ON FANO COMPLETE INTERSECTIONS IN RATIONAL HOMOGENEOUS VARIETIES

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Abstract. Complete intersections inside rational homogeneous varieties provide interesting examples of Fano manifolds. For example, if $X = \cap_{i=1}^r D_i \subset G/P$ is a general complete intersection of r ample divisors such that $K^*_{G/P} \otimes \mathcal{O}_{G/P}(-\sum_i D_i)$ is ample, then X is Fano. We first classify these Fano complete intersections which are locally rigid. It turns out that most of them are hyperplane sections. We then classify general hyperplane sections which are quasi-homogeneous.

1. Introduction

We work within the category of complex projective varieties, unless stated otherwise. Rational homogeneous varieties are among the simplest algebraic varieties, and a better understanding of them is always a motivation for the development of algebraic geometry. For example, the solution by Mori of the Hartshorne conjecture characterizes projective spaces by the ampleness of its tangent bundle, which is a milestone of the minimal model program. A more recent conjecture of Campana-Peternell claims that rational homogeneous varieties are the only smooth rational varieties with nef tangent bundle, which is still far from resolved.

Complete intersections in rational homogeneous varieties provide many interesting examples of Fano varieties. It is expected by Hartshorne that all smooth subvarieties in $\mathbb{P}^n$ of small codimension are complete intersections, which is again far from resolved. In this paper, we will study two geometrical properties of Fano complete intersections in rational homogeneous varieties: local rigidity and quasi-homogeneity.

Recall that a smooth projective variety X is said *locally rigid* if for any smooth deformation $\mathcal{X} \to B$ with $\mathcal{X}_0 \simeq X$, we have $\mathcal{X}_t \simeq X$ for t in a small (analytic) neighborhood of 0. By Kodaira-Spencer deformation theory, if $H^2(X, T_X) = 0$, then X is locally rigid if and only if $H^1(X, T_X) = 0$. For rational homogeneous varieties G/P, it is shown in [B] (Theorem VII) that $H^i(G/P, T_{G/P}) = 0$ for all $i \geq 1$, hence they are locally rigid. In [BB], the local rigidity is proven for Fano regular G-varieties. The case of two-orbits varieties of Picard number one is studied in [PP].

Let G/P be a rational homogeneous variety with G simple and $X = \cap_{i=1}^r D_i \subset G/P$ a smooth irreducible complete intersection of r ample divisors. We assume that $K^*_{G/P} \otimes \mathcal{O}_{G/P}(-\sum_i D_i)$ is ample, which implies that X is Fano. When G/P is of Picard number one, the converse holds, but in general this condition is stronger than the Fanoness of X (cf. Remark 2.6). The main purpose of this paper is to classify such X which are locally rigid. By Kodaira-Nakano vanishing theorem, we have $H^q(X, T_X) = 0$ for all $q \geq 2$. In particular, X is locally rigid if and only if $H^1(X, T_X) = 0$. The main theorem of this paper is the following, which generalizes Proposition 8.4 in [FH3], where a similar result is obtained in the case of hyperplane sections of irreducible Hermitian symmetric spaces.

Theorem 1.1. *Let G/P be a rational homogeneous variety with G simple and $X = \cap_{i=1}^r D_i \subset G/P$ a smooth complete intersection of ample divisors. Assume that $K_{G/P}^* \otimes \mathcal{O}_{G/P}(-\sum_i D_i)$ is ample. Then X is locally rigid if and only if X is isomorphic to one of the following:*

(i) $\mathbb{P}^n$ *or* $\mathbb{Q}^n$*;*

(ii) *a general hyperplane section of the following:*

$$\mathrm{Gr}(2,n), \mathrm{Gr}(3,6), \mathrm{Gr}(3,7), \mathrm{Gr}(3,8),$$

$$\mathbb{S}_5, \mathbb{S}_6, \mathbb{S}_7, \mathrm{Gr}_\omega(2,6), \mathrm{Lag}(3,6), F_4/P_4, E_6/P_1, E_7/P_7;$$

(iii) *a general hypersurface of bidegree* $(1,1)$ *of* $\mathbb{P}(T_{\mathbb{P}^2})$*;*

(iv) *a general codimension 2 linear section of* $\mathrm{Gr}(2, 2k+1)$*,* $k \geq 2$*;*

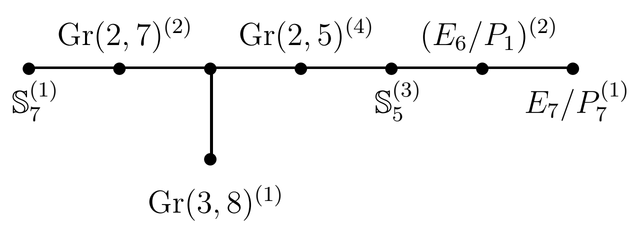
(v) *a general codimension 2 or 3 linear section of* $\mathbb{S}_5$*;*

(vi) *a general codimension 3 or 4 linear section of* $\mathrm{Gr}(2,5)$*.*

Here $\mathbb{Q}^n$ denotes the n-dimensional hyperquadric. $\mathrm{Gr}(a, a+b)$ is the Grassmannian of a-dimensional subspaces in an $(a+b)$-dimensional vector space. $\mathbb{S}_n$ is the spinor variety, parameterizing n-dimensional isotropic linear subspaces in an orthogonal vector space of dimension $2n$. $\mathrm{Gr}_\omega(2,6)$ is the symplectic Grassmanian and $\mathrm{Lag}(3,6)$ is the Lagrangian Grassmannian, which parameterize, respectively, isotropic planes and Lagrangian subspaces in a 6-dimensional symplectic vector space. For a simple Lie group G, we denote by P_i the maximal parabolic subgroup of G corresponding to the i-th root, where we use Bourbaki's numeration of simple roots.

This apparently disparate list can be explained in terms of Vinberg's theory of parabolic prehomogeneous spaces [M]. Briefly, suppose that a node n is chosen on a connected Dynkin diagram D, such that the complement of the node is the disjoint union of a Dynkin diagram of type A_{k-1} (including $k=1$, A_0 being by convention the empty diagram) and a connected Dynkin diagram D_0. The latter comes equipped with a special node n_0, the node which was connected to n in D. The pair (D_0, n_0) encodes a simply connected simple Lie group G and a maximal parabolic subgroup P, hence a homogeneous space G/P embedded in $\mathbf{P}V_P^*$, the projectivization of a (dualized) fundamental representation. The fundamental fact then is that $G \times GL_k$ acts on $V_P \otimes \mathbb{C}^k$ with finitely many orbits. In particular G acts on $Gr(k, V_P)$ with only finitely many orbits, and therefore there exists only a finite number of isomorphism types of codimension k linear sections of G/P. In this situation, the local rigidity of the general section can be expected, and this is exactly what happens.

We illustrate below the cases that originate from $D = E_8$. To each admissible node we attached the corresponding homogeneous space, with a superscript indicating the number k, which is the codimension of the relevant linear sections.



Taking all the connected diagrams we get exactly the list of Theorem 1.1, except the codimension two linear sections of $Gr(2, 2k+1)$ (which for $k \geq 4$ would originate from the non Dynkin diagrams E_{2k+2}). The general hypersurface of bidegree $(1,1)$ of

$\mathbb{P}(T_{\mathbb{P}^2})$ is a complete intersection of two divisors of bidegree $(1,1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$, which can be regarded as the linear section associated to the triple node in E_6.

Severi varieties are extremal projective varieties with remarkable projective geometrical properties, which are classified by Zak as follows: the Veronese surface, minimal embeddings of $\mathbb{P}^2 \times \mathbb{P}^2, \mathrm{Gr}(2,6)$ and E_6/P_1. It is interesting to notice that a general hyperplane section of them is homogeneous, while their general codimension 2 linear sections are locally rigid.

One remarks that in order to prove Theorem 1.1, we may assume that X is a general complete intersection, since special ones have deformations to the general ones, hence they are not locally rigid. On the other hand, special complete intersections may have much richer geometry which remains to be explored systematically. One example is the 10-dimensional spinor variety $\mathbb{S}_5$. Up to projective isomorphism there are only two classes of smooth codimension 2 linear sections of $\mathbb{S}_5$. It is shown in [FH2] (Remark 2.13) that the special ones contain a $\mathbb{P}^4$ and are equivariant compactifications of $\mathbb{C}^8$, which is not the case of the general ones. Of course the special codimension 2 section of $\mathbb{S}_5$ is not locally rigid, while the general one is locally rigid. Surprisingly, we discover that the general codimension 2 linear section of $\mathbb{S}_5$ is one of the two-orbits varieties in [Pa], which is quasi-homogeneous (Proposition 4.9). In particular, we obtain two non-isomorphic quasi-homogeneous varieties (special and general linear sections of codimension 2 of $\mathbb{S}_5$) which have the same VMRT at general points. This makes even more delicate the problem of recognition of Fano varieties of Picard number one from its VMRT.

By [A], a general hyperplane section of G/P (with G simple) is homogeneous if and only if G/P is isomorphic to $\mathbb{P}^n, \mathbb{Q}^n, \mathrm{Gr}(2,2k)$ or E_6/P_1. A natural question is: when is a general hyperplane section X of G/P quasi-homogeneous, i.e. $\mathrm{Aut}(X)$ acts on X with an open orbit? In this paper, we obtain the following classification.

