Extending p-divisible groups and Barsotti-Tate deformation ring in the relative case

Yong Suk Moon

Abstract

Let k be a perfect field of characteristic $p > 2$, and let K be a finite totally ramified extension of $W(k)[\frac{1}{p}]$ of ramification degree e. We consider an unramified base ring R_0 over $W(k)$ satisfying certain conditions, and let $R = R_0 \otimes_{W(k)} \mathcal{O}_K$. Examples of such R include $R = \mathcal{O}_K[\![s_1, \ldots, s_d]\!]$ and $R = \mathcal{O}_K\langle t_1^{\pm 1}, \ldots, t_d^{\pm 1}\rangle$. We show that the generalization of Raynaud's theorem on extending p-divisible groups holds over the base ring R when $e < p-1$, whereas it does not hold when $R = \mathcal{O}_K[\![s]\!]$ with $e \geq p$. As an application, we prove that if R has Krull dimension 2 and $e < p-1$, then the locus of Barsotti-Tate representations of $\mathrm{Gal}(\overline{R}[\frac{1}{p}]/R[\frac{1}{p}])$ cuts out a closed subscheme of the universal deformation scheme. If $R = \mathcal{O}_K[\![s]\!]$ with $e \geq p$, we prove that such a locus is not p-adically closed.

Contents

1 Introduction

Let k be a perfect field of characteristic $p > 2$, and $W(k)$ be its ring of Witt vectors. Let K be a finite totally ramified extension of $W(k)[\frac{1}{p}]$ of ramification degree e, and let $\mathcal{O}_K$ be its ring of integers. We consider an unramified base ring R_0 over $W(k)$ satisfying certain conditions (cf. Section 2), and let $R = R_0 \otimes_{W(k)} \mathcal{O}_K$. Important examples of such R include the formal power series ring $R = \mathcal{O}_K[\![s_1, \dots, s_d]\!]$, and $R = \mathcal{O}_K\langle t_1^{\pm 1}, \dots, t_d^{\pm 1}\rangle$ which is the p-adic completion of $\mathcal{O}_K[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$.

When $R = \mathcal{O}_K$, Raynaud showed the following theorem on extending p-divisible groups.

Theorem 1.1. ([Ray74, Proposition 2.3.1]) *Let G be a p-divisible group over K. Suppose that for each $n \geq 1$, $G[p^n]$ extends to a finite flat group scheme over $\mathcal{O}_K$. Then G extends to a p-divisible group over $\mathcal{O}_K$, and such an extension is unique up to isomorphism.*

In this paper, we prove that the generalization of Raynaud's theorem holds over the relative base R when the ramification is small ($e < p-1$). On the other hand, using an example from [VZ10] on purity of p-divisible groups, we show that such a statement does not hold when the ramification is large.

Theorem 1.2. *Assume $e < p-1$. Let G be a p-divisible group over $R[\frac{1}{p}]$. Suppose that for each $n \geq 1$, $G[p^n]$ extends to a finite locally free group scheme over R. Then G extends to a p-divisible group over R, and such an extension is unique up to isomorphism.*

If $e \geq p$ and $R = \mathcal{O}_K[\![s]\!]$, there exists a p-divisible group G over $R[\frac{1}{p}]$ such that $G[p^n]$ extends to a finite locally free group scheme over R for each n but G does not extend to a p-divisible group over R.

As an application, we study the geometry of the locus of representations arising from p-divisible groups over R when R has Krull dimension 2. Let $\mathcal{G}_R$ be the étale fundamental group of $\mathrm{Spec} R[\frac{1}{p}]$. For a fixed absolutely irreducible $\mathbf{F}_p$-representation V_0 of $\mathcal{G}_R$, there exists a universal deformation ring which parametrizes the deformations of V_0 ([SL97]). We say that a finite continuous $\mathbf{Q}_p$-representation V of $\mathcal{G}_R$ is *Barsotti-Tate* if it arises from a p-divisible group over R, i.e., if there exists a p-divisible group G_R over R such that $V \cong T_p(G_R) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ where $T_p(G_R)$ denotes the Tate module of G_R. For a torsion $\mathbf{Z}_p$-representation T of $\mathcal{G}_R$, we say it is *torsion Barsotti-Tate* if it is a quotient of a finite free $\mathbf{Z}_p$-representation T_1 such that $T_1[\frac{1}{p}]$ is Barsotti-Tate. By using Theorem 1.2, we prove:

Theorem 1.3. *Suppose R has Krull dimension 2 and $e < p-1$. Then the locus of Barsotti-Tate representations of $\mathcal{G}_R$ cuts out a closed subscheme of the universal deformation scheme.*

If $R = \mathcal{O}_K[\![s]\!]$ and $e \geq p$, then the locus of Barsotti-Tate representations is not p-adically closed in the following sense: there exists a finite free $\mathbf{Z}_p$-representation T of $\mathcal{G}_R$ such that T/p^nT is torsion Barsotti-Tate for each integer $n \geq 1$ but $T[\frac{1}{p}]$ is not Barsotti-Tate.

We give a more precise statement of Theorem 1.3 in Section 5.

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2 Relative Breuil-Kisin Classification

We first explain the classification of p-divisible groups and finite locally free group schemes over Spec$R$ via certain Kisin modules, which is proved in [Kis06] when $R = \mathcal{O}_K$ and generalized in [Kim15] for the relative case.

We will work over the relative base rings as considered in [Bri08] with some additional mild assumptions. Denote by $W(k)\langle t_1^{\pm 1}, \ldots, t_d^{\pm 1}\rangle$ the p-adic completion of the polynomial ring $W(k)[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$. Let R_0 be a ring obtained from $W(k)\langle t_1^{\pm 1}, \ldots, t_d^{\pm 1}\rangle$ by iterations of the following operations:

- p-adic completion of an étale extension;
- p-adic completion of a localization;
- completion with respect to an ideal containing p.

We assume that either $W(k)\langle t_1^{\pm 1}, \ldots, t_d^{\pm 1}\rangle \to R_0$ has geometrically regular fibers or R_0 has Krull dimension less than 2, and that $k \to R_0/pR_0$ is geometrically integral and R_0 is an integral domain. Furthermore, we suppose that R_0 is formally smooth formally finite type over some Cohen ring (cf. [Kim15, Section 2.2.2]). In particular, R_0 is a regular ring.

R_0/pR_0 has a finite p-basis given by $\{t_1, \ldots, t_d\}$ in the sense of [DJ95, Definition 1.1.1]. Let $\widehat{\Omega}_{R_0} = \varprojlim_n \Omega_{(R_0/p^n)/W(k)}$ be the module of p-adically continuous Kähler differentials. We have $\widehat{\Omega}_{R_0} \cong \bigoplus_{i=1}^d R_0 \cdot d(\log t_i)$ by [Bri08, Proposition 2.0.2]. The Witt vector Frobenius on $W(k)$ extends (not necessarily uniquely) to R_0. We fix such a Frobenius endomorphism $\varphi : R_0 \to R_0$, and let $R = R_0 \otimes_{W(k)} \mathcal{O}_K$ be our base ring. Examples of such R include $R = \mathcal{O}_K\langle t_1^{\pm 1}, \ldots, t_d^{\pm 1}\rangle$ and $R = \mathcal{O}_K[\![s_1, \ldots, s_d]\!]$ (for example, via $s_i = 1 + t_i$).

It will be useful later to consider the following natural maps between base rings. Let $R_{0,g}$ be the p-adic completion of $\varinjlim_{\varphi}(R_0)_{(p)}$ with the induced Frobenius, and denote by k_g the perfect closure $\varinjlim_{\varphi} \mathrm{Frac}(R_0/pR_0)$ of $\mathrm{Frac}(R_0/pR_0)$. By the universal property of p-adic Witt vectors, we have a unique continuous (with respect to the p-adic topology) morphism $h : W(k_g) \to R_{0,g}$ commuting with their projections to k_g. By unicity, h is compatible with Frobenius endomorphisms. Since h modulo p is an isomorphism and $R_{0,g}$ is p-torsion free and p-adically complete and separated, h is an isomorphism. We will make use of this isomorphism later when we apply results from classical p-adic Hodge theory over p-adic fields, since such results will hold for the base ring $R_{0,g} \otimes_{W(k)} \mathcal{O}_K$. Let $b_g : R_0 \to R_{0,g}$ be the natural morphism compatible with Frobenius. This induces $\mathcal{O}_K$-linearly the base change map $b_g : R \to R_{0,g} \otimes_{W(k)} \mathcal{O}_K$.

Lemma 2.1. *The map $b_g : R_0 \to R_{0,g}$ is injective. Furthermore, for each integer $n \geq 1$, the map $R_0/(p^n) \to R_{0,g}/(p^n)$ induced from b_g is injective.*

Proof. Since $R_0/(p)$ is an integral domain, the map $R_0/(p) \to R_{0,g}/(p) = k_g$ is injective. Thus, $b_g : R_0 \to R_{0,g}$ is injective as R_0 is p-adically separated and $R_{0,g}$ is p-torsion free. It also follows that $R_0/(p^n) \to R_{0,g}/(p^n)$ is injective for each $n \geq 1$. $\square$

Let $\mathfrak{S} = R_0[\![u]\!]$ equipped with the Frobenius extending that on R_0, given by $\varphi : u \mapsto u^p$. Denote by $E(u)$ the Eisenstein polynomial for the extension K over $W(k)[\frac{1}{p}]$.

Definition 2.2. A *quasi-Kisin module of height* 1 is a pair $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ where

- $\mathfrak{M}$ is a finitely generated projective $\mathfrak{S}$-module;
- $\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$ is a φ-semilinear map such that $\mathrm{coker}(1 \otimes \varphi_{\mathfrak{M}} : \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \to \mathfrak{M})$ is annihilated by $E(u)$.

Note that for a quasi-Kisin module $\mathfrak{M}$ of height 1, $1 \otimes \varphi_{\mathfrak{M}} : \varphi^*\mathfrak{M} := \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \to \mathfrak{M}$ is injective since $\mathfrak{M}$ is finite projective over $\mathfrak{S}$ and $\mathrm{coker}(1 \otimes \varphi_{\mathfrak{M}})$ is killed by $E(u)$. Let $\mathrm{Mod}_{\mathfrak{S}}(\varphi)$ denote the category of quasi-Kisin modules of height 1 whose morphisms are $\mathfrak{S}$-module maps compatible with Frobenius.

