SUM FORMULA FOR MULTIPLE ZETA FUNCTION

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Abstract. The sum formula is a well known relation in the field of the multiple zeta values. In this paper, we present its generalization for the Euler-Zagier multiple zeta function.

1. Introduction

The Euler-Zagier multiple zeta function (MZF) is defined by

$$\zeta(s_1,\dots,s_r) := \sum_{1\le n_1<\cdots<n_r} \frac{1}{n_1^{s_1}\cdots n_r^{s_r}},$$

where $s_i \in \mathbb{C}\,(i=1,\dots,r)$ are complex variables. The number of the variables r is called the depth of $\zeta(s_1,\dots,s_r)$. Matsumoto [3] proved that the series is absolutely convergent in the domain

$$\{(s_1,\dots,s_r)\in\mathbb{C}^r \mid \Re(s_l+\cdots+s_r) > r-l+1 \ \ (l=1,\dots,r)\}.$$

Akiyama, Egami, and Tanigawa [1] and Zhao [6] independently proved that $\zeta(s_1,\dots,s_r)$ is meromorphically continued to the whole space $\mathbb{C}^r$. Furthermore, all possible poles of $\zeta(s_1,\dots,s_r)$ are located on $s_l+\cdots+s_r \in \mathbb{Z}_{\le r-l+1}$ for $l=1,\dots,r$. Note that some possible poles are known not to be actual poles, but we do not use this fact in this paper (for details, see [1]).

The special values $\zeta(k_1,\dots,k_r)$ with $k_1,\dots,k_{r-1}\in\mathbb{Z}_{\ge1}$ and $k_r\in\mathbb{Z}_{\ge2}$ are called multiple zeta values (MZVs). The MZVs are real numbers and known to satisfy many kinds of algebraic relations over $\mathbb{Q}$. One of the most fundamental relations is the sum formula:

Proposition 1.1 (Sum formula; Granville [2], Zagier)**.** *For positive integers k, r with $k>r$, we have*

$$\sum_{\substack{k_1+\cdots+k_r=k\\ k_1,\dots,k_{r-1}\ge1,k_r\ge2}} \zeta(k_1,\dots,k_r) = \zeta(k).$$

From the analytic point of view, Matsumoto [4] raised the question whether the known relations among MZVs are valid only for positive integers or not. It is known that the harmonic relations, e.g., $\zeta(s_1)\zeta(s_2)=\zeta(s_1,s_2)+\zeta(s_2,s_1)+\zeta(s_1+s_2)$ are valid not only for positive integers but for complex numbers. Matsumoto and Tsumura [5, Proposition 2.1] gave a relation which is a generalization of Proposition 1.1 with $r=2$. This relation consists not only of the MZFs but also of the Mordell-Tornheim multiple zeta functions. Based on such circumstances, we give a generalization of Proposition 1.1 for the MZF.

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Theorem 1.2. *For $s \in \mathbb{C}$ with $\Re(s) > 1$ and $s \neq 2$, we have*

$$\sum_{n=0}^{\infty}(\zeta(s-n-2, n+2) - \zeta(-n, s+n)) = \zeta(s).$$

Theorem 1.2 is a generalization of Proposition 1.1 with depth $r = 2$. This can be generalized to the arbitrary depth (see Theorem 1.7).

Definition 1.3. For a non-negative integer a and a positive integer b, we define $G_{a,b}(s_1, \ldots, s_a; s)$ inductively by

$$\begin{aligned} G_{a,1}(s_1, \ldots, s_a; s) &:= \zeta(s_1, \ldots, s_a, s), \\ G_{a,b}(s_1, \ldots, s_a; s) &:= \sum_{n=0}^{\infty} G_{a+1,b-1}(s_1, \ldots, s_a, s-n-b; n+b) \\ &\quad - \sum_{n=0}^{\infty} G_{a+1,b-1}(s_1, \ldots, s_a, -n; s+n), \end{aligned}$$

where $s_1, \ldots, s_a, s$ are complex numbers such that $\Re(s) > b, \Re(s+s_a) > 1+b, \ldots, \Re(s+s_a+\cdots+s_1) > a+b$.

Remark 1.4. The convergence will be proved in Lemma 3.4.

