

TRANSLATION INVARIANT FILTERS AND VAN DER WAERDEN'S THEOREM

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ABSTRACT. We present a self-contained proof of a strong version of van der Waerden's Theorem. By using translation invariant filters that are maximal with respect to inclusion, a simple inductive argument shows the existence of "piecewise syndetically"-many monochromatic arithmetic progressions of any length k in every finite coloring of the natural numbers. All the presented constructions are constructive in nature, in the sense that the involved maximal filters are defined by recurrence on suitable countable algebras of sets. No use of the axiom of choice or of Zorn's Lemma is needed.

INTRODUCTION

The importance of maximal objects in mathematics is well-known, starting from the fundamental examples of maximal ideals in algebra, and of ultrafilters in certain areas of topology and of Ramsey theory. In this paper we focus on maximal filters on suitable countable algebras of sets which are stable under translations. By using such maximal objects, along with ultrafilters extending it, we give a proof of a strong version of the following classical result in Ramsey theory:

Theorem (van der Waerden - 1927) *In every finite partition $\mathbb{N} = C_1 \cup \dots \cup C_r$, there exists a piece $C = C_i$ that contains arbitrarily long arithmetic progressions, that is, for every k there exists a progression $x + y, x + 2y, \dots, x + ky \in C$.*

In fact, we will prove the existence of "piecewise syndetically"-many monochromatic arithmetic progressions of any length k .

Usually, van der Waerden's Theorem is proved either by double induction using elementary, but elaborated, combinatorial arguments in the style of the original proof [5], or by using properties of the smallest ideal $K(\beta\mathbb{N}, \oplus)$ in the algebra of ultrafilters (see [4, Ch.14]; see also [2, 1] for stronger versions). In our proof, for any given piecewise set, we restrict to a suitable countable algebra of sets, and explicitly construct by recursion a maximal translation invariant filter, and then an ultrafilter extending it. The desired result is finally obtained by a short proof by induction, that is essentially a simplified version of an argument that was used in [3] in the framework of the compact right-topological semigroup $(\beta\mathbb{N}, \oplus)$. It is worth remarking that, contrarily to the usual ultrafilter proof, we make no explicit use of the algebra in the space of ultrafilters; in fact, we make no use of the axiom of choice nor of Zorn's Lemma.

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1. PRELIMINARY NOTIONS

$\mathbb{N} = \{1, 2, 3, \dots\}$ denotes the set of *positive integers*, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ the set of non-negative integers. For $A \subseteq \mathbb{N}$ and $n \in \mathbb{N}_0$, the *leftward shift* of A by n is the set:

$$A - n := \{m \in \mathbb{N} \mid n + m \in A\}$$

Elemental notions in combinatorics of numbers that we will use in this paper are those of thick set, syndetic set, and piecewise syndetic set. For completeness, let us recall them here.

A set $A \subseteq \mathbb{N}$ is *thick* if it includes arbitrarily long intervals. Equivalently, A is thick if every finite set $F = \{n_1, \dots, n_k\} \subset \mathbb{N}$ has a *rightward shift* included in A , that is, there exists x such that

$$F + x := \{n_1 + x, \dots, n_k + x\} \subseteq A.$$

Notice that such an x can be picked in A . In terms of intersections, the property of thickness of A can be rephrased by saying that the family $\{A - n \mid n \in \mathbb{N}_0\}$ has the *finite intersection property* (FIP for short), that is, $\bigcap_{i=1}^k (A - n_i) \neq \emptyset$ for any n_1, \dots, n_k .

A set $A \subseteq \mathbb{N}$ is *syndetic* if it has “bounded gaps”, that is, there exists $k \in \mathbb{N}$ such that A meets every interval of length k . Equivalently, A is syndetic if a finite number of leftward shifts of A covers all the natural numbers, that is, $\mathbb{N} = \bigcup_{i=1}^k (A - n_i)$ for suitable $n_1, \dots, n_k \in \mathbb{N}_0$.

A set is *piecewise syndetic* if it is the intersection of a thick set with a syndetic set. Equivalently, A is piecewise syndetic if a finite number of leftward shifts cover a thick set, that is, $\bigcup_{i=1}^k (A - n_i)$ is thick for suitable $n_1, \dots, n_k \in \mathbb{N}_0$.

Notice that the families of thick, syndetic, and piecewise syndetic sets are all invariant with respect to shifts. A well-known relevant property of piecewise syndetic sets that is satisfied neither by thick sets nor by syndetic sets, is the *Ramsey property* below. For the sake of completeness, we include here a proof.