Consider the composite $\mathfrak{S} \twoheadrightarrow \mathfrak{S}/u\mathfrak{S} = R_0 \xrightarrow{\varphi} R_0$. Let $\mathrm{Mod}_{\mathfrak{S}}(\varphi, \nabla)$ denote the category whose objects are tuples $(\mathfrak{M}, \varphi_{\mathfrak{M}}, \nabla_{\mathcal{M}})$ where $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ is a quasi-Kisin module of height 1, $\mathcal{M} := \mathfrak{M} \otimes_{\mathfrak{S},\varphi} R_0$, and $\nabla_{\mathcal{M}} : \mathcal{M} \to \mathcal{M} \otimes_{R_0} \widehat{\Omega}_{R_0}$ is a topologically quasi-nilpotent integrable connection commuting with $\varphi_{\mathcal{M}} := \varphi_{\mathfrak{M}} \otimes \varphi_{R_0}$. (Here, $\nabla_{\mathcal{M}}$ being topologically quasi-nilpotent means that the induced connection on $\mathcal{M}/p\mathcal{M}$ is nilpotent). The morphisms in $\mathrm{Mod}_{\mathfrak{S}}(\varphi, \nabla)$ are $\mathfrak{S}$-module maps compatible with Frobenius and connection. The objects in $\mathrm{Mod}_{\mathfrak{S}}(\varphi, \nabla)$ are called *Kisin modules of height* 1. The following theorem is proved in [?].

Theorem 2.3. (cf. [Kim15, Corollary 6.7 and Remark 6.9]) *There exists an exact anti-equivalence of categories*

$$\mathfrak{M}^* : \{p\text{-divisible groups over } R\} \to \mathrm{Mod}_{\mathfrak{S}}(\varphi, \nabla).$$

Let R_0' be another unramifed ring satisfying the conditions as above equipped with a Frobenius, and let $b : R_0 \to R_0'$ be a φ-equivariant map. Then the formation of $\mathfrak{M}^*$ commutes with the base change $R \to R' := R_0' \otimes_{W(k)} \mathcal{O}_K$ induced $\mathcal{O}_K$-linearly from b.

The classification of p-power order finite locally free group schemes over R can be obtained by considering torsion Kisin modules.

Definition 2.4. A *torsion quasi-Kisin module of height* 1 is a pair $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ where

- $\mathfrak{M}$ is a finitely presented $\mathfrak{S}$-module killed by a power of p, and of $\mathfrak{S}$-projective dimension 1;

- $\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$ is a φ-semilinear endomorphism such that $\mathrm{coker}(1 \otimes \varphi_{\mathfrak{M}} : \varphi^*\mathfrak{M} \to \mathfrak{M})$ is killed by $E(u)$.

Let $\mathrm{Mod}_{\mathfrak{S}}^{\mathrm{tor}}(\varphi)$ denote the category of torsion quasi-Kisin modules of height 1 whose morphisms are $\mathfrak{S}$-linear maps compatible with φ. Let $\mathrm{Mod}_{\mathfrak{S}}^{\mathrm{tor}}(\varphi, \nabla)$ denote the category whose objects are tuples $(\mathfrak{M}, \varphi_{\mathfrak{M}}, \nabla_{\mathcal{M}})$ where $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ is a torsion quasi-Kisin module of height 1, $\mathcal{M} := \mathfrak{M} \otimes_{\mathfrak{S},\varphi} R_0$, and $\nabla_{\mathcal{M}} : \mathcal{M} \to \mathcal{M} \otimes_{R_0} \widehat{\Omega}_{R_0}$ is a topologically quasi-nilpotent integrable connection commuting with $\varphi_{\mathcal{M}} := \varphi_{\mathfrak{M}} \otimes \varphi_{R_0}$. The morphisms in $\mathrm{Mod}_{\mathfrak{S}}^{\mathrm{tor}}(\varphi, \nabla)$ are $\mathfrak{S}$-module maps compatible with φ and ∇. The objects are called *torsion Kisin modules of height* 1.

Lemma 2.5. *Let $\mathfrak{M}$ be a torsion quasi-Kisin module of height 1. Then $1 \otimes \varphi_{\mathfrak{M}} : \varphi^*\mathfrak{M} \to \mathfrak{M}$ is injective.*

Proof. Let $\mathfrak{S}_g := R_{0,g}[\![u]\!]$ equipped with the Frobenius given by $\varphi(u) = u^p$. By the local criterion for flatness, $b_g : R_0 \to R_{0,g}$ is flat since $R_0/(p) \to R_{0,g}/(p) = k_g$ is flat and $R_{0,g}$ is p-torsion free, and the map $\mathfrak{S} \to \mathfrak{S}_g$ is flat. Note that $\mathfrak{M}_g := \mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}_g$ equipped with $\varphi_{\mathfrak{M}_g} := \varphi_{\mathfrak{M}} \otimes \varphi_{\mathfrak{S}_g}$ is a torsion Kisin module of height 1 over $\mathfrak{S}_g$.

We first claim that the natural map $b : \mathfrak{M} \to \mathfrak{M}_g$ is injective. Since $\mathfrak{M}$ has projective dimension ≤ 1, there exists a short exact sequence $0 \to \mathfrak{M}_1 \to \mathfrak{M}_2 \to \mathfrak{M} \to 0$ where $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are finite projective $\mathfrak{S}$-modules. $\mathfrak{M}_1$ and $\mathfrak{M}_2$ have the same rank since $\mathfrak{M}$ is killed by a power of p. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{M}_1 & \longrightarrow & \mathfrak{M}_2 & \longrightarrow & \mathfrak{M} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle b} & & \\ 0 & \longrightarrow & \mathfrak{M}_1 \otimes_{\mathfrak{S}} \mathfrak{S}_g & \longrightarrow & \mathfrak{M}_2 \otimes_{\mathfrak{S}} \mathfrak{S}_g & \longrightarrow & \mathfrak{M}_g & \longrightarrow & 0 \end{array}$$

whose rows are exact. Since $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are projective over $\mathfrak{S}$, the left and middle vertical maps are injective. Furthermore, for $i = 1, 2$, we have $\mathrm{coker}(\mathfrak{M}_i \to \mathfrak{M}_i \otimes_{\mathfrak{S}} \mathfrak{S}_g) \cong \mathfrak{M}_i \otimes_{\mathfrak{S}} (\mathfrak{S}_g/\mathfrak{S})$ as $\mathfrak{S}$-modules. On the other hand, all elements in the kernel of the induced map $\mathfrak{M}_1 \otimes_{\mathfrak{S}} (\mathfrak{S}_g/\mathfrak{S}) \to \mathfrak{M}_2 \otimes_{\mathfrak{S}} (\mathfrak{S}_g/\mathfrak{S})$ are killed by some power of p since $\mathfrak{M}_1[\frac{1}{p}] \cong \mathfrak{M}_2[\frac{1}{p}]$. And $\mathfrak{S}_g/\mathfrak{S}$ is p-torsion free since $R_0/(p) \to R_{0,g}/(p) = k_g$ is injective, so $\mathfrak{M}_1 \otimes_{\mathfrak{S}} (\mathfrak{S}_g/\mathfrak{S})$ is p-torsion free as $\mathfrak{M}_1$ is projective over $\mathfrak{S}$. Hence, the map $\mathfrak{M}_1 \otimes_{\mathfrak{S}} (\mathfrak{S}_g/\mathfrak{S}) \to \mathfrak{M}_2 \otimes_{\mathfrak{S}} (\mathfrak{S}_g/\mathfrak{S})$ is injective. By the snake Lemma, we deduce that $b : \mathfrak{M} \to \mathfrak{M}_g$ is injective.

Now, consider the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} & \xrightarrow{1 \otimes \varphi_{\mathfrak{M}}} & \mathfrak{M} \\ \downarrow & & \downarrow{\scriptstyle b} \\ \mathfrak{S}_g \otimes_{\varphi,\mathfrak{S}_g} \mathfrak{M}_g & \xrightarrow{1 \otimes \varphi_{\mathfrak{M}_g}} & \mathfrak{M}_g \end{array}$$

Since $\varphi : \mathfrak{S} \to \mathfrak{S}$ is flat by [Bri08, Lemma 7.1.8], $\mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ has projective dimension 1 as a $\mathfrak{S}$-module and is killed by a power of p. By the same argument as above, the natural map $\mathfrak{S}\otimes_{\varphi,\mathfrak{S}}\mathfrak{M}\to\mathfrak{S}_g\otimes_{\mathfrak{S}}(\mathfrak{S}\otimes_{\varphi,\mathfrak{S}}\mathfrak{M})\cong\mathfrak{S}_g\otimes_{\varphi,\mathfrak{S}_g}\mathfrak{M}_g$ is injective, which is the left vertical map. The bottom map is injective by [Liu07, Proposition 2.3.2] since $R_{0,g}\cong W(k_g)$. Thus, the top map is injective. □

Denote by $(\mathrm{Mod\ FI})_{\mathfrak{S}}(\varphi,\nabla)$ the full subcategory of $\mathrm{Mod}^{\mathrm{tor}}_{\mathfrak{S}}(\varphi,\nabla)$ consisting of objects $\mathfrak{M}$ such that $\mathfrak{M}\cong\bigoplus_i\mathfrak{M}_i$ as $\mathfrak{S}$-modules where $\mathfrak{M}_i$'s are projective over $\mathfrak{S}/(p^{n_i})$ for some positive integers n_i. The following theorem is shown in [Kim15].

Theorem 2.6. (cf. [Kim15, Proposition 9.5 and Theorem 9.8]) *There exists an exact fully faithful functor $\mathfrak{M}^*$ from the category of p-power order finite locally free group schemes over R to $\mathrm{Mod}^{\mathrm{tor}}_{\mathfrak{S}}(\varphi,\nabla)$ with the following properties:*

- *Let H be a p-power order finite locally free group scheme over R. If $H=\ker(h:G^0\to G^1)$ for an isogeny h of p-divisible groups over R, then there exists a natural isomorphism $\mathfrak{M}^*(H)\cong\mathrm{coker}(\mathfrak{M}^*(h))$ of torsion Kisin modules of height 1;*

- *Let R_0' be another unramified ring satisfying the conditions as above equipped with a Frobenius, and let $b:R_0\to R_0'$ be a φ-equivariant map. Then the formation of $\mathfrak{M}^*$ commutes with the base change $R\to R':=R_0'\otimes_{W(k)}\mathcal{O}_K$ induced $\mathcal{O}_K$-linearly from b.*

Moreover, the functor $\mathfrak{M}^*$ induces an anti-equivalence from the category of p-power order finite locally free group schemes H over R such that $H[p^n]$ is locally free over R for all $n\geq 1$ to $(\mathrm{Mod\ FI})_{\mathfrak{S}}(\varphi,\nabla)$.

