

# LECH'S INEQUALITY, THE STÜCKRAD–VOGEL CONJECTURE, AND UNIFORM BEHAVIOR OF KOSZUL HOMOLOGY

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ABSTRACT. Let  $(R, \mathfrak{m})$  be a Noetherian local ring, and let  $M$  be a finitely generated  $R$ -module of dimension  $d$ . We prove that the set  $\left\{ \frac{l(M/IM)}{e(I, M)} \right\}_{\sqrt{I}=\mathfrak{m}}$  is bounded below by  $1/d!e(\overline{R})$  where  $\overline{R} = R/\text{Ann}(M)$ . Moreover, when  $\widehat{M}$  is equidimensional, this set is bounded above by a finite constant depending only on  $M$ . The lower bound extends a classical inequality of Lech, and the upper bound answers a question of Stückrad–Vogel in the affirmative. As an application, we obtain results on uniform behavior of the lengths of Koszul homology modules.

## 1. INTRODUCTION

In [12], Lech proved a simple inequality relating the Hilbert–Samuel multiplicity and the colength of an ideal. It states that if  $(R, \mathfrak{m})$  is a Noetherian local ring of dimension  $d$  and  $I$  is any  $\mathfrak{m}$ -primary ideal of  $R$ , then we have

$$e(I, R) \leq d!e(R)l(R/I),$$

where  $e(I, R)$  denotes the Hilbert–Samuel multiplicity of  $I$  and  $e(R) = e(\mathfrak{m}, R)$ . In the same paper, Lech conjectured that, for every flat local extension  $(R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$  of Noetherian local rings, one has  $e(R) \leq e(S)$ . This conjecture is wide open in general. Using the above inequality, Lech obtained the estimate  $e(R) \leq d!e(S)$  where  $d = \dim R$  [12]. We refer to [9] and [7] for some generalizations of Lech's inequality and to [13] for recent progress on Lech's conjecture.

If we consider the set  $\left\{ \frac{l(R/I)}{e(I, R)} \right\}_{\sqrt{I}=\mathfrak{m}}$  of positive numbers, then Lech's inequality is simply saying that this set is bounded below by  $\frac{1}{d!e(R)}$  (and, thus, is bounded away from 0). The infimum of this set was investigated by Mumford in his study of local stability [14]. In a different direction, in [21] Stückrad and Vogel studied whether  $\left\{ \frac{l(R/I)}{e(I, R)} \right\}_{\sqrt{I}=\mathfrak{m}}$  is bounded from above (see also [16]), and they conjectured the following [21, Theorem 1 and Conjecture]:

**Conjecture 1.1** (Stückrad–Vogel). *Let  $(R, \mathfrak{m})$  be a Noetherian local ring and let  $M$  be a finitely generated  $R$ -module. Let  $e(I, M)$  be the Hilbert–Samuel multiplicity<sup>1</sup> of  $M$  with respect to  $I$ . Set*

$$n(M) = \sup_{\sqrt{I+\text{Ann}(M)}=\mathfrak{m}} \left\{ \frac{l(M/IM)}{e(I, M)} \right\}.$$

*Then  $n(M) < \infty$  if and only if  $M$  is quasi-unmixed (i.e.,  $\widehat{M}$  is equidimensional).*

Stückrad and Vogel proved the “only if” direction in general and a graded version of the “if” direction [21, Theorem 1]. Some other partial results were obtained in [1]. In this paper we settle

<sup>1</sup>In this paper, we define the Hilbert–Samuel multiplicity of a finitely generated module  $M$  with respect to  $I$  to be  $e(I, M) = \lim_{n \rightarrow \infty} t! \frac{l_R(M/I^n M)}{n^t}$  where  $t = \dim M$ . This is always a positive integer even when  $\dim M < \dim R$ . We will simplify our notation when  $I = \mathfrak{m}$  and write  $e(M)$  for  $e(\mathfrak{m}, M)$ .

this conjecture in the affirmative. Furthermore, motivated by Conjecture 1.1, it is quite natural to inquire whether Lech's classical inequality can be extended to all finitely generated modules, i.e., whether there is a lower bound on the set  $\left\{ \frac{l(M/IM)}{e(I, M)} \right\}_{\sqrt{I}=\mathfrak{m}}$  for a finitely generated  $R$ -module  $M$ . We also answer this question in the affirmative. In sum, our main result is the following:

**Theorem A** (Theorem 2.4 and Theorem 3.2). *Let  $(R, \mathfrak{m})$  be a Noetherian local ring, and let  $M$  be a finitely generated  $R$ -module of dimension  $d$ . Set*

$$m(M) = \inf_{\sqrt{I+\text{Ann}(M)}=\mathfrak{m}} \left\{ \frac{l(M/IM)}{e(I, M)} \right\} \quad \text{and} \quad n(M) = \sup_{\sqrt{I+\text{Ann}(M)}=\mathfrak{m}} \left\{ \frac{l(M/IM)}{e(I, M)} \right\}.$$

Then we have

$$m(M) \geq \frac{1}{d!e(\overline{R})}$$

where  $\overline{R} = R/\text{Ann}(M)$ . Moreover, if  $M$  is quasi-unmixed, then we also have

$$n(M) < \infty.$$

As an application of Theorem A, we obtain the following result on Koszul homology:

**Theorem B** (Theorem 4.8). *Let  $R$  be a Noetherian local ring, and let  $M$  be a finitely generated quasi-unmixed  $R$ -module of dimension  $d$ . For every  $\varepsilon > 0$ , there exists  $t_0$  such that, for all  $t \geq t_0$ , all systems of parameters  $\underline{x} := x_1, \dots, x_d$  of  $M$ , and all  $1 \leq i \leq d$ ,*

$$\frac{l(H_i(x_1^t, \dots, x_d^t; M))}{l(M/(x_1^t, \dots, x_d^t)M)} < \varepsilon.$$

In fact, there exists a constant  $K$  such that for all  $t \geq 1$ , all systems of parameters  $\underline{x} := x_1, \dots, x_d$  of  $M$ , and all  $1 \leq i \leq d$ ,

$$\frac{l(H_i(x_1^t, \dots, x_d^t; M))}{l(M/(x_1^t, \dots, x_d^t)M)} \leq \frac{K}{t^i}.$$

It is well known that the ratio in Theorem B tends to 0 for any fixed system of parameters  $\underline{x}$ . What we achieve in Theorem B is a *uniform* convergence. We also point out that our Theorem A says that  $m(M)$  is bounded below by  $\frac{1}{d!e(\overline{R})}$ , which is independent of  $M$  (and only depends on  $R/\text{Ann}(M)$ ). One cannot expect the same for the upper bound  $n(M)$ :

*Example 1.2.* Let  $R = k[[x, y]]$  and let  $M_t = \mathfrak{m}^t = (x, y)^t$ . Then the  $M_t$  are all faithful  $R$ -modules of rank one, but clearly

$$n(M_t) \geq \frac{l(M_t/\mathfrak{m}M_t)}{e(\mathfrak{m}, M_t)} = \frac{l(\mathfrak{m}^t/\mathfrak{m}^{t+1})}{e(R)} = t + 1.$$

Therefore, there cannot exist a constant  $c$  such that  $n(M_t) \leq c$  works for all  $M_t$ .

Nonetheless, inspired by this example, we will see in Remark 4.14 that  $n(M)/\mu(M)$  is indeed bounded above by a constant depending only on  $R/\text{Ann}(M)$ .

This paper is organized as follows. In Section 2 we prove Conjecture 1.1, which is the second part of the Theorem A, and we also prove some results about the behavior of the invariant  $n(M)$  under base change. In Section 3 we extend the classical version of Lech's inequality and prove the first part of Theorem A. In Section 4 we give many applications of Theorem A, prove Theorem B, and obtain an alternative proof of Conjecture 1.1. In the Appendix, we establish global versions of results in Section 4.

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## 2. FINITENESS OF $n(M)$ : RESOLVING THE STÜCKRAD–VOGEL CONJECTURE

To prove the Stückrad–Vogel conjecture, we need the concept of extended degree of a finitely generated module introduced by Vasconcelos in [22, 23].

**Definition 2.1.** Let  $(R, \mathfrak{m})$  be a Noetherian local ring with infinite residue field. Let  $\mathcal{M}(R)$  denote the category of finitely generated  $R$ -modules. An *extended degree* on  $\mathcal{M}(R)$  with respect to an  $\mathfrak{m}$ -primary ideal  $I$  is a numerical function

$$\text{Deg}(I, \bullet): \mathcal{M}(R) \rightarrow \mathbb{R}$$

satisfying the following conditions:

- (1)  $\text{Deg}(I, M) = \text{Deg}(I, \overline{M}) + l(H_{\mathfrak{m}}^0(M))$ , where  $\overline{M} = M/H_{\mathfrak{m}}^0(M)$ ;
- (2)  $\text{Deg}(I, M) \geq \text{Deg}(I, M/xM)$  for every generic element  $x \in I - \mathfrak{m}I$  of  $M$ ;
- (3) If  $M$  is Cohen-Macaulay then  $\text{Deg}(I, M) = e(I, M)$ .

The original definition in [23] only deals with the case  $I = \mathfrak{m}$ . The above definition was taken from [5, Definition 5.3]. The first question is whether, given a Noetherian local ring  $(R, \mathfrak{m})$ , an extended degree function exists. This question was settled in the affirmative by Vasconcelos ([22, 23]), who showed that *homological degree* is an example of extended degree (when the residue field is infinite).<sup>2</sup>

**Definition 2.2.** Let  $(R, \mathfrak{m})$  be a homomorphic image of a Gorenstein local ring  $(S, \mathfrak{n})$  of dimension  $n$ , and let  $M$  be a finitely generated  $R$ -module of dimension  $d$ . Then the *homological degree*,  $\text{hdeg}(I, M)$ , of  $M$  with respect to an  $\mathfrak{m}$ -primary ideal  $I$  is defined by the following recursive formula:

$$\text{hdeg}(I, M) = e(I, M) + \sum_{i=n-d+1}^n \binom{d-1}{i-n+d-1} \text{hdeg}(I, \text{Ext}_S^i(M, S)).$$

We note that the above definition is recursive on dimension since  $\dim \text{Ext}_S^i(M, S) \leq n-i < d = \dim M$  for all  $i = n-d+1, \dots, n$ . For a long time, the homological degree was the only known explicit example of an extended degree. Quite recently in [5], Cuong and the third author discovered another extended degree, this one defined in terms of the *Cohen-Macaulay deviated sequence*  $\{U_i(M)\}_{i=0}^{d-1}$  of  $M$ . Roughly speaking,  $U_i(M)$  is the unmixed component of  $M/(x_{i+2}, \dots, x_d)M$  for a certain carefully chosen system of parameters  $x_1, \dots, x_d$  of  $M$ . It is shown in [5, Theorem 4.4] that this unmixed component is independent of the choice of  $x_1, \dots, x_d$  as long as  $x_1, \dots, x_d$  is a *C-system of parameters of  $M$* , which always exists when  $R$  is a homomorphic image of a Cohen-Macaulay local ring. Thus,  $\{U_i(M)\}_{i=0}^{d-1}$  is a sequence of finitely generated  $R$ -modules depending only on  $M$ . Note that  $U_{d-1}(M)$  is just the unmixed component of  $M$ . We refer to [5, Section 4] for more details.

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<sup>2</sup>Here again, Vasconcelos's papers [22, 23] focus on the case  $I = \mathfrak{m}$ , and in fact the main case Vasconcelos considered is the graded case. However the proofs in [22, 23] work in the general set up, and we refer to [5] for more details.

**Definition 2.3.** Let  $(R, \mathfrak{m})$  be a homomorphic image of a Cohen-Macaulay local ring, let  $M$  be a finitely generated  $R$ -module of dimension  $d$ , and let  $U_i(M)$ ,  $0 \leq i \leq d-1$ , be the Cohen-Macaulay deviated sequence of  $M$ . We define the *unmixed degree* of  $M$  with respect to an  $\mathfrak{m}$ -primary ideal  $I$ , denoted  $\text{udeg}(I, M)$ , as follows:

$$\text{udeg}(I, M) = e(I, M) + \sum_{i=0}^{d-1} \delta_{i, \dim U_i(M)} e(I, U_i(M)).$$

It is shown in [5, Theorem 5.18] that  $\text{udeg}(I, \bullet)$  is an extended degree (when the residue field is infinite). We make an elementary but important observation that, for a fixed finitely generated module  $M$ ,  $\text{hdeg}(I, M)$  (resp.  $\text{udeg}(I, M)$ ) is a *finite* sum  $\sum_i e(I, M_i)$ , where  $\{M_i\}$  only depends on  $M$ : this is clear from the definition for  $\text{udeg}(I, M)$  and is easily seen by induction for  $\text{hdeg}(I, M)$ . Therefore, by the associativity formula for multiplicities, for a fixed finitely generated  $R$ -module  $M$ , there exists a *finite* collection of prime ideals  $\Lambda(M) = \Lambda \subseteq \text{Supp}(M)$  (allowing repetition) such that

$$(2.1) \quad \text{hdeg}(I, M) = \sum_{P \in \Lambda} e(I, R/P), \text{ and similarly for } \text{udeg}(I, M).$$

Now we are ready to state and prove our main result in this section. We recall that a finitely generated  $R$ -module  $M$  is called *quasi-unmixed* if  $\widehat{M}$  is equidimensional. This is equivalent to the condition that  $\widehat{R}$  be equidimensional where  $\overline{R} = R/\text{Ann}(M)$ .