Proposition 1.1. *In every finite partition $A = C_1 \cup \dots \cup C_r$ of a piecewise syndetic set A , one of the pieces C_i is piecewise syndetic.*

Proof. For simplicity, let us say that an interval I is k -good for the set B if for every sub-interval $J \subseteq I$ of length k one has $J \cap B \neq \emptyset$. By the hypothesis of piecewise syndeticity of A , there exists $k \in \mathbb{N}$ and a sequence of intervals $\langle I_n \mid n \in \mathbb{N} \rangle$ with increasing length such that every I_n is k -good for A . It is enough to consider the case when $A = C_1 \cup C_2$ is partitioned into two pieces, because the general case $A = C_1 \cup \dots \cup C_r$ where $r \geq 2$ will then follow by induction. We distinguish two cases.

Case # 1: There exists h such that infinitely many intervals I_n are h -good for C_1 . In this case C_1 is piecewise syndetic.

Case # 2: For every h , there are only finitely many intervals I_n that are h -good for C_1 . So, for every h we can pick an interval I_{n_h} of length $\geq h$ that is not h -good. Let $J_h \subseteq I_{n_h}$ be a sub-interval of length h such that $J_h \cap C_1 = \emptyset$. The sequence of intervals $\langle J_h \mid h \in \mathbb{N} \rangle$ shows that C_2 is piecewise syndetic. Indeed, given h , for every sub-interval $J \subseteq J_h$ of length k we have that $J \cap C_1 \subseteq J_h \cap C_1 = \emptyset$; and so $J \cap C_2 = J \cap A \neq \emptyset$, since $J \subseteq I_{n_h}$ and I_{n_h} is k -good for A . \square

2. MAXIMAL TRANSLATION INVARIANT FILTERS

In the following, by *family* we mean a nonempty collection of subsets of \mathbb{N} .

Definition 2.1. A family \mathcal{G} is *translation invariant* if $A \in \mathcal{G} \Rightarrow A - 1 \in \mathcal{G}$ (and hence, $A - n \in \mathcal{G}$ for all $n \in \mathbb{N}_0$).

An *algebra of sets* (on \mathbb{N}) is a family that contains \mathbb{N} and is closed under finite unions, finite intersections, and complements. The [translation invariant] algebra *generated* by a family \mathcal{G} is the smallest [translation invariant] algebra of sets that contains \mathcal{G} .

Proposition 2.2. *If the family \mathcal{G} is countable, then one can give explicit constructions of both the (countable) algebra generated by \mathcal{G} , and the (countable) translation invariant algebra generated by \mathcal{G} .*

Proof. Let $\langle A_n \mid n \in \mathbb{N} \rangle$ be an enumeration of the sets in \mathcal{G} , and let $\langle F_n \mid n \in \mathbb{N} \rangle$ be an enumeration of the nonempty finite sets of natural numbers.¹ For $A \subseteq \mathbb{N}$, denote $A^{+1} = A$ and $A^{-1} = A^c$. Then the following family $\mathcal{B}_{\mathcal{G}}$ is the smallest algebra of sets that contains \mathcal{G} :

$$\mathcal{B}_{\mathcal{G}} := \left\{ \bigcup_{i=1}^t \left(\bigcap_{k \in F_{n_i}} A_k^{\sigma_i(k)} \right) \mid n_1, \dots, n_t \in \mathbb{N}, \sigma_i : F_{n_i} \rightarrow \{+1, -1\} \right\}.$$

Notice that if \mathcal{G} is translation invariant, then also $\mathcal{B}_{\mathcal{G}}$ is translation invariant. So, the algebra $\mathcal{B}_{\mathcal{G}'}$ generated by the family of shifts $\mathcal{G}' := \{A - n \mid A \in \mathcal{G}, n \in \mathbb{N}_0\}$ is the smallest translation invariant algebra containing \mathcal{G} . \square

A *filter* on an algebra of sets \mathcal{B} is a nonempty family $\mathcal{F} \subseteq \mathcal{B}$ such that:

- \mathcal{F} is closed under finite intersections, that is, $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$;
- \mathcal{F} is closed under supersets, that is, if $B \in \mathcal{B}$ and $B \supseteq A \in \mathcal{F}$ then $B \in \mathcal{F}$.