We end this section by recalling some necessary results on connections explained in [Kim15, Section 10.2], which is based on [Vas13]. Let $(\mathfrak{M},\varphi_{\mathfrak{M}})$ be a quasi-Kisin module of height 1, and let $\mathcal{M}=\mathfrak{M}\otimes_{\mathfrak{S},\varphi}R_0$ equipped with the induced Frobenius $\varphi_{\mathfrak{M}}\otimes\varphi_{R_0}$. From [Kim15, Eq. (6.1), (6.2) and Remark 3.13], we have the R_0-submodule $\mathrm{Fil}^1\mathcal{M}\subset\mathcal{M}$ associated with $\mathfrak{M}$ such that $p\mathcal{M}\subset\mathrm{Fil}^1\mathcal{M}$, $\mathcal{M}/\mathrm{Fil}^1\mathcal{M}$ is projective over $R_0/(p)$, and $(1\otimes\varphi)(\varphi^*\mathrm{Fil}^1\mathcal{M})=p\mathcal{M}$ as R_0-modules (cf. [Kim15, Definition 3.4 and 3.6] for the frame $(R_0,pR_0,R_0/(p),\varphi_{R_0},\frac{\varphi_{R_0}}{p})$). Fix an R_0-direct factor $\mathcal{M}^1\subset\mathcal{M}$ which lifts $\mathrm{Fil}^1\mathcal{M}/p\mathcal{M}\subset\mathcal{M}/p\mathcal{M}$, and let $\tilde{M}:=(\mathcal{M}+\frac{1}{p}\mathcal{M}^1)\otimes_{R_0,\varphi}R_0\subset\mathcal{M}\otimes_{R_0,\varphi}R_0[\frac{1}{p}]$. For each integer $n\geq 1$, suppose $\nabla_n:R_0/(p^n)\otimes_{R_0}\mathcal{M}\to(R_0/(p^n)\otimes_{R_0}\mathcal{M})\otimes_{R_0}\widehat{\Omega}_{R_0}$ is a connection such that the following diagram is commutative:

$$\begin{array}{ccc} R_0/(p^n)\otimes_{R_0}\tilde{\mathcal{M}} & \xrightarrow{\varphi^*(\nabla_n)} & R_0/(p^n)\otimes_{R_0}\tilde{\mathcal{M}}\otimes_{R_0}\widehat{\Omega}_{R_0} \\ {\scriptstyle 1\otimes\varphi}\downarrow & & \downarrow{\scriptstyle (1\otimes\varphi)\otimes\mathrm{id}_{\widehat{\Omega}_{R_0}}} \\ R_0/(p^n)\otimes_{R_0}\mathcal{M} & \xrightarrow{\nabla_n} & R_0/(p^n)\otimes_{R_0}\mathcal{M}\otimes_{R_0}\widehat{\Omega}_{R_0} \end{array} \tag{2.1}$$

Here, $\varphi^*(\nabla_n)$ is given by choosing an arbitrary lift of ∇_n on $R_0/(p^{n+1})\otimes_{R_0}\mathcal{M}$, and $\varphi^*(\nabla_n)$ does not depend on the choice of such a lift (cf. [Vas13, Section 3.1.1 Equation (9)]).

Identify $\widehat{\Omega}_{R_0} = \bigoplus_{i=1}^d R_0 \cdot d(\log t_i)$. By passing to a finite Zariski covering of $\mathrm{Spf}(R_0, p)$, we may assume that $\mathcal{M}^1$ and $\mathcal{M}/\mathcal{M}^1$ are free over R_0. Fix such a choice of the covering, and fix a R_0-basis of $\mathcal{M}$ adapted to the direct factor $\mathcal{M}^1$. By [Vas13, Section 3.2 Basic Theorem] and its proof, the set of connections ∇_1 on $R_0/(p) \otimes_{R_0} \mathcal{M}$ satisfying the commutative diagram (2.1) for $n = 1$ corresponds to the solutions over $R_0/(p)$ of a certain Artin-Schreier system of equations over $R_0/(p)$. In particular, it follows directly that we have finitely many such ∇_1 (cf. [Vas13, Theorem 2.4.1 (b)]). Furthermore, given a connection ∇_n on $R_0/(p^n) \otimes_{R_0} \mathcal{M}$, the set of connections ∇_{n+1} on $R_0/(p^{n+1}) \otimes_{R_0} \mathcal{M}$ which lift ∇_n and satisfy the commutative diagram (2.1) for $n+1$ corresponds the solutions over $R_0/(p)$ of a certain Artin-Schreier system of equations over $R_0/(p)$ by *loc. cit.*, and we have finitely many such ∇_{n+1}.

3 Étale φ-modules and Galois Representations

We recall the results in [Kim15, Section 7] on associating Galois representations with étale φ-modules in the relative setting. The underlying geometry is based on perfectoid spaces (cf. [Sch12]). We will use the results to translate our question on p-divisible groups into a question on Kisin modules and étale φ-modules.

Let $\overline{R}$ denote the union of finite R-subalgebras R' of a fixed separable closure of $\mathrm{Frac}(R)$ such that $R'[\frac{1}{p}]$ is étale over $R[\frac{1}{p}]$. Then $\mathrm{Spec}\overline{R}[\frac{1}{p}]$ is a pro-universal covering of $\mathrm{Spec}R[\frac{1}{p}]$, and $\overline{R}$ is the integral closure of R in $\overline{R}[\frac{1}{p}]$. Let $\mathcal{G}_R := \mathrm{Gal}(\overline{R}[\frac{1}{p}]/R[\frac{1}{p}]) = \pi_1^{\text{ét}}(\mathrm{Spec}R[\frac{1}{p}], \eta)$ with a choice of a geometric point η. Choose a uniformizer $\varpi \in \mathcal{O}_K$. For integers $n \geq 0$, we choose compatibly $\varpi_n \in \overline{R}$ such that $\varpi_0 = \varpi$ and $\varpi_{n+1}^p = \varpi_n$, and let L be the p-adic completion of $\bigcup_{n\geq 0} K(\varpi_n)$. Then L is a perfectoid field and $(\widehat{\overline{R}}[\frac{1}{p}], \widehat{\overline{R}})$ is a perfectoid affinoid L-algebra, where $\widehat{\overline{R}}$ denotes the p-adic completion of $\overline{R}$.

Let $L^\flat$ denote the tilt of L as defined in [Sch12], and let $\underline{\varpi} := (\varpi_n) \in L^\flat$. Let $(\overline{R}^\flat[\frac{1}{\underline{\varpi}}], \overline{R}^\flat)$ be the tilt of $(\widehat{\overline{R}}[\frac{1}{p}], \widehat{\overline{R}})$. Let $E^+_{R_\infty} = \mathfrak{S}/p\mathfrak{S}$, and let $\tilde{E}^+_{R_\infty}$ be the u-adic completion of $\varinjlim_\varphi E^+_{R_\infty}$. Let $E_{R_\infty} = E^+_{R_\infty}[\frac{1}{u}]$ and $\tilde{E}_{R_\infty} = \tilde{E}^+_{R_\infty}[\frac{1}{u}]$. By [Sch12, Proposition 5.9], $(\tilde{E}_{R_\infty}, \tilde{E}^+_{R_\infty})$ is a perfectoid affinoid $L^\flat$-algebra, and we have the natural injection $(\tilde{E}_{R_\infty}, \tilde{E}^+_{R_\infty}) \hookrightarrow (\overline{R}^\flat[\frac{1}{\underline{\varpi}}], \overline{R}^\flat)$ given by $u \mapsto \underline{\varpi}$. Let $(\tilde{R}_\infty[\frac{1}{p}], \tilde{R}_\infty)$ be a perfectoid affinoid L-algebra whose tilt is $(\tilde{E}_{R_\infty}, \tilde{E}^+_{R_\infty})$, and let $\mathcal{G}_{\tilde{R}_\infty} = \pi_1^{\text{ét}}(\mathrm{Spec}\tilde{R}_\infty[\frac{1}{p}], \eta)$. Then we have a continuous map of Galois groups $\mathcal{G}_{\tilde{R}_\infty} \to \mathcal{G}_R$, which is a closed embedding by [GR03, Proposition 5.4.54]. By the almost purity theorem in [Sch12], $\overline{R}^\flat[\frac{1}{\underline{\varpi}}]$ can be canonically identified with the $\underline{\varpi}$-adic completion of the affine ring of a pro-universal covering of $\mathrm{Spec}\tilde{E}_{R_\infty}$, and letting $\mathcal{G}_{\tilde{E}_{R_\infty}}$ be the Galois group corresponding to the pro-universal covering, there exists a canonical isomorphism $\mathcal{G}_{\tilde{E}_{R_\infty}} \cong \mathcal{G}_{\tilde{R}_\infty}$.

Now, let $\mathcal{O}_\mathcal{E}$ be the p-adic completion of $\mathfrak{S}[\frac{1}{u}]$. Note that φ on $\mathfrak{S}$ extends naturally to

$\mathcal{O}_{\mathcal{E}}$.

Definition 3.1. An *étale* $(\varphi, \mathcal{O}_{\mathcal{E}})$*-module* is a pair (M, φ_M) where M is a finitely generated $\mathcal{O}_{\mathcal{E}}$-module and $\varphi_M : M \to M$ is a φ-semilinear endomorphism such that $1 \otimes \varphi_M : \varphi^* M \to M$ is an isomorphism. We say that an étale $(\varphi, \mathcal{O}_{\mathcal{E}})$-module is *projective* (resp. *torsion*) if the underlying $\mathcal{O}_{\mathcal{E}}$-module M is projective (resp. p-power torsion).

Let $\mathrm{Mod}_{\mathcal{O}_{\mathcal{E}}}$ denote the category of étale $(\varphi, \mathcal{O}_{\mathcal{E}})$-modules whose morphisms are $\mathcal{O}_{\mathcal{E}}$-linear maps compatible with Frobenius. Let $\mathrm{Mod}^{\mathrm{pr}}_{\mathcal{O}_{\mathcal{E}}}$ and $\mathrm{Mod}^{\mathrm{tor}}_{\mathcal{O}_{\mathcal{E}}}$ respectively denote the full subcategories of projective and torsion objects.

Note that we have a natural notion of a subquotient, direct sum, and tensor product for étale $(\varphi, \mathcal{O}_{\mathcal{E}})$-modules, and duality is defined for projective and torsion objects. If $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ is a quasi-Kisin module (resp. torsion quasi-Kisin module) of height 1, then $(\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}, \varphi_{\mathfrak{M}} \otimes \varphi_{\mathcal{O}_{\mathcal{E}}})$ is a projective (resp. torsion) étale $(\varphi, \mathcal{O}_{\mathcal{E}})$-module since $1 \otimes \varphi_{\mathfrak{M}}$ is injective (by Lemma 2.5 for torsion quasi-Kisin modules) and its cokernel is killed by $E(u)$ which is a unit in $\mathcal{O}_{\mathcal{E}}$. If we denote by $\mathcal{O}_{\mathcal{E},g}$ the corresponding ring for $R_{0,g}$, then for any étale $(\varphi, \mathcal{O}_{\mathcal{E}})$-module M, $M \otimes_{\mathcal{O}_{\mathcal{E}}, b_g} \mathcal{O}_{\mathcal{E},g}$ with the induced Frobenius is an étale $(\varphi, \mathcal{O}_{\mathcal{E},g})$-module. If M is a torsion object, we define its *length* to be the length of $\mathcal{O}_{\mathcal{E},g}$-module $M \otimes_{\mathcal{O}_{\mathcal{E}}, b_g} \mathcal{O}_{\mathcal{E},g}$.

