Improved Hölder continuity near the boundary of one-dimensional super-Brownian motion

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Abstract

We show that the local time of one-dimensional super-Brownian motion is locally γ-Hölder continuous near the boundary if $0 < \gamma < 3$ and fails to be locally γ-Hölder continuous if $\gamma > 3$.

1 Introduction

Let $M_F = M_F(\mathbb{R}^d)$ be the space of finite measures on $(\mathbb{R}^d, \mathfrak{B}(\mathbb{R}^d))$ equipped with the topology of weak convergence of measures, and write $\mu(\phi) = \int \phi(x)\mu(dx)$ for $\mu \in M_F$. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space. A super-Brownian motion $(X_t,\ t \geq 0)$ starting at $\mu \in M_F$ is a continuous M_F-valued strong $(\mathcal{F}_t)_{t\geq 0}$-Markov process defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ with $X_0 = \mu$ a.s.. It is well known that super-Brownian motion is the solution to the following *martingale problem* (see [Per02], II.5): For any $\phi \in C_b^2(\mathbb{R}^d)$,

$$X_t(\phi) = X_0(\phi) + M_t(\phi) + \int_0^t X_s(\frac{\Delta}{2}\phi)ds,\ \forall t \geq 0, \tag{1.1}$$

where $(M_t(\phi))_{t\geq 0}$ is a continuous $(\mathcal{F}_t)_{t\geq 0}$-martingale such that $M_0(\phi) = 0$ and

$$[M(\phi)]_t = \int_0^t X_s(\phi^2)ds,\ \forall t \geq 0.$$

The above martingale problem uniquely characterizes the law $\mathbb{P}_{X_0}$ of super-Brownian motion X, starting from $X_0 \in M_F$, on $C([0,\infty), M_F)$, the space of continuous functions from $[0,\infty)$ to M_F furnished with the compact open topology. In particular, if we let X_0 be the Dirac mass δ_0, then $\mathbb{P}_{\delta_0}$ denotes the law of super-Brownian motion X starting from δ_0.

Local times of superprocesses have been studied by many authors (cf. [Sug89], [BEP91], [AL92], [Kro93], [Mer06]). We recall that [Sug89] has proved that for $d \leq 3$, there exists a jointly lower semi-continuous local time L_t^x, which is monotone increasing in t for all x, such that

$$\int_0^t X_s(\phi)ds = \int_{\mathbb{R}^d} \phi(x)L_t^x dx,\ \text{for all } t \geq 0 \text{ and non-negative measurable } \phi. \tag{1.2}$$

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Moreover, there is a version of the local time L_t^x which is jointly continuous on the set of continuity points of $X_0 q_t(x)$, where $q_t(x) = \int_0^t p_s(x)ds$, $p_t(x)$ is the transition density of Brownian motion, and $X_0 q_t(x) = \int q_t(y-x) X_0(dy)$ (see Theorem 3 in [Sug89]). Let the extinction time ζ of X be defined as $\zeta = \zeta_X = \inf\{t \geq 0 : X_t(1) = 0\}$. We know that $\zeta < \infty$ a.s. (see Chp. II.5 in [Per02]). Then we have $L^x = L_\infty^x = L_\zeta^x$ is also lower semicontinuous. Note the set $\overline{\{x : L^x > 0\}}$ is defined to be the range of super-Brownian motion X (see [MP17]). Theorem 2.2 of [MP17] gives that for any $\eta > 0$, with $\mathbb{P}_{\delta_0}$-probability one we have L^x is $C^{(4-d)/2-\eta}$-Hölder continuous for x away from 0 if $d \leq 3$. When $d = 1$, L^x is globally continuous (see Proposition 3.1 in [Sug89]).

Definition. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be locally γ-Hölder continuous at $x \in \mathbb{R}$, if there exist $\delta > 0$ and $c > 0$ such that

$$|f(x) - f(y)| \leq c|x-y|^\gamma, \ \forall y \text{ with } |y-x| < \delta.$$

We refer to $\gamma > 0$ as the Hölder index and to $c > 0$ as the Hölder constant.

The problem studied in this paper was originally motivated by a heuristic calculation of the Hausdorff dimension, d_f, of the boundary of $\{x : L^x > 0\}$ in [MP17]. With the following bounds given in Theorem 1.3 of [MP17],

$$\mathbb{P}_{\delta_0}(0 < L^x \leq a) \leq C a^\alpha \text{ for } a \text{ small},$$

and an improved γ-Hölder continuity of L^x for x near its zero set, the two authors derived the upper bound $d_f \leq d - \alpha\gamma$ by a heuristic covering lemma in Section 1 of the same reference. Although these arguments were given for $d = 3$, they work in any dimension. As d_f and α are known from [MP17], one can reverse engineer and find the required γ. This leads to their conjecture [private communication] that for any $\eta > 0$, with $\mathbb{P}_{\delta_0}$-probability one

$$x \to L^x \text{ is locally Hölder continuous of index } 4 - d - \eta \text{ near the zero set of } L^x. \tag{1.3}$$

In [MP17] they reported that they can establish the above for $d = 3$ (and make the argument for the upper bound on d_f work). In this paper we confirm the above conjecture for $d = 1$, as stated in Theorem 1.1 below. This result also gives us confidence on the validity of the $d = 2$ case, which remains an interesting open problem.

To state our main results, we first recall a result from Theorem 1.7 in [MP17].

Theorem A. *([MP17]) If $d = 1$ then $\mathbb{P}_{\delta_0}$-a.s. there are random variables $\mathsf{L} < 0 < \mathsf{R}$ such that*

$$\{x : L^x > 0\} = (\mathsf{L}, \mathsf{R}).$$

As discussed above, we are interested in the decay rate of the local time L^x on the boundary, i.e., at L and R.

Theorem 1.1. *Let $d = 1$. If $0 < \gamma < 3$, then $\mathbb{P}_{\delta_0}$-a.s. the local time L^x is locally γ-Hölder continuous at L and R.*

This result will be proved in Section 2 and it is optimal in the sense of the following theorem, whose proof will be given in Section 3.

