

On the number of edges in some graphs [★]

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Abstract

In 1975, P. Erdős proposed the problem of determining the maximum number $f(n)$ of edges in a graph with n vertices in which any two cycles are of different lengths. The sequence (c_1, c_2, \dots, c_n) is the cycle length distribution of a graph G with n vertices, where c_i is the number of cycles of length i in G . Let $f(a_1, a_2, \dots, a_n)$ denote the maximum possible number of edges in a graph which satisfies $c_i \leq a_i$, where a_i is a nonnegative integer. In 1991, Shi posed the problem of determining $f(a_1, a_2, \dots, a_n)$ which extended the problem due to Erdős. It is clear that $f(n) = f(1, 1, \dots, 1)$. Let $g(n, m) = f(a_1, a_2, \dots, a_n)$, where $a_i = 1$ if i/m is an integer, and $a_i = 0$ otherwise. It is clear that $f(n) = g(n, 1)$. We prove that $\liminf_{n \rightarrow \infty} \frac{f(n)-n}{\sqrt{n}} \geq \sqrt{2 + \frac{40}{99}}$, which is better than the previous bounds $\sqrt{2}$ (Shi, 1988), and $\sqrt{2 + \frac{7654}{19071}}$ (Lai, 2017). We show that $\liminf_{n \rightarrow \infty} \frac{g(n,m)-n}{\sqrt{\frac{n}{m}}} > \sqrt{2.444}$, for all even integers m . We make the following conjecture: $\liminf_{n \rightarrow \infty} \frac{f(n)-n}{\sqrt{n}} > \sqrt{2.444}$.

Key words: Graph, cycle, number of edges.

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1 Introduction

Let $f(n)$ be the maximum number of edges in a graph with n vertices in which no two cycles have the same length. In 1975, Erdős raised the problem of determining $f(n)$ (see Bondy and Murty [1], p.247, Problem 11). Shi [11] proved a lower bound.

Theorem 1 (Shi [11])

$$f(n) \geq n + [(\sqrt{8n - 23} + 1)/2]$$

for $n \geq 3$.

Chen, Lehel, Jacobson and Shreve [3], Jia [4], Lai [5,6,7], Shi [13,14] obtained some additional related results.

Boros, Caro, Füredi and Yuster [2] proved an upper bound as follows.

Theorem 2 (Boros, Caro, Füredi and Yuster [2]) For n sufficiently large,

$$f(n) < n + 1.98\sqrt{n}.$$

Lai [8] improved the lower bound by Shi as follows.

Theorem 3 (Lai [8]) Let $t = 1260r + 169$ ($r \geq 1$), then

$$f(n) \geq n + \frac{107}{3}t + \frac{7}{3}$$

for $n \geq \frac{2119}{4}t^2 + 87978t + \frac{15957}{4}$.

Lai [5] proposed the following conjecture:

Conjecture 4 (Lai [5])

$$\liminf_{n \rightarrow \infty} \frac{f(n) - n}{\sqrt{n}} \leq \sqrt{3}.$$

It would be nice to prove that

$$\liminf_{n \rightarrow \infty} \frac{f(n) - n}{\sqrt{n}} \leq \sqrt{3 + \frac{3}{5}}.$$

Survey papers on this problem can be found in Tian [15], Zhang [16], Lai and Liu [9].

The progress of all 50 problems in [1] can be found in Locke [10].

The sequence (c_1, c_2, \dots, c_n) is the cycle length distribution of a graph G with n vertices, where c_i is the number of cycles of length i in G . Let $f(a_1, a_2, \dots, a_n)$ denote the maximum possible number of edges in a graph which satisfies $c_i \leq a_i$, where a_i is a nonnegative integer. Shi [12] posed the problem of determining $f(a_1, a_2, \dots, a_n)$ which extended the problem due to Erdős. It is clear that $f(n) = f(1, 1, \dots, 1)$. Let $g(n, m) = f(a_1, a_2, \dots, a_n)$, where $a_i = 1$ if i/m is an integer, and $a_i = 0$ otherwise. It is clear that $f(n) = g(n, 1)$.

In this paper, we obtain the following results.

Theorem 5 Let m be even, $s_1 > s_2, s_1 + 3s_2 > k$, then

$$g(n, m) \geq n + (k + s_1 + 2s_2 + 1)t - 1$$

for $n \geq (\frac{3}{4}mk^2 + \frac{1}{2}mks_1 + \frac{3}{2}mks_2 + \frac{1}{2}ms_1^2 + \frac{3}{2}ms_1s_2 + \frac{9}{4}ms_2^2 + mk + ms_1 + 3ms_2 + \frac{1}{2}m)t^2 + (\frac{1}{4}mk + \frac{1}{2}ms_1 + \frac{3}{4}ms_2 - k - s_1 - 2s_2 + \frac{1}{2}m - 1)t + 1$.