Every family $\mathcal{G} \subseteq \mathcal{B}$ with the *finite intersection property* (FIP for short) generates a filter $\langle \mathcal{G} \rangle$, namely

$$\langle \mathcal{G} \rangle := \{B \in \mathcal{B} \mid B \supseteq A_1 \cap \dots \cap A_k \text{ for suitable } A_1, \dots, A_k \in \mathcal{G}\}.$$

An *ultrafilter* \mathcal{U} on the algebra of sets \mathcal{B} is a filter with the additional property that $A \in \mathcal{U}$ whenever $A \in \mathcal{B}$ and the complement $A^c \notin \mathcal{U}$. It is easily verified that a filter \mathcal{U} is an ultrafilter if and only if the *Ramsey property* holds: If $A_1 \cup \dots \cup A_k \in \mathcal{U}$ where all sets $A_i \in \mathcal{B}$, then $A_j \in \mathcal{U}$ for some j . Ultrafilters can also be characterized as those filters that are maximal under inclusion and so, by a straight application of *Zorn's Lemma*, it is proved that every filter can be extended to an ultrafilter.

The following objects are the main ingredient in our proof of van der Waerden's Theorem.

Definition 2.3. A *translation invariant filter* (TIF for short) is a filter \mathcal{F} on a translation invariant algebra \mathcal{B} such that $A \in \mathcal{F} \Rightarrow A - 1 \in \mathcal{F}$ (and hence $A - n \in \mathcal{F}$ for all $n \in \mathbb{N}_0$).

Notice that if the algebra \mathcal{B} is translation invariant, and the family $\mathcal{G} \subseteq \mathcal{B}$ is translation invariant, then the generated filter $\langle \mathcal{G} \rangle$ is a TIF.

The notions of TIF and thick set are closely related.

¹*E.g.*, if $n = \sum_{k=1}^{\infty} a_{nk} 2^{k-1}$ is written in binary expansion where $a_{nk} \in \{0, 1\}$, then we can let $F_n := \{k \mid a_{nk} = 1\}$.

Proposition 2.4. *A set A is thick if and only if it belongs to a TIF \mathcal{F} .*

Proof. Recall that A is thick if and only if the family $\mathcal{G} = \{A - n \mid n \in \mathbb{N}_0\}$ has the FIP. Since \mathcal{G} is translation invariant, the generated filter $\langle \mathcal{G} \rangle$ is a TIF that contains A .

Conversely, assume that $A \in \mathcal{F}$ for some TIF \mathcal{F} . Then trivially the family $\mathcal{G} = \{A - n \mid n \in \mathbb{N}_0\}$ has the FIP because $\mathcal{G} \subseteq \mathcal{F}$. \square

Similarly to ultrafilters, by a straightforward application of *Zorn's Lemma* it can be shown that every TIF can be extended to a maximal TIF. However, in the countable case, recursive constructions suffice to produce both ultrafilters and maximal TIFs, which are thus obtained in a constructive manner, without any use of the axiom of choice.

Proposition 2.5. *Let $\mathcal{B} = \{B_n \mid n \in \mathbb{N}\}$ be a countable algebra of sets.*

- (1) *Given a family $\mathcal{G} \subseteq \mathcal{B}$ with the FIP, inductively define $\mathcal{G}_0 = \mathcal{G}$; $\mathcal{G}_{n+1} = \mathcal{G}_n \cup \{B_n\}$ in case $B_n \cap A \neq \emptyset$ for every $A \in \mathcal{G}_n$; and $\mathcal{G}_{n+1} = \mathcal{G}_n$ otherwise. Then $\mathcal{U} := \bigcup_n \mathcal{G}_n$ is an ultrafilter on \mathcal{B} that extends \mathcal{G} .*
- (2) *Assume that the algebra \mathcal{B} is translation invariant. Given a translation invariant family $\mathcal{G} \subseteq \mathcal{B}$ with the FIP, inductively define $\mathcal{G}_0 = \mathcal{G}$; $\mathcal{G}_{n+1} = \mathcal{G}_n \cup \{B_n - k \mid k \in \mathbb{N}_0\}$ in case that union has the FIP; and $\mathcal{G}_{n+1} = \mathcal{G}_n$ otherwise. Then $\mathcal{M} := \bigcup_n \mathcal{G}_n$ is a maximal TIF that extends \mathcal{G} .*

Proof. (1). By the definition, it is clear that all families \mathcal{G}_n have the FIP, and so also their increasing union \mathcal{U} has the FIP. Now assume by contradiction that $A \in \mathcal{B}$ is such that both $A, A^c \notin \mathcal{U}$. If $A = B_n$ and $A^c = B_m$ then, by the definition of \mathcal{U} , there exist $U \in \mathcal{G}_n$ and $U' \in \mathcal{G}_m$ such that $A \cap U = A^c \cap U' = \emptyset$, and hence $U \cap U' = \emptyset$, against the FIP of \mathcal{U} . Finally, if $B \supseteq A$ where $B \in \mathcal{B}$ and $A \in \mathcal{U}$ then $B \in \mathcal{U}$, as otherwise, by what just proved, $B^c \in \mathcal{U}$, and hence $\emptyset = B^c \cap A \in \mathcal{U}$, a contradiction.