**Theorem 2.4.** *Let  $(R, \mathfrak{m})$  be a Noetherian local ring, and let  $M$  be a finitely generated quasi-unmixed  $R$ -module. Then we have*

$$n(M) = \sup_{\sqrt{I + \text{Ann}(M)} = \mathfrak{m}} \left\{ \frac{l(M/IM)}{e(I, M)} \right\} < \infty.$$

*Proof.* By passing to the  $\mathfrak{m}$ -adic completion, we can assume that  $R$  is a complete local ring and  $M$  is equidimensional. We can assume also that the residue field is infinity. We now consider  $\text{Deg}(I, M) = \text{hdeg}(I, M)$  (or  $\text{Deg}(I, M) = \text{udeg}(I, M)$ ), which is an extended degree. Thus, by Definition 2.1 (2) we know that, for every generic element  $x \in I - \mathfrak{m}I$  of  $M$ , we have

$$\text{Deg}(I, M) \geq \text{Deg}(I, M/xM).$$

Therefore, for a generic sequence of elements  $x_1, \dots, x_d$  of  $M$  (we may choose  $x_i$  sufficiently general such that  $x_1, \dots, x_d$  is a system of parameters of  $M$ ), we have

$$\text{Deg}(I, M) \geq \text{Deg}(I, M/x_1M) \geq \dots \geq \text{Deg}(I, M/(x_1, \dots, x_d)M) = l(M/(x_1, \dots, x_d)M) \geq l(M/IM),$$

where the equality is because  $M/(x_1, \dots, x_d)M$  is Cohen-Macaulay and, thus

$$\text{Deg}(I, M/(x_1, \dots, x_d)M) = e(I, M/(x_1, \dots, x_d)M) = l(M/(x_1, \dots, x_d)M).$$

Thus, it is enough to prove that

$$\sup_{\sqrt{I} = \mathfrak{m}} \left\{ \frac{\text{Deg}(I, M)}{e(I, M)} \right\} < \infty.$$

At this point we invoke (2.1): it is enough to prove that, for every  $P \in \text{Supp}(M)$ ,

$$(2.2) \quad \sup_{\sqrt{I} = \mathfrak{m}} \left\{ \frac{e(I, R/P)}{e(I, M)} \right\} < \infty.$$

In order to prove (2.3), we use induction on  $\dim M$ . If  $\dim M = 0$ , (2.3) is obvious. In the general case, if  $\dim R/P = \dim M$  then  $e(I, R/P) \leq e(I, M)$  by the associativity of multiplicities, so (2.3) is again obvious. Now we assume  $\dim R/P < \dim R$ . We choose a prime ideal  $P_0 \in \text{Ass } M$  such that  $\dim R/P_0 = \dim M$  and  $P_0 \subseteq P$ . We have  $e(I, R/P_0) \leq e(I, M)$  by the associativity of multiplicities again. Therefore, it is enough to prove that, for every  $P \in \text{Spec } R$ ,

$$(2.3) \quad \sup_{\sqrt{I}=\mathfrak{m}} \left\{ \frac{e(I, R/P)}{e(I, R)} \right\} < \infty,$$

where  $R$  is a complete local domain and  $\dim R/P < \dim R$ . We pick  $0 \neq x \in P$  and a minimal prime  $Q$  of  $(x)$  such that  $Q \subseteq P$ . Since  $R$  is a complete local domain,  $R/(x)$  is equidimensional; in particular,  $\dim R/(x) = \dim R/Q$ , and, thus,  $e(I, R/(x)) \geq e(I, R/Q)$ . Now we write

$$\frac{e(I, R/P)}{e(I, R)} = \frac{e(I, R/P)}{e(I, R/Q)} \cdot \frac{e(I, R/Q)}{e(I, R/(x))} \cdot \frac{e(I, R/(x))}{e(I, R)} \leq \frac{e(I, R/P)}{e(I, R/Q)} \cdot \frac{e(I, R/(x))}{e(I, R)}.$$

Since  $\dim R/Q < \dim R$ ,  $\sup_{\sqrt{I}=\mathfrak{m}} \left\{ \frac{e(I, R/P)}{e(I, R/Q)} \right\} < \infty$  by induction, which means there exists a constant  $c_1$  such that  $\frac{e(I, R/P)}{e(I, R/Q)} \leq c_1$  for all  $\mathfrak{m}$ -primary ideals  $I$ . Since  $x$  is a nonzerodivisor in a complete local ring  $R$ , by Lemma 2.5 below, we know that there exists a constant  $c_2$  such that  $\frac{e(I, R/(x))}{e(I, R)} \leq c_2$  for all  $\mathfrak{m}$ -primary ideals  $I$ . Thus, putting  $c = c_1 c_2$  we see that

$$\frac{e(I, R/P)}{e(I, R)} \leq c$$

for all  $\mathfrak{m}$ -primary ideals  $I$ . This finishes the proof.  $\square$

**Lemma 2.5.** *Let  $(R, \mathfrak{m})$  be a Noetherian complete local ring, and let  $x$  be a nonzerodivisor on  $R$ . Then there exists a constant  $k$  such that, for all  $\mathfrak{m}$ -primary ideals  $I$ , we have*

$$e(I, R/(x)) \leq k \cdot e(I, R).$$

*Proof.* We consider the short exact sequence:

$$0 \rightarrow \frac{R}{I^n : x} \xrightarrow{\cdot x} \frac{R}{I^n} \rightarrow \frac{R}{I^n + (x)} \rightarrow 0$$

Note that if  $y \in I^n : x$ , then  $xy \in I^n \cap (x)$ . By Huneke's uniform Artin-Rees lemma [8, Theorem 4.12], there exists a constant  $k$  such that, for all  $I \subseteq R$ ,  $I^n \cap (x) \subseteq I^{n-k}x$ . Thus,  $xy \in I^{n-k}x$ , and so  $y \in I^{n-k}$  since  $x$  is a nonzerodivisor. This shows that  $I^n : x \subseteq I^{n-k}$  for all  $\mathfrak{m}$ -primary ideals  $I$ . By the short exact sequence above, we know that

$$l\left(\frac{R}{I^n + (x)}\right) \leq l\left(\frac{R}{I^n}\right) - l\left(\frac{R}{I^{n-k}}\right)$$

Now we let  $n \rightarrow \infty$  and compute the corresponding Hilbert function to see that

$$e(I, R/(x)) \leq k \cdot e(I, R)$$

for all  $\mathfrak{m}$ -primary ideals  $I$ .  $\square$

**Remark 2.6.** Assume that  $R$  is a (not necessarily local) Noetherian ring such that  $R$  is a homomorphic image of a Noetherian Gorenstein ring  $S$  with  $\dim(S) < \infty$ . Further assume that  $M$  is a finitely generated  $R$ -module such that  $R/\text{Ann}(M)$  is locally equidimensional and satisfies the uniform Artin-Rees property (e.g., [8, Theorem 4.12]) and that, for all  $P \in \text{Supp}(M)$ , the residue field of  $R_P$  is infinite. Then the proof of Theorem 2.4 actually gives us a (global) upper bound of  $n(M_P)$  for all  $P \in \text{Supp}(M)$ . Also see Remark A.11 for an alternative treatment.

Given Theorem 2.4, it is quite natural to ask whether the supremum is actually a maximum, i.e., whether  $n(M)$  is attained at some  $I$ . We do not know the answer to this question. Below we prove a special case. Recall that a finitely generated  $R$ -module  $M$  is called *generalized Cohen-Macaulay* if  $H_{\mathfrak{m}}^i(M)$  has finite length for all  $i < \dim(M)$ .

**Theorem 2.7.** *Let  $(R, \mathfrak{m})$  be a Noetherian local ring with infinite residue field, and let  $M$  be a finitely generated  $R$ -module. If  $M$  is generalized Cohen-Macaulay (e.g.,  $\dim(M) = 1$ ), then  $n(M)$  is attained, i.e.,  $n(M) = l(M/IM)/e(I, M)$  for some  $\mathfrak{m}$ -primary ideal  $I$ .*

*Proof.* Since the residue field of  $R$  is infinite, every  $\mathfrak{m}$ -primary ideal  $I$  has a minimal reduction  $(\underline{x})$  generated by a system of parameters of  $M$ . Because  $\frac{l(M/IM)}{e(I, M)} \leq \frac{l(M/(\underline{x})M)}{e((\underline{x}), M)}$  for any minimal reduction  $(\underline{x})$  of  $I$ , we have

$$n(M) = \sup \left\{ \frac{l(M/(\underline{x})M)}{e((\underline{x}), M)} \mid \underline{x} \text{ is a system of parameter of } M \right\}.$$

When  $M$  is Cohen-Macaulay, it is easy to see that  $n(M) = 1 = \frac{l(M/(\underline{x})M)}{e((\underline{x}), M)}$  for any ideal  $(\underline{x})$  generated by a system of parameters of  $M$ . Thus, we assume that  $M$  is not Cohen-Macaulay; hence,  $n(M) > 1 + \varepsilon$  for some  $\varepsilon > 0$ .

Since  $M$  is generalized Cohen-Macaulay, it is well known that there exists  $C > 0$  (e.g.,  $C = \sum_{i=0}^{d-1} \binom{d-1}{i} l(H_{\mathfrak{m}}^i(M))$  [19, Theorem 3.18]) such that

$$l(M/(\underline{x})M) \leq e((\underline{x}), M) + C \quad \text{hence} \quad \frac{l(M/(\underline{x})M)}{e((\underline{x}), M)} \leq 1 + \frac{C}{e((\underline{x}), M)}$$

for all systems of parameters  $\underline{x}$  of  $M$  (for example see [18] or [20]). This shows that  $n(M) \leq 1 + \frac{C}{e(M)} < \infty$ . There exists a positive integer  $N$  such that  $1 + \frac{C}{n} < 1 + \varepsilon$  for all  $n > N$ . Therefore

$$n(M) = \sup \left\{ \frac{l(M/(\underline{x})M)}{e((\underline{x}), M)} \mid \underline{x} \text{ is a system of parameters of } M \text{ such that } e((\underline{x}), M) \leq N \right\}.$$

However, the set of numbers on the right side is finite, so  $n(M)$  is must be attained at some system of parameters of  $M$ .  $\square$

**2.1. The behavior of  $n(M)$  under base change.** In this subsection we study the behavior of  $n(M)$  under localization, flat local extension, and the killing of a parameter. We begin with a result on localization.

**Theorem 2.8.** *Let  $(R, \mathfrak{m})$  be a Noetherian local ring and  $M$  a finitely generated  $R$ -module. Then for any  $P \in \text{Supp}(M)$ , we have  $n_{R_P}(M_P) \leq n_R(M)$ .*

*Proof.* We can assume  $M$  is quasi-unmixed since otherwise  $n(M) = \infty$  by [20, Theorem 1], in which case there is nothing to prove. We can replace  $R$  by  $R/\text{Ann}(M)$ : this does not affect  $n(M)$  or  $n(M_P)$ . Therefore, we can assume  $R$  is quasi-unmixed. By [15, Theorem 31.6], this implies  $R$  is equidimensional and catenary and that  $R_P$  is also quasi-unmixed. We set  $\dim(R) = d$ .

By induction it is enough to consider the case of  $\dim(R/P) = 1$ . Since the residue field of  $R_P$  is infinite, it suffices to show that

$$n(M) \geq \frac{l_{R_P}(M_P/IM_P)}{e(I, M_P)}$$

for all ideals  $I$  generated by a system of parameters in  $R_P$  (as in the proof of Theorem 2.7). We know  $\dim R_P = \text{ht } P = d - 1$  since  $R$  is equidimensional and catenary; thus, for any such  $I$ , by prime avoidance, we can find elements  $x_1, \dots, x_{d-1} \in R$  that form part of a system of



parameters in  $R$  and that have images in  $R_P$  that generate  $I$ . So, abusing notation a bit, we will call  $I = (x_1, \dots, x_{d-1}) \subseteq R$ .