Theorem 6 Let $t = 1260r + 169$ ($r \geq 1$), then

$$f(n) \geq n + \frac{119}{3}t - \frac{26399}{3}$$

for $n \geq \frac{1309}{2}t^2 - \frac{1349159}{6}t + \frac{6932215}{3}$.

2 Proof of Theorem 5

Proof. Let $n_t = (\frac{3}{4}mk^2 + \frac{1}{2}mks_1 + \frac{3}{2}mks_2 + \frac{1}{2}ms_1^2 + \frac{3}{2}ms_1s_2 + \frac{9}{4}ms_2^2 + mk + ms_1 + 3ms_2 + \frac{1}{2}m)t^2 + (\frac{1}{4}mk + \frac{1}{2}ms_1 + \frac{3}{4}ms_2 - k - s_1 - 2s_2 + \frac{1}{2}m - 1)t + 1$, m be even, $s_1 > s_2, s_1 + 3s_2 > k, n \geq n_t$. It suffice to show that there exists a graph G on n vertices with $n + (k + s_1 + 2s_2 + 1)t - 1$ edges such that all cycles in G have distinct lengths and all the lengths of cycles are the multiple of m .

Now we construct the graph G which consists of a number of subgraphs: B_i , ($0 \leq i \leq s_1t, i = s_1t + j$ ($1 \leq j \leq s_2t$), $i = s_1t + s_2t + j$ ($1 \leq j \leq t$)).

Now we define these B_i s. These subgraphs all only have a common vertex x , otherwise their vertex sets are pairwise disjoint.

For $1 \leq i \leq s_2t$, let the subgraph B_{s_1t+i} consists of a cycle

$$xa_i^1 a_i^2 \dots a_i^{ms_1t+2ms_2t+mi-1} x$$

and a path:

$$xa_{i,1}^1 a_{i,1}^2 \dots a_{i,1}^{\frac{ms_1 t - ms_2 t + mi}{2} - 1} a_i^{\frac{ms_1 t + ms_2 t + mi}{2}}.$$

Based on the construction, $B_{s_1 t + i}$ contains exactly three cycles of lengths:

$$ms_1 t + mi, ms_1 t + ms_2 t + mi, ms_1 t + 2ms_2 t + mi.$$

For $1 \leq i \leq t$, let the subgraph $B_{s_1 t + s_2 t + i}$ consists of a cycle

$$C_{s_1 t + s_2 t + i} = xy_i^1 y_i^2 \dots y_i^{ms_1 t + 3ms_2 t + mk(k+1)t + mi - 1} x$$

and k paths sharing a common vertex x , the other end vertices are on the cycle $C_{s_1 t + s_2 t + i}$:

$$xy_{i,p}^1 y_{i,p}^2 \dots y_{i,p}^{\frac{ms_1 t + 3ms_2 t - mkt + m(p-1)t + mi}{2} - 1} y_i^{\frac{ms_1 t + 3ms_2 t + mk(2p-1)t + m(p-1)t + mi}{2}} \quad (p = 1, 2, \dots, k).$$

As a cycle with k chords contains $\binom{k+2}{2}$ distinct cycles, $B_{s_1 t + s_2 t + i}$ contains exactly $\frac{(k+2)(k+1)}{2}$ cycles of lengths:

$$ms_1 t + 3ms_2 t + mkht + (h + j - 1)mt + mi \quad (j \geq 1, h \geq 0, k + 1 \geq j + h).$$

B_0 is a path with an end vertex x and length $n - n_t$. The other B_i is simply a cycle of length mi .

Then $g(n, m) \geq n + (k + s_1 + 2s_2 + 1)t - 1$, for $n \geq n_t$.

This completes the proof.

From Theorem 5, we have

$$\liminf_{n \rightarrow \infty} \frac{g(n, m) - n}{\sqrt{\frac{n}{m}}} \geq$$

$$\sqrt{\frac{(k + s_1 + 2s_2 + 1)^2}{\left(\frac{3}{4}k^2 + \frac{1}{2}ks_1 + \frac{3}{2}ks_2 + \frac{1}{2}s_1^2 + \frac{3}{2}s_1s_2 + \frac{9}{4}s_2^2 + k + s_1 + 3s_2 + \frac{1}{2}\right)}},$$

for all even integers m .