Theorem 1.2. *Let $d = 1$. For any $\gamma > 3$, we have $\mathbb{P}_{\delta_0}$-a.s. that there is some $\delta(\gamma, \omega) > 0$ such that $L^x \geq 2^{-\gamma/2}(\mathsf{R} - x)^\gamma$ for all $\mathsf{R} - \delta < x < \mathsf{R}$.*

With the lower bound established above, we can get the following result immediately.

Corollary 1.3. *Let $d = 1$. If $\gamma > 3$, then $\mathbb{P}_{\delta_0}$-a.s. the local time L^x fails to be locally γ-Hölder continuous at L and R.*

Proof. By symmetry we may consider only R. For any $\gamma > 3$, define $\gamma' = (3+\gamma)/2$ such that $3 < \gamma' < \gamma$. Then Theorem 1.2 would imply that $\mathbb{P}_{\delta_0}$-a.s. that there is some $\delta(\gamma', \omega) > 0$ such that $L^x \geq 2^{-\gamma'/2}(\mathsf{R} - x)^{\gamma'}$ for all $\mathsf{R} - \delta < x < \mathsf{R}$. For ω as above and $c > 0$, if $x < \mathsf{R}$ is chosen close enough to R, then

$$L^x \geq 2^{-\gamma'/2}(\mathsf{R} - x)^{\gamma'} > c(\mathsf{R} - x)^\gamma,$$

and so the local γ-Hölder continuity at R fails a.s.. ■

Now we continue to study the case under the canonical measure $\mathbb{N}_0$. $\mathbb{N}_{x_0}$ is a σ-finite measure on $C([0,\infty), M_F)$ which arises as the weak limit of $NP^N_{\delta_{x_0}/N}(X^N_\cdot \in \cdot)$ as $N \to \infty$, where $X^N_\cdot$ under $P^N_{\delta_{x_0}/N}$ is the approximating branching particle system starting from a single particle at x_0 (see Theorem II.7.3(a) in [Per02]). In this way it describes the contribution of a cluster from a single ancestor at x_0, and the super-Brownian motion is then obtained by a Poisson superposition of such clusters. In fact, if we let $\Xi = \sum_{i \in I} \delta_{\nu^i}$ be a Poisson point process on $C([0,\infty), M_F)$ with intensity $\mathbb{N}_{x_0}(d\nu)$, then

$$X_t = \sum_{i \in I} \nu^i_t = \int \nu_t \, \Xi(d\nu), \ t > 0,$$

has the law, $\mathbb{P}_{\delta_{x_0}}$, of a super-Brownian motion X starting from δ_{x_0}. We refer the readers to Theorem II.7.3(c) in [Per02] for more details. The existence of the local time L^x under $\mathbb{N}_{x_0}$ will follow from this decomposition and the existence under $\mathbb{P}_{\delta_{x_0}}$. Therefore the local time L^x may be decomposed as

$$L^x = \sum_{i \in I} L^x(\nu^i) = \int L^x(\nu) \Xi(d\nu). \tag{1.4}$$

The continuity of local times L^x under $\mathbb{N}_{x_0}$ is given in Theorem 1.2 of [Hong18]. We first give a version of Theorem A under the canonical measure.

Theorem 1.4. *If $d = 1$ then $\mathbb{N}_0$-a.e. there are random variables $\mathsf{L} < 0 < \mathsf{R}$ such that*

$$\{x : L^x > 0\} = (\mathsf{L}, \mathsf{R}).$$

Theorem 1.5. *Theorem 1.1, Theorem 1.2 and Corollary 1.3 hold if $\mathbb{P}_{\delta_0}$ is replaced with $\mathbb{N}_0$.*

The proofs of these analogous results under $\mathbb{N}_0$ will be given in Section 4.

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2 Upper bound of the local time near the boundary

Let $g_x(y) = |y-x|$. Then $\frac{d^2}{dy^2} g_x(y) = 2\delta_x(y)$ holds in the distributional sense and the martingale problem (1.1) suggests the following result.

Proposition 2.1. *(Tanaka formula for d=1) Let $d = 1$ and fix $x \neq 0$ in $\mathbb{R}^1$. Then we have $\mathbb{P}_{\delta_0}$-a.s. that*

$$L_t^x + |x| = X_t(g_x) - M_t(g_x), \ \forall t \geq 0, \tag{2.1}$$

where $t \mapsto X_t(g_x)$ is continuous for $t \geq 0$ and $(M_t(g_x))_{t\geq 0}$ is a continuous L^2 martingale which is the stochastic integral with respect to the martingale measure associated with super-Brownian motion.

Proof. Let $(P_t)_{t\geq 0}$ be the Markov semigroup of one-dimensional Brownian motion. By cutoff arguments similar to those used in the proof of Proposition 2.4 in [Hong18], we may use the martingale problem (1.1) to see that for any $\varepsilon > 0$, with $\mathbb{P}_{\delta_0}$-probability one we have

$$X_t(P_\varepsilon g_x) = P_\varepsilon g_x(0) + M_t(P_\varepsilon g_x) + \int_0^t X_s(\frac{\Delta}{2} P_\varepsilon g_x) ds, \ \forall t \geq 0. \tag{2.2}$$

One can check that

$$|P_\varepsilon g_x(y) - g_x(y)| \leq \varepsilon^{1/2}, \ \forall x, y \in \mathbb{R}, \tag{2.3}$$

and so it follows that

$$\Big| P_\varepsilon g_x(0) - |x| \Big| \to 0, \ \text{as } \varepsilon \downarrow 0. \tag{2.4}$$

Use (2.3) again to see that for any $T > 0$,

$$\sup_{t\leq T} \Big| X_t(P_\varepsilon g_x) - X_t(g_x) \Big| \leq \varepsilon^{1/2} \sup_{t\leq T} X_t(1) \to 0, \quad \mathbb{P}_{\delta_0} - a.s., \tag{2.5}$$

and

$$\mathbb{E}_{\delta_0} \Big[\Big(\sup_{t\leq T} \Big| M_t(P_\varepsilon g_x) - M_t(g_x) \Big| \Big)^2 \Big] \leq 4\mathbb{E}_{\delta_0} \Big[\int_0^T X_s\Big((P_\varepsilon g_x - g_x)^2 \Big) ds \Big] \to 0. \tag{2.6}$$

