MIXED RAY TRANSFORM ON SIMPLE 2-DIMENSIONAL RIEMANNIAN MANIFOLDS

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Abstract. We characterize the kernel of the mixed ray transform on simple 2-dimensional Riemannian manifolds, that is, on simple surfaces for tensors of any order.

1. Introduction

We provide a characterization of the kernel of the mixed ray transform on simple 2-dimensional Riemannian manifolds for tensors of any order. The key application pertains to elastic qS-wave tomography [3] in weakly anisotropic media.

We let (M, g) be a smooth, compact, connected 2-dimensional Riemannian manifold with smooth boundary ∂M. We assume that (M, g) is simple, that is, ∂M is strictly convex with respect to g and $\exp_p : \exp_p^{-1}(M) \to M$ is a diffeomorphism for every $p \in M$. We let $SM = \{(x, v) \in TM; \|v\|_g = 1\}$ be the unit sphere bundle. We use the notation ν for the outer unit normal vector field to ∂M. We write $\partial_{in}(SM) = \{(x, v) \in SM; x \in \partial M, \langle v, \nu\rangle_g \leq 0\}$ for the vector bundle of inward pointing unit vectors on ∂M. For $(x, v) \in SM$, $\gamma_{x,v}(t)$ is the geodesic starting from x in direction v, and $\tau(x, v)$ is the time when $\gamma_{x,v}$ exits M. Since (M, g) is simple $\tau(x, v) < \infty$ for all $(x, v) \in \partial_{in}(SM)$ and the *exit time function* τ is smooth in $\partial_{in}(SM)$ [15, Section 4.1].

We use the notation S^kM, $k \in \mathbf{N}$, for the space of smooth symmetric tensor fields on M. We also use the notation $S^kM \times S^\ell M$, $k, \ell \geq 1$ for the space of smooth tensor fields that are symmetric with respect to first k and last ℓ variables. *The mixed ray transform* $L_{k,\ell}$ of a tensor field $f \in S^kM \times S^\ell M$ is given by the formula

(1)
$$L_{k,\ell}f(x, v) = \int_0^{\tau(x,v)} f_{i_1,\dots,i_k j_1,\dots,j_\ell}(\gamma(t))\dot\gamma(t)^{i_1} \cdots \dot\gamma(t)^{i_k}\eta(t)^{j_1} \cdots \eta(t)^{j_\ell}\mathrm{d}t,\ (x, v) \in \partial_{in}(SM), \gamma = \gamma_{x,v},$$

where we used the summation convention, while $\eta(t)$ is some unit length vector field on γ that is parallel and perpendicular to $\dot\gamma(t)$ and depends smoothly on $(x, v) \in \partial_{in}(SM)$. We note that the definition of the mixed ray transform is different in higher dimensions, due to the freedom in the choice of η (See [15, Section 7.2]). We consider the choice of $\eta(t)$ and the mapping properties of $L_{k,\ell}$ in dimension 2.

We define two linear operators the images of which are contained in the kernel of $L_{k,\ell}$. For a $(k \times \ell)$-tensor, $f_{i_1,\dots,i_k j_1,\dots,j_\ell}$, we introduce the symmetrization operator as

$$(\mathrm{Sym}(i_1, \dots, i_k)f)_{i_1,\dots,i_k j_1,\dots,j_\ell} := \frac{1}{k!}\sum_\sigma f_{i_{\sigma(1)},\dots,i_{\sigma(k)} j_1,\dots,j_\ell}, \qquad (2)$$

where σ runs over all permutations of $(1, 2, \cdots, k)$. This operator symmetrizes f with respect to the first k indices. We define the symmetrization operator $\mathrm{Sym}(j_1, \dots, j_\ell)$, for the last ℓ indices analogously.