Suppose  $x \in R$  is such that  $(I, x)$  is  $\mathfrak{m}$ -primary. Since  $M$  is faithful,  $x_1, \dots, x_{d-1}, x$  form a system of parameters on both  $R$  and  $M$ . We have

$$l(M/(I, x)M) \geq e(x, M/IM) = \sum_{Q \in \text{Min}(M/IM)} e(x, R/Q)l(M_Q/IM_Q),$$

where the equality holds by the additivity property of multiplicity. By Lech's associativity formula for multiplicities for parameter ideals [11] (see also [17]), we also have

$$e((I, x), M) = \sum_{Q \in \text{Min}(M/IM)} e(x, R/Q)e(IR_Q, M_Q).$$

Therefore, by definition,

$$n(M) \geq \frac{l(M/(I, x)M)}{e((I, x), M)} \geq \frac{\sum_{Q \in \text{Min}(M/IM)} e(x, R/Q)l_{R_Q}(M_Q/IM_Q)}{\sum_{Q \in \text{Min}(M/IM)} e(x, R/Q)e(IR_Q, M_Q)}.$$

Now we let  $y \in \bigcap_{\substack{Q \in \text{Min}(M/IM) \\ Q \neq P}} Q \setminus P$  and  $z \in P \setminus \bigcup_{\substack{Q \in \text{Min}(M/IM) \\ Q \neq P}} Q$ . Observe that, for any  $t \geq 1$ , we can

use  $x = y^t + z$  to complete  $I$  to a full system of parameters. In this case

$$\frac{\sum_Q e(x, R/Q)l_{R_Q}(M_Q/IM_Q)}{\sum_Q e(x, R/Q)e(IR_Q, M_Q)} = \frac{e(y^t, R/P)l_{R_P}(M_P/IM_P) + \sum_{Q \neq P} e(z, R/Q)l_{R_Q}(M_Q/IM_Q)}{e(y^t, R/P)e(IR_P, M_P) + \sum_{Q \neq P} e(z, R/Q)e(IR_Q, M_Q)}.$$

Since  $e(y^t, R/P) = te(y, R/P)$ , if we pass to the limit as  $t$  approaches infinity we obtain

$$n(M) \geq \frac{l_{R_P}(M_P/IM_P)}{e(IR_P, M_P)}. \quad \square$$

As a consequence, we show that the invariant  $n(-)$  is non-decreasing under flat local extensions.

**Corollary 2.9.** *Let  $(R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$  be a flat local extension of Noetherian local rings. Suppose  $M$  is a finitely generated  $R$ -module. Then  $n_R(M) \leq n_S(M \otimes_R S)$ .*

*Proof.* Let  $P$  be a minimal prime of  $\mathfrak{m}S$ . By Theorem 2.8, we have

$$n_{S_P}((M \otimes_R S)_P) \leq n_S(M \otimes_R S).$$

Thus, replacing  $S$  by  $S_P$ , we may assume that  $S$  is local and that  $\mathfrak{m}S$  is  $\mathfrak{n}$ -primary. For any  $\mathfrak{m}$ -primary ideal  $I$ , its extension  $IS$  is an  $\mathfrak{n}$ -primary ideal, and tensoring the composition series with  $S$  shows that

$$l_R(M/IM)l_S(S/\mathfrak{m}S) = l_S((M \otimes_R S)/I(M \otimes_R S))$$

for any finitely generated  $R$ -module  $M$ . Thus,  $e(I, M)l_S(S/\mathfrak{m}S) = e(IS, M \otimes_R S)$  and

$$\begin{aligned} n(M) &= \sup_{\sqrt{I+\text{Ann}(M)}=\mathfrak{m}} \left\{ \frac{l_R(M/IM)}{e(I, M)} \right\} \\ &= \sup_{\sqrt{I+\text{Ann}(M)}=\mathfrak{m}} \left\{ \frac{l_S((M \otimes_R S)/I(M \otimes_R S))}{e(IS, M \otimes_R S)} \right\} \\ &\leq \sup_{\sqrt{IS+\text{Ann}(M)S}=\mathfrak{n}} \left\{ \frac{l_S((M \otimes_R S)/I(M \otimes_R S))}{e(IS, M \otimes_R S)} \right\} = n(M \otimes_R S). \quad \square \end{aligned}$$

Given a flat local extension  $(R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$  and a finitely generated  $R$ -module  $M$ , it would also be interesting to bound  $n(M \otimes_R S)$  in terms of  $n(M)$  and  $n(S/\mathfrak{m}S)$ . We do not know how to obtain such a relation yet. Our last result in this subsection relates  $n(M)$  and  $n(M/xM)$  for a parameter  $x$  on  $M$  (i.e.,  $\dim(M/xM) = \dim(M) - 1$  or, equivalently,  $x$  is a parameter on  $R/\text{Ann}(M)$ ).

**Proposition 2.10.** *Let  $(R, \mathfrak{m})$  be a Noetherian local ring, and let  $M$  be a finitely generated  $R$ -module of dimension  $d$ . Then for any parameter  $x$  of  $M$ , we have  $n(M/xM) \leq n(M)$ .*

*Proof.* Replacing  $R$  by  $R/\text{Ann}(M)$ , we may assume  $M$  is a faithful  $R$ -module. Hence  $x$  is a parameter on  $R$  as well. Since  $e(I, M/xM) \geq e(I, M)$  for every  $\mathfrak{m}$ -primary ideal  $I$  that contains  $x$ , we have

$$\frac{l((M/xM)/I(M/xM))}{e(I, M/xM)} = \frac{l(M/IM)}{e(I, M/xM)} \leq \frac{l(M/IM)}{e(I, M)} \leq n(M).$$

This clearly implies  $n(M/xM) \leq n(M)$ , as desired.  $\square$

### 3. THE LOWER BOUND: A GENERALIZATION OF LECH'S INEQUALITY

Our goal in this section is to generalize Lech's inequality to all finitely generated  $R$ -modules, thus proving the first part of Theorem A in the introduction. We first prove a key lemma.

**Lemma 3.1.** *Let  $(R, \mathfrak{m}, k)$  be a complete local domain with an algebraically closed residue field. Let  $M$  be a finitely generated  $R$ -module with  $\dim(R) = \dim(M)$ , and let  $J$  be an integrally closed  $\mathfrak{m}$ -primary ideal. Then we have*

$$l(M/JM) \geq l(R/J) \dim_K(M \otimes_R K),$$

where  $K$  denotes the fraction field of  $R$ .

*Proof.* First of all, if we let  $T(M)$  denote the torsion submodule of  $M$ , then we have

$$0 \rightarrow T(M) \rightarrow M \rightarrow M' \rightarrow 0$$

where  $M'$  is torsion-free. Since  $l(M/JM) \geq l(M'/JM')$  while  $\dim_K(M \otimes_R K) = \dim_K(M' \otimes_R K)$ , if the lemma holds for  $M'$  then it also holds for  $M$ . Thus, in the rest of the proof we assume  $M$  is torsion-free. In this case  $\dim_K(M \otimes_R K) = \text{rank } M$ .

By [4, Corollary 2.2], we have

$$l(M/JM) \geq \bar{l}(R/J) \cdot \text{rank } M,$$

where  $\bar{l}(R/J)$  denotes the length of the longest chain of integrally closed ideals between  $J$  and  $R$ . Therefore, it is enough to show  $\bar{l}(R/J) = l(R/J)$ . To prove this it is enough to find an integrally closed ideal  $J' \supseteq J$  in  $R$  such that  $l(J'/J) = 1$  because then  $\bar{l}(R/J) = l(R/J)$  follows from an easy induction. Let  $R \rightarrow S$  be the normalization of  $R$ . Since  $R$  is a complete local domain,  $S$  is local by [10, Proposition 4.8.2], and so  $S = (S, \mathfrak{n})$  is a normal local domain with  $R/\mathfrak{m} = S/\mathfrak{n} = k$  since  $k$  is algebraically closed. Now by [24, Theorem 2.1], there exists a chain

$$\overline{JS} = J_0 \subseteq J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n = \mathfrak{n}$$

such that

- (1) Each  $J_i$  is integrally closed in  $S$ ;
- (2)  $l(J_{i+1}/J_i) = 1$  for every  $i$ .



Since  $J$  is integrally closed in  $R$  and  $S$  is integral over  $R$ , by [10, Proposition 1.6.1] we know

$$J_0 \cap R = \overline{JS} \cap R = \bar{J} = J.$$

Let  $t = \max\{i \mid J_i \cap R = J\}$ . Obviously  $0 \leq t < n$ . Set  $J' = J_{t+1} \cap R$ . It is easy to see that  $J' \supseteq J$  is integrally closed in  $R$  (one can use [10, Proposition 1.6.1] again). Moreover,  $l(J'/J) > 0$  by our choice of  $t$  while  $J'/J \hookrightarrow J_{t+1}/J_t$  shows that  $l(J'/J) \leq l(J_{t+1}/J_t) = 1$ . Thus, we have  $l(J'/J) = 1$ .  $\square$

We define  $\text{Assh}(M) = \{P \in \text{Ass}(M) \mid \dim(R/P) = \dim(M)\}$  for a finitely generated  $R$ -module  $M$ . We are now ready to state and prove the following generalization of Lech's inequality.

**Theorem 3.2.** *Let  $(R, \mathfrak{m}, k)$  be a Noetherian local ring, and let  $M$  be a finitely generated  $R$ -module of dimension  $d$ . Then for every ideal  $I$  of  $R$  whose image in  $\bar{R} = R/\text{Ann}(M)$  is  $\mathfrak{m}$ -primary, we have*

$$e(I, M) \leq d!e(\bar{R})l(M/IM).$$

*Proof.* Replacing  $R$  by  $\bar{R}$  does not change either side of the inequality. Therefore, we may assume  $\text{Ann}(M) = 0$  and so that  $\dim(R) = \dim(M) = d$ . We next take a flat local extension  $(R, \mathfrak{m}, k) \rightarrow (R', \mathfrak{m}', k')$  such that  $\mathfrak{m}' = \mathfrak{m}R'$  and  $k' = R'/\mathfrak{m}'$  is the algebraic closure of  $R/\mathfrak{m} = k$ . (Such an  $R'$  always exists: it is a suitable gonflement of  $R$ ; see [3, Corollaire in Appendix 2]). Then  $R \rightarrow R' \rightarrow \widehat{R'}$  is a faithfully flat extension with  $\mathfrak{m}\widehat{R'} = \mathfrak{m}_{\widehat{R'}}$ , so passing from  $R$  to  $\widehat{R'}$  and replacing  $M$  by  $M \otimes_R \widehat{R'}$  do not affect either side of the inequality. Therefore, without loss of generality, we may assume  $(R, \mathfrak{m}, k)$  is a complete local ring with  $k = \bar{k}$  and  $\text{Assh}(R) = \text{Assh}(M)$ .

By the associativity formula of multiplicity, we have

$$e(I, M) = \sum_{P \in \text{Assh}(M)} l_{R_P}(M_P)e(I, R/P) = \sum_{P \in \text{Assh}(R)} l_{R_P}(M_P)e(\overline{IR/P}, R/P).$$

Using Lech's inequality [12, Theorem 3] for each  $R/P$ , we have

$$(3.1) \quad e(I, M) \leq \sum_{P \in \text{Assh}(R)} d!e(R/P)l\left((R/P)/(\overline{IR/P})\right)l_{R_P}(M_P).$$

**Claim 3.3.** *For every minimal prime  $P$  of  $R$ , we have*

$$(3.2) \quad l\left((R/P)/(\overline{IR/P})\right) \cdot l_{R_P}(M_P) \leq l(M/IM) \cdot l_{R_P}(R_P).$$

*Proof of Claim.* Clearly we have  $l_{R_P}(M_P) \leq l_{R_P}(R_P) \cdot l_{R_P}(M_P/PM_P)$  because  $l_{R_P}(M_P/PM_P)$  is the minimal number of generators of  $M_P$  as an  $R_P$ -module. Therefore,

$$l\left((R/P)/(\overline{IR/P})\right) \cdot l_{R_P}(M_P) \leq l\left((R/P)/(\overline{IR/P})\right) \cdot l_{R_P}(M_P/PM_P) \cdot l_{R_P}(R_P).$$

Now  $R/P$  is a complete local domain with algebraically closed residue field  $k = \bar{k}$ , and  $M/PM$  is a finitely generated  $R/P$ -module. Applying Lemma 3.1 and noting that  $\dim_{\kappa(P)}(M/PM) \otimes \kappa(P) = l_{R_P}(M_P/PM_P)$ , we have

$$l\left((R/P)/(\overline{IR/P})\right) \cdot l_{R_P}(M_P/PM_P) \leq l\left(\frac{M/PM}{(\overline{IR/P})(M/PM)}\right) \leq l\left(\frac{M}{(I+P)M}\right) \leq l(M/IM).$$

Putting the two inequalities above together, we get

$$l\left((R/P)/(\overline{IR/P})\right) \cdot l_{R_P}(M_P) \leq l(M/IM) \cdot l_{R_P}(R_P).$$

This finishes the proof of the Claim.  $\square$

Finally, we plug in (3.2) to (3.1) and use additivity of multiplicity to get

$$\begin{aligned} e(I, M) &\leq \sum_{P \in \text{Assh}(R)} d!e(R/P)l_{R_P}(R_P)l(M/IM) \\ &= dl(M/IM) \left( \sum_{P \in \text{Assh}(R)} l_{R_P}(R_P)e(R/P) \right) = d!e(R)l(M/IM). \end{aligned}$$

This finishes the proof.  $\square$

#### 4. APPLICATIONS AND AN ALTERNATIVE APPROACH

In this section we give some applications of our results in Section 2 and Section 3. In the process, we obtain another way to prove Conjecture 1.1 without using Vasconcelos's extended degree (see Remark 4.12).