The last inequality follows by Doob's inequality. Now for the convergence of last term on the right-hand side of (2.2), we apply integration by parts to get for any $\varepsilon > 0$, $\frac{d^2}{dy^2} P_\varepsilon g_x(y) = 2p_\varepsilon(y-x) =: 2p_\varepsilon^x(y)$. Theorem 6.1 in [BEP91] gives us that as $\varepsilon \to 0$,

$$\sup_{t\leq T} \Big| \int_0^t X_s(p_\varepsilon^x) ds - L_t^x \Big| \to 0, \quad \mathbb{P}_{\delta_0} - a.s., \tag{2.7}$$

and hence by taking an appropriate subsequence $\varepsilon_n \downarrow 0$, (2.1) would follow immediately from (2.2), (2.4), (2.5), (2.6) and (2.7). ■

Now we discuss the differentiability of L_t^x in $d = 1$. We denote, by $D_x f(x)$ (resp. $D_x^+ f(x)$, $D_x^- f(x)$), the derivative (resp. right derivative, left derivative) of $f(x)$. Then we have the following result from Theorem 4 of [Sug89].

Theorem B. *([Sug89]) Let $d = 1$ and $X_0 = \mu \in M_F(\mathbb{R})$. Then the following (i) and (ii) hold with $\mathbb{P}_\mu$-probability one.*

(i) $Z(t, x) = L_t^x - \mathbb{E}_\mu(L_t^x)$ is differentiable with respect to x, $\forall t \geq 0$;

(ii) $D_x Z(t, x)$ is jointly continuous in $t \geq 0$ and $x \in \mathbb{R}$, and we have

$$D_x^+ \mathbb{E}_\mu(L_t^x) - D_x^- \mathbb{E}_\mu(L_t^x) = -2\mu(\{x\}),\ t > 0, x \in \mathbb{R}. \tag{2.8}$$

In particular, if we let $H = \{x \in \mathbb{R} : \mu(\{x\}) = 0\}$, then $D_x \mathbb{E}_\mu(L_t^x)$ is jointly continuous on $[0, \infty) \times H$ and so with $\mathbb{P}_\mu$-probability one we have L_t^x is differentiable with respect to x on H and $D_x L_t^x$ is jointly continuous on $[0, \infty) \times H$.

So for the case $X_0 = \delta_0$, we know from the above theorem that L_t^x is continuously differentiable on $\{x \neq 0\}$. Let $sgn(x) = x/|x|$ for $x \neq 0$ and $sgn(0) = 0$. Then $D_y g_x(y) = sgn(y - x)$ for $y \neq x$ and we have the following Tanaka formula for $D_x L_t^x$.

Proposition 2.2. *Let $d = 1$ and fix $x \neq 0$ in $\mathbb{R}^1$. Then we have $\mathbb{P}_{\delta_0}$-a.s. that*

$$D_x L_t^x = -sgn(x) + X_t(sgn(x - \cdot)) - M_t(sgn(x - \cdot)),\ \forall t \geq 0. \tag{2.9}$$

Proof. Fix any $x \neq 0$ and any $t \geq 0$. Choose some positive sequence $\{h_n\}_{n \geq 1}$ such that $h_n \downarrow 0$. Then use (2.1) to see that with $\mathbb{P}_{\delta_0}$-probability one,

$$\frac{1}{h_n}(L_t^{x+h_n} - L_t^x) + \frac{1}{h_n}(|x + h_n| - |x|) = \frac{1}{h_n}(X_t(g_{x+h_n}) - X_t(g_x)) - \frac{1}{h_n}(M_t(g_{x+h_n}) - M_t(g_x)). \tag{2.10}$$

By Theorem B, we conclude that the left hand side converges a.s. to $D_x L_t^x + sgn(x)$ as $h_n \downarrow 0$. For the right hand side, first note that for all $x, y \in \mathbb{R}$, we have $|(|x + h - y| - |x - y|)/h| \leq 1$ for all $h > 0$. Then bounded convergence theorem implies as $h_n \downarrow 0$,

$$\frac{1}{h_n}(X_t(g_{x+h_n}) - X_t(g_{x_n})) = \int \frac{1}{h_n}(|x + h_n - y| - |x - y|)X_t(dy) \to \int sgn(x - y)X_t(dy),$$

and

$$\begin{aligned}
&\mathbb{E}_{\delta_0}\left[\left(\frac{1}{h_n}(M_t(g_{x+h_n}) - M_t(g_x)) - M_t(sgn(x - \cdot))\right)^2\right] \\
\leq &\mathbb{E}_{\delta_0}\left[\int_0^t \int \left(\frac{1}{h_n}(|x + h_n - y| - |x - y|) - sgn(x - y)\right)^2 X_s(dy)ds\right] \\
= &\int_0^t ds \int p_s(y) \left(\frac{1}{h_n}(|x + h_n - y| - |x - y|) - sgn(x - y)\right)^2 dy \to 0.
\end{aligned}$$

In the last equality we use $\mathbb{E}_{\delta_0} X_t(dy) = p_t(y)dy$ from Lemma 2.2 of [KS88]. So every term, except the last term on the right-hand side, in (2.10) converges a.s. and hence the last term converges a.s. as well. Note we have shown that it converges in L^2 to $M_t(sgn(x-\cdot))$. Then it follows that the last term converges a.s. to $M_t(sgn(x-\cdot))$ and so (2.9) for any fix $t \geq 0$ follows from (2.10) .