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We introduce a *first* operator λ the image of which is contained in the kernel of $L_{k,\ell}$. The operator $\lambda : S^{k-1}M \times S^{\ell-1}M \to S^kM \times S^\ell M$ is defined by

$$(\lambda w)_{i_1,\dots,i_k j_1,\dots,j_\ell} := \mathrm{Sym}(i_1,\dots,i_k)\mathrm{Sym}(j_1,\dots,j_\ell)(g_{i_1 j_1} w_{i_2,\dots,i_k j_2,\dots,j_\ell}). \tag{3}$$

Using (2) and (3) it is straightforward to verify that

$$(\lambda w)_{i_1,\dots,i_k j_1,\dots,j_\ell} v^{i_1}\cdots v^{i_k}(v^\perp)^{j_1}\cdots(v^\perp)^{j_\ell} = 0, \quad v \in TM, \tag{4}$$

where $v^\perp$ is any vector orthogonal to v. Therefore (4) implies that

$$\mathrm{Im}(\lambda) \subset \ker(L_{k,\ell}).$$

We use the notation $u_{i_1,\dots,i_k;h}$, for the (h) component functions of the covariant derivative ∇u of the tensor field u. We define the *second* operator, d' say, by the formula,

$$d' : S^{k-1}M \times S^\ell M \to S^kM \times S^\ell M, \quad (d'u)_{i_1,\dots,i_k j_1,\dots,j_\ell} := \mathrm{Sym}(i_1,\dots,i_k) u_{i_2,\dots,i_k j_1,\dots,j_\ell; i_1}. \tag{5}$$

Then the following holds for any $u \in S^{k-1}M \times S^\ell M$,

$$\begin{aligned} &\frac{d}{dt}\Big(u_{i_1,\dots,i_{k-1} j_1,\dots,j_\ell}(\gamma(t))\dot\gamma(t)^{i_1}\cdots\dot\gamma(t)^{i_{k-1}}\eta(t)^{j_1}\cdots\eta(t)^{j_\ell}\Big) \\ &\qquad = (d'u)_{i_1,\dots,i_k j_1,\dots,j_\ell}\dot\gamma(t)^{i_1}\cdots\dot\gamma(t)^{i_k}\eta(t)^{j_1}\cdots\eta(t)^{j_\ell}. \end{aligned} \tag{6}$$

If $u|_{\partial M} = 0$, then $L_{k,\ell}(d'u) = 0$ by the fundamental theorem of calculus. Thus

$$\{d'u : \; u \in S^{k-1}M \times S^\ell M, \; u|_{\partial M} = 0\} \subset \ker(L_{k,\ell}).$$

Our main result shows that the kernel of $L_{k,\ell}$ is spanned by the images of these two linear operators.

Theorem 1. *Let (M,g) be a simple 2-dimensional Riemannian manifold. Let $f \in S^kM \times S^\ell M$, $k,\ell \geq 1$. Then*

$$L_{k,\ell} f(x,v) = 0, \qquad (x,v) \in \partial_{in}(SM)$$

if and only if

$$f = d'u + \lambda w, \quad u \in S^{k-1}M \times S^\ell M, \; u|_{\partial M} = 0, \quad w \in S^{k-1}M \times S^{\ell-1}M.$$

The key observation needed to prove this theorem is that the mixed ray transform and the geodesic ray transform can be transformed to one another, for arbitrary $k,\ell \geq 1$, if (M,g) is a 2-dimensional simple Riemannian manifold. A similar observation has already been obtained for the transverse ray transform by Sharafutdinov [15, Chapter 5]. The work by Paternain, Salo and Uhlmann [9] proved the s-injectivity of the geodesic ray transform on simple manifolds in dimension 2. In Theorem 1, we characterize the kernel of $L_{k,\ell}$ using their results.

2. Relation with elastic qS-wave tomography

We describe a mixed ray transform arising from elastic wave tomography. We follow the presentation in [15, Chapter 7], wherein one can find more details. Let (x^1,x^2) be any curvilinear coordinate system in $\mathbb{R}^2$, where the Euclidean metric is

$$ds^2 = g_{jk}dx^j dx^k.$$

The elastic wave equations

$$\rho\frac{\partial^2 u_j}{\partial t^2} = \sigma_{jk;}{}^k := \sigma_{jk;l}g^{kl} \tag{7}$$

describes the waves traveling in a two-dimensional elastic body $M \subset \mathbf{R}^2$. Here $u(x,t) = (u^1, u^2)$ is the displacement vector. The strain tensor is given by

$$\varepsilon_{jk} = \frac{1}{2}(u_{j;k} + u_{k;j}),$$

while the stress tensor is

$$\sigma_{jk} = C_{jklm}\varepsilon^{lm},$$

where $\mathbf{C}(x) = (C_{jklm})$ is the elastic tensor and $\rho(x)$ is the density of mass. Here ε^{lm} is obtained by raising indices with respect to the metric g_{jk}. The elastic tensor has the following symmetry properties