**Lemma 4.1.** *If  $(R, \mathfrak{m})$  is a Noetherian local ring and  $M$  is a finitely generated quasi-unmixed  $R$ -module of dimension  $d$ , then there exists a constant  $C$  such that for every  $h \in \mathbb{R}$*

$$\frac{h}{C} \leq \inf_{\substack{\sqrt{I}=\sqrt{J}=\mathfrak{m} \\ e(I, M) \geq he(J, M)}} \left\{ \frac{l(M/IM)}{l(M/JM)} \right\} \quad \text{and} \quad \sup_{\substack{\sqrt{I}=\sqrt{J}=\mathfrak{m} \\ e(I, M) \leq he(J, M)}} \left\{ \frac{l(M/IM)}{l(M/JM)} \right\} \leq hC.$$

*In particular, there exists a constant  $C$  such that for all  $\mathfrak{m}$ -primary ideals  $I$  and for all  $n \geq 1$  we have*

$$\frac{n^d}{C} \leq \frac{l(M/I^n M)}{l(M/IM)} \leq Cn^d.$$

*Proof.* Let  $\dim(M) = \dim(\overline{R}) = d$  where  $\overline{R} = R/\text{Ann}(M)$ . We use Theorem 2.4 and Theorem 3.2 to see

$$l(M/IM) \geq e(I, M)/(d!e(\overline{R})) \geq he(J, M)/(d!e(\overline{R})) \geq hl(M/JM)/(n(M)d!e(\overline{R})),$$

which proves the first inequality. By symmetry, we have

$$l(M/IM) \leq n(M)e(I, M) \leq n(M)he(J, M) \leq hn(M)d!e(\overline{R})l(M/JM),$$

which proves the second inequality. So we can take  $C = n(M)d!e(\overline{R})$  in both cases. Finally, taking  $h = n^d$  and noting that  $e(I^n, M) = n^d e(I, M)$  immediately proves the last claim.  $\square$

This lemma has an immediate consequence, which we will need.

**Corollary 4.2.** *If  $(R, \mathfrak{m})$  is a Noetherian local ring and  $M$  is a finitely generated quasi-unmixed  $R$ -module, then*

$$\sup_{\substack{\sqrt{I}=\mathfrak{m} \\ I \subseteq J \subseteq \overline{I}}} \left\{ \frac{l(M/IM)}{l(M/JM)} \right\} < \infty.$$

*Proof.* The condition  $I \subseteq J \subseteq \overline{I}$  implies  $e(I, M) = e(J, M)$ . Therefore we apply Lemma 4.1 with  $h = 1$  to get the desired claim.  $\square$

Next we prove a result that extends Lemma 3.1 at the cost of precision in the inequality.

**Lemma 4.3.** *Let  $(R, \mathfrak{m})$  be a Noetherian local ring and  $N$  be a finitely generated quasi-unmixed  $R$ -module. Then there exists a constant  $C_N > 0$  depending only on  $N$  such that*

$$1/C_N \leq \inf_{\sqrt{I}=\mathfrak{m}} \left\{ \frac{l(M/IM)}{l(N/IN)} \right\} \quad \text{and, equivalently,} \quad \sup_{\sqrt{I}=\mathfrak{m}} \left\{ \frac{l(N/IN)}{l(M/IM)} \right\} \leq C_N$$

for all finitely generated  $R$ -modules  $M$  with  $\text{Supp}(M) \supseteq \text{Supp}(N)$ .

*Proof.* Since  $N$  is quasi-unmixed, it is equidimensional. We set  $c = \max_{P \in \text{Min}(N)} l_{R_P}(N_P)$ . Let  $M$  be any finitely generated  $R$ -module  $M$  with  $\text{Supp}(M) \supseteq \text{Supp}(N)$ . Denote  $\overline{M} = M/\mathfrak{a}M$ , where  $\mathfrak{a} = \cap_{P \in \text{Min}(N)} P$ . Then  $\text{Supp}(\overline{M}) = \text{Supp}(N)$ , and, by the associativity formula for multiplicities,

$$ce(I, \overline{M}) = \sum_{P \in \text{Min}(N)} c \cdot l_{R_P}(\overline{M}_P) e(I, R/P) \geq \sum_{P \in \text{Min}(N)} l_{R_P}(N_P) e(I, R/P) = e(I, N)$$

for all  $\mathfrak{m}$ -primary ideals  $I$ . Now we use Theorem 2.4 and Theorem 3.2 to obtain (with  $d = \dim(N)$ )

$$l(N/IN) \leq n(N)e(I, N) \leq n(N)ce(I, \overline{M}) \leq n(N)cd!e(\overline{R})l(\overline{M}/I\overline{M}) \leq n(N)cd!e(\overline{R})l(M/IM)$$

where  $\overline{R} = R/\mathfrak{a}$  depends only on  $N$ . So we can take  $C_N = n(N)cd!e(\overline{R})$ .  $\square$

Lemma 4.3 allows us to establish the following general result:

**Theorem 4.4.** *Let  $(R, \mathfrak{m})$  be a Noetherian local ring, and let  $M$  and  $N$  denote finitely generated  $R$ -modules. Then*

- (1)  $0 < \inf_{\sqrt{I}=\mathfrak{m}} \left\{ \frac{l(M/IM)}{l(N/IN)} \right\} \iff \sup_{\sqrt{I}=\mathfrak{m}} \left\{ \frac{l(N/IN)}{l(M/IM)} \right\} < \infty \iff \text{Supp}(M) \supseteq \text{Supp}(N)$ .  
(2) *There exists a constant  $C > 0$  depending only on  $N$  such that*

$$1/C \leq \inf_{\sqrt{I}=\mathfrak{m}} \left\{ \frac{l(M/IM)}{l(N/IN)} \right\} \quad \text{and, equivalently,} \quad \sup_{\sqrt{I}=\mathfrak{m}} \left\{ \frac{l(N/IN)}{l(M/IM)} \right\} \leq C$$

for all (finitely generated  $R$ -modules)  $M$  with  $\text{Supp}(M) \supseteq \text{Supp}(N)$ .

- (3)  $0 < \inf_{\sqrt{I}=\mathfrak{m}} \left\{ \frac{l(M/IM)}{l(N/IN)} \right\} \leq \sup_{\sqrt{I}=\mathfrak{m}} \left\{ \frac{l(M/IM)}{l(N/IN)} \right\} < \infty \iff \text{Supp}(M) = \text{Supp}(N)$ .

*Proof.* (1): Clearly, we only need to prove the second equivalence. For the forward direction, assume  $\sup_{\sqrt{I}=\mathfrak{m}} \left\{ \frac{l(N/IN)}{l(M/IM)} \right\} < \infty$  and let  $P \in \text{Supp}(N)$ . Denote  $\overline{M} = M/PM$  and  $\overline{N} = N/PN$ . As  $\left\{ \frac{l(N/IN)}{l(M/IM)} \mid \sqrt{I} = \mathfrak{m} \right\} \supseteq \left\{ \frac{l(\overline{N}/I\overline{N})}{l(\overline{M}/I\overline{M})} \mid \sqrt{I} = \mathfrak{m} \right\}$ , we get  $\sup_{\sqrt{I}=\mathfrak{m}} \left\{ \frac{l(\overline{N}/I\overline{N})}{l(\overline{M}/I\overline{M})} \right\} < \infty$ , which implies  $\dim(\overline{N}) \leq \dim(\overline{M})$  by considering  $\frac{l(\overline{N}/I\overline{N})}{l(\overline{M}/I\overline{M})}$  with  $I = \mathfrak{m}^t$  for  $t \gg 0$ . Note that  $\dim(N/PN) = \dim(R/P)$ , since  $P \in \text{Supp}(N)$ . Thus,  $\dim(M/PM) = \dim(R/P)$ , which forces  $P \in \text{Supp}(M)$ .

For the backward direction, assume  $\text{Supp}(M) \supseteq \text{Supp}(N)$ . We can further assume that  $R$  is complete, which does not affect the statement. We next take a prime cyclic filtration of  $N$  of length  $n$  with factors  $N_i = R/P_i$  such that  $P_i \in \text{Supp}(N)$  for  $i = 1, \dots, n$  (note that the  $P_i$  are not necessarily distinct). As  $l(N/IN) \leq \sum_{i=1}^n l(N_i/IN_i)$  for every  $\mathfrak{m}$ -primary ideal  $I$ , it suffices to show  $\sup_{\sqrt{I}=\mathfrak{m}} \left\{ \frac{l(N_i/IN_i)}{l(M/IM)} \right\} < \infty$  for each  $i = 1, \dots, n$ . But this follows from Lemma 4.3 since each  $N_i = R/P_i$  is quasi-unmixed (since  $R$  is complete) and  $\text{Supp}(M) \supseteq \text{Supp}(N_i)$ . In detail, let  $C_{N_i} > 0$  be as in Lemma 4.3 for each  $i = 1, \dots, n$ . Then

$$\sup_{\sqrt{I}=\mathfrak{m}} \left\{ \frac{l(N/IN)}{l(M/IM)} \right\} \leq \sum_{i=1}^n \sup_{\sqrt{I}=\mathfrak{m}} \left\{ \frac{l(N_i/IN_i)}{l(M/IM)} \right\} \leq \sum_{i=1}^n C_{N_i} < \infty$$

with  $\sum_{i=1}^n C_{N_i} < \infty$  depending only on  $N$ .

(2): From the proof of (1) above (for the backward direction), we can set  $C = \sum_{i=1}^n C_{N_i}$ , which depends only on (the completion of)  $N$ .

(3): This is clear from (1).  $\square$

**Lemma 4.5.** *Let  $(R, \mathfrak{m})$  be a Noetherian local ring and  $M$  a finitely generated  $R$ -module. If  $(y_1, \dots, y_d) \subseteq (x_1, \dots, x_k)$  are  $\mathfrak{m}$ -primary ideals of  $R$ , then for all  $0 \leq i \leq k$ ,*

$$l(H_i(x_1, \dots, x_k; M)) \leq \sum_{j=0}^k \binom{k}{j} l(H_{i-j}(y_1, \dots, y_d; M)) \leq 2^k \max_{0 \leq j \leq k} l(H_{i-j}(y_1, \dots, y_d; M)),$$

with the convention that  $H_{<0}(y_1, \dots, y_d; M) = 0$ .

*Proof.* If  $\underline{f} = f_1, \dots, f_s$  is any sequence of elements of  $R$  and  $\underline{f}^- = f_1, \dots, f_{s-1}$ , then there is a short exact sequence for each  $0 \leq i \leq s-1$

$$0 \rightarrow \frac{H_i(\underline{f}^-; M)}{f_s H_i(\underline{f}^-; M)} \rightarrow H_i(\underline{f}; M) \rightarrow \text{Ann}_{H_{i-1}(\underline{f}^-; M)}(f_s) \rightarrow 0,$$

Using the short exact sequence above, we see that

$$\begin{aligned} l(H_i(x_1, \dots, x_k; M)) &\leq l(H_i(x_1, \dots, x_k; M)) + l(H_{i-1}(x_1, \dots, x_k; M)) \\ &= l(H_i(x_1, \dots, x_k, y_1; M)) \quad (\text{since } y_1 \in (x_1, \dots, x_k)) \\ &\leq \dots \quad (\text{by joining } y_2, \dots, y_d \text{ inductively}) \\ &\leq l(H_i(x_1, \dots, x_k, y_1, \dots, y_d; M)) \\ &= l\left(\frac{H_i(x_1, \dots, x_{k-1}, y_1, \dots, y_d; M)}{x_k H_i(x_1, \dots, x_{k-1}, y_1, \dots, y_d; M)}\right) + l(\text{Ann}_{H_{i-1}(x_1, \dots, x_{k-1}, y_1, \dots, y_d; M)}(x_k)) \\ &\leq l(H_i(x_1, \dots, x_{k-1}, y_1, \dots, y_d; M)) + l(H_{i-1}(x_1, \dots, x_{k-1}, y_1, \dots, y_d; M)) \\ &\leq \dots \quad (\text{by removing } x_{k-1}, \dots, x_1 \text{ inductively}) \\ &\leq \sum_{j=0}^k \binom{k}{j} l(H_{i-j}(y_1, \dots, y_d; M)) \\ &\leq 2^k \max_{0 \leq j \leq k} l(H_{i-j}(y_1, \dots, y_d; M)), \end{aligned}$$

completing the proof.  $\square$

**Remark 4.6** ([19] or [2]). Let  $(R, \mathfrak{m})$  be a local ring of dimension  $d$  that is a homomorphic image of a local Gorenstein ring  $S$  of dimension  $n$ . Then for every finitely generated  $R$ -module  $M$ , every system of parameters  $\underline{x} = x_1, \dots, x_d$  of  $R$ , and every  $i = 1, \dots, d$ , we have

$$l(H_i(x_1, \dots, x_d; M)) \leq \sum_{j=0}^{d-i} l(H_{d-i-j}(x_1, \dots, x_d; \text{Ext}_S^{n-j}(M, S))).^3$$

Note that  $\dim(\text{Ext}_S^{n-j}(M, S)) \leq d-i$  for each  $j = 0, \dots, d-i$ , since  $\text{Ext}_S^{n-j}(M, S)^\vee \cong H_{\mathfrak{m}}^j(M)$  where  $(-)^\vee$  stands for Matlis dual.