Theorem 1.2. *Let G/P be a rational homogeneous variety of Picard number one and $X \subset G/P$ a general hyperplane section. Then X is quasi-homogeneous if and only if G/P is isomorphic to one of the following*

$$\mathbb{P}^n, \mathbb{Q}^n, \mathrm{Gr}(2,n), \mathrm{Gr}(3,6), \mathrm{Gr}(3,7),$$
$$\mathbb{S}_5, \mathbb{S}_6, \mathbb{S}_7, \mathrm{Gr}_\omega(2,6), \mathrm{Lag}(3,6), F_4/P_4, E_6/P_1, E_7/P_7.$$

An observation is that a general hyperplane section of G/P is quasi-homogeneous if and only if it is locally rigid but not a hyperplane section of $\mathrm{Gr}(3,8)$. In general, there is no direct relation between the two properties.

Once the local rigidity is settled, the next question is whether the varieties in Theorem 1.1 are rigid? Namely if we have a smooth Kähler deformation $\mathcal{X} \to B$ such that $\mathcal{X}_t \simeq X$ for all $t \neq 0$, does this imply that $\mathcal{X}_0 \simeq X$? This problem is already difficult for G/P and was solved by Hwang and Mok (cf. [HM]). It seems very interesting to extend their results to the varieties in Theorem 1.1. Note that by the previous discussions, a general codimension 2 linear section of $\mathbb{S}_5$ is locally rigid, but not rigid, as it has deformations to the special section.

Remark 1.3. As is well-known, a smooth Fano complete intersection in $\mathbb{P}^n$ is locally rigid if and only if it is isomorphic to $\mathbb{P}^m$ or $\mathbb{Q}^m$ (cf. Proposition 2.13). If a homogeneous variety G/P is a complete intersection in G'/P', then we only need to consider complete intersections in G'/P'. This is the reason why we introduce the following convention: we say that G/P satisfies ♣ if G/P is not isomorphic to one of the following: $\mathbb{P}^n, \mathbb{Q}^n, C_\ell/P_2 (\ell \geq 3), F_4/P_4, \mathbb{P}(T_{\mathbb{P}^m})$.

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2. Reduction to Picard number one case

Let G be a semi-simple Lie group of rank ℓ with Lie algebra $\mathfrak{g}$. We fix a Borel subgroup and a maximal torus. Let $\{\alpha_1, \cdots, \alpha_\ell\}$ be the set of simple roots. The fundamental weights are denoted by $\{\lambda_1, \cdots, \lambda_\ell\}$. Every standard parabolic subgroup P in G is determined by a subset of indexes $\Delta \subset \{1, \cdots, \ell\}$, with the property that $\alpha_i \notin \mathrm{Lie}(P)$ for all $i \in \Delta$. We have a natural identification

$$\mathrm{Pic}(G/P) = \{\sum_{i\in\Delta} n_i\lambda_i | n_i \in \mathbb{Z}\}.$$

For $\lambda = \sum_{i\in\Delta} n_i\lambda_i$, we denote by L^λ the corresponding line bundle. It is well-known that L^λ is ample if and only if $n_i > 0$ for all $i \in \Delta$ (and in this case, it is very ample). In particular, there exists a minimal ample line bundle L_0, which corresponds to $\sum_{i\in\Delta} \lambda_i$. As a consequence, we have a minimal G-equivariant embedding $G/P \subset \mathbb{P}(V_P^*)$, where $V_P = H^0(G/P, L_0)$.

By Kodaira vanishing theorem, we have

Lemma 2.1. *Let G/P be a rational homogeneous variety and $L \in \mathrm{Pic}(G/P)$ an ample Line bundle. Assume that $K_{G/P}^* \otimes L^*$ is ample. Then $H^q(G/P, L^* \otimes A) = 0$ for all $q > 0$ and nef line bundle A.*

We recall the following theorem from [MS] (Theorem B), which plays a key role in our computations. Note that claim (0) holds for any smooth projective variety by the result of Wahl [W].

Theorem 2.2. *Let G/P be a rational homogeneous variety and $L \in \mathrm{Pic}(G/P)$ an ample line bundle. Then*

(0) $H^0(G/P, T_{G/P} \otimes L^*) = 0$ *except for* $(G/P, L) = (\mathbb{P}^1, \mathcal{O}(2))$ *or* $(\mathbb{P}^n, \mathcal{O}(1))$.
(1) $H^1(G/P, T_{G/P} \otimes L^*) = 0$ *except the following cases*
  - (a) $H^1(\mathbb{P}^1, T(-k)) \simeq \mathrm{Sym}^{k-4}\mathbb{C}^2, k \geq 4$;
  - (b) $H^1(\mathbb{P}^2, T(-3)) \simeq \mathbb{C}$;
  - (c) $H^1(\mathbb{Q}^n, T(-2)) \simeq \mathbb{C}, n \geq 3$;
  - (d) $H^1(C_\ell/P_2, T_{C_\ell/P_2}(-1)) \simeq \mathbb{C}$;
  - (e) $H^1(F_4/P_4, T_{F_4/P_4}(-1)) \simeq \mathbb{C}$;
  - (f) $H^1(\mathbb{P}(T_{\mathbb{P}^m}), T(-1,-1)) \simeq \mathbb{C}$;
  - (g) $H^1(\mathbb{P}^1 \times \mathbb{P}^1, T(-k,-2)) \simeq \mathrm{Sym}^{k-2}\mathbb{C}^2, k \geq 2$;
  - (h) $H^1(\mathbb{P}^1 \times \mathbb{P}^n, T(-k,-1)) \simeq \mathrm{Sym}^{k-2}\mathbb{C}^2 \otimes \mathbb{C}^{n+1}, k \geq 2$.

For any smooth projective variety X, we have $H^q(X, T_X \otimes L^*) = 0$ for all $q \geq 2$ provided that $K_X^* \otimes L^*$ is ample, by Akizuki-Nakano vanishing theorem. As a consequence, we have

Corollary 2.3. *Assume G/P satisfies ♣. Let $L \in \mathrm{Pic}(G/P)$ be an ample line bundle such that $K_{G/P}^* \otimes L^*$ is ample. Then $H^q(G/P, T_{G/P} \otimes L^*) = 0$ for all $q \geq 0$.*

Proof. The only case to be considered is $G/P \simeq \mathbb{P}^1 \times \mathbb{P}^n$, then $K^*_{G/P} = \mathcal{O}(2, n+1)$. By the assumption that $K^*_{G/P} \otimes L^*$ is ample, we have $L = \mathcal{O}(1, a)$ with $a \leq n$. This implies that $H^1(G/P, T_{G/P} \otimes L^*) = 0$ by Theorem 2.2. $\square$

Let $D_i \subset G/P$ be r ample divisors and $X = \cap_{i=1}^r D_i$ their complete intersection. Assume that X is smooth of the expected dimension, and irreducible. Let $D := \sum_i D_i$. Then we have the following Koszul exact sequence

(2.1)
$$0 \to \mathcal{O}_{G/P}(-D) \to \oplus_i \mathcal{O}_{G/P}(-D + D_i) \to \cdots \to \oplus_i \mathcal{O}_{G/P}(-D_i) \to \mathcal{O}_{G/P} \to \mathcal{O}_X \to 0.$$

The following fact is classical, see Lemma 5.7 in [FH3].

Lemma 2.4. *Let $0 \to \mathcal{F}_0 \to \mathcal{F}_1 \to \cdots \to \mathcal{F}_m \to 0$ be an exact sequence of coherent sheaves on a variety X. If $H^{q+j-1}(X, \mathcal{F}_{m-j}) = 0$ for all $j \in \{1, 2, \cdots, m\}$, then $H^q(X, \mathcal{F}_m) = 0$.*

By [D], we may assume that $H^0(G/P, T_{G/P}) \simeq \mathfrak{g}$ up to representing G/P, if necessary, as another quotient G'/P'.

Proposition 2.5. *Assume G/P satisfies ♣ and $H^0(G/P, T_{G/P}) = \mathfrak{g}$. Consider a smooth complete intersection $X = \cap_{i=1}^r D_i \subset G/P$ such that $K^*_{G/P} \otimes \mathcal{O}_{G/P}(-\sum_i D_i)$ is ample. Then*

$$h^0(T_X) - h^1(T_X) = \dim \mathfrak{g} - \sum_{i=1}^r h^0(X, \mathcal{O}_{G/P}(D_i)|_X).$$

Proof. Taking the tensor product of the Koszul exact sequence (2.1) with $T_{G/P}$, and using Corollary 2.3 and Lemma 2.4, we get that

$$H^0(X, T_{G/P}|_X) = \mathfrak{g} \quad \text{and} \quad H^q(X, T_{G/P}|_X) = 0 \ \forall q \geq 1.$$

The exact sequence $0 \to T_X \to T_{G/P}|_X \to \oplus_{i=1}^r \mathcal{O}_{G/P}(D_i)|_X \to 0$ implies that

$$0 \to H^0(T_X) \to H^0(X, T_{G/P}|_X) \to \oplus_{i=1}^r H^0(\mathcal{O}_{G/P}(D_i)|_X) \to H^1(T_X) \to 0$$

is exact, from which the claim follows. $\square$

Remark 2.6. By adjunction, we have $K^*_X = (K^*_{G/P} \otimes \mathcal{O}_{G/P}(-D))|_X$, which is ample by assumption, hence X is Fano. When G/P is of Picard number one (a main case in our discussions), the converse also holds, namely if X is Fano, then $K^*_{G/P} \otimes \mathcal{O}_{G/P}(-D)$ is ample on G/P. But in general, our assumption is stronger than the Fanoness of X. For example, take a general hypersurface X of bidegree $(2, 1)$ in $\mathbb{P}^1 \times \mathbb{P}^2$. Then the map $p : X \to \mathbb{P}^2$ is a finite morphism (of degree 2). By adjunction, $K^*_X = \mathcal{O}(0, 2)|_X = p^*\mathcal{O}_{\mathbb{P}^2}(2)$ which is ample. Hence X is Fano but $\mathcal{O}(0, 2)$ is not ample on $\mathbb{P}^1 \times \mathbb{P}^2$.