$$C_{jklm} = C_{kjlm} = C_{lmjk}. \tag{8}$$

We assume that the elastic tensor is weakly anisotropic, that is, it can be represented as

$$C_{jklm} = \lambda g_{jk} g_{lm} + \mu(g_{jl} g_{km} + g_{jm} g_{kl}) + \delta c_{jklm},$$

where λ and μ are positive functions called the Lamé parameters, and $\mathbf{c} = (c_{jklm})$ is an anisotropic perturbation. Here, δ is a small positive real number. We note here that $\mathbf{c} = 0$ corresponds to an isotropic medium.

We construct geometric optics solutions to system (7) using the parameter $\omega = \omega_0/\delta$,

$$u_j = e^{i\omega\iota} \sum_{m=0}^{\infty} \frac{u_j^{(m)}}{(i\omega)^m}, \quad \varepsilon_{jk} = e^{i\omega\iota} \sum_{m=-1}^{\infty} \frac{\varepsilon_{jk}^{(m)}}{(i\omega)^m}, \quad \sigma_{jk} = e^{i\omega\iota} \sum_{m=-1}^{\infty} \frac{\sigma_{jk}^{(m)}}{(i\omega)^m},$$

where $\iota(x)$ is a real function.

We substitute the above solutions into equation (7), assume $u^{(-1)} = \varepsilon^{(-2)} = \sigma^{(-2)} = 0$ and equate the terms of the order -2 and -1 respectively in ω, to obtain

$$(\lambda + \mu)\langle u^{(0)}, \nabla\iota\rangle_g \nabla\iota + (\mu\|\nabla\iota\|_g^2 - \rho)u^{(0)} = 0.$$

If we take

$$\|\nabla\iota\|_g^2 = \frac{\rho}{\mu}, \tag{9}$$

then

$$\langle u^{(0)}, \nabla\iota\rangle_g = 0.$$

The solutions $u_j^{(0)}$ represent shear waves (S-waves), and the displacement vector $u^{(0)}$ is orthogonal to $\nabla\iota$. We denote $n_s = \rho/\mu$ and $v_s = 1/n_s$. The characteristics of the eikonal equation (9) are geodesics of the Riemannian metric $n_s^2 ds^2 = n_s^2 g_{jk} dx^j dx^k$.

We choose a geodesic γ of metric $n_s^2 ds^2$ and apply the change of variables,

$$u_j^{(0)} = A_s n_s^{-1} \zeta_j,$$

where

$$A_s = \frac{C}{\sqrt{J\rho v_s}}, \quad J^2 = n_s^2 \det(g_{jk}), \quad C \text{ is a constant.}$$

Then it is shown in [15, Section 7.1.5.] that ζ satisfies the following *Rytov's law*

$$\left(\frac{D\zeta}{\mathrm{d}\iota}\right)_j = -i\frac{1}{\rho v_s^6}(\delta_j^q - \dot\gamma_j \dot\gamma^q)\omega_0 c_{qklm} \dot\gamma^k \dot\gamma^m \zeta^l, \tag{10}$$

where $\frac{D}{d\iota}$ is the covariant derivative along γ. We note that $c_{qklm}\dot{\gamma}^k\dot{\gamma}^m$ is quadratic in $\dot{\gamma}$, and symmetric in k, m, so the solution ζ of (10) depends only on the symmetrization

$$f_{jklm} = -i\frac{1}{4\rho v_s^6}(c_{jlkm} + c_{jmkl}).$$

We assume that for every unit speed geodesic $\gamma : [a, b] \to M$ (in Riemannian manifold $(M, n_s^2 ds^2)$) with endpoints in ∂M, the value $\zeta(b)$ of a solution to equation (10) is known as $\zeta(b) = U(\gamma)\zeta(a)$, where $U(\gamma)$ is the solution operator of (10) and $\eta(a)$ is the initial value. We formulate an inverse problem.