We consider $W(\overline{R}^{\flat}[\frac{1}{\underline{\varpi}}])$ as an $\mathcal{O}_{\mathcal{E}}$-algebra via mapping u to the Teichmüller lift $[\underline{\varpi}]$ of $\underline{\varpi}$, and let $\mathcal{O}^{\mathrm{ur}}_{\mathcal{E}}$ be the integral closure of $\mathcal{O}_{\mathcal{E}}$ in $W(\overline{R}^{\flat}[\frac{1}{\underline{\varpi}}])$. Let $\widehat{\mathcal{O}}^{\mathrm{ur}}_{\mathcal{E}}$ be its p-adic completion. Since $\mathcal{O}_{\mathcal{E}}$ is normal, we have $\mathrm{Aut}_{\mathcal{O}_{\mathcal{E}}}(\mathcal{O}^{\mathrm{ur}}_{\mathcal{E}}) \cong \mathcal{G}_{E_{R_\infty}} := \pi_1^{\text{ét}}(\mathrm{Spec} E_{R_\infty})$, and by [GR03, Proposition 5.4.54] and the almost purity theorem, we have $\mathcal{G}_{E_{R_\infty}} \cong \mathcal{G}_{\tilde{E}_{R_\infty}} \cong \mathcal{G}_{\tilde{R}_\infty}$. This induces $\mathcal{G}_{\tilde{R}_\infty}$-action on $\widehat{\mathcal{O}}^{\mathrm{ur}}_{\mathcal{E}}$. The following is shown in [Kim15].

Lemma 3.2. (cf. [Kim15, Lemma 7.5 and 7.6]) *We have* $(\widehat{\mathcal{O}}^{\mathrm{ur}}_{\mathcal{E}})^{\mathcal{G}_{\tilde{R}_\infty}} = \mathcal{O}_{\mathcal{E}}$ *and the same holds modulo* p^n*. Furthermore, there exists a unique* $\mathcal{G}_{\tilde{R}_\infty}$*-equivariant ring endomorphism* φ *on* $\widehat{\mathcal{O}}^{\mathrm{ur}}_{\mathcal{E}}$ *lifting the* p*-th power map on* $\widehat{\mathcal{O}}^{\mathrm{ur}}_{\mathcal{E}}/(p)$ *and extending* φ *on* $\mathcal{O}_{\mathcal{E}}$*. The inclusion* $\widehat{\mathcal{O}}^{\mathrm{ur}}_{\mathcal{E}} \hookrightarrow W(\overline{R}^{\flat}[\frac{1}{\underline{\varpi}}])$ *is* φ*-equivariant where the latter ring is given the Witt vector Frobenius.*

Let $\mathrm{Rep}_{\mathbf{Z}_p}(\mathcal{G}_{\tilde{R}_\infty})$ be the category of finite continuous $\mathbf{Z}_p$-representations of $\mathcal{G}_{\tilde{R}_\infty}$, and let $\mathrm{Rep}^{\mathrm{free}}_{\mathbf{Z}_p}(\mathcal{G}_{\tilde{R}_\infty})$ and $\mathrm{Rep}^{\mathrm{tor}}_{\mathbf{Z}_p}(\mathcal{G}_{\tilde{R}_\infty})$ respectively denote the full subcategories of free and torsion objects. For $M \in \mathrm{Mod}_{\mathcal{O}_{\mathcal{E}}}$ and $T \in \mathrm{Rep}_{\mathbf{Z}_p}(\mathcal{G}_{\tilde{R}_\infty})$, we define $T(M) := (M \otimes_{\mathcal{O}_{\mathcal{E}}} \widehat{\mathcal{O}}^{\mathrm{ur}}_{\mathcal{E}})^{\varphi=1}$ and $M(T) := (T \otimes_{\mathbf{Z}_p} \widehat{\mathcal{O}}^{\mathrm{ur}}_{\mathcal{E}})^{\mathcal{G}_{\tilde{R}_\infty}}$. Then we have the following proposition from [Kim15].

Proposition 3.3. ([Kim15, Proposition 7.7]) *The constructions* $T(\cdot)$ *and* $M(\cdot)$ *give exact quasi-inverse equivalences of* $\otimes$*-categories between* $\mathrm{Mod}_{\mathcal{O}_{\mathcal{E}}}$ *and* $\mathrm{Rep}_{\mathbf{Z}_p}(\mathcal{G}_{\tilde{R}_\infty})$*. Moreover,* $T(\cdot)$ *and* $M(\cdot)$ *restrict to rank-preserving equivalences of categories between* $\mathrm{Mod}^{\mathrm{pr}}_{\mathcal{O}_{\mathcal{E}}}$ *and* $\mathrm{Rep}^{\mathrm{free}}_{\mathbf{Z}_p}(\mathcal{G}_{\tilde{R}_\infty})$*, and length-preserving equivalences between* $\mathrm{Mod}^{\mathrm{tor}}_{\mathcal{O}_{\mathcal{E}}}$ *and* $\mathrm{Rep}^{\mathrm{tor}}_{\mathbf{Z}_p}(\mathcal{G}_{\tilde{R}_\infty})$*. In both cases,* $T(\cdot)$ *and* $M(\cdot)$ *commute with taking duals.*

For M in $\mathrm{Mod}^{\mathrm{pr}}_{\mathcal{O}_{\mathcal{E}}}$ (resp. in $\mathrm{Mod}^{\mathrm{tor}}_{\mathcal{O}_{\mathcal{E}}}$), we define the contravariant functor $T^\vee(\cdot)$ to $\mathrm{Rep}_{\mathbf{Z}_p}(\mathcal{G}_{\tilde{R}_\infty})$ by $T^\vee(M) := \mathrm{Hom}_{\mathcal{O}_{\mathcal{E}},\varphi}(M, \widehat{\mathcal{O}}^{\mathrm{ur}}_{\mathcal{E}})$ (resp. $\mathrm{Hom}_{\mathcal{O}_{\mathcal{E}},\varphi}(M, \widehat{\mathcal{O}}^{\mathrm{ur}}_{\mathcal{E}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p/\mathbf{Z}_p)$). Note that if we have a short exact sequence of étale $(\varphi, \mathcal{O}_{\mathcal{E}})$-modules $0 \to M_1 \to M_2 \to M \to 0$ where M_1, M_2 are projective over $\mathcal{O}_{\mathcal{E}}$ and M is p-power torsion, then it induces a short exact sequence

$$0 \to T^\vee(M_2) \to T^\vee(M_1) \to T^\vee(M) \to 0$$

in $\mathrm{Rep}_{\mathbf{Z}_p}(\mathcal{G}_{\tilde{R}_\infty})$.

Now, if G_R is a p-divisible group over R, we write $T_p(G_R) := \mathrm{Hom}_{\overline{R}}(\mathbf{Q}_p/\mathbf{Z}_p, G_R \times_R \overline{R})$ to be the associated Tate module, which is a finite free $\mathbf{Z}_p$-representation of $\mathcal{G}_R$. By [Kim15, Corollary 8.2], we have a natural $\mathcal{G}_{\tilde{R}_\infty}$-equivariant isomorphism $T^\vee(\mathfrak{M}^*(G_R) \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}) \cong T_p(G_R)$. If H is a p-power order finite locally free group scheme over R, then $H(\overline{R})$ is a finite torsion $\mathbf{Z}_p$-representation of $\mathcal{G}_R$. By [Kim15, Proposiiton 9.10], there exists a natural $\mathcal{G}_{\tilde{R}_\infty}$-equivariant isomorphism $T^\vee(\mathfrak{M}^*(H) \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}) \cong H(\overline{R})$, and if $H = \ker(h : G^0 \to G^1)$ for some isogeny h of p-divisible groups over R, then the isomorphism $T^\vee(\mathfrak{M}^*(H) \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}) \cong H(\overline{R})$ is compatible with the isomorphisms $T^\vee(\mathfrak{M}^*(G^i) \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}) \cong T_p(G^i),\ i = 0, 1$.

Note that any p-divisible group over $R[\frac{1}{p}]$ is étale, so the category of p-divisible groups over $R[\frac{1}{p}]$ is equivalent to the category of finite free $\mathbf{Z}_p$-representations of $\mathcal{G}_R$. If we are given a p-divisible group G over $R[\frac{1}{p}]$, then the corresponding Galois representation is given by $T_p(G) = \mathrm{Hom}_{\overline{R}[\frac{1}{p}]}(\mathbf{Q}_p/\mathbf{Z}_p, G \times_{R[\frac{1}{p}]} \overline{R}[\frac{1}{p}])$. By Proposition 3.3, there exists a unique (up to isomorphism) projective étale $(\varphi, \mathcal{O}_{\mathcal{E}})$-module M such that $T^\vee(M) \cong T_p(G)$ as $\mathcal{G}_{\tilde{R}_\infty}$-representations. We remark that if G extends to a p-divisible group G_R over R, then $T_p(G_R) = T_p(G)$ as $\mathcal{G}_R$-representations.

4 Extending p-divisible Groups

We now prove the generalization of Raynaud's theorem for the relative base R when $e < p-1$, and use an example in [VZ10] on purity of p-divisible groups to show that when the ramification is large, such a generalization does not hold. We first consider the special case when the base ring R_0 as in Section 2 is equal to the formal power series ring over a Cohen ring.

Proposition 4.1. *Suppose $R_0 = \mathcal{O}[\![s_1, \ldots, s_r]\!]$ over a Cohen ring $\mathcal{O}$ and $e < p-1$. Let G be a p-divisible group over $R[\frac{1}{p}]$, and let $n \geq 1$ be an integer. Suppose that $G[p^n]$ extends to a finite flat group scheme $G_{n,R}$ over R. Then for each integer $1 \leq m \leq n$, the group scheme $G_{n,R}[p^m]$ is finite flat over R.*

Furthermore, if H is another finite flat group scheme over R extending $G[p^n]$ and if we identify the associated étale $(\varphi, \mathcal{O}_{\mathcal{E}})$-modules $M_n := \mathfrak{M}^*(G_{R,n}) \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} = \mathfrak{M}^*(H) \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$, then $\mathfrak{M}^*(G_{R,n}) = \mathfrak{M}^*(H)$ as $\mathfrak{S}$-submodules of M_n with compatible Frobenius.