Lemma 2.7. *Let $X = \cap_{i=1}^r D_i \subset G/P$ be a smooth complete intersection such that $K^*_{G/P} \otimes \mathcal{O}_{G/P}(-\sum_i D_i)$ is ample. Let $L_0 \in \mathrm{Pic}(G/P)$ be the minimal ample line bundle. Then $h^0(X, L_0|_X) = \dim V_P - s$, where $s = \sharp\{i | \mathcal{O}_{G/P}(D_i) \simeq L_0\}$.*

Proof. Taking the tensor product of (2.1) with L_0, we get

$$0 \to \mathcal{O}_{G/P}(-D) \otimes L_0 \to \cdots \to \oplus_i \mathcal{O}_{G/P}(-D_i) \otimes L_0 \to L_0 \to L_0|_X \to 0.$$

From Lemmas 2.4 and 2.1, we deduce an exact sequence

$$0 \to \oplus_i H^0(\mathcal{O}_{G/P}(-D_i) \otimes L_0) \to H^0(L_0) \to H^0(L_0|_X) \to 0,$$

which implies the claim since L_0 is the minimal ample line bundle on G/P. $\square$

Note that $h^0(X, \mathcal{O}_{G/P}(D_i)|_X) \geq h^0(X, L_0|_X) = \dim V_P - s \geq \dim V_P - r$. Hence we obtain

Corollary 2.8. *Assume G/P satisfies ♣ and $H^0(G/P, T_{G/P}) = \mathfrak{g}$. Let $X = \cap_{i=1}^r D_i \subset G/P$ be a smooth complete intersection of codimension r such that $K^*_{G/P} \otimes \mathcal{O}_{G/P}(-\sum_i D_i)$ is ample. Then X is not locally rigid if $\dim \mathfrak{g} < r(\dim V_P - r)$.*

From now on, we will assume further that G is simple.

Lemma 2.9. *Let V be an irreducible representation of a simple Lie group G. Then*
(1) $\dim V \neq \dim \mathfrak{g} + 1$.
(2) $\dim V = \dim \mathfrak{g}$ if and only if V is the adjoint representation.

Proof. Assume G is of type A_ℓ. The irreducible representations of G of dimension $\leq (\ell+1)^2$ are classified in [SK] (Proposition 7 on p.45) and the claim follows. For type C_ℓ, we can apply [SK] (Lemma 13 and Proposition 14 on p.50). The case of SO(m) follows from Proposition 20 in [SK] (p. 54). If G is of exceptional type, we can apply Proposition 22 on p. 56 of [SK]. □

Remark 2.10. Note that if G is not simple, then there are exceptions. For example, take $G/P = \mathrm{Gr}(2,5) \times \mathbb{P}^3$, then $\dim G = 39$ and $\dim V_P = 40$

Proposition 2.11. *Let G/P be a rational homogeneous variety with G simple such that $H^0(G/P, T_{G/P}) = \mathfrak{g}$. Assume that $\dim \mathfrak{g} < \dim V_P$. Consider a smooth complete intersection $X = \cap_{i=1}^r D_i$ in G/P such that $K^*_{G/P} \otimes \mathcal{O}_{G/P}(-\sum_i D_i)$ is ample. Then X is not locally rigid.*

Proof. Note that the condition $\dim \mathfrak{g} < \dim V_P$ implies that G/P satisfies ♣, then by Lemma 2.9, the assumption $\dim \mathfrak{g} < \dim V_P$ implies that $\dim V_P \geq \dim \mathfrak{g} + 2$. Now the claim follows from Corollary 2.8, as $r < \dim G/P < \frac{1}{2} \dim \mathfrak{g}$. □

By Proposition 2.11, we are reduced to the case $\dim \mathfrak{g} \geq \dim V_P$. By Lemma 2.9, the case of equality implies that V_P is the adjoint representation and then G/P is the adjoint variety. In this case, G/P has Picard number one except for type A, where $G/P = \mathbb{P}(T_{\mathbb{P}^m})$.

Proposition 2.12. *Let $X = \cap_{i=1}^r D_i \subset Z := \mathbb{P}(T_{\mathbb{P}^m})$ be a general complete intersection of r ample divisors. Assume that $K^*_Z \otimes \mathcal{O}_Z(-\sum_{i=1}^r D_i)$ is ample. Then X is locally rigid if and only if X is a general hypersurface of bidegree $(1,1)$ of $\mathbb{P}(T_{\mathbb{P}^2})$. In this case, X is isomorphic to the blowup of $\mathbb{P}^2$ at 3 general points.*

Proof. Note that $K^*_{\mathbb{P}(T_{\mathbb{P}^m})} = \mathcal{O}(m,m)$, hence $r < m$ by the ampleness of $K^*_Z \otimes \mathcal{O}_Z(-\sum_{i=1}^r D_i)$. By a similar argument as in the proof of Proposition 2.5, we have

$$h^0(T_X) - h^1(T_X) = \dim \mathfrak{g} + s - \sum_{i=1}^r h^0(X, \mathcal{O}_Z(D_i)|_X),$$

where s is the number $\sharp\{j | D_j \text{ is of bidegree } (1,1)\}$. By Lemma 2.7, we have

$$h^0(X, \mathcal{O}_Z(1,1)|_X) = h^0(Z, \mathcal{O}_Z(1,1)) - s = m^2 + 2m - s.$$

This gives that $h^0(T_X) - h^1(T_X) \leq m^2 + 2m + s - r(m^2 + 2m - s)$, which is negative if $r \geq 2$.

Now assume $X \subset Z$ is a hypersurface of bidegree (a,b). If $(a,b) \neq (1,1)$, we have $h^0(T_X) - h^1(T_X) = \dim \mathfrak{g} - h^0(X, \mathcal{O}_Z(a,b)|_X)$. As

$$h^0(X, \mathcal{O}_Z(a,b)|_X) \geq h^0(X, \mathcal{O}_Z(2,1)|_X) \geq h^0(Z, \mathcal{O}_Z(2,1)) - 1 = \frac{(m+1)(m^2+2m-2)}{2} - 1,$$

we obtain that $h^0(T_X) - h^1(T_X) < 0$, which implies that X is not locally rigid. If $X \subset Z$ is of bidegree $(1,1)$, then $h^0(T_X) - h^1(T_X) = m^2+2m+1-(m^2+2m-1) = 2$. This implies that X is locally rigid if and only if $h^0(T_X) = 2$. On the other hand, for the adjoint action of $G = \mathrm{PGL}_{m+1}$ on $\mathfrak{g} = \mathfrak{sl}_{m+1}$, its stabilizer at a general point has dimension m, hence $h^0(T_X) \geq m$, which gives $m \leq 2$.

Let $X \subset \mathbb{P}(T_{\mathbb{P}^2})$ be a general hypersurface of bidegree $(1,1)$. The projection $X \to \mathbb{P}^2$ is birational, with three fibers isomorphic to $\mathbb{P}^1$, hence it is the blowup of $\mathbb{P}^2$ along 3 points, which are in general position as X is Fano. □

The following is probably well-known, but we do not find an explicit reference.

Proposition 2.13. *Let $X \subset \mathbb{P}^N$ be a smooth Fano complete intersection. Then X is locally rigid if and only if X is isomorphic to a projective space or a hyperquadric.*

Proof. Let $(d_1, \cdots, d_r)$ be the multi-degree of X such that $2 \leq d_1 \leq \cdots \leq d_r$. We may assume $\dim X \geq 2$ as the only Fano curve is $\mathbb{P}^1$. If X is not a hyperquadric, then $h^0(X, T_X) = 0$ (see for example Lemma 7.3 [FH3]). By a similar argument as that in Proposition 2.5, we have

$$h^1(X, T_X) = \sum_{i=1}^{r} h^0(X, \mathcal{O}_X(d_i)) - (N^2+2N) \geq r h^0(X, \mathcal{O}_X(2)) - (N^2+2N),$$

while $h^0(X, \mathcal{O}_X(2)) = \binom{N+2}{2} - s$ with $s = \sharp\{j | d_j = 2\}$. This implies that $h^1(X, T_X) > 0$ if $r \geq 2$. Now if $X \subset \mathbb{P}^N$ is a hypersurface of degree $d \geq 3$, then $h^1(X, T_X) = \binom{N+d}{d} - (N^2+2N) \geq \binom{N+3}{3} - (N^2+2N) > 0$, which concludes the proof. □

In [E], irreducible representations V of G with $\dim G > \dim V$ are classified. It turns out that they are all fundamental representations (see Table 1 in the following section) except for $G/P = \mathbb{P}^n$. By Proposition 2.12 and Proposition 2.13, we may assume from now on that G is simple and G/P is of Picard number one satisfying ♣.

Remark 2.14. When G is semi-simple but not simple, a classification of the irreducible representations V of G such that $\dim G + 1 \geq \dim V$ is given in Section 3 of [SK](Note that in the notation therein, G has 1 dimensional center, hence their classification gives all V with $\dim G + 1 \geq \dim V$.) With this, a similar result as Theorem 1.1 can be obtained for any G semi-simple. We leave this to the reader.

3. Rigidity of hypersurfaces in G/P

In this section, G/P is a rational homogeneous variety of Picard number one. Recall that for an irreducible representation V of G, there exists an open subset U such that the stationary subalgebras $\mathfrak{g}_v$ of all the points $v \in U$ are conjugate to a single subalgebra $\mathfrak{h} \subset \mathfrak{g}$ by [Ri] (Theorem A).