<sup>3</sup>This is written down in [2]. We point out that this also follows from [19, Theorem 3.16] as follows: since  $\underline{x}$  is a system of parameters of  $R$ , we can pick  $\underline{y} = y_1, \dots, y_d$  with  $(\underline{y}) = (\underline{x})$  such that they form a strong filter regular sequence on  $R$  and  $M$  by prime avoidance. Replacing  $\underline{x}$  by  $\underline{y}$  does not affect the Koszul homology, so we can assume  $\underline{x}$  is a strong filter regular sequence on  $R$  and  $M$  and then note that we have a canonical isomorphism  $H^j(x_1, \dots, x_d; N^\vee) \cong H_j(x_1, \dots, x_d; N)^\vee$  for all finitely generated  $R$ -modules  $N$  by [19, bottom of page 286]. (In particular, they have the same length.). Therefore, the displayed formula is a restatement of [19, Theorem 3.16].

**Theorem 4.7.** *Let  $(R, \mathfrak{m})$  be a Noetherian local ring and  $M$  a finitely generated  $R$ -module of dimension  $d$ . Then there exists a constant  $C$  depending on  $M$  such that, for every  $k \geq d$ , we have*

$$\sup_{\substack{\sqrt{(x_1, \dots, x_k) + \text{Ann}(M)} = \mathfrak{m} \\ 0 \leq i \leq k}} \left\{ \frac{l(H_i(x_1, \dots, x_k; M))}{l(M/(x_1, \dots, x_k)M)} \right\} \leq 2^k C.$$

*Proof.* As in the first paragraph in the proof of Theorem 3.2, we can replace  $R$  by  $R/\text{Ann}(M)$ , enlarge the residue field of  $R$ , and then complete  $R$ . Therefore, we can assume that  $R$  is a complete local ring with infinite residue field and that  $M$  is a faithful  $R$ -module. (The rest of the proof only relies on the fact that  $(R, \mathfrak{m})$  is a homomorphic image of a Gorenstein ring  $S$  with infinite residue field.)

We proceed by induction on  $d = \dim(M)$ . When  $\dim(M) = 0$ , clearly  $C = l(M)$  works. Assume that the theorem holds for modules of dimension  $< d$ . Now let  $\dim(M) = d$ . Let  $R/P_1, \dots, R/P_r$  be the (not necessarily distinct) factors appearing in a prime cyclic filtration of  $M$ . We note that, for each  $0 \leq i \leq d$ ,

$$\frac{l(H_i(x_1, \dots, x_k; M))}{l(M/(x_1, \dots, x_k)M)} \leq \frac{\sum_{j=1}^r l(H_i(x_1, \dots, x_k; R/P_j))}{l(M/(x_1, \dots, x_k)M)} \leq \sum_{j=1}^r \frac{l(H_i(x_1, \dots, x_k; R/P_j))}{l(M/((x_1, \dots, x_k) + P_j)M)}.$$

Now each  $M/P_j M$  is a finitely generated faithful module over the local domain  $R/P_j$  with infinite residue field. It then follows from Lemma 4.3 or Theorem 4.4 (applied to  $M/P_j M$  and  $R/P_j$ ) that we may replace each term  $l(M/((x_1, \dots, x_k) + P_j)M)$  by  $l(R/((x_1, \dots, x_k) + P_j)R)$  without affecting the claim of the theorem. We have now reduced to the case of  $M = R/P_j$  over the local domain  $R/P_j$  of dimension  $\leq d$ .

Therefore, it suffices to verify the case of  $M = R$  where  $R$  is a domain with  $\dim(R) = d$ . (We still have that  $R$  is a homomorphic image of a Gorenstein local ring  $S$  with infinite residue field.)

Let  $(y_1, \dots, y_d)$  be a minimal reduction of  $(x_1, \dots, x_k)$ . By Corollary 4.2 and Lemma 4.5, it suffices to find a constant  $D$  such that

$$\frac{l(H_i(y_1, \dots, y_d; R))}{l(R/(y_1, \dots, y_d))} \leq D$$

for all systems of parameters  $\underline{y} := y_1, \dots, y_d$  of  $R$  and for all  $i = 0, \dots, d$ . Now by Remark 4.6, it suffices to show that, for any fixed finitely generated  $R$ -module  $L$  with  $\dim(L) < d$ , there exists a constant  $D_L$  such that

$$\frac{l(H_i(y_1, \dots, y_d; L))}{l(R/(y_1, \dots, y_d))} \leq D_L$$

independent of  $\underline{y}$  and  $i$ . Indeed, as  $l(L/(y_1, \dots, y_d)L) \leq \mu(L)l(R/(y_1, \dots, y_d))$ , we have

$$\frac{l(H_i(y_1, \dots, y_d; L))}{l(R/(y_1, \dots, y_d))} \leq \mu(L) \frac{l(H_i(y_1, \dots, y_d; L))}{l(L/(y_1, \dots, y_d)L)}.$$

Since  $\dim(L) < d$ , the right hand side of the above inequality is bounded above (independent of  $\underline{y}$  and  $i$ ) by the inductive hypothesis (noting that  $2^d$  is a constant as well).  $\square$

It is well known that (for example, see [6]) if  $R$  is a complete local domain of characteristic  $p > 0$  and dimension  $d \geq 1$ , then, for every system of parameters  $(x_1, \dots, x_d)$  of  $R$ ,  $\frac{l(H_i(x_1^e, \dots, x_d^e; R))}{l(H_0(x_1^e, \dots, x_d^e; R))} \xrightarrow{e \rightarrow \infty} 0$  for each  $1 \leq i \leq d$ . This classical result is essentially saying that the length of higher Koszul homology modules tends to 0 compared to the length of the 0-th Koszul homology module when we raise any system of parameters to high Frobenius powers. Our final result is a generalization of

this result to a characteristic-free version. More importantly, the convergence to 0 occurs in a way that is independent of the system of parameters!

**Theorem 4.8.** *Let  $R$  be a Noetherian local ring and  $M$  be a finitely generated quasi-unmixed  $R$ -module with  $\dim(M) = d$ . For every  $\varepsilon > 0$ , there exists  $t_0$  such that, for all  $t \geq t_0$ , all systems of parameters  $\underline{x} := x_1, \dots, x_d$  of  $M$ , and all  $1 \leq i \leq d$ ,*

$$\frac{l(H_i(x_1^t, \dots, x_d^t; M))}{l(M/(x_1^t, \dots, x_d^t)M)} < \varepsilon.$$

*In fact, there exists a constant  $K$  such that for all  $t \geq 1$ , all systems of parameters  $\underline{x} := x_1, \dots, x_d$  of  $M$ , and all  $1 \leq i \leq d$ ,*

$$\frac{l(H_i(x_1^t, \dots, x_d^t; M))}{l(M/(x_1^t, \dots, x_d^t)M)} \leq \frac{K}{t^i}.$$

*Proof.* As usual, we replace  $R$  by  $\overline{R} = R/\text{Ann}(M)$  and complete  $R$  to assume that  $R$  is complete and that  $M$  is faithful over  $R$ . Our hypothesis then implies that both  $M$  and  $R$  are equidimensional. (The rest of the proof only relies on the fact that  $M$  is equidimensional and that  $(R, \mathfrak{m})$  is a homomorphic image of a Gorenstein local ring  $S$ .)

Because we consider only finitely many  $i$ , it is sufficient to fix some  $1 \leq i \leq d$ . By Remark 4.6, it suffices to show that, for any fixed finitely generated  $R$ -module  $L$  with  $\dim(L) \leq d - i$  and any fixed  $j = 0, \dots, d - i$ , there exists a constant  $K$  such that, for all  $t \geq 1$  and all  $\underline{x}$ ,

$$\frac{l(H_j(x_1^t, \dots, x_d^t; L))}{l(M/(x_1^t, \dots, x_d^t)M)} \leq \frac{K}{t^i}.$$

By taking a prime cyclic filtration of  $L$ , it suffices to show that, for any fixed  $P \in \text{Spec}(R)$  such that  $\dim(R/P) = d' \leq d - i$ , there exists a constant  $K$  such that, for all  $t \geq 1$  and all  $\underline{x}$ ,

$$\frac{l(H_j(x_1^t, \dots, x_d^t; R/P))}{l(M/(x_1^t, \dots, x_d^t)M)} \leq \frac{K}{t^i}.$$

Denote  $D := R/P$ . By Theorem 4.7, we fix

$$C = \sup_{\sqrt{(x_1, \dots, x_d)} = \mathfrak{m}} \left\{ \frac{l(H_j(x_1, \dots, x_d; D))}{l(D/(x_1, \dots, x_d)D)} \right\} < \infty.$$

According to Theorem 2.4, we let

$$B = \sup_{\sqrt{I} = \mathfrak{m}} \left\{ \frac{l(D/ID)}{e(I, D)} \right\} < \infty.$$

Moreover, as  $M$  is equidimensional, there exists  $Q \in \text{Min}(R) = \text{Min}(M)$  such that  $Q \subsetneq P$  so that  $D$  is a proper homomorphic image of  $R/Q$ , in which case  $\dim(R/Q) = d > d' = \dim(D)$  and  $e(I, R/Q) \leq e(I, M)$  for all  $\mathfrak{m}$ -primary ideals  $I$  (since  $\dim(R/Q) = \dim(M)$ ). In light of Equation (2.3) in the proof of Theorem 2.4, we set

$$A = \sup_{\sqrt{I} = \mathfrak{m}} \left\{ \frac{e(I, D)}{e(I, R/Q)} \right\} < \infty.$$



Finally, we see

$$\begin{aligned}
\frac{l(H_j(x_1^t, \dots, x_d^t; D))}{l(M/(x_1^t, \dots, x_d^t)M)} &\leq \frac{l(H_j(x_1^t, \dots, x_d^t; D))}{e((x_1^t, \dots, x_d^t), M)} \\
&\leq C \frac{l(D/(x_1^t, \dots, x_d^t)D)}{e((x_1^t, \dots, x_d^t), M)} \\
&\leq BC \frac{e((x_1^t, \dots, x_d^t), D)}{e((x_1^t, \dots, x_d^t), M)} \\
&= BC \frac{t^{d'} e((x_1, \dots, x_d), D)}{t^d e((x_1, \dots, x_d), M)} \\
&\leq \frac{BC}{t^{d-d'}} \frac{e((x_1, \dots, x_d), D)}{e((x_1, \dots, x_d), R/Q)} \leq \frac{ABC}{t^{d-d'}} \leq \frac{ABC}{t^i},
\end{aligned}$$

whose convergence to 0, as  $t \rightarrow \infty$ , is independent of systems of parameters  $\underline{x} := x_1, \dots, x_d$ .  $\square$

**Lemma 4.9.** *Let  $R$  be a (Noetherian) ring,  $M$  a finitely generated  $R$ -module,  $\underline{x} := x_1, \dots, x_d$  a sequence of elements of  $R$  such that  $l(M/(\underline{x})M) < \infty$ , and  $t_j \geq s_j \geq 1$  for  $1 \leq j \leq d$ . Then for all  $i$ , we have*

$$l(H_i(x_1^{t_1}, \dots, x_d^{t_d}; M)) \leq l(H_i(x_1^{s_1}, \dots, x_d^{s_d}; M)) \prod_{j=1}^d \frac{t_j}{s_j}$$

*Proof.* By symmetry, it suffices to show  $l(H_i(x_1^t, x_2, \dots, x_d; M)) \leq l(H_i(x_1^s, x_2, \dots, x_d; M)) \frac{t}{s}$  for all  $t \geq s \geq 1$ . For each  $i$ , denote  $H_i = H_i(x_2, \dots, x_d; M)$ . From the exact sequence

$$H_i \xrightarrow{x_1^t} H_i \rightarrow H_i(x_1^t, x_2, \dots, x_d; M) \rightarrow H_{i-1} \xrightarrow{x_1^t} H_{i-1}$$

we see that

$$\begin{aligned}
l(H_i(x_1^t, x_2, \dots, x_d; M)) &= l(H_i/x_1^t H_i) + l((0 :_{H_{i-1}} x_1^t)) \\
&= \sum_{j=1}^t l(x_1^{j-1} H_i / x_1^j H_i) + \sum_{j=1}^t l((0 :_{H_{i-1}} x_1^j) / (0 :_{H_{i-1}} x_1^{j-1})).
\end{aligned}$$

Now, as we can do the above to  $l(H_i(x_1^s, x_2, \dots, x_d; M))$  as well, it suffices to show the sequences  $\{l(x_1^{j-1} H_i / x_1^j H_i)\}_j$  and  $\{l((0 :_{H_{i-1}} x_1^j) / (0 :_{H_{i-1}} x_1^{j-1}))\}_j$  are both non-increasing. But this follows because the following maps induced by multiplication by  $x_1$ :

$$\frac{x_1^{j-1} H_i}{x_1^j H_i} \xrightarrow{x_1} \frac{x_1^j H_i}{x_1^{j+1} H_i} \quad \text{and} \quad \frac{(0 :_{H_{i-1}} x_1^{j+1})}{(0 :_{H_{i-1}} x_1^j)} \xrightarrow{x_1} \frac{(0 :_{H_{i-1}} x_1^j)}{(0 :_{H_{i-1}} x_1^{j-1})},$$

are onto and 1-1 respectively.  $\square$

The next theorem generalizes the uniform convergence established in Theorem 4.8.