Now take countable union of null sets to see that with $\mathbb{P}_{\delta_0}$-probability one, we have (2.9) holds for all rational $t \geq 0$. Note by Theorem B we have $t \mapsto D_x L_t^x$ is continuous for all $t \geq 0$ $\mathbb{P}_{\delta_0}$-a.s.. For the right-hand side terms of (2.9), since $X_t(\{x\}) = 0$ for all $t \geq 0$ $\mathbb{P}_{\delta_0}$-a.s., the weak continuity of $t \mapsto X_t$ for all $t \geq 0$ would give us the continuity of $t \mapsto X_t(sgn(x-\cdot))$ for all $t \geq 0$. Next since $sgn(x-\cdot)$ is a bounded function and $M_t(sgn(x-\cdot)) = \int_0^t \int sgn(x-y)M(dyds)$ is an integral with respect to the martingale measure, it follows immediately that $t \mapsto M_t(sgn(x-\cdot))$ is continuous for all $t \geq 0$. Therefore we can upgrade the rational $t \geq 0$ to all $t \geq 0$ and the proof is complete. ■

Now we will turn to the proof of Theorem 1.1. By symmetry we can consider the case $x > 0$. Since $X_t(1) = 0$ for $t = \zeta$, $\mathbb{P}_{\delta_0}$-a.s., we use Proposition 2.2 with $t = \zeta$ to see that for any $x > 0$, with $\mathbb{P}_{\delta_0}$-probability one we have

$$L'(x) := D_x L^x = -1 - \int_0^\infty \int sgn(x-z)M(dzds).$$

Define $N_t^{x,y} = \int_0^t \int (sgn(y-z) - sgn(x-z))M(dzds)$ for $x, y > 0$ and $t \geq 0$. Then we have

$$L'(x) - L'(y) = N_\infty^{x,y} = \int_0^\infty \int (sgn(y-z) - sgn(x-z))M(dzds), \tag{2.11}$$

and its quadratic variation is

$$\begin{aligned} [N^{x,y}]_\infty &= \int_0^\infty \int (sgn(y-z) - sgn(x-z))^2 X_s(dz)ds \\ &= \int (sgn(y-z) - sgn(x-z))^2 L^z dz = 4 \left| \int_x^y L^z dz \right|. \end{aligned} \tag{2.12}$$

The second equality is by (1.2) and the last follows since $(sgn(y-z) - sgn(x-z))^2 \equiv 4$ for z between x and y, and $\equiv 0$ otherwise.

The following theorem, which is a generalization of Theorem 4.1 of [MPS06], carries out the main bootstrap idea we use to prove Theorem 1.1: we start from a lower order of Hölder continuity, say ξ_0, of the local time L^x and then upgrade to a higher order of Hölder continuity $\xi_1 \approx (3 + \xi_0)/2$. By iterating we can reach the highest possible order 3.

Theorem 2.3. *Let Z_N be the random set $[\mathsf{R} - 2^{-N}, \mathsf{R}] \cap (0, \infty)$ for any positive integer $N \geq 1$, where R is the r.v. from Theorem A. Assume $\xi_0 \in (0, 3)$ satisfies*

$$\begin{aligned} &\exists 1 \leq N_{\xi_0}(\omega) < \infty \text{ a.s. such that } \forall N \geq N_{\xi_0}(\omega), x \in Z_N, \\ &\forall |y - x| \leq 2^{-N} \Rightarrow |L^x - L^y| \leq 2^{-\xi_0 N}. \end{aligned} \tag{2.13}$$

Then for all $0 < \xi_1 < (3 + \xi_0)/2$,

$$\begin{aligned} &\exists 1 \leq N_{\xi_1}(\omega) < \infty \text{ a.s. such that } \forall N \geq N_{\xi_1}(\omega), x \in Z_N, \\ &\forall |y - x| \leq 2^{-N} \Rightarrow |L^x - L^y| \leq 2^{-\xi_1 N}. \end{aligned} \tag{2.14}$$

Proof. Note that $\mathsf{R} \in Z_N$ for all $N \geq 1$. By (2.13), we have

$$|L^z| = |L^z - L^{\mathsf{R}}| \leq 2^{-\xi_0(N-1)}, \text{ if } z \in Z_{N-1}, \ N \geq N_{\xi_0} + 1. \tag{2.15}$$

Let $N \geq N_{\xi_0} + 1$. For $x \in Z_N$ and $|y - x| \leq 2^{-N}$, we have $y \in Z_{N-1}$ and $z \in Z_{N-1}$ for any z between x and y. Therefore (2.12) implies

$$[N^{x,y}]_\infty = 4 \, |\int_x^y L^z dz| \leq 4 \cdot 2^{-\xi_0(N-1)}|y - x| \leq 2^5 \cdot 2^{-\xi_0 N}|y - x|, \tag{2.16}$$

the first inequality by (2.15) with $z \in Z_{N-1}$.

Pick $1/4 < \eta < 1/2$ such that

$$\eta(1 + \xi_0) + 1 > \xi_1. \tag{2.17}$$

By using the Dubins-Schwarz theorem (see [RY94], Theorem V1.6 and V1.7), with an enlargement of the underlying probability space, we can construct some Brownian motion $(B(t), t \geq 0)$ in $\mathbb{R}$ such that $L'(x) - L'(y) = N^{x,y}_\infty = B([N^{x,y}]_\infty)$. So for any $N \in \mathbb{N}$, we have

$$\begin{aligned}
&\mathbb{P}_{\delta_0}(|L'(x) - L'(y)| \geq 2^5 \cdot 2^{-\eta\xi_0 N}|y - x|^\eta, \ x \in Z_N, \ |y - x| \leq 2^{-N}, \ N \geq N_{\xi_0} + 1) \\
\leq &P(\sup_{s \leq 2^5 \cdot 2^{-\xi_0 N}|y-x|} |B(s)| \geq 2^5 \cdot 2^{-\eta\xi_0 N}|y - x|^\eta) \ \text{ (by (2.16))} \\
\leq &2 \exp(-2^5 \cdot 2^{\xi_0 N(1-2\eta)}|y - x|^{2\eta - 1}).
\end{aligned} \tag{2.18}$$