(2). By induction, it directly follows from the definition that all families \mathcal{G}_n have the FIP and are translation invariant; so, the same properties hold for \mathcal{M} . Now let $B \supseteq A$ where $A \in \mathcal{M}$ and $B \in \mathcal{B}$, say $B = B_n$. Notice that $\mathcal{G}_n \cup \{B - k \mid k \in \mathbb{N}_0\}$ has the FIP because $A - k \subseteq B - k$ for all k and $\mathcal{G}_n \cup \{A - k \mid k \in \mathbb{N}_0\} \subseteq \mathcal{M}$ has the FIP. Then $B \in \mathcal{G}_{n+1} \subseteq \mathcal{M}$, and we can conclude that \mathcal{M} is a TIF. As for the maximality, let $\mathcal{M}' \supseteq \mathcal{M}$ be a TIF. Given $A \in \mathcal{M}'$, pick n with $A = B_n$. The family $\mathcal{G}_n \cup \{A - n \mid n \in \mathbb{N}_0\}$ has the FIP, since it is included in the filter \mathcal{M}' , and so $A \in \mathcal{G}_{n+1}$. This shows that $\mathcal{M}' \subseteq \mathcal{M}$, and hence the two TIFs are equal. \square

Two properties of maximal TIFs that will be relevant to our purposes are the following.

Proposition 2.6. *Let \mathcal{B} be a translation invariant algebra, and let \mathcal{U} be an ultrafilter on \mathcal{B} that includes a maximal TIF \mathcal{M} . Then:*

- (1) *Every $B \in \mathcal{U}$ is piecewise syndetic.*
- (2) *For every $B \in \mathcal{U}$, the set $B_{\mathcal{U}} := \{n \in \mathbb{N} \mid B - n \in \mathcal{U}\}$ is syndetic.²*

Proof. Notice first that for every $B \in \mathcal{U}$ there exist n_1, \dots, n_k such that the union $\bigcup_{i=1}^k (B - n_i) \in \mathcal{M}$. Indeed, if $\Lambda := \{B^c - n \mid n \in \mathbb{N}_0\}$ then the union $\mathcal{M} \cup \Lambda$ does not have the FIP, as otherwise $\mathcal{M} \cup \Lambda$ would generate a TIF that properly

² We remark that in general the set $B_{\mathcal{U}}$ does not belong to the algebra of sets \mathcal{B} .

extends \mathcal{M} (since it would contain B^c while $B^c \notin \mathcal{M}$), against the maximality. So, there exist $A \in \mathcal{M}$ and n_1, \dots, n_k such that $A \cap \bigcap_{i=1}^k (B^c - n_i) = \emptyset$. But then $\bigcup_{i=1}^k (B - n_i) \in \mathcal{M}$, because it is a superset of $A \in \mathcal{M}$.

(1). Pick a finite union of shifts $\bigcup_{i=1}^k (B - n_i) \in \mathcal{M}$. By Proposition 2.4, that union is thick because it is as an element of a TIF, and hence B is piecewise syndetic.

(2). As above, pick a finite union of shifts $\bigcup_{i=1}^k (B - n_i) \in \mathcal{M}$. By translation invariance, for every $m \in \mathbb{N}$ one has that $\bigcup_{i=1}^k (B - n_i - m) \in \mathcal{M} \subseteq \mathcal{U}$ and so, by the Ramsey property of ultrafilters, there exists i such that $B - n_i - m \in \mathcal{U}$, that is, $m \in B_{\mathcal{U}} - n_i$. This shows that $\mathbb{N} = \bigcup_{i=1}^k (B_{\mathcal{U}} - n_i)$ is a finite union of shifts of $B_{\mathcal{U}}$, and hence $B_{\mathcal{U}}$ is syndetic. \square

3. A STRONG VERSION OF VAN DER WAERDEN'S THEOREM

The following property of piecewise syndetic sets was first proved by exploiting the properties of ultrafilters in the smallest ideal of the right-topological semigroup $(\beta\mathbb{N}, \oplus)$ (see [2, 1]).