The next table is taken from [E] (table 1) and it gives all the fundamental representations V_P with $\dim V_P < \dim \mathfrak{g}$. In the column headed $\mathfrak{h}$ is given the generic stationary subalgebra. In those cases when $\mathfrak{h}$ is the direct sum of ideals $\mathfrak{h}_1, \ldots, \mathfrak{h}_k$, we write $\mathfrak{h} = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_k$. If $\mathfrak{h}$ decomposes into the semidirect sum of a subalgebra P and an ideal U, we write $\mathfrak{h} = P + U$ and in parentheses we specify the action of P on U. Furthermore, U_k is a k-dimensional commutative Lie algebra.

type	k	$\dim V^{\lambda_k}$	$\mathfrak{h}$	$\dim \mathfrak{h}$
A_ℓ	1	$\ell+1$	$A_{\ell-1}+U_\ell(R(\lambda_1))$	$\ell^2+\ell-1$
A_{2j-1}	2	$j(2j-1)$	C_j	$2j^2+j$
A_{2j}	2	$j(2j+1)$	$C_j+U_{2j}(R(\lambda_1))$	$2j^2+3j$
A_5	3	20	$A_2\oplus A_2$	16
A_6	3	35	G_2	14
A_7	3	56	A_2	8
B_ℓ	1	$2\ell+1$	D_ℓ	$2\ell^2-\ell$
B_3	3	8	G_2	14
B_4	4	16	B_3	21
B_5	5	32	A_4	24
B_6	6	64	$A_2\oplus A_2$	16
C_ℓ	1	2ℓ	$C_{\ell-1}+U_{2\ell-1}(R(\lambda_1)+1)$	$2\ell^2-\ell$
C_ℓ	2	$2\ell^2-\ell-1$	$\underbrace{A_1\oplus\ldots\oplus A_1}_{\ell}$	3ℓ
C_3	3	14	A_2	8
D_ℓ	1	2ℓ	$B_{\ell-1}$	$2\ell^2-3\ell+1$
D_5	4	16	$B_3+U_8(R(\lambda_3))$	29
D_6	5	32	A_5	35
D_7	6	64	$G_2\oplus G_2$	28
G_2	1	7	A_2	8
F_4	4	26	D_4	28
E_6	1	27	F_4	52
E_7	7	56	E_6	78

Table 1: Èlašvili's list

Lemma 3.1. *Assume G/P satisfies ♣ and $H^0(G/P, T_{G/P}) = \mathfrak{g}$. Let $X \subset G/P$ be a smooth Fano hypersurface of degree $d \geq 2$. Then X is not locally rigid.*

Proof. We may assume $\dim \mathfrak{g} \geq \dim V_P$ by Proposition 2.11. By Proposition 2.5, we have $h^0(X, T_X) - h^1(X, T_X) = \dim \mathfrak{g} - h^0(X, \mathcal{O}_X(d))$ while $h^0(X, \mathcal{O}_X(d)) = h^0(G/P, \mathcal{O}_{G/P}(d)) - 1$. Thus if $h^0(G/P, \mathcal{O}_{G/P}(d)) > \dim \mathfrak{g} + 1$, then $H^1(X, T_X) \neq 0$, which implies that X is not locally rigid.

By Lemma 2.9, if $\dim \mathfrak{g} = \dim V_P$, then V_P is the adjoint representation and G/P is the adjoint variety. Then $h^0(G/P, \mathcal{O}_{G/P}(d)) > \dim \mathfrak{g} + 1$ if $d \geq 2$.

Finally, in [E], all irreducible representations V of G with $\dim \mathfrak{g} > \dim V$ are listed, from which we deduce that $\dim \mathfrak{g} > \dim h^0(G/P, \mathcal{O}_{G/P}(d))$ is only possible for $G/P \simeq \mathbb{P}^n$ and $d = 2$, which is excluded by our assumption ♣. □

Proposition 3.2. *Assume $G/P \subset \mathbb{P}(V_P^*)$ satisfies ♣ and $H^0(G/P, T_{G/P}) = \mathfrak{g}$. Let $L \subset \mathbb{P}(V_P^*)$ be a linear subspace of codimension r such that $X = G/P \cap L$ is smooth Fano. We denote by $\mathfrak{aut}(G/P, L)$ the Lie algebra of automorphisms of G/P preserving the linear space L. Then*

(i) *$H^0(X, T_X) \simeq \mathfrak{aut}(G/P, L)$.*
(ii) *X is locally rigid if and only if the G-orbit of $[L^\perp] \in \mathrm{Gr}(r, V_P)$ is open.*

Proof. (i) By Lemma 2.7, we have $H^0(X, \mathcal{O}_X(1)) = L$, hence L is the linear span of X. By the proof of Proposition 2.5, we have $H^0(X, T_{G/P}|_X) \simeq H^0(G/P, T_{G/P}) = \mathfrak{g}$.

By the normal bundle exact sequence $0 \to T_X \to T_{G/P}|_X \to \mathcal{N}_{X|G/P} \to 0$, we get that

$$H^0(X, T_X) = \mathrm{Ker}(H^0(X, T_{G/P}|_X) \simeq H^0(G/P, T_{G/P}) \to H^0(X, \mathcal{N}_{X|G/P})),$$

namely $H^0(X, T_X)$ identifies with the set of vector fields on G/P which preserves X (hence its linear span L).

(ii) By Proposition 2.5, we have

$$h^1(X, T_X) = r(\dim V_P - r) - (\dim \mathfrak{g} - \dim \mathfrak{aut}(G/P, L)) = \dim \mathrm{Gr}(r, V_P) - \dim G \cdot [L^\perp],$$

which vanishes if and only if the G-orbit of $[L^\perp] \in \mathrm{Gr}(r, V_P)$ is open. □

In the rest of this section, we will only consider hyperplane sections of G/P. Let $L \subset \mathbb{P}V_P^*$ be a general hyperplane, projectivization of the affine hyperplane $\widehat{L} \subset V_P^*$. Let $X = G/P \cap L$ be the corresponding hyperplane section of G/P. We will denote by $\mathbf{l} \subset V_P$ the line orthogonal to the hyperplane $\widehat{L} \subset V_P^*$. Recall that V_P^* is an irreducible representation of G and so is V_P. We denote by $G_\mathbf{l}$ the subgroup of G preserving $\mathbf{l}$ and $\mathfrak{g}_\mathbf{l}$ its Lie algebra. For $v \in \mathbf{l}$ a non-zero element, the stabilizer G_v is a subgroup of $G_\mathbf{l}$. The quotient $Q := G_\mathbf{l}/G_v$ acts on $\mathbf{l}$ by a subgroup of $\mathbb{C}^*$.

Lemma 3.3. *If G_v is reductive, then Q is a finite group, hence $\mathfrak{g}_\mathbf{l} = \mathfrak{g}_v$.*

Proof. As L is a general hyperplane, the point $v \in \mathbf{l}$ is a general point of V_P. If G_v is reductive, then by [P], the orbit $G \cdot v$ is closed in V_P. If $Q \simeq \mathbb{C}^*$, then Q acts on $\mathbf{l}$ by scalars, hence $G \cdot v \supset Q \cdot v = \mathbf{l} \setminus \{\mathbf{0}\}$. By the closedness of the orbit, we get that $0 \in G \cdot v$, which is absurd. □

Proposition 3.4. *Assume G/P satisfies ♣ and $H^0(G/P, T_{G/P}) = \mathfrak{g}$. Let $v \in \mathbf{l}$ be a non-zero point and $\mathfrak{g}_v$ the Lie algebra of the stabilizer G_v. Then*

$$\mathfrak{aut}(X) = \begin{cases} \mathfrak{g}_v \oplus \mathbb{C} & \text{if } G/P = \mathrm{Gr}(2, 2k+1) \text{ or } \mathbb{S}_5, \\ \mathfrak{g}_v & \text{otherwise.} \end{cases}$$

Proof. First note that $\mathfrak{aut}(G/P, L)$ is exactly $\mathfrak{g}_\mathbf{l}$. If $\dim V_P > \dim G$, then by [AVE] (Corollary on p.260), the stabilizer G_v is discrete, hence by Lemma 3.3, we have $\mathfrak{g}_v = \mathfrak{g}_\mathbf{l} = 0$. If $\dim V_P = \dim G$, then V_P is the adjoint representation (cf. Lemma 2.9) and in this case $\mathfrak{g}_v$ is a Cartan subalgebra, hence by Lemma 3.3, we have $\mathfrak{g}_v = \mathfrak{g}_\mathbf{l}$.

Now assume $\dim V_P < \dim G$, then the stabilizer $\mathfrak{g}_v$ is computed in [E] (table 1). One checks that $\mathfrak{g}_v$ is not reductive only for $G/P = \mathrm{Gr}(2, 2k+1)$ or $\mathbb{S}_5$.