Inverse Problem 1. *Determine tensor field f from $U(\gamma)$.*

We linearize this problem as in [15, Chapter 5]. Take a unit vector $\xi(t) \perp \dot{\gamma}(t)$, which is also parallel along γ. Then $e_1(t) = \xi(t)$ and $e_2(t) = \dot{\gamma}(t)$ form an orthonormal frame along γ. In this basis, equation (10) is

$$\dot{\zeta}_1 = -i\frac{1}{\rho v_s^6}\omega_0 c_{1l1m}\dot{\gamma}^l\dot{\gamma}^m\zeta^1, \quad \dot{\zeta}_2 = 0. \tag{11}$$

We denote $F(t) = -i\frac{1}{\rho v_s^6}\omega_0 c_{1l1m}(\gamma(t))\dot{\gamma}^l(t)\dot{\gamma}^m(t)$. Since (11) is a separable first order ordinary differential equation, its solution is

$$\zeta_1(b) = e^{\int_a^b F(t)dt}\zeta_1(a).$$

We take the first-order Taylor expansion of the right-hand side of the equation above to obtain

$$\zeta_1(b) - \zeta_1(a) \sim \int_a^b F(t)\zeta^1(a)dt.$$

Multiplying this equation by $\eta^1(a)$, we get

$$(\zeta_1(b) - \zeta_1(a))\zeta^1(a) \sim \int_a^b F(t)\zeta^1(a)\zeta^1(a)dt = \int_a^b \omega_0 f_{11lm}(\gamma(t))\zeta^1(a)\zeta^1(a)\dot{\gamma}^l(t)\dot{\gamma}^m(t)dt. \tag{12}$$

We denote the vector field $\eta(t) = \zeta^i(a)e_i(t)$, $\zeta^2(a) = 0$, and observe that it is parallel along γ and perpendicular to $\dot{\gamma}(t)$. The right-hand side of (12) then takes the form

$$\int_a^b \omega_0 f_{11lm}(\gamma(t))\eta^1(t)\eta^1(t)\dot{\gamma}^l(t)\dot{\gamma}^m(t)dt,$$

We arrive at the inverse problem.

Inverse Problem 2. *Determine the tensor field f from*

$$L_{2,2}(f) = \int_a^b f_{jklm}(\gamma(t))\eta^j(t)\eta^k(t)\dot{\gamma}^l(t)\dot{\gamma}^m(t)dt$$

for all γ and $\eta \perp \gamma$, where η is parallel along γ.

Remark 1. *The tensor field f possesses the same symmetry properties* (8) *as* $\mathbf{C}$*. Therefore $f \in S^2M \times S^2M$. Since*

$$L_{2,2}(f + d'u + \lambda w) = L_{2,2}(f), \quad \text{for any } u \in S^1M \times S^2M,\ w \in S^1M \times S^1M,$$

we can only recover the tensor f up to the kernel of $L_{2,2}$. Thus the Inverse Problem 2 is a special case of Theorem 1.

3. Context and previous work

We note that if $\ell = 0$ in (1), the operator $L_{k,0}$ is the geodesic ray transform I_k for a symmetric k-tensor f. It is well known that $\mathrm{Sym}(i_1, \dots, i_k)\nabla u$ is in the kernel of I_k, where u is a symmetric $(k-1)$-tensor with $u|_{\partial\Omega} = 0$. If $I_k f = 0$ implies $f = \mathrm{Sym}(i_1, \dots, i_k)\nabla u$, we say I_k is s-injective.

When (M, g) is a 2-dimensional simple manifold, Paternain, Salo and Uhlmann [9] proved the s-injectivity of I_k for arbitrary k. The standard way to prove s-injectivity of I_0 and I_1 is to use an energy identity known as the Pestov identity. If $k \geq 2$ this identity alone is not sufficient to prove the s-injectivity. The special case $k = 2$ was proved earlier [14] using the proof for boundary rigidity [13].

In dimension three or higher, it has been proved that I_0 is injective [6, 7], and I_1 is s-injective [2]. The s-injectivity of I_k for $k \geq 2$ is still open for simple Riemannian manifolds. Under certain curvature conditions, the s-injectivity of I_k, $k \geq 2$ has been proved in [4, 11, 12, 15]. Without any curvature condition, it has been proved that I_2 has a finite-dimensional kernel [16]. If g is in a certain open and dense subset of simple metrics in $C^r, r \gg 1$, containing analytic metrics, the s-injectivity is proved by analytic microlocal analysis for $k = 2$ [17]. Under a different assumption that M can be foliated by strictly convex hypersurfaces, the s-injectivity has been established for $m = 0$ [20], and $m = 1, 2$ [18].