Theorem 3.1. *Let A be a piecewise syndetic set. Then for every $k \in \mathbb{N}$, the set $\text{AP}_k(A) := \{x \in A \mid \exists y \in \mathbb{N} \text{ s.t. } x + iy \in A \text{ for } i = 1, \dots, k\}$ is piecewise syndetic.*

Notice that, as a straight consequence, one obtains the following strong version of van der Waerden's Theorem.

Theorem 3.2. *In every finite partition $\mathbb{N} = C_1 \cup \dots \cup C_r$ there exists a piece $C = C_i$ such that, for every $k \in \mathbb{N}$, the set $\text{AP}_k(C)$ is piecewise syndetic.*

Proof. By the Ramsey property of piecewise syndetic sets (see Proposition 1.1), we can pick a color C_i which is piecewise syndetic. \square

In this section we will give a new proof of the above theorem which relies on the existence of an ultrafilter \mathcal{U} on the appropriate translation invariant algebra \mathcal{B} , which extends a maximal TIF and contains a shift of A .

of Theorem 3.1. Let \mathcal{B} be the (countable) translation invariant algebra of sets generated by the translation invariant family $\{A - n \mid n \in \mathbb{N}_0\}$. By the property of piecewise syndeticity, a finite union of shifts $T = \bigcup_{j=1}^m (A - n_j)$ is thick. Then the translation invariant family $\mathcal{G} := \{T - n \mid n \in \mathbb{N}_0\} \subseteq \mathcal{B}$ has the FIP, and by Proposition 2.5 we can pick a maximal TIF \mathcal{M} on \mathcal{B} with $\mathcal{M} \supseteq \mathcal{G}$, and an ultrafilter \mathcal{U} on \mathcal{B} with $\mathcal{U} \supseteq \mathcal{M}$. The desired result is a consequence of the following general property.

Claim. *Let \mathcal{U} be an ultrafilter that extends a maximal TIF. If a shift $B - \ell \in \mathcal{U}$ for some $\ell \in \mathbb{N}_0$, then $B_{\mathcal{U}} - \ell$ contains arbitrarily long arithmetic progressions.*

Indeed, since the finite union $T = \bigcup_{j=1}^m (A - n_j) \in \mathcal{G} \subseteq \mathcal{U}$, by the Ramsey property of ultrafilters there exists n_j such that $A - n_j \in \mathcal{U}$. By the Claim, for every $k \in \mathbb{N}$ there exist x and y such that $x + iy \in A_{\mathcal{U}} - n_j$ for $i = 0, 1, \dots, k$. But then $B := \bigcap_{i=0}^k (A - n_j - x - iy) \in \mathcal{U}$, and hence also the superset $\text{AP}_k(A) - n_j - x \supseteq B$ belongs to \mathcal{U} , as one can easily verify. Now recall that all sets in \mathcal{U} are piecewise syndetic by Proposition 2.6, and so we can conclude that $\text{AP}_k(A)$ is piecewise syndetic because it is a shift of a member of \mathcal{U} .

We are left to prove the Claim. We proceed by induction on k , and prove that if $B - \ell \in \mathcal{U}$ for some $\ell \in \mathbb{N}_0$, then $B_{\mathcal{U}} - \ell$ contains a k -term arithmetic progression.³

If $B - \ell \in \mathcal{U}$, then the set $(B - \ell)_{\mathcal{U}} = B_{\mathcal{U}} - \ell$ is syndetic by Proposition 2.6. In particular, $B_{\mathcal{U}} - \ell \neq \emptyset$, and this proves the induction base $k = 1$.

Let us turn to the inductive step $k+1$, and assume that $B - \ell \in \mathcal{U}$. Let $\ell_0 = \ell$. By syndeticity of $B_{\mathcal{U}} - \ell_0$, there exists a finite $F \subset \mathbb{N}_0$ such that for every $n \in \mathbb{N}$ there exists $x \in F$ with $\ell_0 + n + x \in B_{\mathcal{U}}$. For convenience, let us assume that $0 \in F$. By the inductive hypothesis, there exist $\ell_1 \in \mathbb{N}_0$ and $y_1 \in \mathbb{N}$ such that $\ell_1 + iy_1 \in B_{\mathcal{U}} - \ell_0$ for $i = 1, \dots, k$, that is, $\ell_0 + \ell_1 + x_0 + iy_1 \in B_{\mathcal{U}}$ where $x_0 = 0 \in F$. Pick $x_1 \in F$ with $\ell_0 + \ell_1 + x_1 \in B_{\mathcal{U}}$. If $x_1 = x_0$ then we already found a $(k+1)$ -term arithmetic progression in $B_{\mathcal{U}} - \ell_0$, as desired. Otherwise, let us consider the intersection