When $G/P = \mathrm{Gr}(2, 2k+1)$, then X is the so-called odd symplectic Grassmanian. Its automorphism group is computed in [PV], from which one checks that $\mathfrak{aut}(X) \simeq \mathfrak{g}_v \oplus \mathbb{C}$. When $G/P = \mathbb{S}_5$, its Lie algebra of automorphism group is well-known (see for example Proposition 3.9 [FH1]) and one checks directly the claim. □

By Propositions 2.5 and 3.4, we obtain the following

Corollary 3.5. *Assume G/P satisfies ♣ and $H^0(G/P, T_{G/P}) = \mathfrak{g}$. Let $X = G/P \cap L$ be a general hyperplane section and $v \in \mathbf{l}$ a nonzero point. Then*

$$h^1(X, T_X) = \begin{cases} \dim \mathfrak{g}_v + \dim V_P - \dim \mathfrak{g} & \text{if } G/P = \mathrm{Gr}(2, 2k+1) \text{ or } \mathbb{S}_5, \\ \dim \mathfrak{g}_v + \dim V_P - \dim \mathfrak{g} - 1 & \text{otherwise.} \end{cases}$$

Lemma 3.6. *Let $X \subset \mathrm{Gr}(2, n+1) (n \geq 4)$ be a general codimension 2 linear section, then X is locally rigid if and only if either n is even or $n = 5$.*

Proof. By Proposition 2.5, we have $h^0(T_X)-h^1(T_X)=(n^2+2n)-2(n(n+1)/2-2)=n+4$. By [PV], we have

$$\dim \mathrm{Aut}(X)=\begin{cases} n+4 & n \text{ even} \\ 3(n+1)/2 & n \text{ odd.} \end{cases}$$

The claim follows immediately. □

Remark 3.7. By [PV], the general codimension 2 linear section $X \subset \mathrm{Gr}(2,2k+1)$ is quasi-homogeneous if and only if $k \leq 3$. In even dimension, a codimension 2 linear section $X \subset \mathrm{Gr}(2,2k)$ is defined by a pencil of skew-symmetric forms, and those that are not of maximal rank define in general a k-tuple of points on $\mathbb{P}^1$, well-defined up to PGL_2. This k-tuple of points has the same number of moduli as X. Moreover it is easy to see that X is quasi-homogeneous only when $k \leq 3$. For $k=3$ it is a compactification of $\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2/\mathrm{diag}(\mathrm{SL}_2)$.

Remark 3.8. Linear sections of $\mathrm{Gr}(2,5)$ have been studied classically and appear in the classification of del Pezzo manifolds. It is well-known that there is a unique isomorphism class of del Pezzo manifolds of degree five in each dimension between 2 and 6, hence they are all locally rigid.

Theorem 3.9. *Let G/P be a rational homogeneous variety of Picard number one. Let $X \subset G/P$ be a general hyperplane. Then X is locally rigid if and only if G/P is isomorphic to one of the following*

$$\mathbb{P}^n, \mathbb{Q}^n, \mathrm{Gr}(2,n), \mathrm{Gr}(3,6), \mathrm{Gr}(3,7), \mathrm{Gr}(3,8),$$
$$\mathbb{S}_5, \mathbb{S}_6, \mathbb{S}_7, \mathrm{Gr}_\omega(2,6), \mathrm{Lag}(3,6), F_4/P_4, E_6/P_1, E_7/P_7.$$

Proof. If G/P does not satisfy ♣, we need to consider two cases: C_ℓ/P_2 and F_4/P_4. A general hyperplane section of C_ℓ/P_2 is a codimension 2 linear sections of $A_{2\ell-1}/P_2$, which is locally rigid if and only if $\ell=3$ by Lemma 3.6. For F_4/P_4, its general hyperplane section X is a codimension 2 linear section of E_6/P_1. By Proposition 2.5, we have $h^0(T_X)-h^1(T_X)=28$. By Proposition 48 ([SK], p. 139), the stabilizer $\mathfrak{a}ut(E_6/P_1, L)$ is $\mathfrak{so}(8)$, which implies $h^0(X,T_X)=28$ by Proposition 3.2, hence X is locally rigid.

Now we assume G/P satisfies ♣. By Proposition 2.11, we may assume $\dim V_P \leq \dim \mathfrak{g}$. If $\dim V_P = \dim G$, then V_P is the adjoint representation by Lemma 2.9. As a consequence, $\dim \mathfrak{g}_v = \mathrm{rk}(\mathfrak{g})$. By Corollary 3.5, $\dim H^1(X,T_X)=\mathrm{rk}(\mathfrak{g})-1$, which is non-zero except for type A_1.

Now assume $\dim V_P < \dim G$, then the stabilizer $\mathfrak{g}_v$ is computed in Table 1. Then a case-by-case check using Corollary 3.5 concludes the proof. □

As an application, we recover the following well-known fact from [A].

Corollary 3.10. *A general hyperplane section of G/P of Picard number one is homogeneous if and only if G/P is isomorphic to one of the following*

$$\mathbb{P}^n, \mathbb{Q}^n, \mathrm{Gr}(2,2k), E_6/P_1.$$

4. Rigidity of complete intersections in G/P

4.1. **End of the classification.**

Let G/P be a rational homogeneous variety of Picard number one. Let $X \subset G/P$ be a smooth complete intersection of codimension $r \geq 2$. We may assume $\dim X \geq 2$.

By Corollary 2.8, if X is locally rigid, then $\dim \mathfrak{g} \geq r(\dim V_P - r)$. The following lists all such possibilities (by using Table 1).

Lemma 4.1. *Assume G/P satisfies ♣ and $H^0(G/P, T_{G/P}) = \mathfrak{g}$. Then $\dim \mathfrak{g} \geq r(\dim V_P - r)$ for some $r \geq 2$ holds only for the following cases:*

(1) *$G/P = A_\ell/P_2$ and $r = 2$, or $\ell = 4$ and $r = 3, 4$;*
(2) *$G/P = \mathbb{S}_5$ and $r = 2, 3$;*
(3) *$G/P = \mathbb{S}_6$ and $r = 2$;*
(4) *$G/P = E_6/P_1$ and $r = 2, 3$;*
(5) *$G/P = E_7/P_7$ and $r = 2$.*

By a similar argument as in Lemma 3.1, the only possible complete intersections which are locally rigid in these cases are linear sections. Case (1) is done by Lemma 3.6 and Remark 3.8. We will consider case (2) in the following subsection. Concerning case (4), the codimension 2 linear section of E_6/P_1 is a hyperplane section of F_4/P_4 and has been studied in the previous section. The remaining three cases are treated by the following result. Alltogether, this will complete the proof of Theorem 1.1.

Proposition 4.2. *A general codimension 2 (resp. 3) linear section of $\mathbb{S}_6$ or E_7/P_7 (resp. E_6/P_1) is not locally rigid.*

Proof. For X a smooth codimension 2 (resp. 3) linear section of $\mathbb{S}_6$ or E_7/P_7 (resp. E_6/P_1), we have $h^0(T_X) - h^1(T_X) = 6, 25$ (resp. 6) by Proposition 2.5. On the other hand, for X general, the stabilizer of its linear span in $\mathbb{P}V_P^*$ is of type $3A_1, D_4$ (resp. A_2) by [V] (table on p. 491-492), so $H^0(T_X)$ has dimension $9, 28$ (resp. 8) by Proposition 3.2. This gives that $h^1(T_X) = 3, 3$ (resp. 2), concluding the proof. □

4.2. **Sections of the 10-dimensional spinor variety.**

Let $S = \mathbb{S}_5 \subset \mathbb{P}^{15}$. For $k \geq 1$, we denote by $S_k \subset \mathbb{P}^{15-k}$ a smooth linear section of S of codimension k. The hyperplane section S_1 is a horospherical variety with Picard number one and non reductive automorphism group, it appears as case 2 of [Pa], Theorem 0.1. It is uniquely defined up to isomorphism, but this is no longer the case of S_k for $k = 2, 3$.

Lemma 4.3. *(1) We have $h^0(T_{S_2}) - h^1(T_{S_2}) = 17$ and $h^0(T_{S_3}) - h^1(T_{S_3}) = 6$.*
(2) The general S_2 and S_3 are locally rigid.

Proof. Claim (1) is immediate from Proposition 2.5. Let $k = 2, 3$. The action of $\mathrm{GL}_k \times \mathrm{Spin}_{10}$ on $\mathbb{C}^k \otimes V_{16}$ (where V_{16} is a spin representation) is known to be quasi-homogeneous by [SK] (Propositions 32 and 33, p. 124-126), and therefore the action of Spin_{10} on the Grassmannian $\mathrm{Gr}(k, V_{16})$ is also quasi-homogeneous. By Proposition 3.2, the general S_2 and S_3 are locally rigid. □

In the following, we will study the quasi-homogeneity of the sections $S_k (k = 2, 3)$, and show that it does not always imply their local rigidity. To that purpose, we first introduce some results which allow us to determine $\mathfrak{aut}(S_k)$.

For a smooth projective subvariety $Z \subset \mathbb{P}^N$ covered by lines, the variety of lines on Z through a point $z \in Z$ is called the VMRT of Z at z. For equivariant compactifications of affine spaces, we can describe the infinitesimal automorphisms in terms of prolongations of the VMRT. For that purpose, we recall the following

Definition 4.4. Let V be a complex vector space and $\mathfrak{g} \subset \mathrm{End}(V)$ a Lie subalgebra. The *k-th prolongation* (denoted by $\mathfrak{g}^{(k)}$) of $\mathfrak{g}$ is the space of symmetric multi-linear homomorphisms $A : \mathrm{Sym}^{k+1} V \to V$ such that for any fixed $v_1, \cdots, v_k \in V$, the endomorphism $A_{v_1,\ldots,v_k} : V \to V$ defined by

$$v \in V \mapsto A_{v_1,\ldots,v_k,v} := A(v, v_1, \cdots, v_k) \in V$$

is in $\mathfrak{g}$. In other words, $\mathfrak{g}^{(k)} = \mathrm{Hom}(\mathrm{Sym}^{k+1} V, V) \cap \mathrm{Hom}(\mathrm{Sym}^k V, \mathfrak{g})$.