**Theorem 4.10.** *Let  $R$  be a Noetherian local ring and  $M$  a finitely generated quasi-unmixed  $R$ -module with  $\dim(M) = d$ . For every  $\varepsilon > 0$ , there exists  $t_0$  such that, for all  $t_j \geq t_0$  with  $1 \leq j \leq d$ , all systems of parameters  $\underline{x} := x_1, \dots, x_d$  of  $M$ , and all  $1 \leq i \leq d$ ,*

$$\frac{l(H_i(x_1^{t_1}, \dots, x_d^{t_d}; M))}{l(M/(x_1^{t_1}, \dots, x_d^{t_d})M)} < \varepsilon.$$

In fact, there exists a constant  $K$  such that for all  $t_j \geq 1$  with  $1 \leq j \leq d$ , all systems of parameters  $\underline{x} := x_1, \dots, x_d$  of  $M$ , and all  $1 \leq i \leq d$ ,

$$\frac{l(H_i(x_1^{t_1}, \dots, x_d^{t_d}; M))}{l(M/(x_1^{t_1}, \dots, x_d^{t_d})M)} \leq \frac{K}{(\min_j t_j)^i}.$$

*Proof.* With  $t_j \geq 1$  for  $1 \leq j \leq d$ , we denote  $t = \min_j t_j$ . Then for all systems of parameters  $\underline{x} := x_1, \dots, x_d$  of  $M$  and all  $1 \leq i \leq d$ , we have

$$\frac{l(H_i(x_1^{t_1}, \dots, x_d^{t_d}; M))}{l(M/(x_1^{t_1}, \dots, x_d^{t_d})M)} \leq \frac{l(H_i(x_1^t, \dots, x_d^t; M))}{l(M/(x_1^t, \dots, x_d^t)M)} \prod_{j=1}^d \frac{t_j}{t} \quad (\text{Lemma 4.9})$$

$$\leq \frac{l(H_i(x_1^t, \dots, x_d^t; M))}{e((x_1^t, \dots, x_d^t), M)} d!e(\bar{R}) \prod_{j=1}^d \frac{t_j}{t} \quad (\text{Theorem 3.2})$$

$$= \frac{l(H_i(x_1^t, \dots, x_d^t; M))}{e((x_1^t, \dots, x_d^t), M)} d!e(\bar{R})$$

$$\leq \frac{l(H_i(x_1^t, \dots, x_d^t; M))}{l(M/(x_1^t, \dots, x_d^t)M)} n(M) d!e(\bar{R}) \quad (\text{Theorem 2.4})$$

in which  $\bar{R} = R/\text{Ann}(M)$ . Now Theorem 4.8 completes the proof.  $\square$

**Remark 4.11.** We would like to mention that, in Theorem 4.8 (hence in Theorem 4.10), the assumption that  $M$  is quasi-unmixed is necessary (at least when  $R$  has infinite residue field). In fact, the conclusion of Theorem 4.8 (i.e., the uniform convergence to 0) for  $M$  implies  $n(M) < \infty$  provided that  $R$  has infinite residue field, which forces  $M$  to be quasi-unmixed by [21, Theorem 1]. For details, the existence of  $t$  such that  $\frac{\sum_{i=1}^d (-1)^{i-1} l(H_i(x_1^t, \dots, x_d^t; M))}{l(M/(x_1^t, \dots, x_d^t)M)} < \varepsilon < 1$  for all systems of parameters  $\underline{x} := x_1, \dots, x_d$  of  $M$  implies

$$\begin{aligned} \frac{e((x_1, \dots, x_d), M)}{l(M/(x_1, \dots, x_d)M)} &= \frac{e((x_1^t, \dots, x_d^t), M)}{t^d l(M/(x_1^t, \dots, x_d^t)M)} \geq \frac{e((x_1^t, \dots, x_d^t), M)}{t^d l(M/(x_1^t, \dots, x_d^t)M)} \\ &\geq \frac{1}{t^d} \left( \frac{l(M/(x_1^t, \dots, x_d^t)M)}{l(M/(x_1^t, \dots, x_d^t)M)} - \frac{\sum_{i=1}^d (-1)^{i-1} l(H_i(x_1^t, \dots, x_d^t; M))}{l(M/(x_1^t, \dots, x_d^t)M)} \right) > \frac{1 - \varepsilon}{t^d} \end{aligned}$$

for all systems of parameters  $\underline{x} := x_1, \dots, x_d$  of  $M$ , which implies  $n(M) < \infty$  (given that  $R$  has infinite residue field).

**Remark 4.12.** Evidently the results of this section rely on Theorem 2.4. However, a careful analysis of the proofs in this section reveals an alternative proof of Theorem 2.4 by induction on the dimension of the quasi-unmixed  $R$ -module  $M$  without explicit usage of homological degree or unmixed degree. Without loss of generality, assume that  $(R, \mathfrak{m})$  is complete with infinite residue field (thus,  $R$  is a homomorphic image of a Gorenstein ring  $S$  with  $\dim(S) = n$ ). When  $\dim(M) = 0$ , it is clear that Theorem 4.8, Theorem 2.4, Lemma 4.1, Corollary 4.2, Lemma 4.3, and Theorem 4.7 all hold. Now assume that *all* these results hold in dimension  $< d$ , and consider the case of dimension  $d$ . Then Theorem 4.8 holds in dimension  $d$  (because the proof of Theorem 4.8 only requires the aforementioned results in dimension  $< d$ ), which implies Theorem 2.4 in dimension  $d$  as we explained in Remark 4.11. Then we have Lemma 4.1, Corollary 4.2, Lemma 4.3 and Theorem 4.7 in dimension  $d$ , completing the induction. Alternatively, Theorem 4.7 and Theorem 2.4 in dimension  $< d$  implies

Theorem 2.4 in dimension  $d$  as follows: It suffice to consider the case of  $M = R$  being a domain. For an arbitrary system of parameters  $\underline{y} := y_1, \dots, y_d$  of  $R$ , we have

$$\frac{l(R/(\underline{y}))}{e((\underline{y}), R)} \leq \frac{e((\underline{y}), R) + l(H_1(\underline{y}; R))}{e((\underline{y}), R)} = 1 + \frac{l(H_1(\underline{y}; R))}{e((\underline{y}), R)}.$$

Similar to the reasoning in the proof of Theorem 4.7 (plus taking prime cyclic filtration), it suffices to consider  $R/P$  with  $0 \neq P \in \text{Spec}(R)$  and to find an upper bound for  $\frac{l(H_i(\underline{y}; R/P))}{e((\underline{y}), R)}$  for all systems of parameters  $\underline{y}$ . By Theorem 4.7 and Theorem 2.4 in dimension  $< d$ , it suffices to find an upper bound for  $\frac{e(\underline{y}, R/P)}{e((\underline{y}), R)}$  for all systems of parameters  $\underline{y}$ . But this is Equation (2.3) in the proof of Theorem 2.4.

Even though the alternative proof sketched above does not use extended degree explicitly, its approach is very similar to that of homological degree: the alternative approach relies on Remark 4.6 to reduce the dimension from  $\dim(M) = d$  to  $\dim(\text{Ext}_S^{n-j}(M, S)) < d$ ,  $0 \leq j < d$ , while the homological degree involves the same modules in its definition.

The following is an easy consequence of Theorem 4.10. It says that for all systems of parameters  $\underline{x} = x_1, \dots, x_d$  on  $M$ , the rate of convergence of  $\frac{l(M/(x_1^{t_1}, \dots, x_d^{t_d})M)}{\prod_{j=1}^d t_j}$  to  $e((\underline{x}), M)$  is uniformly controlled by  $l(M/(\underline{x})M)$  and  $\min_j t_j$  only.

**Corollary 4.13.** *Let  $(R, \mathfrak{m})$  be a Noetherian local ring and  $M$  be a finitely generated quasi-unmixed  $R$ -module. Then for every constant  $C > 0$  and every  $\epsilon > 0$ , there exists  $t_0 \in \mathbb{N}$  such that, for all  $t_j \geq t_0$  with  $1 \leq j \leq d$ , all systems of parameters  $\underline{x} = x_1, \dots, x_d$  on  $M$  such that  $e((\underline{x}), M) \leq C$ , we have*

$$0 \leq \frac{l(M/(x_1^{t_1}, \dots, x_d^{t_d})M)}{\prod_{j=1}^d t_j} - e((\underline{x}), M) < \epsilon.$$

In fact, there exists a constant  $K$  such that, for all  $t_j \geq 1$  with  $1 \leq j \leq d$  and all systems of parameters  $\underline{x} = x_1, \dots, x_d$  on  $M$ , we have

$$0 \leq \frac{l(M/(x_1^{t_1}, \dots, x_d^{t_d})M)}{\prod_{j=1}^d t_j} - e((\underline{x}), M) \leq e((\underline{x}), M) \frac{K}{\min_j t_j} \leq l(M/(\underline{x})M) \frac{K}{\min_j t_j}.$$

*Proof.* For all all systems of parameters  $\underline{x} = x_1, \dots, x_d$  on  $M$ , we have (with  $\underline{x}^{[t]} := x_1^{t_1}, \dots, x_d^{t_d}$ )

$$\begin{aligned} 0 \leq \frac{l(M/(\underline{x}^{[t]})M)}{\prod_{j=1}^d t_j} - e((\underline{x}), M) &= \frac{l(M/(\underline{x}^{[t]})M) - e((\underline{x}^{[t]}), M)}{\prod_{j=1}^d t_j} \\ &= \frac{l(M/(\underline{x}^{[t]})M) \sum_{i=1}^d (-1)^{i-1} l(H_i(\underline{x}^{[t]}; M))}{\prod_{j=1}^d t_j \cdot l(M/(\underline{x}^{[t]})M)} \\ &\leq \frac{e((\underline{x}^{[t]}), M) n(M) \sum_{i=1}^d (-1)^{i-1} l(H_i(\underline{x}^{[t]}; M))}{\prod_{j=1}^d t_j \cdot l(M/(\underline{x}^{[t]})M)} \\ &= e((\underline{x}), M) n(M) \frac{\sum_{i=1}^d (-1)^{i-1} l(H_i(\underline{x}^{[t]}; M))}{l(M/(\underline{x}^{[t]})M)} \\ &\leq e((\underline{x}), M) \left( n(M) \sum_{i=1}^d (-1)^{i-1} \frac{l(H_i(\underline{x}^{[t]}; M))}{l(M/(\underline{x}^{[t]})M)} \right). \end{aligned}$$

Now Theorem 4.10 completes the proof.  $\square$

Finally, we remark that even though  $n(M)$  could tend to  $\infty$  as  $M$  varies (see Example 1.2),  $\frac{n(M)}{\mu(M)}$  has an upper bound that depends only on  $R/\text{Ann}(M)$  (and not on  $M$ ). We would like to thank the referee for pointing out this question.