$$B_1 := (B - x_1) \cap \bigcap_{i=1}^k (B - x_0 - iy_1).$$

Since $\ell_0 + \ell_1 + x_1 \in B_{\mathcal{U}}$ and $\ell_0 + \ell_1 + x_0 + iy_1 \in B_{\mathcal{U}}$ for all $i = 1, \dots, k$, the shift $B_1 - \ell_0 - \ell_1 \in \mathcal{U}$ and so, by the inductive hypothesis, there exist $\ell_2 \in \mathbb{N}_0$ and $y_2 \in \mathbb{N}$ such that $\ell_2 + iy_2 \in (B_1)_{\mathcal{U}} - \ell_0 - \ell_1$ for $i = 1, \dots, k$. In consequence, $\ell_0 + \ell_1 + \ell_2 + x_0 + i(y_1 + y_2) \in B_{\mathcal{U}}$ and $\ell_0 + \ell_1 + \ell_2 + x_1 + iy_2 \in B_{\mathcal{U}}$ for every $i = 1, \dots, k$. Pick $x_2 \in F$ such that $\ell_0 + \ell_1 + \ell_2 + x_2 \in B_{\mathcal{U}}$. Notice that if $x_2 = x_0$ or $x_2 = x_1$ then we have a $(k+1)$ -term arithmetic progression in $B_{\mathcal{U}} - \ell_0$. Otherwise, let us consider the intersection

$$B_2 := (B - x_2) \cap \bigcap_{i=1}^k (B - x_1 - iy_2) \cap \bigcap_{i=1}^k (B - x_0 - i(y_1 + y_2)).$$

Similarly as above, one can easily verify that $B_2 - \ell_0 - \ell_1 - \ell_2 \in \mathcal{U}$ and so, by the inductive hypothesis, we can pick an arithmetic progression in $B_{\mathcal{U}} - \ell_0 - \ell_1 - \ell_2$ of length k . We iterate the procedure. As the set F is finite, after finitely many steps we will find elements $x_n = x_m$ where $n > m$, and finally obtain the following arithmetic progression of length $k+1$:

$$\ell_0 + \ell_1 + \dots + \ell_n + x_n + i(y_{m+1} + \dots + y_n) \quad i = 0, 1, \dots, k.$$

□

4. TIFS AND LEFT IDEALS IN THE SPACE OF ULTRAFILTERS

The usual ultrafilter proof of van der Waerden's Theorem (see [4, §14.1]) is grounded on the existence of minimal ultrafilters, that is, on those ultrafilters that belong to some minimal left ideals of the compact right-topological semigroup $(\beta\mathbb{N}, \oplus)$. In this final section, we show how (maximal) translation invariant filters are in fact related to the closed (minimal) left ideals of $(\beta\mathbb{N}, \oplus)$. Let us recall here the involved notions.

The space $\beta\mathbb{N}$ is the topological space of all ultrafilters \mathcal{U} over the full algebra of sets $\mathcal{B} = \mathcal{P}(\mathbb{N})$ where a base of (cl)open sets is given by the family $\{\mathcal{O}_A \mid A \subseteq \mathbb{N}\}$, with $\mathcal{O}_A := \{\mathcal{U} \in \beta\mathbb{N} \mid A \in \mathcal{U}\}$. The space $\beta\mathbb{N}$ is Hausdorff and compact, and coincides with the *Stone-C ech compactification* of the discrete space \mathbb{N} .

The *pseudosum* $\mathcal{U} \oplus \mathcal{V}$ of ultrafilters $\mathcal{U}, \mathcal{V} \in \beta\mathbb{N}$ is defined by letting:

$$A \in \mathcal{U} \oplus \mathcal{V} \iff \{n \in \mathbb{N} \mid A - n \in \mathcal{V}\} \in \mathcal{U}.$$

³ This inductive construction uses a simplified version of an argument in [3].

The operation \oplus is associative (but not commutative), and for every \mathcal{V} the map $\mathcal{U} \mapsto \mathcal{U} \oplus \mathcal{V}$ is continuous. This makes $(\beta\mathbb{N}, \oplus)$ a *right-topological semigroup*.