The mixed ray transform ($\ell \neq 0$, $k \neq 0$) is not studied as extensively as the geodesic ray transform. In dimension two or higher, a result similar to Theorem 1 has been obtained under a restrictive curvature condition [15].

When $k = 0$, $L_{0,\ell}$ is called the transverse ray transform, also denoted by J_ℓ. For J_ℓ, the situations are quite different for dimension two and higher dimensions. In dimension three or higher, J_ℓ is injective for $\ell < \dim M$ under certain curvature conditions [15]. However, J_ℓ has a nontrivial kernel in dimension 2. This problem is related to *polarization* tomography, for which some results are given under different conditions [5, 8, 10].

4. Proof of Theorem 1

Since (M, g) is a 2-dimensional simple Riemannian manifold, there exists a diffeomorphism ϕ from M onto a closed unit disc $\overline{\mathbb{D}}$ of $\mathbf{R}^2$. If g' is the pullback of metric g under ϕ^{-1} on $\overline{\mathbb{D}}$ then g' is conformally Euclidean, meaning that there exists a change of coordinates after which $g' = he$, where h is some positive function and e is the Euclidean metric; this was shown in [1, Theorem 4] and [19, Proposition 1.3]. Therefore there exists global isothermal coordinates (x_1, x_2) on M, so that the metric g can be written as $e^{2\alpha(x)}(\mathrm{d}x_1^2 + \mathrm{d}x_2^2)$ where $\alpha(x)$ is a smooth real-valued function of x.

The global isothermal coordinate structure makes it possible to define a smooth rotation,

$$\sigma : TM \to TM, \quad \sigma(v) := (v_2, -v_1),$$

where $v = (v_1, v_2)$ in these coordinates. This map satisfies

$$v \perp \sigma(v) \quad \text{and} \quad \|v\|_g = \|\sigma(v)\|_g. \tag{13}$$

Moreover, there exists a linear map

$$\Phi : S^k M \times S^\ell M \to C^\infty(SM), \quad (\Phi f)(x, v) := f_{i_1,\dots,i_k j_1,\dots,j_\ell}(x) v^{i_1} \cdots v^{i_k} \sigma(v)^{j_1} \cdots \sigma(v)^{j_\ell}. \tag{14}$$

Thus each tensor field $f \in S^k M \times S^\ell M$ is related to a smooth function on SM via (14). We note that Φ is not one-to-one since $\Phi(\lambda w) = 0$ for any $w \in S^{k-1} M \times S^{\ell-1} M$, where λ is as in (3). We have the following

Lemma 1. *For any $f \in S^kM \times S^\ell M$ it holds that*

$$L_{k,\ell}f(x,v) = \int_0^{\tau(x,v)} (\Phi f)(\gamma_{x,v}(t), \dot\gamma_{x,v}(t))\mathrm{d}t, \qquad (x,v) \in \partial_{in}(SM) \tag{15}$$

and

$$L_{k,\ell} : S^kM \times S^\ell M \to C^\infty(\partial_{in}SM),$$

if we assume that

$$\eta(0) = \sigma(v), \quad (x,v) \in \partial_{in}(SM).$$

Proof. Let $(x,v) \in \partial_{in}SM$. We define $\eta = \sigma(v)$. Let $P_t(\eta)$ be the parallel transport of η from T_xM to $T_{\gamma_{x,v}(t)}M$, $t \in [0, \tau(x,v)]$. By the property of parallel translation, $P_t : T_xM \to T_{\gamma_{x,v}(t)}M$ is an isometry, whence $\|P_t\eta\|_g = 1$ and $\langle P_t\eta, \dot\gamma(t)\rangle_g = 0$. Since M is 2-dimensional, the continuity of $P_t\eta$ in t with (13) imply

$$P_t\eta = \sigma(\dot\gamma_{x,v}(t)).$$

Because the functions Φf and τ are smooth in $\partial_{in}(SM)$, the function $L_{k,\ell}(f)$ is smooth in $\partial_{in}(SM)$ due to (15). □