Proof. Let M be the projective étale $(\varphi, \mathcal{O}_{\mathcal{E}})$-module such that $T^\vee(M) = T_p(G)$ as $\mathcal{G}_{\tilde{R}_\infty}$-representations. Denote $\mathfrak{M}_n = \mathfrak{M}^*(G_{n,R})$. Since $T_p(G[p^n]) \cong T_p(G)/p^n T_p(G)$, we have $M_n = \mathfrak{M}_n \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} \cong M/p^n M$ as étale $(\varphi, \mathcal{O}_{\mathcal{E}})$-modules.

For proving the first statement, we can make the following choice of Frobenius on R_0 without loss of generality. Let $k' = \mathcal{O}/(p)$. Note that since $R_0/pR_0 \cong k'[\![s_1, \ldots, s_r]\!]$ has a finite p-basis, we have $[k' : k'^p] < \infty$, i.e., k' has a finite p-basis. Choose a Frobenius $\varphi_{\mathcal{O}} : \mathcal{O} \to \mathcal{O}$ lifting the natural Frobenius on $W(k)$, and equip R_0 with Frobenius given by $\varphi_{\mathcal{O}}$ and $\varphi(s_i) = s_i^p$. Let $b_0 : R_0 \to \mathcal{O}$ be the $\mathcal{O}$-linear map given by $s_i \mapsto 0$, which is φ-equivariant. Let $b_g : R_0 \to R_{0,g} \cong W(k_g)$ be the φ-equivariant map considered in Section 2. Note that $\mathfrak{M}_n \otimes_{\mathfrak{S}, b_g} W(k_g)[\![u]\!]$ and $\mathfrak{M}_n \otimes_{\mathfrak{S}, b_0} \mathcal{O}[\![u]\!]$ with the induced diagonal Frobenius are torsion quasi-Kisin modules of height 1 over $W(k_g)[\![u]\!]$ and $\mathcal{O}[\![u]\!]$ respectively. Denote by I_j the j-th Fitting ideal of $\mathfrak{M}_n$ over $\mathfrak{S}_n := \mathfrak{S}/p^n\mathfrak{S}$. Let $I_{j,0}$ and $I_{j,g}$ be the j-th Fitting ideal of $\mathfrak{M}_n \otimes_{\mathfrak{S}, b_g} W(k_g)[\![u]\!]$ and $\mathfrak{M}_n \otimes_{\mathfrak{S}, b_0} \mathcal{O}[\![u]\!]$ over $W(k_g)[\![u]\!]/(p^n)$ and $\mathcal{O}[\![u]\!]/(p^n)$ respectively. Then $I_{j,0}$ and $I_{j,g}$ are given by the images of I_j under the corresponding maps b_0 and b_g respectively.

Let h be the height of G. Since $e < p - 1$, we deduce from [Liu07, Lemma 4.3.1 and Corollary 4.2.5] that $\mathfrak{M}_n \otimes_{\mathfrak{S}, b_g} W(k_g)[\![u]\!]$ is free of rank h over $W(k_g)[\![u]\!]/(p^n)$. Furthermore, if we denote by $\mathcal{O}_g$ the p-adic completion of $\varinjlim_{\varphi} \mathcal{O}_{(p)}$ with the induced Frobenius and $\kappa := \varinjlim_{\varphi} \mathcal{O}/(p)$, then by the universal property of p-adic Witt vectors as in Section 2, $\mathcal{O}_g \cong W(\kappa)$ compatibly with Frobenius endomorphisms. The map $\mathcal{O}[\![u]\!]/(p^n) \to W(\kappa)[\![u]\!]/(p^n)$ is faithfully flat, and the induced torsion Kisin module $(\mathfrak{M}_n \otimes_{\mathfrak{S}, b_0} \mathcal{O}[\![u]\!]) \otimes_{\mathcal{O}[\![u]\!]} W(\kappa)[\![u]\!]$ is free of rank h over $W(\kappa)[\![u]\!]/(p^n)$ by *loc. cit.* Hence, $\mathfrak{M}_n \otimes_{\mathfrak{S}, b_0} \mathcal{O}[\![u]\!]$ is free of rank h over $\mathcal{O}[\![u]\!]/(p^n)$. We obtain

$$I_{j,g} = \begin{cases} 0 & \text{if } j < h \\ W(k_g)[\![u]\!]/(p^n) & \text{if } j \geq h, \end{cases}$$

$$I_{j,0} = \begin{cases} 0 & \text{if } j < h \\ \mathcal{O}[\![u]\!]/(p^n) & \text{if } j \geq h. \end{cases}$$

By Lemma 2.1, the map $\mathfrak{S}_n \to W(k_g)[\![u]\!]/(p^n)$ induced from b_g is injective. For $j < h$, the image of I_j under b_g in $W(k_g)[\![u]\!]/(p^n)$ is equal to $I_{j,g}$ which is 0. Thus, $I_j = 0$ if $j < h$. Suppose $j \geq h$. If I_j is contained in the maximal ideal $(p, s_1, \ldots, s_r, u)$ of $\mathfrak{S}_n$, then the image of I_j under b_0 would be contained in the maximal ideal of $\mathcal{O}[\![u]\!]/(p^n)$. Since $I_{j,0} = \mathcal{O}[\![u]\!]/(p^n)$, we have $I_j = \mathfrak{S}_n$. Hence, $\mathfrak{M}_n$ is projective and thus free of rank h over $\mathfrak{S}_n$. By Theorem 2.6, $G_{n,R}[p^m]$ is finite flat over R for each $m \geq 1$.

Now we show the second statement, for any choice of Frobenius on R_0. Suppose that $G[p^n]$ extends to another finite flat group scheme H over R, and let $\mathfrak{N} := \mathfrak{M}^*(H)$ be the associated torsion Kisin module. Identify $\mathfrak{N} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} = \mathfrak{M}_n \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} = M_n$ as étale $(\varphi, \mathcal{O}_{\mathcal{E}})$-modules, and consider both $\mathfrak{N}$ and $\mathfrak{M}_n$ as $\mathfrak{S}_n$-submodules of M_n. Since $G_{n,R}[p^m]$ is finite flat over R for each $m \geq 1$ and similarly for H, and since M_n is projective over $\mathcal{O}_{\mathcal{E},n} := \mathcal{O}_{\mathcal{E}}/(p^n)$, we have by Theorem 2.6 that $\mathfrak{M}_n$ and $\mathfrak{N}$ are projective and thus flat over $\mathfrak{S}_n$. By [Liu07, Corollary 4.2.5], we have $\mathfrak{M}_n \otimes_{\mathfrak{S},b_g} W(k_g)[\![u]\!] = \mathfrak{N} \otimes_{\mathfrak{S},b_g} W(k_g)[\![u]\!]$ as $W(k_g)[\![u]\!]$-submodules of $M_n \otimes_{\mathfrak{S}} W(k_g)[\![u]\!]$. Note that by Lemma 2.1, the induced map $\mathcal{O}_{\mathcal{E},n} \to W_n(k_g)[\![u]\!][\frac{1}{u}]$ is injective, and $\mathcal{O}_{\mathcal{E},n} \cap W_n(k_g)[\![u]\!] = \mathfrak{S}_n$ as subrings of $W_n(k_g)[\![u]\!][\frac{1}{u}]$. Since $\mathfrak{M}_n$ is flat over $\mathfrak{S}_n$, we deduce

$$(\mathfrak{M}_n \otimes_{\mathfrak{S}_n} \mathcal{O}_{\mathcal{E},n}) \bigcap (\mathfrak{M}_n \otimes_{\mathfrak{S}_n} W_n(k_g)[\![u]\!]) = \mathfrak{M}_n \otimes_{\mathfrak{S}_n} (\mathcal{O}_{\mathcal{E},n} \bigcap W_n(k_g)[\![u]\!]) = \mathfrak{M}_n \otimes_{\mathfrak{S}_n} \mathfrak{S}_n = \mathfrak{M}_n$$

as $\mathfrak{S}_n$-submodules of $\mathfrak{M}_n \otimes_{\mathfrak{S}_n} W_n(k_g)[\![u]\!][\frac{1}{u}] = M_n \otimes_{\mathfrak{S}} W(k_g)[\![u]\!]$, and similarly

$$(\mathfrak{N} \otimes_{\mathfrak{S}_n} \mathcal{O}_{\mathcal{E},n}) \bigcap (\mathfrak{N} \otimes_{\mathfrak{S}_n} W_n(k_g)[\![u]\!]) = \mathfrak{N}$$

as $\mathfrak{S}_n$-submodules of $\mathfrak{N} \otimes_{\mathfrak{S}_n} W_n(k_g)[\![u]\!][\frac{1}{u}] = M_n \otimes_{\mathfrak{S}} W(k_g)[\![u]\!]$. Since $\mathfrak{M}_n \otimes_{\mathfrak{S}_n} \mathcal{O}_{\mathcal{E},n} = M_n = \mathfrak{N} \otimes_{\mathfrak{S}_n} \mathcal{O}_{\mathcal{E},n}$ and $\mathfrak{M}_n \otimes_{\mathfrak{S}_n} W_n(k_g)[\![u]\!] = \mathfrak{N} \otimes_{\mathfrak{S}_n} W_n(k_g)[\![u]\!]$ as submodules of $M_n \otimes_{\mathfrak{S}} W(k_g)[\![u]\!]$, we obtain $\mathfrak{M}_n = \mathfrak{N}$ with compatible Frobenius. □

We remark that in the second statement of above Proposition 4.1, we do not know whether $\mathfrak{M}^*(G_{R,n}) \cong \mathfrak{M}^*(H)$ as Kisin modules, i.e., whether the connections on both sides are compatible.

Now we consider the general base ring R as in Section 2.

Theorem 4.2. *Assume $e < p-1$. Let G be a p-divisible group over $R[\frac{1}{p}]$. Suppose that for each n, $G[p^n]$ extends to a finite locally free group scheme $G_{n,R}$ over R. Then G extends to a p-divisible group over R, and such an extension is unique up to isomorphism.*

If $e \geq p$ and $R = \mathcal{O}_K[\![s]\!]$, then there exists a p-divisible group G over $R[\frac{1}{p}]$ such that $G[p^n]$ extends to a finite locally free group scheme $G_{n,R}$ over R for each n but G does not extend to a p-divisible group over R.