For $k \geq N$, define

$$M_{k,N} = \max\Big\{ \ |L'(\mathsf{R} - \frac{i+1}{2^k}) - L'(\mathsf{R} - \frac{i}{2^k})| : \ 0 \leq i \leq 2^{k-N}\Big\},$$

and

$$A_N = \Big\{\omega : \exists \ k \geq N \ s.t. \ M_{k,N} \geq 2^5 \cdot 2^{-\eta\xi_0 N} 2^{-\eta k}, N \geq N_{\xi_0} + 1\Big\}.$$

Note for each $0 \leq i \leq 2^{k-N}$, we have $\mathsf{R} - i2^{-k} \in Z_N$. Let $x = \mathsf{R} - i2^{-k}$ and $y = \mathsf{R} - (i+1)2^{-k}$ in (2.18) to get

$$\begin{aligned}
&\mathbb{P}_{\delta_0}(|L'(\mathsf{R} - \frac{i}{2^k}) - L'(\mathsf{R} - \frac{i+1}{2^k})| \geq 2^5 \cdot 2^{-\eta\xi_0 N} 2^{-\eta k}, k \geq N \geq N_{\xi_0} + 1) \\
\leq &2 \exp(-2^5 \cdot 2^{\xi_0 N(1-2\eta)} 2^{k(1-2\eta)}),
\end{aligned} \tag{2.19}$$

and hence

$$\mathbb{P}_{\delta_0}\Big(\bigcup_{N'=N}^\infty A_{N'}\Big) \leq \sum_{N'=N}^\infty \sum_{k=N'}^\infty (2^{k-N'}+1) \cdot 2 \exp(-2^5 \cdot 2^{\xi_0 N'(1-2\eta)} 2^{k(1-2\eta)}) \leq c_0 \exp(-c_1 2^{N(1+\xi_0)(1-2\eta)})$$

for some constants $c_0, c_1 > 0$. Let

$$N_1 = \min\{N \in \mathbb{N} : \omega \in \bigcap_{N'=N}^\infty A^c_{N'}\}.$$

The above implies

$$\mathbb{P}_{\delta_0}(N_1 > N) = \mathbb{P}_{\delta_0}\Big(\bigcup_{N'=N}^{\infty} A_{N'}\Big) \le c_0 \exp(-c_1 2^{N(1+\xi_0)(1-2\eta)}),$$

and so N_1 is an a.s. finite random variable. Define

$$N_{\xi_1} = N_1 \vee (N_{\xi_0}+1) \vee \frac{12}{\eta(1+\xi_0)+1-\xi_1} \vee 1, \tag{2.20}$$

where the third one is well defined by (2.17). For all $N \ge N_{\xi_1}$, $k \ge N$, $x \in Z_N$ and $|y-x| \le 2^{-N}$, let $x_k = \mathsf{R} - \lfloor 2^k(\mathsf{R}-x)\rfloor 2^{-k} \downarrow x$ and $y_k = \mathsf{R} - \lfloor 2^k(\mathsf{R}-y)\rfloor 2^{-k} \downarrow y$. Then $|x_k - x_{k+1}| \le 2^{-(k+1)}$ and $|y_k - y_{k+1}| \le 2^{-(k+1)}$. Note $x_N, y_N \in \{\mathsf{R}, \mathsf{R}-2^{-N}, \mathsf{R}-2^{1-N}\}$ and $|x_N - y_N| \le 2^{-N}$ since $|y-x| \le 2^{-N}$. The continuity of $L'(x)$ gives

$$L'(x) = -L'(x_N) + \sum_{k=N}^{\infty}\big(L'(x_k) - L'(x_{k+1})\big),$$

and

$$L'(y) = -L'(y_N) + \sum_{k=N}^{\infty}\big(L'(y_k) - L'(y_{k+1})\big).$$

So

$$\begin{aligned} &|L'(x) - L'(y)| \\ \le&|L'(x_N) - L'(y_N)| + \sum_{k=N+1}^{\infty}\Big(|L'(x_k) - L'(x_{k+1})| + |L'(y_k) - L'(y_{k+1})|\Big) \\ \le&M_{N,N} + \sum_{k=N}^{\infty} 2M_{k+1,N} \le 2^5 \cdot 2^{-\eta\xi_0 N}2^{-\eta N} + 2\sum_{k=N}^{\infty} 2^5 \cdot 2^{-\eta\xi_0 N}2^{-\eta(k+1)} \\ \le&2^{10}\cdot 2^{-\eta N(\xi_0+1)}, \end{aligned} \tag{2.21}$$

where we have used the definitions of $M_{k,N}$ and A_N and $N \ge N_{\xi_1} \ge N_1 \vee (N_{\xi_0}+1)$ by (2.20) in the third line. Let $x = z \in Z_N$ and $y = \mathsf{R}$ in above. Then use $L'(\mathsf{R}) = 0$ to see that

$$|L'(z)| \le 2^{10}\cdot 2^{-N\eta(1+\xi_0)},\ \forall z \in Z_N, N \ge N_{\xi_1}. \tag{2.22}$$

Let $N \ge N_{\xi_1}+1$. For $x \in Z_N$ and $|y-x| \le 2^{-N}$, we have $y \in Z_{N-1}$ and $z \in Z_{N-1}$ for any z between x and y. Use (2.22) to get

$$|L(y) - L(x)| = |L'(z)||y-x| \le 2^{10}\cdot 2^{-(N-1)\eta(1+\xi_0)}2^{-N} \le 2^{-\xi_1 N},$$

the last by $N > N_{\xi_1} > 12/(\eta(1+\xi_0)+1-\xi_1)$ and (2.17). ■

Theorem 1.1 follows from the following corollary of the above result.