It is shown in [HM] that for a smooth non-degenerate $C \subsetneq \mathbb{P}^{n-1}$, the second prolongation satisfies $\mathfrak{aut}(\hat{C})^{(2)} = 0$, where $\mathfrak{aut}(\hat{C}) \subset \mathfrak{gl}(n)$ is the Lie algebra of infinitesimal automorphisms of $\hat{C}$. The following result is a combination of Propositions 5.10, 5.14 and 6.13 in [FH1].

Proposition 4.5. *Let X be an n-dimensional smooth uniruled projective variety of Picard number one. Assume that the VMRT at a general point is isomorphic to a smooth irreducible non-degenerate projective subvariety $C \subsetneq \mathbb{P}^{n-1}$. Then*

$$\dim \mathfrak{aut}(X) \leq n + \dim \mathfrak{aut}(\hat{C}) + \dim \mathfrak{aut}(\hat{C})^{(1)},$$

with equality if and only if X is an equivariant compactification of $\mathbb{C}^n$. In case of equality, we have $\mathfrak{aut}(X) \simeq \mathbb{C}^n \rtimes \mathfrak{aut}(\hat{C}) \oplus \mathfrak{aut}(\hat{C})^{(1)}$.

Lemma 4.6. *Let $C \subset \mathrm{Gr}(2,5)$ be a general codimension 2 linear section. Then $\mathfrak{aut}(\hat{C})^{(1)} \simeq \mathbb{C}$.*

Proof. First notice that $C \subset \mathbb{P}^7$ is quadratically symmetric by Proposition 7.6 [FH2], hence $\mathfrak{aut}(\hat{C})^{(1)} \neq 0$ by Proposition 7.11 [FH2]. By the proof of Theorem 6.15 [FH2] (whose conclusion is not correct, since there is an error in the proof of Proposition 2.9 *loc. cit.*), we get that $\mathfrak{aut}(\hat{C})^{(1)} \simeq \mathbb{C}$ as the VMRT of C (namely a twisted cubic in $\mathbb{P}^3$) has no prolongations. □

By Corollary 6.17 [K], there are exactly two isomorphic classes of smooth codimension 2 linear sections of $\mathbb{S}_5$. By Remark 2.13 [FH2], the special section is an equivariant compactification of $\mathbb{C}^8$ while the general one is not.

Proposition 4.7. *If S_2 is special, then $h^0(T_{S_2}) = 18$ and $h^1(T_{S_2}) = 1$.*

Proof. The VMRT C of S_2 is a codimension 2 linear section of $\mathrm{Gr}(2,5)$, so $\mathfrak{aut}(C)$ has dimension 8 by [PV]. If S_2 is special, then it is an equivariant compactification of $\mathbb{C}^8$, hence by Proposition 4.5, we have $\mathfrak{aut}(S_2) \simeq \mathbb{C}^8 \rtimes \mathfrak{aut}(\hat{C}) \oplus \mathfrak{aut}(\hat{C})^{(1)}$. Since $\mathfrak{aut}(\hat{C}_1)^{(1)}$ is one-dimensional by Lemma 4.6, we obtain that $\dim \mathfrak{aut}(S_2) = 18$. As $H^0(\mathcal{O}_{S_2}(1))$ has dimension 14, this gives that $H^1(T_{S_2}) = \mathbb{C}$, proving (1).

□

The general section S_2 has a very different automorphism group from that of the special one. By [SK] (Propositions 32, p. 124), the former is of type $G_2 \times SL_2$. This can be understood from the following construction: recall that if $n = p + q$, a half-spin representation of $\mathfrak{spin}_{2n}$, when restricted to the subalgebra $\mathfrak{spin}_{2p+1} \times \mathfrak{spin}_{2q-1}$, is isomorphic to the tensor product of the spin representations of $\mathfrak{spin}_{2p+1}$ and $\mathfrak{spin}_{2q-1}$. In particular, for $n = 5$, $p = 1$, $q = 4$, the half-spin representation Δ_{10} of $\mathfrak{spin}_{10}$ restricts to $\Delta_3 \otimes \Delta_7$. Of course $\mathfrak{spin}_3 \simeq \mathfrak{sl}_2$ and its spin representation Δ_3 is just the natural representation V_2 of $\mathfrak{sl}_2$. Now take another copy of $\mathfrak{sl}_2$ with its natural representation U_2, and consider the representation $U_2 \otimes \Delta_{10}$ of $\mathfrak{sl}_2 \times \mathfrak{spin}_{10}$. When restricted to $\mathfrak{sl}_2 \times \mathfrak{spin}_3 \times \mathfrak{spin}_7 \simeq \mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{spin}_7$, and then to the subalgebra $\delta(\mathfrak{sl}_2) \times \mathfrak{g}_2$, where $\delta(\mathfrak{sl}_2) \subset \mathfrak{sl}_2 \times \mathfrak{sl}_2$ denotes the diagonal subalgebra, one gets

$$U_2 \otimes \Delta_{10} \simeq U_2 \otimes U_2 \otimes (\mathbb{C} \oplus V_7),$$

where V_7 is the natural representation of $\mathfrak{g}_2 \subset \mathfrak{spin}_7$. In particular the line $L = \wedge^2 U_2 \subset U_2 \otimes \Delta_{10}$ is fixed by $\delta(\mathfrak{sl}_2) \times \mathfrak{g}_2$.

Lemma 4.8. *The stabilizer of L in $\mathfrak{sl}_2 \times \mathfrak{spin}_{10}$ is exactly $\delta(\mathfrak{sl}_2) \times \mathfrak{g}_2$.*

Proof. Let (u_1, u_2) and (v_1, v_2) be basis of U_2 and V_2, respectively. Since $\mathfrak{g}_2 \subset \mathfrak{spin}_7$ is the stabilizer of a generic point p in the spin representation Δ_7, we may suppose that the line L is generated by $q = (u_1 \otimes v_2 - u_2 \otimes v_1) \otimes p$. Let us compute the stabilizer of q in $\mathfrak{sl}_2 \times \mathfrak{spin}_{10}$. We can decompose

$$\mathfrak{spin}_{10} = \mathfrak{sl}_2 \times \mathfrak{spin}_7 \oplus \mathrm{End}_0(V_2) \otimes V_7.$$

Consider in $\mathrm{End}_0(V_2)$ the standard basis $(\theta_1, \theta_2, \theta_3) = (v_1^* \otimes v_1 - v_2^* \otimes v_2, v_2^* \otimes v_1, v_1^* \otimes v_2)$. We write an element of $\mathfrak{sl}_2 \otimes \mathfrak{spin}_{10}$ as $M = X + Y + Z + \theta_1 \otimes w_1 + \theta_2 \otimes w_2 + \theta_3 \otimes w_3$ for $X, Y \in \mathfrak{sl}_2$, $Z \in \mathfrak{spin}_7$ and $w_1, w_2, w_3 \in V_7$. The condition that $Mq = 0$ is equivalent to the equations $Zp = 0$ and

$$w_1 * p = (X_{11} + Y_{22})p, \quad w_2 * p = (X_{21} - Y_{21})p, \quad w_3 * p = (X_{12} - Y_{12})p,$$

where we denote by $*$ the Clifford multiplication map from $V_7 \otimes \Delta_7$ to Δ_7. In terms of the Cayley algebra $\mathbb{O}$, we can identify Δ_7 with $\mathbb{O}$, p with the unit octonion and V_7 with $\mathrm{Im}(\mathbb{O})$, and $*$ is then just the octonionic multiplication. In particular, $w * p$ identifies with w, and can never a be a non zero multiple of p. The previous equations therefore reduce to $w_1 = w_2 = w_3 = 0$, $X = Y$, and $Zp = 0$, that is, Z must belong to $\mathfrak{g}_2$. □

As a consequence, the $\mathrm{SL}_2 \times \mathrm{Spin}_{10}$-orbit of L in $\mathbf{P}(U_2 \otimes \Delta_{10})$ is open, and so must be the orbit of the corresponding plane of Δ_{10}. With the notations we have just used, this plane is nothing else than $V_2 \otimes p \subset V_2 \otimes \Delta_7 = \Delta_{10}$. We identify this subspace with its orthogonal (V_2 and Δ_7 are naturally self-dual), and we aim at describing the corresponding linear section S_2 of the spinor variety. For this we use the fact that the spinor variety is defined by its quadratic equations, which are parametrized by $V_{10} = \mathrm{Sym}^2 V_2 \oplus V_7$. These equations can be understood as follows: we have

$$\mathrm{Sym}^2(V_2 \otimes \Delta_7) = \mathrm{Sym}^2 V_2 \otimes \mathrm{Sym}^2 \Delta_7 \oplus \wedge^2 V_2 \otimes \wedge^2 \Delta_7.$$