Example 1.5. We show some examples of $G_{a,b}(s_1, \ldots, s_a; s)$:

$$\begin{aligned} &G_{a,2}(s_1, \ldots, s_a; s) \\ &= \sum_{n=0}^{\infty} \zeta(s_1, \ldots, s_a, s-n-2, n+2) - \sum_{n=0}^{\infty} \zeta(s_1, \ldots, s_a, -n, s+n), \\ &G_{a,3}(s_1, \ldots, s_a; s) \\ &= \sum_{n_1=0}^{\infty} \left(\sum_{n_2=0}^{\infty} \zeta(s_1, \ldots, s_a, s-n_1-3, n_1-n_2+1, n_2+2) \right. \\ &\qquad \left. - \sum_{n_2=0}^{\infty} \zeta(s_1, \ldots, s_a, s-n_1-3, -n_2, n_1+n_2+3) \right) \\ &\quad - \sum_{n_1=0}^{\infty} \left(\sum_{n_2=0}^{\infty} \zeta(s_1, \ldots, s_a, -n_1, s+n_1-n_2-2, n_2+2) \right. \\ &\qquad \left. - \sum_{n_2=0}^{\infty} \zeta(s_1, \ldots, s_a, -n_1, -n_2, s+n_1+n_2) \right). \end{aligned}$$

Remark 1.6. We note that $G_{a,b}(s_1, \ldots, s_a; s)$ is a sum of MZFs of depth $a + b$.

Theorem 1.7. *Let b be a positive integer. For $s \in \mathbb{C}$ with $\Re(s) > b$, we have*

$$G_{0,b}(s) = \zeta(s).$$

Remark 1.8. If $s \in \mathbb{Z}_{>b}$, $G_{a,b}(s_1, \ldots, s_a; s)$ is equal to

$$\sum_{\substack{m_1+\cdots+m_b=s \\ m_1,\ldots,m_{b-1}\geq 1 \\ m_b \geq 2}} \zeta(s_1, \ldots, s_a, m_1, \ldots, m_b)$$

by definition. Thus the case $s \in \mathbb{Z}_{>b}$ of the theorem implies Proposition 1.1.

2. Proof of theorem 1.2

Let $\sigma := \Re(s)$ and $\sigma_i := \Re(s_i)$ for $i \in \mathbb{Z}_{\geq 1}$. The proof of Theorem 1.2 is divided into two parts. For $\sigma > 2$, we use series transformation. For $\sigma > 1$, we prove the theorem by showing the regularness of the infinite series on the left-hand side and using analytic continuation.

Proof of Theorem 1.2 for $\sigma > 2$. Since the functions $\zeta(s-n-2, n+2)$ and $\zeta(-n, s+n)$ are absolutely convergent in $\sigma > 2$, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}(\zeta(s-n-2,n+2)-\zeta(-n,s+n)) \\
&=\sum_{n=0}^{\infty}\sum_{0<m_1<m_2}\left(\frac{1}{m_1^{s-n-2}m_2^{n+2}}-\frac{1}{m_1^{-n}m_2^{s+n}}\right) \\
&=\sum_{n=0}^{\infty}\sum_{0<m_1<m_2}\left(\frac{m_1}{m_2}\right)^n\left(\frac{1}{m_1^{s-2}m_2^2}-\frac{1}{m_2^s}\right) \\
&=\sum_{0<m_1<m_2}\frac{1}{m_2^2-m_1m_2}\left(\frac{1}{m_1^{s-2}}-\frac{1}{m_2^{s-2}}\right).
\end{aligned}$$

Here we note that interchanging the order of summations $\sum_{n=0}^{\infty}$ and $\sum_{0<m_1<m_2}$ is valid because the last sum is absolutely convergent in $\sigma > 2$. Since