Corollary 2.4. *Let $\gamma \in (0,3)$. Then $\mathbb{P}_{\delta_0}$-a.s. there is a random variable $\delta(\gamma,\omega) > 0$ such that for any $0 < \mathsf{R} - x < \delta$, we have $L^x \le 2^\gamma(\mathsf{R}-x)^\gamma$.*

Proof. By Theorem 2.2 in [MP17], for any $0 < \xi_0 < 1$, with $\mathbb{P}_{\delta_0}$-probability one, there is some $0 < \rho(\omega) \le 1$ such that

$$|L^y - L^x| < |y - x|^{\xi_0}, \quad \text{for } x, y > 0 \text{ with } |y - x| < \rho. \tag{2.23}$$

Note we may set $\varepsilon_0 = 0$ in Theorem 2.2 of [MP17] due to the global continuity of L^x in $d = 1$. Pick $\xi_0 = 1/2$, then (2.13) in Theorem 2.3 holds for $N \ge N_{\xi_0}(\omega) = 1 \vee \log_2(\rho(\omega)^{-1})$. Inductively, define $\xi_{n+1} = \frac{1}{2}(3 + \xi_n)(1 - \frac{1}{n+3})$ so that $\xi_{n+1} \uparrow 3$. Pick n_0 such that $\xi_{n_0} \ge \gamma > \xi_{n_0-1}$. Apply Theorem 2.3 inductively n_0 times to get (2.13) for $\xi_0 = \xi_{n_0-1}$ and hence, (2.14) with $\xi_1 = \xi_{n_0}$.

Consider $0 < \mathsf{R} - x \le 2^{-N_{\xi_{n_0}}}$. Choose $N \ge N_{\xi_{n_0}}$ such that $2^{-(N+1)} < \mathsf{R} - x \le 2^{-N}$. Then $x \in Z_N$ and (2.14) with $\xi_1 = \xi_{n_0}$ implies

$$|L^x| = |L^x - L^{\mathsf{R}}| \le 2^{-N\xi_{n_0}} \le 2^{-N\gamma} \le (2(\mathsf{R} - x))^\gamma = 2^\gamma (\mathsf{R} - x)^\gamma. \tag{2.24}$$

The proof is completed by choosing $\delta = 2^{-N_{\xi_{n_0}}} > 0$. ■

3 Lower bound of the local time near the boundary

Proof of Theorem 1.2. The proof of the lower bound on the local time near the boundary requires an application of Dynkin's exit measures of super-Brownian motion X. The exit measure of X from an open set G under $\mathbb{P}_{X_0}$ is denoted by X_G (see Chp. V of [Leg99a] for the construction of the exit measure). Intuitively X_G is a random finite measure supported on ∂G, which corresponds to the mass started at X_0 which is stopped at the instant it leaves G. The Laplace functional of X_G is given by

$$\mathbb{E}_{X_0}(\exp(-X_G(g)) = \exp\Big(-\int U^g(x) X_0(dx)\Big), \tag{3.1}$$

where $g : \partial G \to [0, \infty)$ is continuous and $U^g \ge 0$ is the unique continuous function on $\overline{G}$ which is C^2 on G and solves

$$\Delta U^g = (U^g)^2 \text{ on } G, \quad U^g = g \text{ on } \partial G. \tag{3.2}$$

Now we work with a one-dimensional super-Brownian motion X with initial condition $y_0\delta_0$. For $r > 0$ we let $Y_r\delta_r$ denote the exit measure $X_{(-\infty,r)}$ from $(-\infty, r)$ and set $Y_0 = y_0$. Then Proposition 4.1 of [MP17] implies under $\mathbb{P}_{y_0\delta_0}$ there is a cadlag version of Y which is a stable continuous state branching process (SCSBP) starting at y_0 with parameter $3/2$, and so is an $(\mathcal{F}_r^Y)_{r\ge0}$-martingale with $\mathcal{F}_r^Y = \sigma(Y_s, s \le r)$ (see Section II.1 of [Leg99a] for the definition of (SCSBP)). In particular (4.6) in [MP17] gives

$$\mathbb{E}_{y_0\delta_0}(\exp(-\lambda Y_r)) = \exp(-6y_0(r + \sqrt{6/\lambda})^{-2}), \quad \forall \lambda \ge 0, r \ge 0.$$

Let $\lambda \uparrow \infty$, we have

$$\mathbb{P}_{y_0\delta_0}(Y_r = 0) = \exp(-6y_0 r^{-2}), \quad \forall r \ge 0. \tag{3.3}$$

Let $R_n = \inf\{r \geq 0 : Y_r \leq 2^{-n}\} \uparrow \mathsf{R} = \inf\{r \geq 0 : Y_r = 0\}$ as $n \to \infty$. Note the R defined here will give the same R in Theorem A. By repeating the arguments in the proof of Theorem 1.7 in [MP17], for any $\beta > 3/2$, we have

$$\text{w.p.1 } \exists N_0(\omega) < \infty, \text{ so that } \inf_{0<x<R_n} L^x > 2^{-n\beta}, \ \forall n > N_0. \tag{3.4}$$

Note again we may set $\varepsilon_0 = 0$ in Theorem 2.2 of [MP17] due to the global continuity of L^x in $d = 1$ to get the above. The definition of R_n implies $Y(R_n) = 2^{-n}, \mathbb{P}_{\delta_0}$-a.s. as Y is a SCSBP and hence it only has positive jumps, i.e. it is spectrally positive (see [CLB09]). So for any $0 < \xi < 1/2$, recalling that the non-negative martingale Y stops at 0 when it hits 0 at time R, we see that

$$\begin{aligned}
\mathbb{P}_{\delta_0}(|R_n - \mathsf{R}| > (2^{-n})^\xi) =& \mathbb{P}_{\delta_0}(\mathsf{R} > R_n + (2^{-n})^\xi) \leq \mathbb{P}_{\delta_0}(Y_{R_n+2^{-n\xi}} > 0) \\
=& \mathbb{E}_{\delta_0}(\mathbb{P}_{\delta_0}(Y_{R_n+2^{-n\xi}} > 0 | \mathcal{F}^Y_{R_n})) = \mathbb{E}_{\delta_0}(\mathbb{P}_{Y_{R_n}\delta_0}(Y_{2^{-n\xi}} > 0)) \\
=& \mathbb{E}_{\delta_0}(1 - \exp(-6Y_{R_n} 2^{2n\xi})) \\
\leq& \mathbb{E}_{\delta_0}(6Y_{R_n} 2^{2n\xi}) = 6(\frac{1}{2^n})^{1-2\xi},
\end{aligned}$$

where the second line holds by the strong Markov property of Y, and the third line uses (3.3). By Borel-Cantelli Lemma, w.p.1 there is some $N_1(\omega) < \infty$ such that

$$|R_n - \mathsf{R}| \leq (\frac{1}{2^n})^\xi, \ \forall n \geq N_1. \tag{3.5}$$

For any fixed $\gamma > 3$, pick $0 < \xi < 1/2$ such that $\gamma\xi > 3/2$. Let $\beta = \gamma\xi > 3/2$ in (3.4) and define $N(\omega) = N_0(\omega) \vee N_1(\omega) < \infty$. Then it follows from (3.5) that

$$|R_n - \mathsf{R}|^\gamma \leq (\frac{1}{2^n})^{\gamma\xi}, \ \forall n \geq N \geq N_1. \tag{3.6}$$