Note that $\mathrm{Sym}^2 \Delta_7$ has a unique invariant (up to scalar), with an irreducible complement, while $\wedge^2 \Delta_7 = \mathfrak{spin}_7 \oplus V_7$. This means that an element $v_1 \otimes p_1 + v_2 \otimes p_2$ of Δ_{10} belongs to (the cone over) the spinor variety if and only if

$$Q(p_1) = Q(p_2) = Q(p_1, p_2) = 0 \qquad \text{and} \qquad \Omega(p_1, p_2) = 0,$$

where Q is the unique invariant quandratic form on Δ_7, and $\Omega : \wedge^2 \Delta_7 \longrightarrow V_7$ the unique invariant map (up to scalar). Now we restrict to the codimension two linear section S_2 orthogonal to $V_2 \otimes p$, which just means that p_1, p_2 must be orthogonal to p. Recall that as $\mathfrak{g}_2$-modules, $\Delta_7 \simeq \mathbb{C}p \oplus V_7$. Moreover, if we identify V_7 with the space of imaginary octonions, the restriction of Q must be (a multiple of) the standard quadratic form, and the unique $\mathfrak{g}_2$-invariant map from $\wedge^2 V_7$ to V_7 is given by the imaginary part of the octonionic multiplication. We conclude that $v_1 \otimes p_1 + v_2 \otimes p_2$ belongs to (the cone over) S_2 if and only if, either p_1 and p_2 are colinear and of norm zero, or they generate what is called a *null plane* in [LM], that is, a plane of imaginary octonions in restriction to which the octonionic product vanishes identically. Since G_2 acts transitively on the space of null-planes, we can finally conclude that S_2 is quasi-homogeneous under the action of $G_2 \times \mathrm{SL}_2$. Note moreover that it follows from this explicit description that S_2 is isomorphic with the two-orbits variety denoted X_2

in [Pa] (Definition 2.12). Taking into account Theorem 0.2 of [Pa], we summarize our discussion as follows:

Proposition 4.9. *The general codimension two linear section S_2 of the spinor variety $S = \mathbb{S}_5 \subset \mathbb{P}^{15}$ is a two-orbits variety, which is quasi-homogeneous under its connected automorphism group $G_2 \times \mathrm{PSL}_2$.*

Remark 4.10. Here is another way to prove that $X_2 \simeq S_2$: by [Pa] (Section 2.2.1), the connected automorphism group $\mathrm{Aut}^0(X_2)$ is isomorphic to $G_2 \times \mathrm{PSL}_2$, which acts on X_2 with two orbits. The closed orbit Y is isomorphic to $\mathbb{Q}^5 \times \mathbb{P}^1$ and the normal bundle $\mathcal{N}_{Y|X_2}$ is isomorphic to the vector bundle of rank 2 on Y associated to the irreducible representation with weights $\lambda_2 - \lambda_1 + 2\lambda_0$ and $-\lambda_2 + 2\lambda_1 + 2\lambda_0$ (where λ_1, λ_2 are fundamental weights of G_2 and λ_0 is that of PSL_2.) Note that in [Pa], he used the convention ω_i in stead of λ_i, which leads to a sign change. This gives that the first Chern class $c_1(\mathcal{N}_{Y|X_2}) = L^{\lambda_1+4\lambda_0} = \mathcal{O}_Y(1,4)$. By adjunction formula, we have $K^*_{X_2}|_Y = K^*_Y \otimes c_1(\mathcal{N}_{Y|X_2}) = \mathcal{O}_Y(6,6)$, which implies that $K^*_{X_2} = \mathcal{O}(6)$. This shows that X_2 is a Mukai variety, hence it must be a codimension 2 linear section of $\mathbb{S}_5$. As its automorphism group has dimension 17, it must be the general codimension 2 linear section S_2.

Now we consider S_3. From the classification results in [KW] (page 40), one can easily check that there exists exactly four isomorphism classes. By Remark 2.13 [FH2], although this is not the case of the general one, it can happen that S_3 is an equivariant compactification of $\mathbb{C}^7$; in this case we will say it is very special.

Proposition 4.11. *If S_3 is very special, then $h^0(T_{S_3}) = 11$ and $h^1(T_{S_3}) = 5$.*

Proof. If S_3 is an equivariant compactification of $\mathbb{C}^7$, its VMRT is a codimension 3 linear section of $\mathrm{Gr}(2,5)$, whose automorphism group is PGL_2. Hence $h^0(T_{S_3}) = 11$ by Proposition 4.5, and the claim follows from Lemma 4.3. □

In particular its automorphism group acts on the very special S_3 quasi-homogeneously, while the automorphism group of the general S_3, which is of type $SL_2 \times SL_2$ ([SK], Proposition 33 p. 126), is too small for that.

5. Quasi-homogeneous hyperplane sections

Let G/P be a rational homogeneous variety of Picard number one and $X \subset G/P$ a general hyperplane section. We consider the following question: when is X quasi-homogeneous, in the sense that $\mathrm{Aut}(X)$ acts on X with an open orbit?

If $\dim V_P \geq \dim \mathfrak{g}$, then X cannot be quasi-homogeneous, because $\dim \mathfrak{aut}(X)$ is smaller than $\dim X$. When $\dim V_P < \dim \mathfrak{g}$, we are in the list of Table 1 and we do a case-by-case check.

(i) If $G/P = \mathrm{Gr}(2,n)$, then X is either homogeneous (for n even) or it is an odd symplectic Grassmanian, which is quasi-homogeneous.

(ii) If $G/P = \mathrm{Gr}(3,6)$ (resp. $\mathrm{Lag}(3,6), \mathbb{S}_6$, E_7/P_7), then by [Ru] (Theorem 3), the connected automorphism group $\mathrm{Aut}^0(X)$ is isomorphic to $\mathrm{SL}_3 \times \mathrm{SL}_3$ (resp. SL_3, SL_6, E_6), which acts on X with an open orbit isomorphic to $\mathrm{SL}_3 \times \mathrm{SL}_3/\mathrm{diag}(\mathrm{SL}_3)$ (resp. $\mathrm{SL}_3/\mathrm{SO}_3$, $\mathrm{SL}_6/\mathrm{Sp}_6$, E_6/F_4), hence it is quasi-homogeneous.

(iii) For $G/P = \mathbb{S}_5 = D_5/P_5$, X is well-known to be quasi-homogeneous. In fact, it is one of the two-orbits varieties studied in [PP].

(iv) If $G/P = C_\ell/P_2$, then G/P is a general hyperplane section of $\mathrm{Gr}(2, 2\ell)$, hence X is a general codimension 2 linear section of $\mathrm{Gr}(2, 2\ell)$. By [PV], X is quasi-homogeneous if and only if $\ell \le 3$.

(v) For $G/P = G_2/P_1, E_6/P_1$, the general hyperplane sections are homogeneous.

(vi) For $G/P = \mathrm{Gr}(3, 8)$, then $\mathfrak{aut}(X) = \mathfrak{sl}_3$ has too small dimension for X to be quasi-homogeneous.

The remaining cases are $\mathrm{Gr}(3, 7)$, $\mathbb{S}_7$ and F_4/P_4. In the following subsections, we will prove that their general hyperplane sections are quasi-homogeneous. This will conclude the proof of Theorem 1.2.

5.1. $\mathrm{Gr}(3, 7)$.

Let V_7 be a seven-dimensional vector space. The stabilizer in $\mathrm{SL}(V_7)$ of a generic three-form $\omega \in \wedge^3 V_7^*$ is a subgroup isomorphic to G_2 [SK] (Proposition 8, p. 86). This can be understood by letting $V_7 = \mathrm{Im}\mathbb{O}$, the space of imaginary octonions. There is a natural three-form on this space, defined by

$$\omega(x, y, z) = \mathrm{Re}((xy)z), \qquad \forall x, y, z \in \mathrm{Im}\mathbb{O}.$$

This three-form ω is invariant under $G_2 = \mathrm{Aut}(\mathbb{O})$, and it is known to be generic in the sense that its $\mathrm{GL}(V_7)$-orbit is open in $\wedge^3 V_7^*$.

Proposition 5.1. *The hyperplane section X_ω of $\mathrm{Gr}(3, V_7)$ defined by the generic three-form $\omega \in \wedge^3 V_7^*$ is a compactification of G_2/O_3.*

Proof. We denote by $e_0 = 1, e_1, \dots, e_7$ the standard basis of $\mathbb{O}$, whose multiplication table is given by an oriented Fano plane, as in [M] (p. 105). A point of X_ω is the three-plane $L = \langle e_1, e_2, e_4 \rangle$, and we claim that the stabilizer of L in G_2 is isomorphic to O_3. Indeed let g_1, g_2, g_4 be any orthonormal basis of L. Define $g_0 = 1$, $g_3 = g_1 g_2$, $g_6 = g_2 g_4$, $g_7 = g_4 g_1$, and $g_5 = (g_1 g_2) g_4$.

Lemma 5.2. *The endomorphism of $\mathbb{O}$ sending e_i to g_i for $0 \le i \le 7$, belongs to G_2.*

This proves that an element of G_2 that stabilizes L is uniquely defined by its restriction to L, which can be any element in the orthogonal group $O(L) \simeq O_3$. □

5.2. $\mathbb{S}_7$.

Let V_{14} be a fourteen-dimensional vector space, endowed with a nondegenerate quadratic form. We split $V_{14} = E \oplus F$ into two maximal isotropic spaces. These two spaces are in duality with respect to the quadratic form. We choose a basis $e_1, \dots, e_7$ of E and denote by $f_1, \dots, f_7$ the dual basis of F.

Recall that the two half-spin representations of $\mathrm{Spin}(V_{14}) \simeq \mathrm{Spin}_{14}$ can be defined as $S_+ = \wedge^+ E$ and $S_- = \wedge^- E$, the even and odd parts of the exterior algebra of E. The action of the spin group is induced by the action of the Clifford algebra of V_{14} on $\wedge E$, where E acts by exterior product and F by contraction. Here S_+ and S_- are dual one to the other; alternatively, one can therefore define S_- as $\wedge^+ F$.

The spinor variety $\mathbb{S}_7$ is the closed orbit of the spin group inside $\mathbb{P}S_+$. It parametrizes the set of maximal isotropic subspaces of V_{14} that meet E in even dimension. This is done by associating to any such isotropic subspace W a pure spinor s_W, uniquely defined (up to scalar) by the fact that its annihilator (for the Clifford multiplication) is precisely W. For example $s_F = 1$.

