for 1 and $1'$, and the coherence conditions for these 2-morphisms are similarly implied by that for $\mathbf{G}$ and $\mathbf{G}'$.

Inverse. The inverse map $\tilde{\imath}\colon G \times_H^{(w)} G' \to G \times_H^{(w)} G'$ is essentially (i, i'), where i is the inverse map for G and i' that for G'. More precisely, on objects

$$\tilde{\imath}_0(g_0, h_1, g'_0) := (i_0(g_0), \gamma'^{-1} \circ i_1(h_1) \circ \gamma, i'_0(g'_0)),$$

where

$$\phi_0(i_0(g_0)) \xrightarrow{\gamma} i_0(\phi_0(g_0)) \xrightarrow{i_1(h_1)} i_0(\phi_0(g'_0)) \xrightarrow{\gamma'^{-1}} \phi_0.$$

And on arrows

$$\tilde{\imath}_1(g_1, h_1, g'_1) := (i_1(g_1), \gamma'^{-1} \circ h_1^{-1} \circ \gamma, i'_1(g'_1)).$$

Here γ is uniquely determined by the 2-isomorphism data of ϕ and satisfies certain coherence laws; see [1, § 6]. The 2-morphisms involving $\tilde{\imath}$ for $\mathbf{G} \times_{\mathbf{H}} \mathbf{G}'$ are similarly defined by those for i and i', and the coherence conditions for these 2-morphisms are similarly implied by that for $\mathbf{G}$ and $\mathbf{G}'$.

Projections. From the construction of the group structure maps $\tilde{m}, \tilde{1}, \tilde{\imath}$ for $\mathbf{K}$, we see that the projections to $\mathbf{G}$ and $\mathbf{G}'$ are naturally strict homomorphisms of Lie group stacks. QED

Notation Index

References

1. J. Baez and A. Lauda, *Higher-dimensional algebra V: 2-groups*, Theory Appl. Categ. **12** (2004), 423–491. 23, 54, 62
2. F. Battaglia and E. Prato, *Nonrational symplectic toric reduction*, eprint, 2018, `arXiv:1806.10431`. 45
3. K. Behrend and P. Xu, *Differentiable stacks and gerbes*, J. Symplectic Geom. **9** (2011), no. 3, 285–341. 5, 8, 19, 57
4. D. Berwick-Evans and E. Lerman, *Lie 2-algebras of vector fields*, eprint, 2016, `arXiv:1609.03944`. 8, 16, 18, 19, 58
5. C. Blohmann, *Stacky Lie groups*, Int. Math. Res. Not. (2008), 51 pp. 5, 7, 8
6. N. Bourbaki, *Groupes et algèbres de Lie*, Éléments de mathématique, Springer-Verlag, 2007. 27, 45
7. R. Brown and O. Mucuk, *Covering groups of nonconnected topological groups revisited*, Math. Proc. Cambridge Philos. Soc. **115** (1994), no. 1, 97–110. 27
8. R. Brown and C. Spencer, *G-groupoids, crossed modules and the fundamental groupoid of a topological group*, Indag. Math. **38** (1976), no. 4, 296–302. 23
9. H. Bursztyn, F. Noseda, and C. Zhu, *Principal actions of stacky Lie groupoids*, Int. Math. Res. Not., to appear, `arXiv:1510.09208`. 37, 46, 49, 56
10. D. Calaque, *Three lectures on derived symplectic geometry and topological field theories*, Indag. Math. (N.S.) **25** (2014), no. 5, 926–947. 3
11. D. Carchedi, *Categorical properties of topological and differentiable stacks*, Ph.D. thesis, Utrecht University, 2011, `https://dspace.library.uu.nl/handle/1874/208971`. 5, 7, 20
12. ———, *An étalé space construction for stacks*, Algebr. Geom. Topol. **13** (2013), no. 2, 831–903, *Correction*, ibid. **16** (2016), no. 1, 541–546. 5, 7
13. M. Crainic and J. Mestre, *Measures on differentiable stacks*, eprint, 2016, `arXiv:1603.01673`. 50
14. M. Crainic and I. Moerdijk, *Foliation groupoids and their cyclic homology*, Adv. Math. **157** (2001), no. 2, 177–197. 5, 8, 9
15. T. Delzant, *Hamiltoniens périodiques et images convexes de l'application moment*, Bull. Soc. Math. France **116** (1988), no. 3, 315–339. 3, 5
16. J. J. Duistermaat and G. Heckman, *On the variation in the cohomology of the symplectic form of the reduced phase space*, Invent. Math. **69** (1982), no. 2, 259–268, *Addendum*, ibid. **72** (1983), no. 1, 153–158. 50
17. A. Geraschenko and M. Satriano, *Toric stacks I: The theory of stacky fans*, Trans. Amer. Math. Soc. **367** (2015), no. 2, 1033–1071, *Toric stacks II: Intrinsic characterization of toric stacks*, ibid. **367** (2015), no. 2, 1073–1094. 3
18. E. Getzler, *Differential forms on stacks*, `http://www.math.northwestern.edu/\~getzler/`, 2014. 3
19. V. Guillemin and S. Sternberg, *Birational equivalence in the symplectic category*, Invent. Math. **97** (1989), no. 3, 485–522. 50