**Remark 4.14.** Let  $(R, \mathfrak{m})$  be a Noetherian local ring, and let  $M$  be a finitely generated quasi-unmixed  $R$ -module. Set  $\overline{R} = R/\text{Ann}(M)$ , which is quasi-unmixed by assumption and we have  $\text{Assh}(M) = \text{Assh}(\overline{R})$ . Let  $c = \max_{P \in \text{Assh}(\overline{R})} l_{\overline{R}_P}(\overline{R}_P)$ . Clearly  $l(M/IM) \leq \mu(M)l(\overline{R}/I\overline{R})$ , and by the associativity formula for multiplicities  $e(I, \overline{R}) \leq ce(I, M)$ . Now for all  $\mathfrak{m}$ -primary ideals  $I$  we have

$$\frac{l(M/IM)}{e(I, M)\mu(M)} = \frac{l(M/IM)}{\mu(M)l(\overline{R}/I\overline{R})} \cdot \frac{l(\overline{R}/I\overline{R})}{e(I, \overline{R})} \cdot \frac{e(I, \overline{R})}{e(I, M)} \leq cn(\overline{R}).$$

Therefore  $\frac{n(M)}{\mu(M)} \leq cn(\overline{R})$ , and the latter depends only on  $\overline{R} = R/\text{Ann}(M)$ . Also note that if we take  $R = k[[x, y]]$ ,  $I_t = (x) \cap (x, y)^n$ . Then  $R/I_t$  is quasi-unmixed and  $n(R/I_t) \geq t$ : the ideal  $(y)$  has multiplicity 1 and colength  $t$  in  $R/I_t$ . Therefore, in general the  $n(\overline{R})$  (as  $M$  varies) are not bounded in terms of invariants of  $R$ .

#### APPENDIX A. GLOBAL VERSION OF THE RESULTS

In this appendix we briefly explain that our results and methods in Section 4 work globally. Most of the results rely on  $\sup\{e(\overline{R}_P) \mid P \in \text{Spec}(\overline{R})\} < \infty$ , with  $\overline{R} = R/\text{Ann}(M)$ , and rely on the uniform Artin-Rees property (cf. [8]). We observe that  $\sup\{e(\overline{R}_P) \mid P \in \text{Spec}(\overline{R})\} < \infty$  if the regular loci of all quotient domains of  $\overline{R}$  are open (e.g.,  $\overline{R}$  is excellent). By [8, Theorem 4.12], the uniform Artin-Rees property holds for  $R$  (hence holds for all its localizations  $R_P$  with the same constant) if  $R$  is essentially of finite type over a Noetherian local ring or  $\mathbb{Z}$ , or if  $R$  is an F-finite Noetherian ring of prime characteristic  $p$ . We will also use the fact that a homomorphic image of a Cohen-Macaulay local ring is quasi-unmixed if and only if it is equidimensional.

We start with the global version of 4.1–4.4.

**Lemma A.1.** *Let  $R$  be a Noetherian ring that is a homomorphic image of a Noetherian Gorenstein ring  $S$  with  $\dim(S) < \infty$ , and let  $M$  be a finitely generated  $R$ -module such that  $\overline{R} = R/\text{Ann}(M)$  is locally equidimensional and satisfies the uniform Artin-Rees property with  $\sup\{e(\overline{R}_P) \mid P \in \text{Spec}(\overline{R})\} < \infty$ . Suppose the residue field of  $R_P$  is infinite for all  $P \in \text{Supp}(M)$ . Then there exists a constant  $C$  such that, for every  $h \in \mathbb{R}$  and every  $P \in \text{Supp}(M)$ ,*

$$\frac{h}{C} \leq \inf_{\substack{\sqrt{I}=\sqrt{J}=P_P \\ e(I, M_P) \geq he(J, M_P)}} \left\{ \frac{l(M_P/IM_P)}{l(M_P/JM_P)} \right\} \quad \text{and} \quad \sup_{\substack{\sqrt{I}=\sqrt{J}=P_P \\ e(I, M_P) \leq he(J, M_P)}} \left\{ \frac{l(M_P/IM_P)}{l(M_P/JM_P)} \right\} \leq hC.$$

By  $\sqrt{I} = \sqrt{J} = P_P$ , we regard  $I$  and  $J$  as  $(P_P$ -primary) ideals of  $R_P$ .

*Proof.* For each  $P$ , do the same proof for  $M_P$  over  $R_P$  as in Lemma 4.1, and note that  $n(M_P)$ ,  $\dim(M_P)$  and  $e(\overline{R}_P)$  have global upper bounds (see Remark 2.6).  $\square$

**Corollary A.2.** *With notation and assumptions as in Lemma A.1, we have*

$$\sup_{\substack{P \in \text{Supp}(M) \\ \sqrt{I}=P_P \\ I \subseteq J \subseteq \overline{I}}} \left\{ \frac{l(M_P/IM_P)}{l(M_P/JM_P)} \right\} < \infty.$$

*Proof.* Apply Lemma A.1 with  $h = 1$  to get the desired claim.  $\square$

**Lemma A.3.** *Let  $R$  be a Noetherian ring that is a homomorphic image of a Noetherian Gorenstein ring  $S$  with  $\dim(S) < \infty$ , and let  $N$  be a finitely generated  $R$ -module such that  $R/\text{Ann}(N)$  is locally equidimensional and satisfies the uniform Artin-Rees property with  $\sup\{e(\overline{R}_P) \mid P \in \text{Spec}(\overline{R})\} < \infty$ . Suppose the residue field of  $R_P$  is infinite for all  $P \in \text{Supp}(N)$ . Then there exists a constant  $C_N > 0$  depending only on  $N$  such that*

$$\sup_{\sqrt{I}=P_P} \left\{ \frac{l(N_P/IN_P)}{l(M_P/IM_P)} \right\} \leq C_N$$

for all  $P \in \text{Supp}(N)$ , all  $P_P$ -primary ideals  $I$ , and all finitely generated  $R$ -modules  $M$  such that  $\text{Supp}(M_P) \supseteq \text{Supp}(N_P)$ . (Note that such  $M_P$  covers all finitely generated  $R_P$ -modules whose supports contain the support of  $N_P$ .)

*Proof.* For each  $P$ , do the same proof for  $N_P$  over  $R_P$  as in Lemma 4.3, and note that the constant  $n(N_P)c \dim(N_P)!e(\overline{R}_P)$  has a global upper bound (see Remark 2.6): we can take  $c = \max_{P \in \text{Min}(N)} l_{R_P}(N_P)$ , which is enough for each local consideration.  $\square$

**Theorem A.4.** *Let  $R$  be a Noetherian ring that is a homomorphic image of a Noetherian Gorenstein ring  $S$  with  $\dim(S) < \infty$ . Suppose  $R$  is locally equidimensional and satisfies the uniform Artin-Rees property and that  $\sup\{e(\overline{R}_P) \mid P \in \text{Spec}(\overline{R})\} < \infty$ . Suppose the residue field of  $R_P$  is infinite for all  $P \in \text{Supp}(R)$ . Then, for all finitely generated  $R$ -modules  $M$  and  $N$ , we have*

- (1)  $\sup_{\sqrt{I}=P_P} \left\{ \frac{l(N_P/IN_P)}{l(M_P/IM_P)} \right\} < \infty$  for all  $P \in \text{Supp}(N) \iff \text{Supp}(M) \supseteq \text{Supp}(N)$ .
- (2) There exists a constant  $C > 0$  depending only on  $N$  such that

$$1/C \leq \inf_{\sqrt{I}=P_P} \left\{ \frac{l(M_P/IM_P)}{l(N_P/IN_P)} \right\} \quad \text{and, equivalently,} \quad \sup_{\sqrt{I}=P_P} \left\{ \frac{l(N_P/IN_P)}{l(M_P/IM_P)} \right\} \leq C$$

for all  $P \in \text{Supp}(N)$  and all (finitely generated  $R$ -modules)  $M$  with  $\text{Supp}(M_P) \supseteq \text{Supp}(N_P)$ .

- (3)  $0 < \inf_{\sqrt{I}=P_P} \left\{ \frac{l(M_P/IM_P)}{l(N_P/IN_P)} \right\} \leq \sup_{\sqrt{I}=P_P} \left\{ \frac{l(M_P/IM_P)}{l(N_P/IN_P)} \right\} < \infty \iff \text{Supp}(M) = \text{Supp}(N)$ .

*Proof.* For each  $P$ , do the same proof as in Theorem 4.4, replacing Lemma 4.3 by Lemma A.3.  $\square$

The next lemma is the same as the local version; we record it here only to preserve the numbering.

**Lemma A.5.** *Let  $(R, \mathfrak{m})$  be a Noetherian local ring and  $M$  a finitely generated  $R$ -module. If  $(y_1, \dots, y_d) \subseteq (x_1, \dots, x_k)$  are  $\mathfrak{m}$ -primary ideals of  $R$ , then for all  $0 \leq i \leq k$ ,*

$$l(H_i(x_1, \dots, x_k; M)) \leq \sum_{j=0}^k \binom{k}{j} l(H_{i-j}(y_1, \dots, y_d; M)) \leq 2^k \max_{0 \leq j \leq k} l(H_{i-j}(y_1, \dots, y_d; M)),$$

with the convention that  $H_{<0}(y_1, \dots, y_d; M) = 0$ .

The next remark follows immediately from the local version.

**Remark A.6** ([19] or [2]). *Let  $R$  be a Noetherian ring that is a homomorphic image of a Noetherian Gorenstein ring  $S$ . Then, for every finitely generated  $R$ -module  $M$ , every  $P \in \text{Spec}(R)$ , every system of parameters  $\underline{x} = x_1, \dots, x_{\dim(R_P)}$  of  $R_P$ , and every  $i = 1, \dots, \dim(R_P)$ , we have*

$$l(H_i(x_1, \dots, x_{\dim(R_P)}; M_P)) \leq \sum_{j=0}^{\dim(R_P)-i} l(H_{\dim(R_P)-i-j}(x_1, \dots, x_{\dim(R_P)}; \text{Ext}_{S_P}^{\dim(S_P)-j}(M_P, S_P))).$$

Note that  $\dim((\text{Ext}_S^{\dim(S_P)-j}(M, S))_P) \leq \dim(R_P) - i$  for each  $j = 0, \dots, \dim(R_P) - i$ . Here, by abuse of notation,  $S_P$  stands for  $S_Q$  where  $Q$  is the pre-image of  $P$  under the onto map  $S \rightarrow R$ .

**Theorem A.7.** *Let notation and assumptions be as in Lemma A.1. Then there exists  $C$  depending on  $M$  such that, for every  $P \in \text{Supp}(M)$  and every  $k \geq \dim(M_P)$ , we have*

$$\sup_{\substack{\sqrt{(x_1, \dots, x_k) + \text{Ann}(M_P)} = P_P \\ 0 \leq i \leq k}} \left\{ \frac{l(H_i(x_1, \dots, x_k; M_P))}{l(M_P/(x_1, \dots, x_k)M_P)} \right\} \leq 2^k C.$$

By  $\sqrt{(x_1, \dots, x_k) + \text{Ann}(M)}_P = P_P$ , the understanding is that  $x_i \in P_P$ .

*Proof.* We use induction on  $d$  to prove the following claim: *For every  $d \geq 0$ , there exists  $C_{M,d}$  depending on  $M$  and  $d$  such that, for every  $P \in \text{Supp}(M)$  with  $\dim(M_P) = d$ , every  $k \geq d$ , we have*

$$\sup_{\substack{\sqrt{(x_1, \dots, x_k) + \text{Ann}(M_P)} = P_P \\ 0 \leq i \leq k}} \left\{ \frac{l(H_i(x_1, \dots, x_k; M_P))}{l(M_P/(x_1, \dots, x_k)M_P)} \right\} \leq 2^k C_{M,d}.$$

As  $\dim(M) < \infty$ , we can take  $C = \max_{0 \leq i \leq \dim(M)} C_i$  to complete the proof of the theorem. We can replace  $R$  by  $R/\text{Ann}(M)$ , so  $M$  is a faithful  $R$ -module. In this sense, we are doing induction on the height of  $P$ .

When  $d = 0$ ,  $C_{M,0} = \max_{P \in \text{Min}(M)} l_{R_P}(M_P)$  works. Now let  $d > 0$ , and assume that the claim holds for  $\leq d-1$ . We fix a prime cyclic filtration of  $M$  over  $R$  (which is independent of the choice of  $P$  at which we are localizing). Localizing this prime cyclic filtration of  $M$  at  $P$ , applying a similar argument as in the proof of Theorem 4.7 in light of Lemma A.3 or Theorem A.4, we may further assume that  $M = R$  and that  $R$  is a domain.