For all $R_N \leq x < \mathsf{R}$, there is some $n \geq N$ such that $R_n \leq x < R_{n+1}$. Now use (3.4) with $n \geq N \geq (N_0 \vee N_1)$ to get

$$\begin{aligned}
|L^x - L^{\mathsf{R}}| = L^x \geq& \inf_{0<y<R_{n+1}} L^y > 2^{-\gamma\xi(n+1)} \geq 2^{-\gamma/2}(\frac{1}{2^n})^{\gamma\xi} \\
\geq& 2^{-\gamma/2}|R_n - \mathsf{R}|^\gamma \geq 2^{-\gamma/2}|x - \mathsf{R}|^\gamma,
\end{aligned}$$

where the second last inequality is by (3.6). The proof is completed by choosing $\delta = \mathsf{R} - R_N > 0$. ∎

4 The case under the canonical measure

In this paper we use Le Gall's Brownian snake approach to study super-Brownian motion under the canonical measure. Define $\mathcal{W} = \cup_{t\geq 0} C([0,t], \mathbb{R}^d)$, equipped with the metric given in Chp IV.1 of [Leg99a], and denote by $\zeta(w) = t$ the lifetime of $w \in C([0,t], \mathbb{R}^d) \subset \mathcal{W}$. The Brownian snake $W = (W_t, t \geq 0)$ constructed in Ch. IV of [Leg99a] is a $\mathcal{W}$-valued continuous strong Markov process and we denote by $\mathbb{N}_{x_0}$ the excursion measure of W away from the trivial path x_0 for $x_0 \in \mathbb{R}^d$ with zero lifetime. The law of $X = X(W)$ under $\mathbb{N}_{x_0}$, constructed in

Theorem IV.4 of [Leg99a], is the canonical measure of super-Brownian motion described in the introduction (also denoted by $\mathbb{N}_{x_0}$). For our purpose it suffices to note that if $\Xi = \sum_{i\in I} \delta_{W_i}$ is a Poisson point process on the space of continuous $\mathcal{W}$-valued paths with intensity $\mathbb{N}_{x_0}(dW)$, then

$$X_t(W) = \sum_{i\in I} X_t(W_i) = \int X_t(W)\Xi(dW),\ t > 0,$$

has the law, $\mathbb{P}_{\delta_{x_0}}$, of a super-Brownian motion X starting from δ_{x_0}. Compared to (1.4), (2.19) of [MP17] implies that the local time L^x may also be decomposed as

$$L^x(W) = \sum_{i\in I} L^x(W_i) = \int L^x(W)\Xi(dW). \tag{4.1}$$

Under the excursion measure $\mathbb{N}_{x_0}$, let $\sigma(W) = \inf\{t \geq 0 : \zeta_t = 0\} > 0$ be the length of the excursion path where $\zeta_t = \zeta(W_t)$ is the life time of W_t and $\hat{W}_t = W_t(\zeta_t)$ be the "tip" of the snake at time t. Then (2.20) of [MP17] implies that for any measurable function $\phi \geq 0$,

$$\int_0^\infty X_s(\phi)ds = \int L^x\phi(x)dx = \int_0^\sigma \phi(\hat{W}_s)ds. \tag{4.2}$$

Proof of Theorem 1.4. Let $\mathsf{R} = \sup\{x \geq 0 : L^x > 0\}$ and $\mathsf{L} = \inf\{x \leq 0 : L^x > 0\}$. First we show that $L^0 > 0$, $\mathbb{N}_0$-a.e., and then by Theorem 1.2 of [Hong18], the continuity of local times under $\mathbb{N}_0$ in $d = 1$ would imply that $\mathsf{L} < 0 < \mathsf{R}$, $\mathbb{N}_0$-a.e..

Define the occupation measure $\mathcal{Z}$ by $\mathcal{Z}(A) = \int_0^\sigma 1_A(\hat{W}_s)ds$ for all Borel measurable set A on $\mathbb{R}$. Then (4.2) implies that under $\mathbb{N}_{x_0}$, the local time L^x coincides with the density function of the occupation measure $\mathcal{Z}$, which we denote by $L^x(\mathcal{Z})$. By the Palm measure formula for $\mathcal{Z}$ (see Proposition 16.2.1 of [Leg99b]) with $F(y, \mathcal{Z}) = \exp(-\lambda L^0(\mathcal{Z}))$ for any $\lambda > 0$, we see that

$$\begin{aligned}\mathbb{N}_0\Big(\mathcal{Z}(1)1(L^0 = 0)\Big) &= \lim_{\lambda\to\infty} \mathbb{N}_0\Big(\mathcal{Z}(1)\exp(-\lambda L^0(\mathcal{Z}))\Big) \\ &= \lim_{\lambda\to\infty} \int_0^\infty da \int P_0^a(dw)E^{(w)}\Big(\exp(-\lambda \int L^0(\mathcal{Z}(\omega))\mathcal{N}(dtd\omega)\Big) \\ &= \lim_{\lambda\to\infty} \int_0^\infty da \int P_0^a(dw)\exp\Big(-\int_0^{\zeta(w)} \mathbb{N}_{w(t)}\big(1 - \exp(-\lambda L^0)\big)dt\Big),\end{aligned} \tag{4.3}$$