$$\begin{aligned}
\sum_{0<m_1<m_2}\frac{1}{m_2^2-m_1m_2}\frac{1}{m_1^{s-2}} &= \sum_{0<m_1}\frac{1}{m_1^{s-2}}\sum_{m_1<m_2}\frac{1}{m_2^2-m_1m_2} \\
&=\sum_{0<m_1}\frac{1}{m_1^{s-2}}\sum_{m_1<m_2}\frac{1}{m_1}\left(\frac{1}{m_2-m_1}-\frac{1}{m_2}\right) \\
&=\sum_{0<m_1}\frac{1}{m_1^{s-1}}\left(1+\frac{1}{2}+\cdots+\frac{1}{m_1}\right)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{0<m_1<m_2}\frac{1}{m_2^2-m_1m_2}\frac{1}{m_2^{s-2}} &= \sum_{1<m_2}\frac{1}{m_2^{s-1}}\sum_{0<m_1<m_2}\frac{1}{m_2-m_1} \\
&=\sum_{0<m_2}\frac{1}{m_2^{s-1}}\left(1+\frac{1}{2}+\cdots+\frac{1}{m_2-1}\right),
\end{aligned}$$

we have

$$\sum_{n=0}^{\infty}(\zeta(s-n-2,n+2)-\zeta(-n,s+n)) = \sum_{0<m}\frac{1}{m^s} = \zeta(s).$$

Then we find the result. □

Next, we prove Theorem 1.2 for $1 < \sigma \leq 2$.

Lemma 2.1. *For $s \in \mathbb{C}$ with $s \neq 2$ and $\sigma > 1$, and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 1$, the analytic continuation of $\zeta(s-\alpha, \alpha)$ can be given by*

$$\zeta(s-\alpha,\alpha) = \sum_{0<m_1}\frac{1}{m_1^{s-\alpha}}\sum_{m_1<m_2}\left(\frac{1}{m_2^{\alpha}}-\int_0^1\frac{dt}{(m_2-t)^{\alpha}}\right)+\frac{\zeta(s-1)}{\alpha-1}.$$

Proof. Since

$$\frac{1}{m_2^\alpha} - \int_0^1 \frac{dt}{(m_2-t)^\alpha} = O\left(\frac{1}{m_2^{\alpha+1}}\right) \qquad (m_2 \to \infty),$$

the first term on the right-hand side converges absolutely for $\sigma > 1$. Furthermore, if $\sigma > 2$, we have

$$\begin{aligned}
&\sum_{0<m_1} \frac{1}{m_1^{s-\alpha}} \sum_{m_1<m_2} \left(\frac{1}{m_2^\alpha} - \int_0^1 \frac{dt}{(m_2-t)^\alpha}\right) \\
&= \sum_{0<m_1} \frac{1}{m_1^{s-\alpha}} \left(\sum_{m_1<m_2} \frac{1}{m_2^\alpha} - \int_{m_1}^\infty \frac{du}{u^\alpha}\right) \\
&= \zeta(s-\alpha,\alpha) - \frac{1}{\alpha-1}\zeta(s-1).
\end{aligned}$$

□

We denote by $[t]$ the greatest integer less than or equal to t.

Lemma 2.2. *For $s \in \mathbb{C}$ with $s \neq 2$ and $\sigma > 1$, and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 1$, we have*

$$\zeta(s-\alpha,\alpha) = -\sum_{0<m} \frac{\alpha}{m^{s+1}} \int_0^\infty \left(\frac{m}{m+t}\right)^{\alpha+1} (t-[t])dt + \frac{\zeta(s-1)}{\alpha-1}.$$

Proof. We have

$$\begin{aligned}
\frac{1}{m_2^\alpha} - \int_0^1 \frac{dt}{(m_2-t)^\alpha} &= \int_0^1 \left(\frac{1}{m_2^\alpha} - \frac{1}{(m_2-t)^\alpha}\right) dt \\
&= -\int_{0<u<t<1} \frac{\alpha}{(m_2-u)^{\alpha+1}} du dt \\
&= -\int_0^1 \frac{\alpha(1-u)}{(m_2-u)^{\alpha+1}} du.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\zeta(s-\alpha,\alpha) &= -\sum_{0<m_1} \frac{1}{m_1^{s-\alpha}} \sum_{m_1<m_2} \int_0^1 \frac{\alpha(1-t)}{(m_2-t)^{\alpha+1}} dt + \frac{\zeta(s-1)}{\alpha-1} \\
&= -\sum_{0<m_1} \frac{\alpha}{m_1^{s+1}} \sum_{m_1<m_2} \int_0^1 \left(\frac{m_1}{m_2-t}\right)^{\alpha+1} (1-t)dt + \frac{\zeta(s-1)}{\alpha-1} \\
&= -\sum_{0<m_1} \frac{\alpha}{m_1^{s+1}} \sum_{m_1\le m_2} \int_0^1 \left(\frac{m_1}{m_2+t}\right)^{\alpha+1} tdt + \frac{\zeta(s-1)}{\alpha-1} \\
&= -\sum_{0<m} \frac{\alpha}{m^{s+1}} \int_0^\infty \left(\frac{m}{m+t}\right)^{\alpha+1} (t-[t])dt + \frac{\zeta(s-1)}{\alpha-1}.
\end{aligned}$$