A *left ideal* $L \subseteq \beta\mathbb{N}$ is a nonempty set such that $\mathcal{V} \in L$ implies $\mathcal{U} \oplus \mathcal{V} \in L$ for all $\mathcal{U} \in \beta\mathbb{N}$. The notion of *right ideal* is defined similarly. Left ideals that are minimal with respect to inclusion are particularly relevant objects, as they satisfy special properties. For instance, their union $K(\beta\mathbb{N}, \oplus)$ is shown to be the smallest bilater ideal (*i.e.* it is both a left and a right ideal). Moreover, all ultrafilters \mathcal{U} in $K(\beta\mathbb{N}, \oplus)$, named *minimal ultrafilters*, have the property that every set $A \in \mathcal{U}$ includes arbitrarily long arithmetic progressions.⁴

It is well-known that there are natural correspondences between families with the finite intersection property on the full algebra $\mathcal{P}(\mathbb{N})$, and closed nonempty subsets of $\beta\mathbb{N}$. Indeed, the following properties are directly verified from the definitions.

- If $\mathcal{G} \subseteq \mathcal{P}(\mathbb{N})$ is a family with the FIP then $\mathfrak{C}(\mathcal{G}) := \{\mathcal{V} \in \beta\mathbb{N} \mid \mathcal{V} \supseteq \mathcal{G}\}$ is a nonempty closed subspace.
- If $X \subseteq \beta\mathbb{N}$ is nonempty then $\mathfrak{F}(X) := \bigcap \{\mathcal{V} \mid \mathcal{V} \in X\}$ is a filter on $\mathcal{P}(\mathbb{N})$.
- $\mathfrak{C}(\mathfrak{F}(X)) = \overline{X}$ (the topological closure of X) for every nonempty $X \subseteq \beta\mathbb{N}$.
- $\mathfrak{F}(\mathfrak{C}(\mathcal{G})) = \langle \mathcal{G} \rangle$ (the filter generated by \mathcal{G}) for every family $\mathcal{G} \subseteq \mathcal{P}(\mathbb{N})$ with the FIP.

Proposition 4.1. *If \mathcal{F} is a TIF on $\mathcal{P}(\mathbb{N})$ then $\mathfrak{C}(\mathcal{F})$ is a closed left ideal of $(\beta\mathbb{N}, \oplus)$; and conversely, if L is a left ideal of $(\beta\mathbb{N}, \oplus)$ then $\mathfrak{F}(L)$ is a TIF on $\mathcal{P}(\mathbb{N})$. Moreover, \mathcal{M} is a maximal TIF on $\mathcal{P}(\mathbb{N})$ if and only if $\mathfrak{C}(\mathcal{M})$ is a minimal left ideal of $(\beta\mathbb{N}, \oplus)$; and L is a minimal left ideal of $(\beta\mathbb{N}, \oplus)$ if and only if $\mathfrak{F}(L)$ is a maximal TIF on $\mathcal{P}(\mathbb{N})$.*

Proof. Let $\mathcal{V} \in \mathfrak{C}(\mathcal{F})$ and let $\mathcal{U} \in \beta\mathbb{N}$ be any ultrafilter. For every $A \in \mathcal{F}$, by translation invariance we know that $A - n \in \mathcal{F}$ for all n , and so $\{n \mid A - n \in \mathcal{V}\} = \mathbb{N} \in \mathcal{U}$. This shows that $A \in \mathcal{U} \oplus \mathcal{V}$. As this is true for every $A \in \mathcal{F}$, we conclude that $\mathcal{U} \oplus \mathcal{V} \in \mathfrak{C}(\mathcal{F})$, and so $\mathfrak{C}(\mathcal{F})$ is a closed left ideal.

Now let L be a left ideal, and let $A \in \mathfrak{F}(L)$ be in the filter determined by L . For every $\mathcal{V} \in L$, we have that $\mathfrak{U}_1 \oplus \mathcal{V} \in L$, where $\mathfrak{U}_1 := \{B \subseteq \mathbb{N} \mid 1 \in B\}$ is the principal ultrafilter generated by 1. Then $A \in \mathfrak{U}_1 \oplus \mathcal{V}$, which is equivalent to $A - 1 \in \mathcal{V}$. As this holds for every $\mathcal{V} \in L$, we have proved that $A - 1 \in \mathfrak{F}(L)$, and so $\mathfrak{F}(L)$ is a TIF, as desired.