20. Z. He, *Odd-dimensional symplectic manifolds*, Ph.D. thesis, Massachusetts Institute of Technology, 2010, http://hdl.handle.net/1721.1/60189. 2
21. R. Hepworth, *Vector fields and flows on differentiable stacks*, Theory Appl. Categ. **22** (2009), 542–587. 8, 18, 32
22. B. Hoffman, *Toric symplectic stacks*, eprint, arXiv:1903.05632. 3, 42
23. H. Ishida, *Torus invariant transverse Kähler foliations*, Trans. Amer. Math. Soc. **369** (2017), no. 7, 5137–5155. 2
24. F. Kirwan, *Convexity properties of the moment mapping, III*, Invent. Math. **77** (1984), no. 3, 547–552. 2
25. E. Lerman, *Orbifolds as stacks?*, Enseign. Math. (2) **56** (2010), no. 3-4, 315–363. 5, 8
26. ———, *Left-invariant vector fields on a Lie 2-group*, eprint, 2018, arXiv:1808.02920. 26
27. E. Lerman and A. Malkin, *Hamiltonian group actions on symplectic Deligne-Mumford stacks and toric orbifolds*, Adv. Math. **229** (2012), no. 2, 984–1000. 2, 3, 8, 20, 32, 38, 42, 43, 45
28. E. Lerman and S. Tolman, *Hamiltonian torus actions on symplectic orbifolds and toric varieties*, Trans. Amer. Math. Soc. **349** (1997), no. 10, 4201–4230. 42
29. Y. Lin and R. Sjamaar, *Cohomological localization for transverse Lie algebra actions on Riemannian foliations*, eprint, arXiv:1806.01908. 43
30. ———, *Convexity properties of presymplectic moment maps*, J. Symplectic Geom., to appear, arXiv:1706.00520. 2, 4, 5, 42, 52
31. K. Mackenzie and P. Xu, *Classical lifting processes and multiplicative vector fields*, Quart. J. Math. Oxford Ser. (2) **49** (1998), no. 193, 59–85. 15
32. J. Marsden and A. Weinstein, *Reduction of symplectic manifolds with symmetry*, Rep. Mathematical Phys. **5** (1974), no. 1, 121–130. 49
33. ———, *Comments on the history, theory, and applications of symplectic reduction*, Quantization of singular symplectic quotients (N. Landsman et al., ed.), Progress in Mathematics, vol. 198, Birkhäuser, Basel, 2001, pp. 1–19. 49
34. D. Metzler, *Topological and smooth stacks*, eprint, 2003, arXiv:math/0306176. 5, 7
35. K. Meyer, *Symmetries and integrals in mechanics*, Dynamical systems (Bahia, Salvador, Brazil, 1971) (M. Peixoto, ed.), Academic Press, New York, 1973, pp. 259–272. 49
36. I. Moerdijk and J. Mrčun, *Introduction to foliations and Lie groupoids*, Cambridge Studies in Advanced Mathematics, vol. 91, Cambridge University Press, Cambridge, 2003. 5, 7, 8, 39, 46
37. B. Noohi, *Foundations of topological stacks I*, eprint, 2005, arXiv:math/0503247. 5, 7, 8, 29
38. ———, *Fibrations of topological stacks*, Adv. Math. **252** (2014), 612–640. 35
39. T. Pantev, B. Toën, M. Vaquié, and G. Vezzosi, *Shifted symplectic structures*, Publ. Math. Inst. Hautes Études Sci. **117** (2013), 271–328. 3, 17
40. J. Pecharich, *The derived Marsden-Weinstein quotient is symplectic*, eprint, 2012, arXiv:1205.6519. 3
41. E. Prato, *Simple non-rational convex polytopes via symplectic geometry*, Topology **40** (2001), no. 5, 961–975. 3, 5, 35, 42
42. D. Pronk, *Etendues and stacks as bicategories of fractions*, Compositio Math. **102** (1996), no. 3, 243–303. 7, 58
43. T. Ratiu and N. T. Zung, *Presymplectic convexity and (ir)rational polytopes*, eprint, 2017, arXiv:1705.11110. 2, 42
44. M. Romagny, *Group actions on stacks and applications*, Michigan Math. J. **53** (2005), no. 1, 209–236. 56
45. P. Safronov, *Quasi-Hamiltonian reduction via classical Chern-Simons theory*, Adv. Math. **287** (2016), 733–773. 3
46. G. Trentinaglia and C. Zhu, *Strictification of étale stacky Lie groups*, Compos. Math. **148** (2012), no. 3, 807–834. 21, 29, 35
47. J. Villatoro, *Stacks in Poisson geometry*, Ph.D. thesis, University of Illinois at Urbana/Champaign, 2018, arXiv:1806.01939. 5, 8
48. J. Watts and S. Wolbert, *Diffeological coarse moduli spaces of stacks over manifolds*, eprint, 2014, arXiv:1406.1392. 19, 29
49. A. Weinstein, *Symplectic groupoids and Poisson manifolds*, Bull. Amer. Math. Soc. (N.S.) **16** (1987), no. 1, 101–104. 3, 17
50. P. Xu, *Momentum maps and Morita equivalence*, J. Differential Geom. **67** (2004), no. 2, 289–333. 3

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