This finishes the proof. □

From the previous lemma, we have

$$\frac{\partial}{\partial\alpha}\zeta(s-\alpha,\alpha) = -H_1(\alpha) + \alpha H_2(\alpha) - H_3(\alpha),$$

where

$$H_1(\alpha) = \sum_{0<m} \frac{1}{m^{s+1}} \int_0^\infty \left(\frac{m}{m+t}\right)^{\alpha+1} (t-[t])dt,$$

$$H_2(\alpha) = \sum_{0<m} \frac{1}{m^{s+1}} \int_0^\infty \log\left(\frac{m+t}{m}\right)\left(\frac{m}{m+t}\right)^{\alpha+1} (t-[t])dt,$$

$$H_3(\alpha) = \frac{\zeta(s-1)}{(\alpha-1)^2}.$$

Lemma 2.3. *Assume that $s \in \mathbb{R}$ and $1 < s < 2$. Then, for $\alpha \in \mathbb{R}_{>1}$, we have*

$$H_1(\alpha) = O\left(\frac{1}{\alpha^s}\right) \qquad (\alpha \to \infty).$$

Proof. Since

$$\left(\frac{m}{m+t}\right)^{\alpha+1} \leq \left(\frac{z}{z+t}\right)^{\alpha+1},$$

$$\frac{1}{m^{s+1}} \leq \frac{2^{s+1}}{(m+1)^{s+1}} \leq \frac{2^{s+1}}{z^{s+1}}$$

for $t > 0, m \in \mathbb{Z}_{>0}$, and $m \leq z \leq m+1$, we have

$$\begin{aligned}
H_1(\alpha) &\leq 2^{s+1} \int_1^\infty \left(\frac{1}{z^s}\int_0^\infty \left(\frac{z}{z+t}\right)^{\alpha+1} (t-[t])dt\right)\frac{dz}{z} \\
&\leq 2^{s+1} \int_0^\infty \left(\frac{1}{z^s}\int_0^\infty \left(\frac{z}{z+t}\right)^{\alpha+1} (t-[t])dt\right)\frac{dz}{z} \\
&= 2^{s+1} \int_0^\infty \frac{z^{\alpha-s}}{(1+z)^{\alpha+1}}dz \int_0^\infty \frac{t-[t]}{t^s}dt \\
&= 2^{s+1} B(\alpha+1-s, s) \int_0^\infty \frac{t-[t]}{t^s}dt,
\end{aligned}$$

where B is the beta function. Since the integral

$$\int_0^\infty \frac{t-[t]}{t^s}dt$$

is convergent and

$$B(\alpha+1-s, s) = O\left(\frac{1}{\alpha^s}\right),$$

we have the result. □

Lemma 2.4. *Assume that $s \in \mathbb{R}$ and $1 < s < 2$. Then, for $\alpha \in \mathbb{R}_{>1}$, we have*

$$H_2(\alpha) = O\left(\frac{1}{\alpha^{s+1}}\right) \qquad (\alpha \to \infty).$$

Proof. Since $\log x \leq x - 1$ for $x \geq 1$, we have

$$\begin{aligned}
H_2(\alpha) &\leq \sum_{0<m} \frac{1}{m^{s+1}} \int_0^\infty \left(\frac{m+t}{m} - 1\right)\left(\frac{m}{m+t}\right)^{\alpha+1} (t-[t])dt \\
&= \sum_{0<m} \frac{1}{m^{s+2}} \int_0^\infty \left(\frac{m}{m+t}\right)^{\alpha+1} t(t-[t])dt.
\end{aligned}$$