Now we prove the claim for  $d$ . By Corollary A.2 and Lemma A.5, it suffices to find a constant  $D$  such that

$$\frac{l(H_i(y_1, \dots, y_d; R_P))}{l(R_P/(y_1, \dots, y_d))} \leq D$$

for all  $P \in \text{Spec}(R)$  such that  $\dim(R_P) = d$ , all systems of parameters  $\underline{y} := y_1, \dots, y_d$  of  $R_P$ , and all  $i = 0, \dots, d$ . Now by Remark A.6, it suffices to show that, for any fixed finitely generated  $R$ -module  $L$  satisfying  $\dim(L_P) < d$  whenever  $\dim(R_P) = d$  (think of  $L$  as one of those (finitely many) global Ext modules over  $R$ ), there exists a constant  $D_L$  such that

$$\frac{l(H_i(y_1, \dots, y_d; L_P))}{l(R_P/(y_1, \dots, y_d))} \leq D_L$$

independent of  $\underline{y}$ ,  $i$ , and  $P \in \text{Spec}(R)$  such that  $\dim(R_P) = d$ . Indeed, as  $l(L_P/(y_1, \dots, y_d)L_P) \leq \mu_R(L)l(R_P/(y_1, \dots, y_d))$ , we have

$$\frac{l(H_i(y_1, \dots, y_d; L_P))}{l(R_P/(y_1, \dots, y_d))} \leq \mu(L) \frac{l(H_i(y_1, \dots, y_d; L_P))}{l(L_P/(y_1, \dots, y_d)L_P)}.$$

Since  $\dim(L_P) < d$  when  $\dim(R_P) = d$ , the right hand side of the above inequality is bounded above (independent of  $P \in \text{Spec}(R)$  such that  $\dim(R_P) = d$ ,  $\underline{y}$  and  $i$ ) by the inductive hypothesis (noting that  $2^d$  is a constant as well).  $\square$

**Theorem A.8.** *Let notation and assumptions be as in Lemma A.1. Then for every  $\varepsilon > 0$ , there exists  $t_0$  such that, for all  $t \geq t_0$ , all  $P \in \text{Supp}(M)$ , all systems of parameters  $\underline{x} := x_1, \dots, x_{\dim(M_P)}$*



of  $M_P$ , and all  $1 \leq i \leq \dim(M_P)$ ,

$$\frac{l(H_i(x_1^t, \dots, x_{\dim(M_P)}^t); M_P)}{l(M_P/(x_1^t, \dots, x_{\dim(M_P)}^t)M_P)} < \varepsilon.$$

In fact, there exists a constant  $K$  such that for all  $t \geq 1$ , all  $P \in \text{Supp}(M)$ , all systems of parameters  $\underline{x} := x_1, \dots, x_{\dim(M_P)}$  of  $M_P$ , and all  $1 \leq i \leq \dim(M_P)$ ,

$$\frac{l(H_i(x_1^t, \dots, x_{\dim(M_P)}^t); M_P)}{l(M_P/(x_1^t, \dots, x_{\dim(M_P)}^t)M_P)} \leq \frac{K}{t^i}.$$

*Proof.* As usual, we replace  $R$  by  $\bar{R} = R/\text{Ann}(M)$  to assume that  $M$  is faithful over  $R$ . Our hypothesis then implies that both  $M$  and  $R$  are locally equidimensional.

Denote  $\mathcal{L} = \{\text{Ext}_S^j(M, S) \mid 0 \leq j \leq \dim(S)\}$ , which is a finite set. For each  $L \in \mathcal{L}$ , fix a prime cyclic filtration  $F_L$  of  $L$ , and denote  $\mathcal{D} = \{\text{all the } R/\mathfrak{a} \text{ appearing as a factor of } F_L \text{ for some } L \in \mathcal{L}\}$ , which is finite.

Fix  $\varepsilon > 0$ . Because we consider only finitely many  $i$ , it is sufficient to fix some  $1 \leq i \leq \dim(R)$ . By Remark A.6, it suffices to show that there exists a constant  $K$  such that, for all  $P \in \text{Supp}(M)$ , for all  $L \in \mathcal{L}$  satisfying  $\dim(L_P) \leq \dim(R_P) - i$ , for all  $t \geq 1$ , for all systems of parameters  $\underline{x}$  of  $R_P$ , and for all  $j = 0, \dots, \dim(R_P) - 1$ ,

$$\frac{l(H_j(x_1^t, \dots, x_{\dim(R_P)}^t); L_P)}{l(M_P/(x_1^t, \dots, x_{\dim(R_P)}^t)M_P)} \leq \frac{K}{t^i}.$$

Then, via  $F_L$ , it suffices to show that there exists a constant  $K$  such that, for all  $P \in \text{Supp}(M)$ , for all  $D \in \mathcal{L}$  satisfying  $\dim(D_P) \leq \dim(R_P) - i$ , for all  $t \geq 1$ , for all systems of parameters  $\underline{x}$  of  $R_P$ , and for all  $j = 0, \dots, \dim(R_P) - 1$ ,

$$(\dagger) \quad \frac{l(H_j(x_1^t, \dots, x_{\dim(R_P)}^t); D_P)}{l(M_P/(x_1^t, \dots, x_{\dim(R_P)}^t)M_P)} \leq \frac{K}{t^i}.$$

By Theorem A.7 and noting that  $\mathcal{D}$  is finite, we fix

$$C = \sup_{\substack{D \in \mathcal{D}, P \in \text{Supp}(D) \\ \sqrt{(x_1, \dots, x_d)} = P_P}} \left\{ \frac{l(H_j(x_1, \dots, x_d; D_P))}{l(D_P/(x_1, \dots, x_d)D_P)} \right\} < \infty.$$

According to Remark 2.6 (i.e., the global version of Theorem 2.4), we let

$$B = \sup_{\substack{D \in \mathcal{D}, P \in \text{Supp}(D) \\ \sqrt{I} = P_P}} \left\{ \frac{l(D_P/ID_P)}{e(I, D_P)} \right\} < \infty.$$

Moreover, for every  $D = R/\mathfrak{a} \in \mathcal{D}$ , fix  $Q(D) \in \text{Min}(R) = \text{Min}(M)$  such that  $Q(D) \subseteq \mathfrak{a}$ , so that  $D = R/\mathfrak{a}$  is a homomorphic image of  $R/Q(D)$ . Note that  $e(I, R_P/Q(D)_P) \leq e(I, M_P)$  for all  $P_P$ -primary ideals  $I$  of  $R_P$  because  $R$  is locally equidimensional (remember we already reduced to the case  $R = R/\text{Ann}(M)$ ). In light of the global version of Equation (2.3) (since we assume that  $R/\text{Ann}(M)$  satisfies the uniform Artin-Rees property, this equation holds by the same argument as in Lemma 2.5), we set

$$A = \sup_{\substack{D \in \mathcal{D}, P \in \text{Supp}(D) \\ \sqrt{I} = P_P}} \left\{ \frac{e(I, D_P)}{e(I, (R/Q(D))_P)} \right\} < \infty.$$

At this point we see that, for all  $P \in \text{Supp}(M)$ , for all  $D \in \mathcal{L}$  satisfying  $\dim(D_P) = d' \leq d - i = \dim(R_P) - i = \dim((R/Q(D))_P) - i$ , for all  $t \geq 1$ , for all systems of parameters  $\underline{x}$  of  $R_P$ , and for all  $j = 0, \dots, \dim(R_P) - 1$ , we have (denoting  $d := \dim((R/Q(D))_P)$ )

$$\begin{aligned} \frac{l(H_j(x_1^t, \dots, x_d^t; D_P))}{l(M/(x_1^t, \dots, x_d^t)M_P)} &\leq \frac{l(H_j(x_1^t, \dots, x_d^t; D_P))}{e((x_1^t, \dots, x_d^t), M_P)} \\ &\leq C \frac{l(D_P/(x_1^t, \dots, x_d^t)D_P)}{e((x_1^t, \dots, x_d^t), M_P)} \\ &\leq BC \frac{e((x_1^t, \dots, x_d^t), D_P)}{e((x_1^t, \dots, x_d^t), M_P)} \\ &= BC \frac{t^{d'} e((x_1, \dots, x_d), D_P)}{t^d e((x_1, \dots, x_d), M_P)} \\ &\leq \frac{BC}{t^{d-d'}} \frac{e((x_1, \dots, x_d), D_P)}{e((x_1, \dots, x_d), (R/Q(D))_P)} \leq \frac{ABC}{t^{d-d'}} \leq \frac{ABC}{t^i}, \end{aligned}$$

whose convergence to 0, as  $t \rightarrow \infty$ , gives what we need in  $(\dagger)$  to complete the proof.  $\square$

Similarly, we can generalize Theorem A.8 as follows:

**Theorem A.9.** *Let notation and assumptions be as in Lemma A.1. Then there exists a constant  $K$  such that, for all  $P \in \text{Supp}(M)$ , all  $t_j \geq 1$  with  $1 \leq j \leq \dim(M_P)$ , all systems of parameters  $\underline{x} := x_1, \dots, x_{\dim(M_P)}$  of  $M_P$ , and all  $1 \leq i \leq \dim(M_P)$ ,*

$$\frac{l(H_i(x_1^{t_1}, \dots, x_{\dim(M_P)}^{t_{\dim(M_P)}}; M_P))}{l(M_P/(x_1^{t_1}, \dots, x_{\dim(M_P)}^{t_{\dim(M_P)}})M_P)} \leq \frac{K}{(\min_j t_j)^i}.$$

*Proof.* We carry out the same proof as in Theorem 4.10 for each  $M_P$ , using the fact that  $\dim(M_P)$ ,  $e(R_P)$ , and  $n(M_P)$  have global upper bounds for all  $P \in \text{Supp}(M)$  (see Remark 2.6 or Remark A.11).  $\square$

**Remark A.10.** We would like to mention that, in Theorem A.8, the assumption that  $M$  is locally equidimensional is necessary. Localized at each  $P$ , it is the same argument as Remark 4.11. Moreover, the conclusion of Theorem A.8 implies the conclusion of Remark 2.6 (i.e., the global version of Theorem 2.4) following the same argument as in Remark 4.11 applied to  $M_P$  for all  $P$ .

**Remark A.11.** Evidently the results of this section rely on the global version of Theorem 2.4 (i.e., Remark 2.6). However, a careful analysis of the proofs in this section reveals an alternative proof of the global version of Theorem 2.4 by induction on dimension of the  $R$ -module  $M$ . We use notation as in Lemma A.1. When  $\dim(M) = 0$ , it is clear that Theorem A.8, Remark 2.6, Lemma A.1, Corollary A.2, Lemma A.3 and Theorem A.7 all hold. Now assume that all these results hold in dimension  $< d$ ; and consider the case of dimension  $d$ . Then Theorem A.8 holds in dimension  $d$  (because the proof of Theorem A.8 only requires the aforementioned results in dimension  $< d$ ), which implies Remark 2.6 in dimension  $d$  as we explained in Remark A.10. Then we have Lemma A.1, Corollary A.2, Lemma A.3 and Theorem A.7 in dimension  $d$ , completing the induction.

The following is an easy consequence of Theorem A.9.

**Corollary A.12.** *Let notation and assumptions be as in Lemma A.1. Then, for every constant  $C > 0$  and every  $\epsilon > 0$ , there exists  $t_0 \in \mathbb{N}$  such that, for all  $P \in \text{Supp}(M)$ , all  $t_j \geq t_0$  with  $1 \leq j \leq$*

$\dim(M_P)$ , and all systems of parameters  $\underline{x} = x_1, \dots, x_{\dim(M_P)}$  on  $M_P$  such that  $e((\underline{x}), M_P) \leq C$ , we have

$$0 \leq \frac{l(M_P/(x_1^{t_1}, \dots, x_{\dim(M_P)}^{t_{\dim(M_P)}})M_P)}{\prod_{j=1}^{\dim(M_P)} t_j} - e((\underline{x}), M_P) < \epsilon.$$

In fact, there exists a constant  $K$  such that for all  $P \in \text{Supp}(M)$ , all  $t_j \geq 1$  with  $1 \leq j \leq \dim(M_P)$ , and all systems of parameters  $\underline{x} = x_1, \dots, x_{\dim(M_P)}$  on  $M_P$ , we have

$$0 \leq \frac{l(M_P/(x_1^{t_1}, \dots, x_{\dim(M_P)}^{t_{\dim(M_P)}})M_P)}{\prod_{j=1}^{\dim(M_P)} t_j} - e((\underline{x}), M_P) \leq e((\underline{x}), M_P) \frac{K}{\min_j t_j} \leq l(M_P/(\underline{x})M_P) \frac{K}{\min_j t_j}.$$

*Proof.* We carry out the same proof as in Corollary 4.13 for each  $M_P$ , replacing Theorem 4.10 by Theorem A.9.  $\square$

For example, (the first part of) Corollary A.12 applies to all minimal reductions of  $P_P/\text{Ann}(M_P)$  in  $R_P/\text{Ann}(M_P)$  for all  $P \in \text{Supp}(M)$ , since there is an upper bound for  $\{e(M_P) \mid P \in \text{Supp}(M)\}$  under the assumption of Corollary A.12.

Finally, similar to Remark 4.14, we have the following

**Remark A.13.** Let notation and assumptions be as in Lemma A.1 and let  $c = \max_{P \in \text{Min}(\bar{R})} l_{\bar{R}_P}(\bar{R}_P)$ . By Remark 4.14 and Remark 2.6 (or Remark A.11), we have

$$\frac{\sup_{P \in \text{Supp}(M)} n(M_P)}{\mu(M)} \leq \sup_{P \in \text{Supp}(M)} \frac{n(M_P)}{\mu(M_P)} \leq c \sup_{P \in \text{Supp}(M)} n(\bar{R}_P) < \infty,$$

in which  $c \sup_{P \in \text{Supp}(M)} n(\bar{R}_P)$  depends only on  $\bar{R} = R/\text{Ann}(M)$ .

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