Let \mathcal{F} be a TIF. If the left ideal $\mathfrak{C}(\mathcal{F})$ is not minimal, pick a minimal $L \subsetneq \mathfrak{C}(\mathcal{F})$. Then $\mathcal{F} \subsetneq \mathfrak{F}(L)$, and hence \mathcal{F} is not maximal. Indeed, $L \subseteq \mathfrak{C}(\mathcal{F}) \Rightarrow \mathfrak{F}(L) \supseteq \mathfrak{F}(\mathfrak{C}(\mathcal{F})) = \mathcal{F}$; moreover, $\mathcal{F} \neq \mathfrak{F}(L)$, as otherwise $\mathfrak{C}(\mathcal{F}) = \mathfrak{C}(\mathfrak{F}(L)) = \overline{L} = L$, against our assumptions. (Recall that a minimal left ideal L is necessarily closed because, by minimality, $L = \beta\mathbb{N} \oplus \mathcal{V} := \{\mathcal{U} \oplus \mathcal{V} \mid \mathcal{U} \in \beta\mathbb{N}\}$ for every given $\mathcal{V} \in L$, and $\beta\mathbb{N} \oplus \mathcal{V}$ is closed as it is the image of the compact Hausdorff space $\beta\mathbb{N}$ under the continuous function $\mathcal{U} \mapsto \mathcal{U} \oplus \mathcal{V}$.) In a similar way, one shows the converse implication: If the TIF \mathcal{F} is not maximal then the left ideal $\mathfrak{C}(\mathcal{F})$ is not minimal. In consequence, $L = \mathfrak{C}(\mathfrak{F}(L))$ is minimal if and only if $\mathfrak{F}(L)$ is maximal, and also the last equivalence is proved. \square

As a straight consequence, we obtain the desired characterization.

⁴ For all notions and basic results on the space of ultrafilters $\beta\mathbb{N}$ and on its algebraic structure, including properties of the smallest ideal $K(\beta\mathbb{N}, \oplus)$, we refer the reader to the book [4].

Proposition 4.2. *An ultrafilter \mathcal{U} on $\mathcal{P}(\mathbb{N})$ includes a maximal TIF if and only if \mathcal{U} belongs to the smallest ideal $K(\beta\mathbb{N}, \oplus)$.*

Proof. Recall that $\mathcal{U} \in K(\beta\mathbb{N}, \oplus)$ if and only if \mathcal{U} belongs to some minimal left ideal. Now let $\mathcal{U} \supseteq \mathcal{M}$ where \mathcal{M} is a maximal TIF. Since $\mathcal{M} = \mathfrak{F}(\mathfrak{C}(\mathcal{M}))$, we have that $\mathcal{U} \in \mathfrak{C}(\mathcal{M})$, where $\mathfrak{C}(\mathcal{M})$ is a minimal left ideal. Conversely, let $\mathcal{U} \in L$ where L is a minimal left ideal. Then $\mathfrak{F}(L)$ is a maximal TIF and $\mathcal{U} \supseteq \mathfrak{F}(L)$, since $\mathcal{U} \in L = \mathfrak{C}(\mathfrak{F}(L))$. \square

Remark 4.3. One can generalize the contents of this paper from the natural numbers to arbitrary countable semigroups (S, \cdot) . Indeed, the notion of translation invariant filter also makes sense in that more general framework.⁵ Precisely, for $A \subseteq S$ and $s \in S$, denote by $s^{-1}A := \{t \in S \mid s \cdot t \in A\}$. We say that an algebra \mathcal{B} of subsets of S is *translation invariant* if $B \in \mathcal{B} \Rightarrow s^{-1}B \in \mathcal{B}$ for all $s \in S$. Then one defines a TIF on a translation invariant algebra \mathcal{B} as a filter \mathcal{F} such that $A \in \mathcal{F} \Rightarrow s^{-1}A \in \mathcal{F}$ for all $s \in S$. By the same arguments as the ones used in this paper, one can prove that a reformulation of Theorem 3.1 holds, provided one adopts the appropriate generalization of the notion of piecewise syndetic set.⁶

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⁵ Actually, our techniques also apply for uncountable semigroups, but in that case one needs Zorn’s Lemma to prove the existence of maximal TIFs and of ultrafilters.

⁶ In an arbitrary semigroup (S, \cdot) , one defines a subset $T \subseteq S$ to be *thick* if for every finite F there exists $s \in S$ with $F \cdot s := \{x \cdot s \mid x \in F\} \subseteq T$; a set $A \subseteq S$ is *syndetic* if a suitable finite union $\bigcup_{i=1}^k s_i^{-1}A = S$ covers the whole semigroup; and finally a set $A \subseteq S$ is *piecewise syndetic* if a suitable finite union $\bigcup_{i=1}^k s_i^{-1}A$ is thick (see [4, §4.4 and §4.5]).