Since

$$\left(\frac{m}{m+t}\right)^{\alpha+1} \leq \left(\frac{z}{z+t}\right)^{\alpha+1},$$
$$\frac{1}{m^{s+2}} \leq \frac{2^{s+2}}{(m+1)^{s+2}} \leq \frac{2^{s+2}}{z^{s+2}}$$

for any $0 < t, m \in \mathbb{Z}_{>0}$ and $m \leq z \leq m+1$, we have

$$\begin{aligned}
\frac{H_2(\alpha)}{2^{s+2}} &\leq \frac{1}{2^{s+2}} \sum_{0<m} \frac{1}{m^{s+2}} \int_0^\infty \left(\frac{m}{m+t}\right)^{\alpha+1} t(t-[t])dt \\
&\leq \int_1^\infty \left(\frac{1}{z^{s+2}} \int_0^\infty \left(\frac{z}{z+t}\right)^{\alpha+1} t(t-[t])dt\right) dz \\
&\leq \int_0^\infty \left(\frac{1}{z^{s+1}} \int_0^\infty \left(\frac{z}{z+t}\right)^{\alpha+1} t(t-[t])dt\right) \frac{dz}{z} \\
&= \int_0^\infty \frac{z^{\alpha-s-1}}{(1+z)^{\alpha+1}} dz \int_0^\infty \frac{t-[t]}{t^s} dt \\
&= B(\alpha-s, s+1) \int_0^\infty \frac{t-[t]}{t^s} dt.
\end{aligned}$$

Similar to the proof of the previous lemma, we have the result. □

Lemma 2.5. *Let $M > 0$. For $0 < \delta < \min(1, \sigma-1) \in \mathbb{R}$, we have*

$$\frac{\partial}{\partial \alpha} \zeta(s-\alpha, \alpha) = O\left(\frac{1}{\Re(\alpha)^{1+\delta}}\right) \qquad (\Re(\alpha) \to \infty, |\Im(\alpha)| < M).$$

Proof. Since

$$H_3(\alpha) = O\left(\frac{1}{\Re(\alpha)^2}\right) \qquad (\Re(\alpha) \to \infty),$$

we need to show

(1) $$H_1(\alpha) = O\left(\frac{1}{\Re(\alpha)^{1+\delta}}\right) \qquad (\Re(\alpha) \to \infty)$$

and

(2) $$\alpha H_2(\alpha) = O\left(\frac{1}{\Re(\alpha)^{1+\delta}}\right) \qquad (\Re(\alpha) \to \infty, |\Im(\alpha)| < M).$$

Note that

$$|H_1(\alpha)| \leq \sum_{0<m} \frac{1}{m^{(1+\delta)+1}} \int_0^\infty \left(\frac{m}{m+t}\right)^{\Re(\alpha)+1} (t-[t])dt,$$
$$|H_2(\alpha)| \leq \sum_{0<m} \frac{1}{m^{(1+\delta)+1}} \int_0^\infty \log\left(\frac{m+t}{m}\right) \left(\frac{m}{m+t}\right)^{\Re(\alpha)+1} (t-[t])dt$$

hold. Then, by Lemmas 2.3 and 2.4, we obtain (1) and (2). □

Proposition 2.6. *Let $M > 0$ and $s \in \mathbb{C}$ with $s \neq 2$, $\sigma > 1$, and $|\Im(s)| < M$. For $0 < \delta < \min(1, \sigma-1) \in \mathbb{R}$, we have*

$$|\zeta(s-n-2, n+2) - \zeta(-n, s+n)| \leq O\left(\frac{1}{n^{1+\delta}}\right).$$

Proof. From

$$\begin{aligned}\zeta(s-n-2,n+2)-\zeta(-n,s+n) &= \int_{n+s}^{n+2}\left(\frac{\partial}{\partial\alpha}\zeta(s-\alpha,\alpha)\right)d\alpha \\ &= \int_{n+s}^{n+2} O\left(\frac{1}{\Re(\alpha)^{1+\delta}}\right)d\alpha \\ &= \int_{n+s}^{n+2} O\left(\frac{1}{\min(n+2,n+\sigma)^{1+\delta}}\right)d\alpha\end{aligned}$$

by Lemma 2.5, we find the result. ☐

Proof of Theorem 1.2. It follows from Proposition 2.6 that the sum $\sum_{n=0}^{\infty}(\zeta(s-n-2,n+2)-\zeta(-n,s+n))$ uniformly converges on any compact subsets of $\{s\in\mathbb{C}\mid s\neq 2,\sigma>1\}$, and holomorphic on the region. By the identity theorem, this finishes the proof. ☐