The action of the spin group on $\mathbb{P}S_+$ and its dual $\mathbb{P}S_-$ is known to be quasihomogeneous. An explicit element in the open orbit of $\mathbb{P}S_-$ is given [SK](pp. 131-132) by the class of the spinor

$$s^* = 1 + f_1f_2f_3f_7 + f_4f_5f_6f_7 + f_1f_2f_3f_4f_5f_6.$$

The stabilizer H of s^* in $\mathrm{Spin}(V_{14})$ is locally isomorphic to $G_2\times G_2$. In fact, according to [SK], there are two distinguished seven dimensional subspaces of V_{14} that are preserved by H, namely

$$I_1 = \langle e_7 - f_7, e_1, e_2, e_3, f_1, f_2, f_3\rangle, \qquad I_2 = \langle e_7 + f_7, e_4, e_5, e_6, f_4, f_5, f_6\rangle.$$

Note that the restriction of the quadratic form to either I_1 and I_2 is nondegenerate, so that each of these spaces can be interpreted as a copy of the imaginary octonions. Moreover the infinitesimal action of $\mathfrak{g}_2$ on I_1 is explicitely given by the matrices of the form

$$M = \begin{pmatrix} 0 & 2u & 2v & 2w & 2a & 2b & 2c \\ a & & & & 0 & w & -v \\ b & & A & & -w & 0 & u \\ c & & & & v & -u & 0 \\ u & 0 & -c & b & & & \\ v & c & 0 & -a & & -{}^tA & \\ w & -b & a & 0 & & & \end{pmatrix},$$

where A belongs to $\mathfrak{sl}_3$. Indeed, $\mathfrak{g}_2$ contains $\mathfrak{sl}_3$ as a subalgebra (generated by the long root vectors), and decomposes as an $\mathfrak{sl}_3$ module as $\mathfrak{g}_2 = \mathfrak{sl}_3 \oplus V_3 \oplus V_3^*$. Moreover the natural action of $\mathfrak{g}_2$ on $V_7 = \mathbb{C}\oplus V_3\oplus V_3^*$ is given by the previous matrices.

Similarly, the infinitesimal action of $\mathfrak{g}_2$ on I_2 is given by the matrices

$$N = \begin{pmatrix} 0 & 2\lambda & 2\mu & 2\nu & 2d & 2e & 2f \\ d & & & & 0 & \nu & -\mu \\ e & & B & & -\nu & 0 & \lambda \\ f & & & & \mu & -\lambda & 0 \\ \lambda & 0 & -f & e & & & \\ \mu & f & 0 & -d & & -{}^tB & \\ \nu & -e & d & 0 & & & \end{pmatrix}$$

where B belongs to $\mathfrak{sl}_3$.

Now consider the space $W = \langle e_1 - e_4, f_1 + f_4, e_2 - e_5, f_2 + f_5, e_3 - e_6, f_3 + f_6, e_7\rangle$. This is a maximal isotropic subspace of V_{14}, that intersects E in four dimensions. The associated pure spinor is

$$s_W = (e_1 - e_4)(e_2 - e_5)(e_3 - e_6)e_7 = e_1e_2e_3e_7 - e_4e_5e_6e_7 + \dots.$$

One has $\langle s_W, s^*\rangle = 0$: so s_W is a pure spinor in the hyperplane section of $\mathbb{S}_7$ defined by s^*. Moreover, a straightforward computation shows that the endomorphism of V_{14} defined by a pair (M, N) as above, preserves W if and only if $A = B$ and all the other coefficients vanish. In other words, the infinitesimal stabilizer of s_W in $\mathfrak{g}_2\times\mathfrak{g}_2$ is isomorphic to $\mathfrak{sl}_3$.

This implies that the $G_2\times G_2$-orbit of s_W has dimension 20, hence it must be open in the hyperplane section of the spinor variety. We have proved:

Proposition 5.3. *The general hyperplane section of $\mathbb{S}_7$ is a compactification of $G_2\times G_2/K$, where K is locally isomorphic to the diagonal SL_3.*

Remark 5.4. Along the same line, we can get yet another proof of Proposition 4.9: by [SK] (p. 122), the general codimension 2 linear section S_2 of $\mathbb{S}_5$ is defined by $s_1^* = 1 + f_1f_2f_3f_4$ and $s_2^* = f_1f_5 + f_2f_3f_4f_5$. Consider the space $W = \langle f_1, e_2 - e_4, f_2 +$

$f_4, e_3 - e_5, f_3 + f_5\rangle$ with associated pure spinor $s_W = (e_2 - e_4)(e_3 - e_5)$ which satisfies $\langle s_W, s_i^* \rangle = 0, i = 1, 2$, hence s_W is on S_2. By Proposition 3.2, $\mathfrak{aut}(S_2) = \mathfrak{g}_2 \times \mathfrak{sl}_2$, which can be represented by the matrices in (5.40) [SK] (p.122). It is now straightforward to compute that the stabilizer of s_W is defined by 8 linear equations, hence the orbit $\mathrm{Aut}^0(S_2) \cdot s_W$ is of dimension 8 and hence open in S_2, which proves that S_2 is quasi-homogeneous. Now if we take $U = \langle e_1 - e_3, f_1 + f_3, e_2 - e_4, f_2 + f_4, f_5 \rangle$, then in the same way, we can show that $s_U \in S_2$ and $\mathrm{Aut}^0(S_2) \cdot s_U$ is of dimension 6, which is the closed orbit in S_2.

5.3. F_4/P_4.

As was already noticed, F_4/P_4 is a generic hyperplane section of the Cayley plane E_6/P_1, embedded in the projectivization of the minimal representation V_{27} of E_6. So we will consider a generic codimension two linear section of the Cayley plane.

Recall that V_{27} can be identified with the exceptional Jordan algebra $H_3(\mathbb{O})$ of 3×3 Hermitian matrices with octonionic coefficients:

$$M = \begin{pmatrix} r_1 & x_3 & x_2 \\ \overline{x}_3 & r_2 & x_1 \\ \overline{x}_2 & \overline{x}_1 & r_3 \end{pmatrix}, \qquad x_1, x_2, x_3 \in \mathbb{O},\ r_1, r_2, r_3 \in \mathbb{C}.$$

Moreover, one can understand the Cayley plane $E_6/P_1 \subset \mathbb{P}H_3(\mathbb{O})$ as the Zariski closure of the set of matrices of the form

$$\begin{pmatrix} 1 & x & y \\ \overline{x} & \overline{x}x & \overline{x}y \\ \overline{y} & \overline{y}x & \overline{y}y \end{pmatrix}, \qquad x, y \in \mathbb{O}.$$

The full action of E_6 on the Cayley plane would be complicated to describe in full. Let us just mention that the part of it that acts trivially on diagonal matrices respects the nondiagonal blocks: it is made of transformations of type

$$\begin{pmatrix} r_1 & x_3 & x_2 \\ \overline{x}_3 & r_2 & x_1 \\ \overline{x}_2 & \overline{x}_1 & r_3 \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} r_1 & g_3(x_3) & g_2(x_2) \\ \overline{g_3(x_3)} & r_2 & g_1(x_1) \\ \overline{g_2(x_2)} & \overline{g_1(x_1)} & r_3 \end{pmatrix},$$

where g_1, g_2, g_3 belong to $\mathrm{GL}(\mathbb{O})$. For such a transformation to preserve the Cayley plane, one needs the condition that

$$g_1(\overline{x}y) = \overline{g_3(x)}g_2(y) \qquad \forall x, y \in \mathbb{O}.$$

By the celebrated triality principle, the set of such triples (g_1, g_2, g_3) form a group isomorphic to Spin_8.

By [SK](p. 138), the action of E_6 on the Grassmannian of codimension two subspaces of V_{27} is quasihomogeneous. Moreover, one defines a point in the open orbit by the linear equations

$$r_1 + r_2 + r_3 = r_1 - r_3 = 0,$$

and the stabilizer of this point in E_6 is precisely the copy of Spin_8 that we have just described (up to a finite group). This Spin_8 acts on the linear section X_0 of the Cayley plane defined by our two linear equations. Consider the point p of X_0 defined by the matrix

$$\begin{pmatrix} 1 & i\sqrt{2}e_0 & e_0 \\ i\sqrt{2}e_0 & -2 & i\sqrt{2}e_0 \\ i\sqrt{2}e_0 & i\sqrt{2}e_0 & 1 \end{pmatrix}.$$

The stabilizer of p in Spin_8 is the set of triples (g_1, g_2, g_3) in $\mathrm{GL}(\mathbb{O})$ such that $g_i(e_0) = e_0, i = 1, 2, 3,$ and $g_1(\overline{x}y) = \overline{g_3(x)}g_2(y)$. Letting $x = e_0$ we get $g_1(y) = g_2(y)$, while letting $y = e_0$ we get $g_1(\overline{x}) = \overline{g_3(x)}$. This means that the triple (g_1, g_2, g_3) is uniquely determined by g_1, which is submitted to the condition that $g_1(\overline{x}y) = g_1(\overline{x})g_1(y)$. In other words, g_1 must belong to $\mathrm{Aut}(\mathbb{O}) = G_2$, and the stabilizer of p in Spin_8 is isomorphic to G_2.

Since Spin_8/G_2 has the same dimension 14 as the codimension two section X_0 of the Cayley plane, we conclude that:

Proposition 5.5. *The generic hyperplane section of F_4/P_4, which is also a generic codimension two linear section of the Cayley plane E_6/P_1, is a compactification of* Spin_8/G_2.

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