Proof. Suppose $e < p-1$. Let M be the projective étale $(\varphi, \mathcal{O}_{\mathcal{E}})$-module such that $T^\vee(M) = T_p(G)$ as $\mathcal{G}_{\tilde{R}_\infty}$-representations. For each $n \geq 1$, let $\mathfrak{M}_n \coloneqq \mathfrak{M}^*(G_{n,R}) \in \mathrm{Mod}_{\mathfrak{S}}^{\mathrm{tor}}(\varphi, \nabla)$ be the torsion Kisin module of height 1 corresponding to $G_{n,R}$. We have $\mathfrak{M}_n \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} \cong M_n \coloneqq M/p^n M$ as étale $(\varphi, \mathcal{O}_{\mathcal{E}})$-modules. Let h be the height of G.

For each maximal ideal $\mathfrak{q}$ of R, denote $\mathfrak{q}_0 \coloneqq \mathfrak{q} \cap R_0 \subset R_0$ the corresponding maximal ideal of R_0, and let $b_{\mathfrak{q}} : R_0 \to \widehat{R}_{0,\mathfrak{q}_0}$ be the natural φ-equivariant map where $\widehat{R}_{0,\mathfrak{q}_0}$ denotes the $\mathfrak{q}_0$-adic completion of $R_{0,\mathfrak{q}_0}$. By the structure theorem for complete regular local rings, $\widehat{R}_{0,\mathfrak{q}_0}$ is isomorphic to a formal power series ring $\widehat{R}_{0,\mathfrak{q}_0} \cong \mathcal{O}[\![s_1, \dots, s_r]\!]$ over a Cohen ring $\mathcal{O}$. We have the induced base change $b_{\mathfrak{q}} : R \to \widehat{R}_{\mathfrak{q}} \cong \widehat{R}_{0,\mathfrak{q}_0} \otimes_{W(k)} \mathcal{O}_K$, where $\widehat{R}_{\mathfrak{q}}$ is the $\mathfrak{q}$-adic completion of $R_{\mathfrak{q}}$. Denote $\mathfrak{S}_{\mathfrak{q}} \coloneqq \widehat{R}_{0,\mathfrak{q}_0}[\![u]\!]$. For the p-divisible group $G \times_{R[\frac{1}{p}],b_{\mathfrak{q}}} \widehat{R}_{\mathfrak{q}}[\frac{1}{p}]$ over $\widehat{R}_{\mathfrak{q}}[\frac{1}{p}]$, note that $(G \times_{R[\frac{1}{p}]} \widehat{R}_{\mathfrak{q}}[\frac{1}{p}])[p^n]$ extends to the finite locally free group scheme $G_{n,\mathfrak{q}} \coloneqq G_{n,R} \times_{R,b_{\mathfrak{q}}} \widehat{R}_{\mathfrak{q}}$ over $\widehat{R}_{\mathfrak{q}}$ for each $n \geq 1$. By Proposition 4.1, $G_{n,\mathfrak{q}}[p^m]$ is finite locally free over $\widehat{R}_{\mathfrak{q}}$ for each $m \geq 1$, and thus $\mathfrak{M}^*(G_{n,\mathfrak{q}}) = \mathfrak{M}_n \otimes_{\mathfrak{S},b_{\mathfrak{q}}} \mathfrak{S}_{\mathfrak{q}}$ is projective over $\mathfrak{S}_{\mathfrak{q}}/(p^n)$

by Theorem 2.6. Since this holds for each maximal ideal $\mathfrak{q}$ of R, we deduce that $\mathfrak{M}_n$ is projective over $\mathfrak{S}/(p^n)$ of rank h. In particular, $G_{n,R}[p^m]$ is finite locally free over R for each $m \geq 1$. Note that $G_{n,R}[p^m] \times_R R[\frac{1}{p}] \cong (G_{n,R} \times_R R[\frac{1}{p}])[p^m] \cong G[p^m]$, and $G_{n,R}[p^m]$ has order p^{mh} for each $1 \leq m \leq n$.

By considering the orders of the groups, we see that the natural sequence of finite locally free group schemes

$$0 \to G_{n+1,R}[p] \to G_{n+1,R} \to G_{n+1,R}[p^n] \to 0,$$

where the map $G_{n+1,R} \to G_{n+1,R}[p^n]$ is induced by multiplication by p, is short exact. Furthermore, it follows easily from the construction of the functor $\mathfrak{M}^*(\cdot)$ in [Kim15, Proof of Proposition 9.5] using isogeny of p-divisible groups that $\mathfrak{M}^*(G_{n+1,R}[p]) \cong \mathfrak{M}_{n+1}/p\mathfrak{M}_{n+1}$ as torsion Kisin modules, where $\mathfrak{M}_{n+1}/p\mathfrak{M}_{n+1}$ is equipped with Frobenius and connection induced from $\mathfrak{M}_{n+1}$. Since $\mathfrak{M}^*(\cdot)$ is exact, we have $\mathfrak{M}^*(G_{n+1,R}[p^n]) \cong p\mathfrak{M}_{n+1}$ where $p\mathfrak{M}_{n+1}$ is equipped with Frobenius and connection induced from $\mathfrak{M}_{n+1}$. We claim that $\mathfrak{M}_n \cong p\mathfrak{M}_{n+1}$ as torsion quasi-Kisin modules with compatible Frobenius. Identify $p\mathfrak{M}_{n+1} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} = M_n = \mathfrak{M}_n \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$ as étale $(\varphi, \mathcal{O}_{\mathcal{E}})$-modules, and consider both $p\mathfrak{M}_{n+1}$ and $\mathfrak{M}_n$ as $\mathfrak{S}_n$-submodules of M_n. For the natural injective map $\mathfrak{M}_n \hookrightarrow \mathfrak{M}_n + p\mathfrak{M}_{n+1}$ of $\mathfrak{S}$-modules, consider the induced map $\mathfrak{M}_n \otimes_{\mathfrak{S},b_{\mathfrak{q}}} \mathfrak{S}_{\mathfrak{q}} \to (\mathfrak{M}_n + p\mathfrak{M}_{n+1}) \otimes_{\mathfrak{S},b_{\mathfrak{q}}} \mathfrak{S}_{\mathfrak{q}}$ for each maximal ideal $\mathfrak{q}$ of R. Since $b_{\mathfrak{q}} : \mathfrak{S} \to \mathfrak{S}_{\mathfrak{q}}$ is flat, we have $(\mathfrak{M}_n + p\mathfrak{M}_{n+1}) \otimes_{\mathfrak{S},b_{\mathfrak{q}}} \mathfrak{S}_{\mathfrak{q}} = \mathfrak{M}_n \otimes_{\mathfrak{S},b_{\mathfrak{q}}} \mathfrak{S}_{\mathfrak{q}} + p\mathfrak{M}_{n+1} \otimes_{\mathfrak{S},b_{\mathfrak{q}}} \mathfrak{S}_{\mathfrak{q}}$, and by Proposition 4.1, $\mathfrak{M}_n \otimes_{\mathfrak{S},b_{\mathfrak{q}}} \mathfrak{S}_{\mathfrak{q}} = p\mathfrak{M}_{n+1} \otimes_{\mathfrak{S},b_{\mathfrak{q}}} \mathfrak{S}_{\mathfrak{q}}$ as submodules of $M_n \otimes_{\mathfrak{S},b_{\mathfrak{q}}} \mathfrak{S}_{\mathfrak{q}}$. Thus, $\mathfrak{M}_n \otimes_{\mathfrak{S}} \mathfrak{S}_{\mathfrak{q}} \xrightarrow{\cong} (\mathfrak{M}_n + p\mathfrak{M}_{n+1}) \otimes_{\mathfrak{S}} \mathfrak{S}_{\mathfrak{q}}$ for each $\mathfrak{q}$, which implies that injective map $\mathfrak{M}_n \hookrightarrow \mathfrak{M}_n + p\mathfrak{M}_{n+1}$ is also surjective. Thus, $p\mathfrak{M}_{n+1} \subset \mathfrak{M}_n$, and similarly $\mathfrak{M}_n \subset p\mathfrak{M}_{n+1}$. This shows the claim $\mathfrak{M}_n = p\mathfrak{M}_{n+1}$ with compatible Frobenius.

Thus, $\mathfrak{M} := \varprojlim_n \mathfrak{M}_n$ with the induced Frobenius is a quasi-Kisin module of height 1 over $\mathfrak{S}$. We now equip $\mathcal{M} := \mathfrak{M} \otimes_{\mathfrak{S},\varphi} R_0$ with a connection. Denote by $\nabla_{\mathfrak{M}_n} : \mathfrak{M}_n \otimes_{\mathfrak{S},\varphi} R_0 \to (\mathfrak{M}_n \otimes_{\mathfrak{S},\varphi} R_0) \otimes_{R_0} \widehat{\Omega}_{R_0}$ the connection for the torsion Kisin module $\mathfrak{M}_n$, and let $\mathcal{M}_n = \mathcal{M} \otimes_{R_0} R_0/(p^n)$. Consider the multiset

$$S_n = \{\nabla_{\mathfrak{M}_k} \otimes_{R_0} R_0/(p^n) \mid k \geq n+1\}$$

of connections on $\mathcal{M}_n$. Note that for each $k \geq n+1$, the connection $\nabla_{\mathfrak{M}_k} \otimes_{R_0} R_0/(p^n)$ satisfies the commutative diagram (2.1) in Section 2. Using the result discussed at the end of Section 2, we choose a compatible system of connections ∇_n on $\mathcal{M}_n$ inductively as follows. Identify $\widehat{\Omega}_{R_0} = \bigoplus_{i=1}^d R_0 \cdot d(\log t_i)$. Let $\mathcal{M}^1 \subset \mathcal{M}$ be a direct factor lifting $\mathrm{Fil}^1\mathcal{M}/p\mathcal{M} \subset \mathcal{M}/p\mathcal{M}$ as in Section 2, and we fix a choice of a finite Zariski covering of $\mathrm{Spf}(R_0, p)$ over which $\mathcal{M}^1$ and $\mathcal{M}/\mathcal{M}^1$ are free, and fix a basis of $\mathcal{M}$ adapted to $\mathcal{M}^1$ after passing to the covering. For $n = 1$, S_1 is finite as a set of connections on $\mathcal{M}_1$, and we choose a connection ∇_1 on $\mathcal{M}_1$ which has infinite multiplicity in the multiset S_1. When we are given a choice of connection ∇_n on $\mathcal{M}_n$, the elements in S_{n+1} which lift ∇_n are contained in a finite set of connections, and we choose a connection ∇_{n+1} on $\mathcal{M}_{n+1}$ which has infinite multiplicity in S_{n+1}. Let $\nabla \coloneqq \varprojlim_n \nabla_n$ be the induced connection on $\mathcal{M}$. Then ∇ is compatible with Frobenius, integrable, and topologically quasi-nilpotent. Hence, $(\mathfrak{M}, \nabla)$ is a Kisin module of height 1, and the corresponding p-divisible group over R extends G. The uniqueness of extending G up to isomorphism follows from [Tat67, Theorem 4].