3. Proof of Theorem 1.7

Definition 3.1. For a positive integer d, a non-negative integer D, and $s\in\mathbb{C}$, we define

$$\begin{aligned}F_d(D;s) &:= \sum_{D<m}\frac{1}{m^{s-d}}\sum_{m-D\le x_1\le\cdots\le x_d\le m}\frac{1}{x_1\cdots x_d}, \\ F_d^{(1)}(D;s) &:= \sum_{D<t<m}\frac{1}{t^{s-d-1}(m-t)}\sum_{m-t\le x_1\le\cdots\le x_d\le m}\frac{1}{x_1\cdots x_d}, \\ F_d^{(2)}(D;s) &:= \sum_{D<t<m}\frac{1}{t^{s-d-1}m}\sum_{m-t\le x_1\le\cdots\le x_d\le m}\frac{1}{x_1\cdots x_d}, \\ F_d^{(3)}(D;s) &:= \sum_{D<t<m}\frac{1}{m^{s-d-1}(m-t)}\sum_{m-t\le x_1\le\cdots\le x_d\le m}\frac{1}{x_1\cdots x_d}.\end{aligned}$$

Lemma 3.2. *If $\sigma>1$, the function $F_d(D;s)$ converges absolutely. In addition, if $\sigma>d+2$, the functions $F_d^{(1)}(D;s)$, $F_d^{(2)}(D;s)$, and $F_d^{(3)}(D;s)$ converge absolutely. Moreover, we have*

$$\left|F_d^{(i)}(D;s)\right| \ll \sum_{D<t}\frac{(\log t)^{d+1}}{t^{\sigma-d-1}} \tag{3}$$

for $i=1,2,3$, where the implicit constant does not depend on D and s.

Proof. The convergence of $F_d(D;s)$ is immediate. The convergence of $F_d^{(i)}(D;s)$ follows from (3). Since

$$\left|F_d^{(i)}(D;s)\right| \le F_d^{(1)}(D;\sigma),$$

it is enough to prove (3) for $i=1$ and $s\in\mathbb{R}_{>d+2}$. Write $F_d^{(1)}(D;s)$ as $A+B$ where

$$\begin{aligned}A &:= \sum_{D<t\le m/2}\frac{1}{t^{s-d-1}(m-t)}\sum_{m-t\le x_1\le\cdots\le x_d\le m}\frac{1}{x_1\cdots x_d}, \\ B &:= \sum_{\substack{D<t\\ m/2<t<m}}\frac{1}{t^{s-d-1}(m-t)}\sum_{m-t\le x_1\le\cdots\le x_d\le m}\frac{1}{x_1\cdots x_d}.\end{aligned}$$

Since

$$\sum_{m-t\le x_1\le\cdots\le x_d\le m}\frac{1}{x_1\cdots x_d}\ll\frac{t^d}{(m-t)^d},$$

we have

$$\begin{aligned}A&\ll\sum_{D<t\le m/2}\frac{1}{t^{s-2d-1}(m-t)^{d+1}}\ll\sum_{D<t\le m/2}\frac{1}{t^{s-2d-1}m^{d+1}}\\&\ll\sum_{D<t}\frac{1}{t^{s-d-1}}.\end{aligned}$$

We also have

$$B\ll\sum_{\substack{D<t\\ m/2<t<m}}\frac{(\log m)^d}{t^{s-d-1}(m-t)}\ll\sum_{D<t<m<2t}\frac{(\log m)^d}{t^{s-d-1}(m-t)}.$$

Thus, putting $n=m-t$, we get

$$B\ll\sum_{\substack{D<t\\ 1\le n<t}}\frac{(\log(2t))^d}{t^{s-d-1}n}\ll\sum_{D<t}\frac{(\log(2t))^d\log t}{t^{s-d-1}}.$$

This finishes the proof. □

We need Lemmas 3.3 and 3.4 to prove Theorem 1.7.