Let $f \in S^kM \times S^\ell M$. Simplifying the notation, from here on we do not distinguish tensor f from function $\Phi(f)$. We notice first that

$$f(x,v) = (-1)^{\ell - N(j_1,\dots,j_\ell)} f_{i_1,\dots,i_k j_1,\dots,j_\ell}(x) v^{i_1} \cdots v^{i_k} v_1^{\ell - N(j_1,\dots,j_\ell)} v_2^{N(j_1,\dots,j_\ell)}, \quad (x,v) \in SM, \tag{16}$$

where $N(j_1,\dots,j_\ell)$ is the number of 1s in $(j_1,\dots,j_\ell)$. We let δ be the map that maps 1s in $(j_1,\dots,j_\ell)$ to 2s and vice versa. We denote by $\delta(j_1,\dots,j_\ell)$ the ℓ-tuple obtained from applying δ to $(j_1,\dots,j_\ell)$. Then we define a linear operator

$$A : S^kM \times S^\ell M \to S^kM \times S^\ell M, \quad (Af)_{i_1,\dots,i_k j_1,\dots,j_\ell} = (-1)^{\ell - N(j_1,\dots,j_\ell)} f_{i_1,\dots,i_k \delta(j_1,\dots,j_\ell)}. \tag{17}$$

We note that if $\ell = 1$, then A and the Hodge star operator coincide. Formula (17) implies that A is invertible with the following inverse

$$A^{-1} = (-1)^\ell A. \tag{18}$$

We then point out that

$$(Af)_{i_1,\dots,i_k j_1,\dots,j_\ell}(x) v^{i_1} \dots v^{i_k} v^{j_1} \cdots v^{j_\ell} = (\mathrm{Sym}Af)_{i_1,\dots i_k j_1,\dots,j_\ell}(x) v^{i_1} \dots v^{i_k} v^{j_1} \cdots v^{j_\ell}. \tag{19}$$

The notation $\mathrm{Sym}h$ stands for the full symmetrization of the tensor field h.

Using equations (16), (17) and (19), we find that

$$L_{k,\ell}(f) = I_{k+\ell}(\mathrm{Sym}(Af)), \tag{20}$$

where $I_{k+\ell}$ is the geodesic ray transform on symmetric tensor field $h \in S^{k+\ell}M$, defined by the formula

$$I_{k+\ell}(h)(x,v) = \int_0^{\tau(x,v)} h_{i_1,\dots,i_{k+\ell}}(\gamma_{x,v}(t)) \dot\gamma_{x,v}(t)^{i_1} \cdots \dot\gamma_{x,v}(t)^{i_{k+\ell}} \mathrm{d}t, \qquad (x,v) \in \partial_{in}(SM).$$

By (20) and [9, Theorem 1.1] it holds that for any $h \in S^kM \times S^\ell M$,

$$L_{k,\ell}(h) = 0 \text{ if and only if } \mathrm{Sym}Ah = d^s v, \quad v \in S^{k+\ell-1}M, \quad v|_{\partial M} = 0. \tag{21}$$

In the above, d^s stands for the inner derivative, that is, the symmetrization of the covariant derivative

$$d^s u = \mathrm{Sym}(\nabla u), \quad u \in S^{k+\ell-1}M. \tag{22}$$

If $L_{k,\ell}(f) = 0$ then, with (18) and (21), we can write

$$f = (-1)^\ell A(\mathrm{Sym}(Af) + (Af - \mathrm{Sym}(Af))) = (-1)^\ell A(d^s u) + f + (-1)^{\ell+1} A(\mathrm{Sym}(Af)).$$

We conclude that the claim of Theorem 1 holds if

$$f + (-1)^{\ell+1} A(\mathrm{Sym}(Af)) = \lambda w, \quad A(d^s u - d' u) = \lambda w', \quad d' A = A d',$$

for some $w, w' \in S^{k-1}M \times S^{\ell-1}M$ and $u \in S^{k+\ell-1}M$. These equations will be proved in the following subsections.