On the other hand, assume $e \geq p$ and $R_0 = W(k)[\![s]\!]$. Let $U = \mathrm{Spec}R\backslash\{\mathfrak{m}\}$ be the open subscheme of $\mathrm{Spec}R$, where $\mathfrak{m}$ is the closed point given by the maximal ideal of R. By [VZ10, Theorem 28], there exists a p-divisible group G_U over U which does not extend to a p-divisible group over R. By [FC90, Chapter V. Lemma 6.2], for each $n \geq 1$, the finite locally free group scheme $G_U[p^n]$ extends uniquely to a finite locally free group scheme over R (if A denotes the Hopf algebra for $G_U[p^n] \times_U R[\frac{1}{p}]$ and B denotes the Hopf algebra for $G_U[p^n] \times_U R[\frac{1}{s}]$, then identifying $C \coloneqq A[\frac{1}{s}] = B[\frac{1}{p}]$ as the Hopf algebra for $G_U[p^n] \times_U R[\frac{1}{p}][\frac{1}{s}]$, the unique extension is given by $A \cap B$ with the induced Hopf algebra structure over R). Let $G = G_U \times_U R[\frac{1}{p}]$ be the p-divisible group over $R[\frac{1}{p}]$, and suppose G extends to a p-divisible group G_R over R. Since $G_U \times_U (R[\frac{1}{s}])[\frac{1}{p}] = G_R \times_R (R[\frac{1}{s}])[\frac{1}{p}]$, we have by [Tat67, Theorem 4] that $G_U \times_U R[\frac{1}{s}] = G_R \times_R R[\frac{1}{s}]$. Thus, $G_R \times_R U = G_U$, which contradicts to that G_U does not extend over R. This shows that G cannot be extended to a p-divisible group over R. $\square$

5 Barsotti-Tate Deformation Ring for Relative Base of Dimension 2

Throughout this section, we assume that the Krull dimension of R is equal to 2. For a finite $\mathbf{Q}_p$-representation V of $\mathcal{G}_R$, we say it is *Barsotti-Tate* if there exists a p-divisible group G_R over R such that $V = T_p(G_R) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ as $\mathcal{G}_R$-representations.

Proposition 5.1. *Assume $e < p - 1$. Let T be a finite free $\mathbf{Z}_p$-representation of $\mathcal{G}_R$ such that $T[\frac{1}{p}]$ is Barsotti-Tate. Then there exists a p-divisible group G_R over R such that $T = T_p(G_R)$.*

Proof. Since $T[\frac{1}{p}]$ is Barsotti-Tate, there exists a p-divisible group G'_R over R such that $T_p(G'_R)[\frac{1}{p}] = T[\frac{1}{p}]$. Denote $T' = T_p(G'_R)$, $G' = G'_R \times_R R[\frac{1}{p}]$, and let G be the p-divisible group over $R[\frac{1}{p}]$ corresponding to the representation T.

Since $p^nT \subset T'$ and $p^nT' \subset T$ for some positiver integer n, we have an isogeny $f : G' \to G$. Let $H \coloneqq \ker(f)$, which is a finite locally free group scheme over $R[\frac{1}{p}]$. Then we have a closed immersion $h : H \hookrightarrow G'[p^m]$ for some positive integer m. Note that $G'[p^m]$ extends to the finite locally free group scheme $G'_R[p^m]$ over R.

Let H_R be the scheme theoretic closure of H over R obtained from h and $G'_R[p^m]$, given similarly as in [Ray74, Section 2.1]. By the construction of the scheme theoretic closure, H_R is a finite group scheme. We claim that it is locally free over R. For that, let $\mathfrak{q}$ be a maximal ideal of R and let $\mathfrak{q}_0 = \mathfrak{q} \cap R_0$, and consider the base change map $b_{\mathfrak{q}} : R \to \widehat{R}_{\mathfrak{q}}$ as in the proof of Theorem 4.2. Since R has Krull dimension 2, we have $\widehat{R}_{\mathfrak{q}} \cong \mathcal{O}_{\mathfrak{q}_0}[\![s]\!] \otimes_{W(k)} \mathcal{O}_K$ for some Cohen ring $\mathcal{O}_{\mathfrak{q}_0}$ with the maximal ideal (p). Let $U_{\mathfrak{q}} \subset \mathrm{Spec}\widehat{R}_{\mathfrak{q}}$ be the closed subscheme obtained by deleting the closed point given by $\mathfrak{q}$. Since $U_{\mathfrak{q}}$ is a Dedekind scheme, $(H_R \times_R \widehat{R}_{\mathfrak{q}}) \otimes_{\widehat{R}_{\mathfrak{q}}} U_{\mathfrak{q}}$ is locally free over $U_{\mathfrak{q}}$ as the corresponding sheaf of Hopf algebras is torsion free. It extends uniquely to a finite locally free group scheme $H_{\mathfrak{q}}$ over $\widehat{R}_{\mathfrak{q}}$ by [FC90, Chapter V. Lemma 6.2]. On the other hand, since $e < p-1$, note that $p \notin (\mathfrak{q}\widehat{R}_{\mathfrak{q}})^{p-1}$. Since h is a monomorphism, we deduce from [VZ10, Proposition 15] applied for $\widehat{R}_{\mathfrak{q}}$ that the map $H_{\mathfrak{q}} \to G'_R[p^m] \times_R \widehat{R}_{\mathfrak{q}}$ of finite flat group schemes is a monomorphism and hence a closed immersion. Thus, $H_R \times_R \widehat{R}_{\mathfrak{q}} = H_{\mathfrak{q}}$. Since this holds for every maximal ideal $\mathfrak{q}$ of R, H_R is locally free over R.

The map h induces a closed immersion $H_R \hookrightarrow G'_R[p^m]$, and $G_R \coloneqq G'_R/H_R$ is a p-divisible group over R. It is clear from the construction that $T_p(G_R) = T$ as $\mathbf{Z}_p[\mathcal{G}_R]$-modules. □

For a finite free $\mathbf{Z}_p$-representation T of $\mathcal{G}_R$, it makes sense by Proposition 5.1 to say that T is *Barsotti-Tate* if there exists a p-divisible group G_R over R such that $T = T_p(G_R)$.

Lemma 5.2. *Assume $e < p-1$. Let H_R be a p-power order finite locally free group scheme over R, and let $T = H_R(\overline{R})$ be the corresponding torsion $\mathbf{Z}_p$-representation of $\mathcal{G}_R$. If we have a short exact sequence of $\mathbf{Z}_p[\mathcal{G}_R]$-modules*

$$0 \to T_1 \to T \to T_2 \to 0,$$

then there exist p-power order finite locally free group schemes $H_{1,R}$ and $H_{2,R}$ over R such that $T_i = H_{i,R}(\overline{R})$ for $i = 1, 2$ as $\mathcal{G}_R$-representations.

Proof. Let $H \coloneqq H_R \times_R R[\frac{1}{p}]$. Let H_i for $i = 1, 2$ be finite locally free group schemes over $R[\frac{1}{p}]$ such that $H_i(\overline{R}[\frac{1}{p}]) = T_i$ as $\mathcal{G}_R$-representations. The given exact sequence of $\mathcal{G}_R$-representations induce the short exact sequence

$$0 \to H_1 \to H \to H_2 \to 0$$

of finite locally free group schemes. Let $H_{1,R}$ be the scheme theoretic closure of H_1 over R obtained from the closed embedding $H_1 \hookrightarrow H$ and H_R. By the same argument as in the proof of Proposition 5.1, $H_{1,R}$ is a finite locally free group scheme over R extending H_1. Furthermore, $H_{2,R} \coloneqq H_R/H_{1,R}$ is a finite locally free group scheme over R extending H_2 (cf. [Ray67]). It is clear that $T_i = H_{i,R}(\overline{R})$ for $i = 1, 2$. □

Corollary 5.3. *Assume $e < p-1$. Let $A_1 \hookrightarrow A_2$ be an injective map of finite free $\mathbf{Z}_p$-algebras. Let T_{A_1} be a finite free A_1-module given the p-adic topology and equipped with a continuous A_1-linear $\mathcal{G}_R$-action. Let $T_{A_2} \coloneqq T_{A_1} \otimes_{A_1} A_2$ be the induced representation with the A_2-linear $\mathcal{G}_R$-action. Then T_{A_1} is Barsotti-Tate if and only if T_{A_2} is Barsotti-Tate.*

Proof. Let G_2 be the p-divisible group over $R[\frac{1}{p}]$ corresponding to T_{A_2}. Suppose first that T_{A_1} is Barsotti-Tate. Note that there exist finitely many elements $x_1, \ldots, x_m \in A_2$ generating A_2 as an A_1-module. We have a surjective map of $\mathbf{Z}_p[\mathcal{G}_R]$-modules $T_{A_1}^m \twoheadrightarrow T_{A_2}$ sending the canonical basis elements e_i of $T_{A_1}^m$ to x_i. Note that the direct sum representation $T_{A_1}^m$ is Barsotti-Tate. For each integer $n \geq 1$, T_{A_2}/p^n is therefore a quotient of $T_{A_1}^m/p^n$, and by Lemma 5.2, $G_2[p^n]$ extends to a finite locally free group scheme over R. Thus, T_{A_2} is Barsotti-Tate by Theorem 4.2.

Conversely, suppose T_{A_2} is Barsotti-Tate. Let B_3 be the quotient of the induced injection $A_1[\frac{1}{p}] \hookrightarrow A_2[\frac{1}{p}]$ of $\mathbf{Q}_p$-algebras, and let $T \subset T_{A_2}$ be the kernel of the induced map of representations $T_{A_2} \to T_{A_2} \otimes_{A_2} B_3$. Then for each integer $n \geq 1$, the map $T/p^n \to T_{A_2}/p^n$ is injective. Hence, by Lemma 5.2 and Theorem 4.2 similarly as above, T is Barsotti-Tate. Since $T[\frac{1}{p}] = T_{A_1}[\frac{1}{p}]$, T_{A_1} is Barsotti-Tate by Proposition 5.1. □

We now study the geometry of the locus of Barsotti-Tate representations. Denote by $\mathcal{C}$ the category of topological local $\mathbf{Z}_p$-algebras A satisfying the following conditions:

- the natural map $\mathbf{Z}_p \to A/\mathfrak{m}_A$ is surjective, where $\mathfrak{m}_A$ denotes the maximal ideal of A;
- the map from A to the projective limit of its discrete artinian quotients is a topological isomorphism.

Note that the first condition implies that the residue field of A is $\mathbf{F}_p$. The second condition is equivalent to the condition that A is complete and its topology can be given by a collection of open ideals $\mathfrak{a} \subset A$ for which $A/\mathfrak{a}$ is aritinian. Morphisms in $\mathcal{C}$ are continuous $\mathbf{Z}_p$-algebra morphisms. The following proposition is shown in [SL97].