Lemma 3.3. *If $\sigma>d+2$, we have*

$$F_{d+1}(D;s)=F_d^{(1)}(D;s)-F_d^{(2)}(D;s)-F_d^{(3)}(D;s).$$

Proof. By the previous lemma, $F_d^{(1)}(D;s)$, $F_d^{(2)}(D;s)$, and $F_d^{(3)}(D;s)$ converge absolutely. Since

$$\begin{aligned}F_d^{(1)}(D;s)&=\sum_{D<t}\frac{1}{t^{s-d-1}}\sum_{0<x_0\le x_1\le\cdots\le x_d\le x_0+t}\frac{1}{x_0x_1\cdots x_d}\\&=\sum_{D<t}\frac{1}{t^{s-d-1}}\sum_{\substack{0<x_0\le x_1\le\cdots\le x_d\\ x_d\le x_0+t}}\frac{1}{x_0x_1\cdots x_d},\\F_d^{(2)}(D;s)&=\sum_{D<t}\frac{1}{t^{s-d-1}}\sum_{0<x_{d+1}-t\le x_1\le\cdots\le x_d\le x_{d+1}}\frac{1}{x_1\cdots x_dx_{d+1}}\\&=\sum_{D<t}\frac{1}{t^{s-d-1}}\sum_{\substack{0<x_0\le x_1\le\cdots\le x_d\\ x_d>t\\ x_d\le x_0+t}}\frac{1}{x_0x_1\cdots x_d},\\F_d^{(3)}(D;s)&=\sum_{D<t<N}\frac{1}{N^{s-d-1}(N-t)}\sum_{N-t\le x_1\le\cdots\le x_d\le N}\frac{1}{x_1\cdots x_d}\\&=\sum_{D<N}\frac{1}{N^{s-d-1}}\sum_{\substack{0<x_0\le x_1\le\cdots\le x_d\le N\\ x_0<N-D}}\frac{1}{x_0x_1\cdots x_d}\\&=\sum_{D<t}\frac{1}{t^{s-d-1}}\sum_{\substack{0<x_0\le\cdots\le x_d\\ x_d\le t\\ x_0<t-D}}\frac{1}{x_0\cdots x_d},\end{aligned}$$

we have

$$\begin{aligned}
&F_d^{(1)}(D;s)-F_d^{(2)}(D;s)-F_d^{(3)}(D;s)\\
&=\sum_{D<t}\frac{1}{t^{s-d-1}}\sum_{t-D\le x_0\le x_1\le\cdots\le x_d\le t}\frac{1}{x_0x_1\cdots x_d}\\
&=F_{d+1}(D;s).
\end{aligned}$$

Hence we find the result. □

Lemma 3.4. *Let a be a non-negative integer and b an integer with $b\ge 2$. If $\sigma>b,\sigma+\sigma_a>1+b,\dots,\sigma+\sigma_a+\cdots+\sigma_1>a+b$, then $G_{a,b}(s_1,\dots,s_a;s)$ is well-defined and equal to*

$$\sum_{0<m_1<\cdots<m_a<m}\frac{1}{m_1^{s_1}\cdots m_a^{s_a}m^{s-b+1}}\sum_{m-m_a\le x_1\le\cdots\le x_{b-1}\le m}\frac{1}{x_1\cdots x_{b-1}}. \tag{4}$$

Here we understand that $m-m_a=m$ if $a=0$.

Proof. Note that (4) converges absolutely. The proof is by induction on b. We can prove the case $b=2$ in the similar manner as in the proof of Theorem 1.2 for $\sigma>2$. For $b\ge 3$, by the induction hypothesis, we have