where P_0^a is the law of Brownian motion in $\mathbb{R}$ started at 0 and stopped at time a and for each w under P_0^a, the probability measure $P^{(w)}$ is defined on an auxiliary probability space and such that under $P^{(w)}$, $\mathcal{N}(dtd\omega)$ is a Poisson point measure with intensity $1_{[0,\zeta(w)]}(t)dt\mathbb{N}_{w(t)}(d\omega)$. Note here we have taken our branching rate for X to be one and so our constants will differ from those in [Leg99b]. For each w under P_0^a, we have $\zeta(w) = a$. Therefore the left-hand side of (4.3) is equal to

$$\int_0^\infty da \int P_0^a(dw)\exp\Big(-\int_0^a \mathbb{N}_{w(t)}\big(L^0 > 0\big)dt\Big) = \int_0^\infty da \int P_0^a(dw)\exp\Big(-\int_0^a \frac{6}{|w(t)|^2}dt\Big),$$

the last by (2.12) of [MP17]. By Levy's modulus of continuity, we have $\int_0^a 6/|w(t)|^2dt = \infty$, P_0^a-a.s. for each $a > 0$ and hence the above implies $\mathbb{N}_0\Big(\mathcal{Z}(1)1(L^0 = 0)\Big) = 0$. Since $\mathcal{Z}(1) = \sigma > 0$, $\mathbb{N}_0$-a.e., we have

$$L^0 > 0, \quad \mathbb{N}_0 - a.e.. \tag{4.4}$$

Now we will show that L^x is strictly positive on (L, R). Fix $\varepsilon > 0$ and let $L = (L^x, x > \varepsilon)$. Note that $\mathsf{R} \le \varepsilon$ implies $L^x \equiv 0$ for all $x > \varepsilon$ by definition. Then the canonical decomposition (4.1) implies that under $\mathbb{P}_{\delta_0}$, (L, N_ε) is equal in law to $(\sum_{i=1}^{N_\varepsilon} L_i, N_\varepsilon)$, where N_ε is a Poisson random variable with parameter $\mathbb{N}_0(\mathsf{R} > \varepsilon) < \infty$ and given N_ε, $(L_i = (L_i^x, x > \varepsilon))_{i \in \mathbb{N}}$ are i.i.d. with law $\mathbb{N}_0(L \in \cdot \big| \mathsf{R} > \varepsilon)$. Theorem A implies that

$$0 = \mathbb{P}_{\delta_0}(N_\varepsilon = 1; \exists \varepsilon < x < \mathsf{R}, L^x = 0) = \mathbb{P}_{\delta_0}(N_\varepsilon = 1)\mathbb{N}_0(\exists \varepsilon < x < \mathsf{R}, L^x = 0 \big| \mathsf{R} > \varepsilon).$$

Therefore we have $\mathbb{N}_0(\exists \varepsilon < x < \mathsf{R}, L^x = 0; \mathsf{R} > \varepsilon) = 0$ for all $\varepsilon > 0$. Let $\varepsilon \downarrow 0$ to see that $\mathbb{N}_0(\exists 0 < x < \mathsf{R}, L^x = 0; \mathsf{R} > 0) = 0$. Since $\mathsf{R} > 0$, $\mathbb{N}_0$-a.e., we have $L^x > 0, \forall 0 < x < \mathsf{R}$, $\mathbb{N}_0$-a.e.. Use symmetry to conclude for L. ■

Proof of Theorem 1.5. Fix $\varepsilon > 0$ and let $L = (L^x, x > \varepsilon)$. Use the same canonical decomposition above to see that under $\mathbb{P}_{\delta_0}$, (L, N_ε) is equal in law to $(\sum_{i=1}^{N_\varepsilon} L_i, N_\varepsilon)$, where N_ε and $(L_i = (L_i^x, x > \varepsilon))_{i \in \mathbb{N}}$ are as above. For any $\gamma \in (0, 3)$, use Corollary 2.4 to see that

$$\begin{aligned} 0 &= \mathbb{P}_{\delta_0}(N_\varepsilon = 1; \exists x_n > \varepsilon,\ x_n \uparrow \mathsf{R},\ s.t.\ L^{x_n} > 2^3(\mathsf{R} - x_n)^\gamma \text{ i.o.}) \\ &= \mathbb{P}_{\delta_0}(N_\varepsilon = 1)\mathbb{N}_0(\exists x_n > \varepsilon,\ x_n \uparrow \mathsf{R},\ s.t.\ L^{x_n} > 2^3(\mathsf{R} - x_n)^\gamma \text{ i.o.} \big| \mathsf{R} > \varepsilon), \end{aligned}$$

where i.o. represents infinitely often. Therefore we have $\mathbb{N}_0(\exists x_n > \varepsilon,\ x_n \uparrow \mathsf{R},\ s.t.\ L^{x_n} > 2^3(\mathsf{R} - x_n)^\gamma \text{ i.o. }; \mathsf{R} > \varepsilon) = 0$ for all $\varepsilon > 0$. Let $\varepsilon \downarrow 0$ to see that $\mathbb{N}_0(\exists x_n > 0,\ x_n \uparrow \mathsf{R},\ s.t.\ L^{x_n} > 2^3(\mathsf{R} - x_n)^\gamma \text{ i.o. }; \mathsf{R} > 0) = 0$. Since $\mathsf{R} > 0$, $\mathbb{N}_0$-a.e., we have $\mathbb{N}_0$-a.e. that $\exists \delta > 0,\ s.t.\ \forall 0 < \mathsf{R} - x < \delta,\ L^x \le 2^3(\mathsf{R} - x)^\gamma$. Use symmetry to conclude for L and hence Theorem 1.1 holds if $\mathbb{P}_{\delta_0}$ is replaced with $\mathbb{N}_0$. The proof of Theorem 1.2 under $\mathbb{N}_0$ follows by similar arguments and Corollary 1.3 under $\mathbb{N}_0$ follows immediately from Theorem 1.2 under $\mathbb{N}_0$. ■

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