Proposition 5.4. (cf. [SL97, Proposition 2.4]) *Suppose A is a Noetherian ring in $\mathcal{C}$. Then the topology on A is equal to the $\mathfrak{m}_A$-adic topology.*

For $A \in \mathcal{C}$, we mean by an *A-representation of $\mathcal{G}_R$* a finite free A-module equipped with a continuous A-linear $\mathcal{G}_R$-action. We fix an $\mathbf{F}_p$-representation V_0 of $\mathcal{G}_R$ which is absolutely irreducible. For $A \in \mathcal{C}$, a *deformation* of V_0 in A is an isomorphism class of A-representations of V of $\mathcal{G}_R$ satisfying $V \otimes_A \mathbf{F}_p \cong V_0$ as $\mathbf{F}_p[\mathcal{G}_R]$-modules. We denote by $\mathrm{Def}(V_0, A)$ the set of such deformations. A morphism $f : A \to A'$ in $\mathcal{C}$ induces a map $f_* : \mathrm{Def}(V_0, A) \to \mathrm{Def}(V_0, A')$ sending the class of an A-representation V to the class of $V \otimes_{A,f} A'$. The following theorem on universal deformation ring is proved in [SL97].

Theorem 5.5. (cf. [SL97, Theorem 2.3]) *There exists a universal deformation ring $A_{\mathrm{univ}} \in \mathcal{C}$ and a deformation $V_{\mathrm{univ}} \in \mathrm{Def}(V_0, A_{\mathrm{univ}})$ such that for all $A \in \mathcal{C}$, we have a bijection*

$$\mathrm{Hom}_{\mathcal{C}}(A_{\mathrm{univ}}, A) \xrightarrow{\cong} \mathrm{Def}(V_0, A) \tag{5.1}$$

given by $f \mapsto f_*(V_{\mathrm{univ}})$.

We remark that A_{univ} is Noetherian if and only if $\dim_{\mathbf{F}_p} H^1(\mathcal{G}_R, \mathrm{End}_{\mathbf{F}_p}(V_0))$ is finite (cf. *loc. cit.*). Thus, A_{univ} is not Noetherian in general, even when $R = \mathcal{O}_K$ if $K/\mathbf{Q}_p$ is infinite.

Let $\mathcal{C}^0$ be the full subcategory of $\mathcal{C}$ consisting of artinian rings. Abusing the notation, we write $V \in \mathrm{Def}(V_0, A)$ for an A-representation V to mean that $V \otimes_A \mathbf{F}_p \cong V_0$. For $A \in \mathcal{C}^0$ and a representation $V_A \in \mathrm{Def}(V_0, A)$, we say V_A is *torsion Barsotti-Tate* if there exists a p-power order finite locally free group scheme H_R over R such that $V_A \cong H_R(\overline{R})$ as $\mathbf{Z}_p[\mathcal{G}_R]$-modules. We remark that if R is local, then every p-power order finite locally free group scheme over R embeds into a p-divisible group over R, and thus V_A is torsion Barsotti-Tate if and only if it is a quotient of a finite free $\mathbf{Z}_p$-representation which is Barsotti-Tate. For $A \in \mathcal{C}$, denote by $\mathrm{BT}(V_0, A)$ the subset of $\mathrm{Def}(V_0, A)$ consisting of the isomorphism classes of representations V_A such that $V_A \otimes_A A/\mathfrak{a}$ is torsion Barsotti-Tate for all open ideals $\mathfrak{a} \subsetneq A$.

Proposition 5.6. *Assume $e < p - 1$. For any $\mathcal{C}$-morphism $f : A \to A'$, we have $f_*(\mathrm{BT}(V_0, A)) \subset \mathrm{BT}(V_0, A')$. Furthermore, there exists a closed ideal $\mathfrak{a}_{\mathrm{BT}}$ of the universal deformation ring A_{univ} such that the map (5.1) induces a bijection $\mathrm{Hom}_{\mathcal{C}}(A_{\mathrm{univ}}/\mathfrak{a}_{\mathrm{BT}}, A) \xrightarrow{\cong} \mathrm{BT}(V_0, A)$.*

Proof. We check the conditions in [SL97, Section 6]. Let $f : A \hookrightarrow A'$ be an injective morphism of artinian rings in $\mathcal{C}$, and let $V_A \in \mathrm{Def}(V_0, A)$ be a representation. We first claim that $V_A \in \mathrm{BT}(V_0, A)$ if and only if $V_{A'} := V_A \otimes_{A,f} A' \in \mathrm{BT}(V_0, A')$. Suppose that $V_A \in \mathrm{BT}(V_0, A)$. Note that A' is a finite A-module. Let $x_1, \ldots, x_m$ generate A' over A. Then we have a surjective map of $\mathbf{Z}_p[\mathcal{G}_R]$-modules $V_A^m \twoheadrightarrow V_{A'}$ sending the canonical basis elements e_i of V_A^m for $i = 1, \ldots, m$ to x_i. Since V_A^m is the direct sum of m-copies of V_A, it is torsion Barsotti-Tate. Thus, by Lemma 5.2, $V_{A'} \in \mathrm{BT}(V_0, A')$. Conversely, suppose $V_{A'} \in \mathrm{BT}(V_0, A')$. Since we have an injective map of $\mathbf{Z}_p[\mathcal{G}_R]$-modules $V_A \hookrightarrow V_{A'}$, we get $V_A \in \mathrm{BT}(V_0, A)$ by Lemma 5.2.

Now, for $A \in \mathcal{C}$ and a representation $V_A \in \mathrm{Def}(V_0, A)$, suppose $\mathfrak{a}_1, \mathfrak{a}_2 \subsetneq A$ are open ideals such that $V_A \otimes_A (A/\mathfrak{a}_i) \in \mathrm{BT}(V_0, A/\mathfrak{a}_i)$ for $i = 1, 2$. The natural map $A/(\mathfrak{a}_1 \cap \mathfrak{a}_2) \to A/\mathfrak{a}_1 \oplus A/\mathfrak{a}_2$ is injective, and it induces the injective map of $\mathbf{Z}_p[\mathcal{G}_R]$-modules

$$V_A \otimes_A A/(\mathfrak{a}_1 \cap \mathfrak{a}_2) \hookrightarrow (V_A \otimes_A A/\mathfrak{a}_1) \oplus (V_A \otimes_A A/\mathfrak{a}_2).$$

Since the direct sum $(V_A \otimes_A A/\mathfrak{a}_1) \oplus (V_A \otimes_A A/\mathfrak{a}_2)$ is torsion Barsotti-Tate, we see from Lemma 5.2 that $V_A \otimes_A A/(\mathfrak{a}_1 \cap \mathfrak{a}_2) \in \mathrm{BT}(V_0, A/(\mathfrak{a}_1 \cap \mathfrak{a}_2))$.

The assertion then follows from [SL97, Proposition 6.1]. □

We now show that when $e < p - 1$, the locus of Barsotti-Tate representations cuts out a closed subscheme of the universal deformation scheme $\mathrm{Spec}(A_{\mathrm{univ}})$:

Theorem 5.7. *Suppose $e < p - 1$ (and recall that the Krull dimension of R is assumed to be equal to 2). Let A be a finite flat $\mathbf{Z}_p$-algebra equipped with the p-adic topology, and let $f : A_{\mathrm{univ}} \to A$ be a continuous $\mathbf{Z}_p$-algebra homomorphism. Then the induced representation $V_{\mathrm{univ}} \otimes_{A_{\mathrm{univ}}, f} A[\frac{1}{p}]$ of $\mathcal{G}_R$ is Barsotti-Tate if and only if f factors through the quotient $A_{\mathrm{univ}}/\mathfrak{a}_{\mathrm{BT}}$.*

Proof. Let $A_1 := \mathrm{im}(f) \subset A$, and let $T_{A_1} = V_{\mathrm{univ}} \otimes_{A_{\mathrm{univ}},f} A_1$. Then $T_{A_1} \otimes_{A_1} A = V_{\mathrm{univ}} \otimes_{A_{\mathrm{univ}},f} A$, and by Proposition 5.1 and Corollary 5.3, it suffices to show that T_{A_1} is Barsotti-Tate if and only if f factors through $A_{\mathrm{univ}}/\mathfrak{a}_{\mathrm{BT}}$. Note that $A_1 \in \mathcal{C}$, and since A_1 is finite flat over $\mathbf{Z}_p$, the topology on A_1 is equivalent to the p-adic topology and $f : A_{\mathrm{univ}} \to A_1$ is continuous by Proposition 5.4. Suppose first that T_{A_1} is Barsotti-Tate, so that there exists a p-divisible group G_R over R such that $T_p(G_R) \cong T_{A_1}$. For each integer $n \geq 1$, we then have $(V_{\mathrm{univ}} \otimes_{A_{\mathrm{univ}},f} A_1) \otimes_{A_1} A_1/(p^n) = T_{A_1}/p^n \cong (G_R[p^n])(\overline{R})$, so $V_{\mathrm{univ}} \otimes_{A_{\mathrm{univ}},f} A_1/(p^n) \in \mathrm{BT}(V_0, A_1/(p^n))$. Hence, by Proposition 5.6, f factors through $A_{\mathrm{univ}}/\mathfrak{a}_{\mathrm{BT}}$.

Conversely, suppose f factors through $A_{\mathrm{univ}}/\mathfrak{a}_{\mathrm{BT}}$. Let G be the p-divisible group over $R[\frac{1}{p}]$ corresponding to T_{A_1}. For each $n \geq 1$, T_{A_1}/p^n is torsion Barsotti-Tate by Proposition 5.6, so $G[p^n]$ extends to a finite locally free group scheme over R. Then by Theorem 4.2, T_{A_1} is Barsotti-Tate. ∎

On the other hand, if the ramification is large, we can deduce that the locus of Barsotti-Tate representations is not p-adically closed in general:

Proposition 5.8. *Let $R = \mathcal{O}_K[\![s]\!]$ and suppose $e \geq p$. There exists a $\mathbf{Z}_p$-representation T of $\mathcal{G}_R$ such that T/p^nT is torsion Barsotti-Tate for each $n \geq 1$ but T is not Barsotti-Tate.*

Proof. By Theorem 4.2, there exists a p-divisible group G over $R[\frac{1}{p}]$ such that $G[p^n]$ extends to a finite locally free group scheme $G_{n,R}$ over R but G does not extend to a p-divisible group over R. Let T be the representation corresponding to G. Then for each n, we have $T/p^nT \cong G_{n,R}(\overline{R})$ so it is torsion Barsotti-Tate. However, T is not Barsotti-Tate since G does not extend over R. ∎

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