$$\begin{aligned}
&G_{a,b}(s_1,\dots,s_a;s)\\
&=\sum_{n=0}^{\infty}G_{a+1,b-1}(s_1,\dots,s_a,s-n-b;n+b)-\sum_{n=0}^{\infty}G_{a+1,b-1}(s_1,\dots,s_a,-n;s+n)\\
&=\sum_{n=0}^{\infty}\sum_{0<m_1<\cdots<m_{a+1}<m}\frac{1}{m_1^{s_1}\cdots m_a^{s_a}m_{a+1}^{s-n-b}m^{n+2}}\sum_{m-m_{a+1}\le x_1\le\cdots\le x_{b-2}\le m}\frac{1}{x_1\cdots x_{b-2}}\\
&\quad-\sum_{n=0}^{\infty}\sum_{0<m_1<\cdots<m_{a+1}<m}\frac{1}{m_1^{s_1}\cdots m_a^{s_a}m_{a+1}^{-n}m^{s+n-b+2}}\sum_{m-m_{a+1}\le x_1\le\cdots\le x_{b-2}\le m}\frac{1}{x_1\cdots x_{b-2}}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{0<m_1<\cdots<m_{a+1}<m}\frac{1}{m_1^{s_1}\cdots m_a^{s_a}m_{a+1}^{s-n-b}m^{n+2}}\sum_{m-m_{a+1}\le x_1\le\cdots\le x_{b-2}\le m}\frac{1}{x_1\cdots x_{b-2}}\\
&=\sum_{0<m_1<\cdots<m_{a+1}<m}\frac{1}{m_1^{s_1}\cdots m_a^{s_a}m_{a+1}^{s-b+1}}\left(\frac{1}{m-m_{a+1}}-\frac{1}{m}\right)\sum_{m-m_{a+1}\le x_1\le\cdots\le x_{b-2}\le m}\frac{1}{x_1\cdots x_{b-2}}\\
&=\sum_{0<m_1<\cdots<m_a}\frac{1}{m_1^{s_1}\cdots m_a^{s_a}}\left(F_{b-2}^{(1)}(m_a;s)-F_{b-2}^{(2)}(m_a;s)\right)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{0<m_1<\cdots<m_{a+1}<m}\frac{1}{m_1^{s_1}\cdots m_a^{s_a}m_{a+1}^{-n}m^{s+n-b+2}}\sum_{m-m_{a+1}\le x_1\le\cdots\le x_{b-2}\le m}\frac{1}{x_1\cdots x_{b-2}}\\
&=\sum_{0<m_1<\cdots<m_{a+1}<m}\frac{1}{m_1^{s_1}\cdots m_a^{s_a}m^{s-b+1}(m-m_{a+1})}\sum_{m-m_{a+1}\le x_1\le\cdots\le x_{b-2}\le m}\frac{1}{x_1\cdots x_{b-2}}\\
&=\sum_{0<m_1<\cdots<m_a}\frac{1}{m_1^{s_1}\cdots m_a^{s_a}}F_{b-2}^{(3)}(m_a;s).
\end{aligned}$$

Since the series

$$\sum_{0<m_1<\cdots<m_a} \frac{1}{m_1^{s_1}\cdots m_a^{s_a}} F_{b-2}^{(i)}(m_a;s) \qquad (i=1,2,3)$$

are absolutely convergent by Lemma 3.2, we see that $G_{a,b}(s_1,\ldots,s_a;s)$ is also absolutely convergent, and furthermore, we have

$$\begin{aligned} &G_{a,b}(s_1,\ldots,s_a;s)\\ &=\sum_{0<m_1<\cdots<m_a} \frac{1}{m_1^{s_1}\cdots m_a^{s_a}}\Big(F_{b-2}^{(1)}(m_a;s)-F_{b-2}^{(2)}(m_a;s)-F_{b-2}^{(3)}(m_a;s)\Big)\\ &=\sum_{0<m_1<\cdots<m_a} \frac{1}{m_1^{s_1}\cdots m_a^{s_a}}F_{b-1}(m_a;s)\\ &=\sum_{0<m_1<\cdots<m_a<m} \frac{1}{m_1^{s_1}\cdots m_a^{s_a}m^{s-b+1}} \sum_{m-m_a\le x_1\le\cdots\le x_{b-1}\le m} \frac{1}{x_1\cdots x_{b-1}} \end{aligned}$$

from Lemma 3.3. Hence the lemma is proved. □

Proof of Theorem 1.7. From Lemma 3.4, we have

$$\begin{aligned} G_{0,b}(s) &= \sum_{0<m} \frac{1}{m^{s-b+1}} \sum_{m\le x_1\le\cdots\le x_{b-1}\le m} \frac{1}{x_1\cdots x_{b-1}}\\ &=\zeta(s). \end{aligned}$$

□

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