Let $s_1 = 28499066, s_2 = 4749839, k = 14249542$, then

$$\liminf_{n \rightarrow \infty} \frac{g(n, m) - n}{\sqrt{\frac{n}{m}}} > \sqrt{2.444},$$

for all even integers m .

3 Proof of Theorem 6

Proof. Let $n_t = \frac{1309}{2}t^2 - \frac{1349159}{6}t + \frac{6932215}{3}$, $t = 1260r + 169, r \geq 1, n \geq n_t$. It suffice to show that there exists a graph G on n vertices with $n + \frac{119}{3}t - \frac{26399}{3}$ edges such that all cycles in G have distinct lengths.

Now we construct the graph G which consists of a number of subgraphs: B_i , ($0 \leq i \leq 22t, i = 22t + j$ ($1 \leq j \leq \frac{5t-8}{3}$), $i = 23t + \frac{2t-2}{3} + j$ ($1 \leq j \leq \frac{5t-8}{3}$), $i = 32t + j - 60$ ($58 \leq j \leq t - 742$)).

Now we define these B_i s. These subgraphs all only have a common vertex x , otherwise their vertex sets are pairwise disjoint.

For $1 \leq i \leq \frac{5t-8}{3}$, let the subgraph B_{22t+i} consists of a cycle

$$xa_i^1 a_i^2 \dots a_i^{28t + \frac{2t-2}{3} + 2i - 3} x$$

and a path:

$$xa_{i,1}^1 a_{i,1}^2 \dots a_{i,1}^{\frac{56t-2}{6}} a_{i,1}^{\frac{76t-4}{6} + i}.$$

Based on the construction, B_{22t+i} contains exactly three cycles of lengths:

$$22t + i, 25t + \frac{t-1}{3} + i - 1, 28t + \frac{2t-2}{3} + 2i - 2.$$

For $1 \leq i \leq \frac{5t-8}{3}$, let the subgraph $B_{23t + \frac{2t-2}{3} + i}$ consists of a cycle

$$xb_i^1 b_i^2 \dots b_i^{28t + \frac{2t-2}{3} + 2i - 2} x$$

and a path:

$$xb_{i,1}^1 b_{i,1}^2 \dots b_{i,1}^{11t-1} b_{i,1}^{\frac{76t-4}{6} + i}.$$

Based on the construction, $B_{23t + \frac{2t-2}{3} + i}$ contains exactly three cycles of lengths:

$$23t + \frac{2t-2}{3} + i, 27t + i - 1, 28t + \frac{2t-2}{3} + 2i - 1.$$

For $58 \leq i \leq t - 742$, let the subgraph $B_{32t+i-60}$ consists of a cycle

$$C_{32t+i-60} = xy_i^1 y_i^2 \dots y_i^{137t+11i+890} x$$

and ten paths sharing a common vertex x , the other end vertices are on the cycle $C_{32t+i-60}$:

$$xy_{i,1}^1 y_{i,1}^2 \dots y_{i,1}^{11t-2} y_i^{21t-59+i}$$

$$xy_{i,2}^1 y_{i,2}^2 \dots y_{i,2}^{12t-2} y_i^{31t-53+2i}$$

$$xy_{i,3}^1 y_{i,3}^2 \dots y_{i,3}^{12t-2} y_i^{41t+156+3i}$$

$$xy_{i,4}^1 y_{i,4}^2 \dots y_{i,4}^{13t-2} y_i^{51t+155+4i}$$

$$xy_{i,5}^1 y_{i,5}^2 \dots y_{i,5}^{13t-2} y_i^{61t+155+5i}$$

$$xy_{i,6}^1 y_{i,6}^2 \dots y_{i,6}^{14t-2} y_i^{71t+154+6i}$$

$$xy_{i,7}^1 y_{i,7}^2 \dots y_{i,7}^{14t-2} y_i^{81t+153+7i}$$

$$xy_{i,8}^1 y_{i,8}^2 \dots y_{i,8}^{15t-2} y_i^{91t+147+8i}$$

$$xy_{i,9}^1 y_{i,9}^2 \dots y_{i,9}^{15t-2} y_i^{101t+149+9i}$$

$$xy_{i,10}^1 y_{i,10}^2 \dots y_{i,10}^{16t-2} y_i^{111t+151+10i}.$$

As a cycle with d chords contains $\binom{d+2}{2}$ distinct cycles, $B_{32t+i-60}$ contains exactly 66 cycles of lengths:

$$\begin{array}{cccc}
32t + i - 60, & 33t + i + 4, & 34t + i + 207, & 35t + i - 3, \\
36t + i - 2, & 37t + i - 3, & 38t + i - 3, & 39t + i - 8, \\
40t + i, & 41t + i, & 42t + i + 739, & 43t + 2i - 54, \\
43t + 2i + 213, & 45t + 2i + 206, & 45t + 2i - 3, & 47t + 2i - 3, \\
47t + 2i - 4, & 49t + 2i - 9, & 49t + 2i - 6, & 51t + 2i + 2, \\
51t + 2i + 741, & 53t + 3i + 155, & 54t + 3i + 212, & 55t + 3i + 206, \\
56t + 3i - 4, & 57t + 3i - 4, & 58t + 3i - 10, & 59t + 3i - 7, \\
60t + 3i - 4, & 61t + 3i + 743, & 64t + 4i + 154, & 64t + 4i + 212, \\
66t + 4i + 205, & 66t + 4i - 5, & 68t + 4i - 10, & 68t + 4i - 8, \\
70t + 4i - 5, & 70t + 4i + 737, & 74t + 5i + 154, & 75t + 5i + 211, \\
76t + 5i + 204, & 77t + 5i - 11, & 78t + 5i - 8, & 79t + 5i - 6, \\
80t + 5i + 736, & 85t + 6i + 153, & 85t + 6i + 210, & 87t + 6i + 198, \\
87t + 6i - 9, & 89t + 6i - 6, & 89t + 6i + 735, & 95t + 7i + 152, \\
96t + 7i + 204, & 97t + 7i + 200, & 98t + 7i - 7, & 99t + 7i + 735, \\
106t + 8i + 146, & 106t + 8i + 206, & 108t + 8i + 202, & 108t + 8i + 734, \\
116t + 9i + 148, & 117t + 9i + 208, & 118t + 9i + 943, & 127t + 10i + 150, \\
127t + 10i + 949, & 137t + 11i + 891. & &
\end{array}$$

B_0 is a path with an end vertex x and length $n - n_t$. The other B_i is simply a cycle of length i .

Then $f(n) \geq n + \frac{119}{3}t - \frac{26399}{3}$, for $n \geq n_t$.

This completes the proof.

From Theorem 6, we have

$$\liminf_{n \rightarrow \infty} \frac{f(n) - n}{\sqrt{n}} \geq \sqrt{2 + \frac{40}{99}},$$

which is better than the previous bounds $\sqrt{2}$ (see [11]), and $\sqrt{2 + \frac{7654}{19071}}$ (see [8]).

Combining this with Boros, Caro, Füredi and Yuster's upper bound, namely Theorem 2, we get

$$1.98 \geq \limsup_{n \rightarrow \infty} \frac{f(n) - n}{\sqrt{n}} \geq \liminf_{n \rightarrow \infty} \frac{f(n) - n}{\sqrt{n}} \geq \sqrt{2 + \frac{40}{99}}.$$

From the proof of Theorem 6, we have

$$\liminf_{n \rightarrow \infty} \frac{g(n, m) - n}{\sqrt{\frac{n}{m}}} \geq \sqrt{2 + \frac{40}{99}},$$

for all integers m .

If $m = 1$, $1 \leq i \leq t$, there exists the subgraph similar to $B_{s_1t+s_2t+i}$ consists of a cycle $C_{s_1t+s_2t+i}$ and k paths sharing a common vertex x , the other end vertices are on the cycle $C_{s_1t+s_2t+i}$ such that all cycles in $B_{s_1t+s_2t+i}$ have distinct lengths, then we could obtain

$$\liminf_{n \rightarrow \infty} \frac{f(n) - n}{\sqrt{n}} > \sqrt{2.444} > \sqrt{2 + \frac{40}{99}}.$$

But we only for $m = 1$, $58 \leq i \leq t - 742$, construct a subgraph similar to $B_{s_1t+s_2t+i}$ consists of a cycle $C_{s_1t+s_2t+i}$ and ten paths sharing a common vertex x , the other end vertices are on the cycle $C_{s_1t+s_2t+i}$ such that all cycles in $B_{s_1t+s_2t+i}$ have distinct lengths and obtain

$$\liminf_{n \rightarrow \infty} \frac{f(n) - n}{\sqrt{n}} \geq \sqrt{2 + \frac{40}{99}}.$$

We make the following conjecture:

Conjecture 7

$$\liminf_{n \rightarrow \infty} \frac{f(n) - n}{\sqrt{n}} > \sqrt{2.444}.$$

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