4.1. **Analysis of operator ASym$A$.** In this subsection, we prove the following identity for any $f \in S^k M \times S^\ell M$:

$$f + (-1)^{\ell+1} A(\mathrm{Sym}(Af)) = \lambda w \quad \text{for some } w \in S^{k-1}M \times S^{\ell-1}M. \tag{23}$$

We start with a lemma that characterizes the kernel of $A\mathrm{Sym}A$

Lemma 2. *For the linear maps $A\mathrm{Sym}A : S^k M \times S^\ell M \to S^k M \times S^\ell M$ and $\lambda : S^{k-1}M \times S^{\ell-1}M \to S^k M \times S^\ell M$ the following holds*

$$\ker(A\mathrm{Sym}A) = \mathrm{Im}(\lambda).$$

Proof. We use the notation $\otimes_s$ for the symmetric product of tensors. We note that operator A maps a basis element $\big((\otimes^h dx^1) \otimes_s (\otimes^{k-h} dx^2)\big) \otimes \big((\otimes^a dx^1) \otimes_s (\otimes^{\ell-a} dx^2)\big)$, $h \in \{0, \dots, k\}, a \in \{0, \dots, \ell\}$ of $S^k M \times S^\ell M$ to

$$(-1)^{\ell-a}\big((\otimes^h dx^1) \otimes_s (\otimes^{k-h} dx^2)\big) \otimes \big((\otimes^{\ell-a} dx^1) \otimes_s (\otimes^a dx^2)\big).$$

We also note that the choice of isothermal coordinates implies

$$\lambda(a \otimes b) = e^{2\alpha(x)}\big((dx^1 \otimes_s a) \otimes (dx^1 \otimes_s b) + (dx^2 \otimes_s a) \otimes (dx^2 \otimes_s b)\big), \quad a \otimes b \in S^{k-1}M \times S^{\ell-1}M. \tag{24}$$

Since A is a bijection, it suffices to prove

$$\mathrm{Im}(\lambda) = \ker(\mathrm{Sym}A). \tag{25}$$

We prove first that $\mathrm{Im}(\lambda) \subset \ker(\mathrm{Sym}A)$. In view of the linearity of λ, it suffices to prove that $\lambda w \in \ker \mathrm{Sym}A$ when

$$w = r(x)\big((\otimes^{h-1} dx^1) \otimes_s (\otimes^{k-h} dx^2)\big) \otimes \big((\otimes^{a-1} dx^1) \otimes_s (\otimes^{\ell-a} dx^2)\big), \quad h \in \{1, \dots, k\}, a \in \{1, \dots, \ell\}.$$

Then

$$\begin{aligned} e^{-2\alpha(x)} A\lambda w = (-1)^{\ell-a} r(x)\Bigg(&\big((\otimes^h dx^1) \otimes_s (\otimes^{k-h} dx^2)\big) \otimes \big((\otimes^{\ell-a} dx^1) \otimes_s (\otimes^a dx^2)\big) \\ &- \big((\otimes^{h-1} dx^1) \otimes_s (\otimes^{k-h+1} dx^2)\big) \otimes \big((\otimes^{\ell-a+1} dx^1) \otimes_s (\otimes^{a-1} dx^2)\big)\Bigg). \end{aligned} \tag{26}$$

Since Sym is a linear operator, we have $\mathrm{Sym}A(\lambda w) = 0$. Therefore $\mathrm{Im}(\lambda) \subset \ker(\mathrm{Sym}A)$

Now we prove that $\ker(\mathrm{Sym}A) \subset \mathrm{Im}(\lambda)$. We assume first that $f = \sum_{m=1}^M u_m$, where

$$u_m = r_m(x)\big((\otimes^h dx^1) \otimes_s (\otimes^{k-h} dx^2)\big) \otimes \big((\otimes^{\ell-a} dx^1) \otimes_s (\otimes^a dx^2)\big), \quad h + a \le \min\{k, \ell\}. \tag{27}$$

Then we can write $f = \sum_{H=0}^{k+\ell} f_H$, where $f_H = 0$, if $H \ge \min\{k, \ell\}$ and otherwise

$$f_H = \sum_{h=0}^{H} a_{H,h} f_{H,h}, \quad f_{H,h} := \big((\otimes^h dx^1) \otimes_s (\otimes^{k-h} dx^2)\big) \otimes \big((\otimes^{\ell-(H-h)} dx^1) \otimes_s (\otimes^{H-h} dx^2)\big).$$

Moreover $f \in \ker(\mathrm{Sym}A)$ if and only if $f_H \in \ker(\mathrm{Sym}A)$ for every $H \in \{1,\dots,\min\{k,\ell\}\}$. In the following we study the tensor f_H, for a given $H \in \{1,\dots,\min\{k,\ell\}\}$.

For $h \in \{1,\dots,H\}$ we define $w_h \in S^{k-1}M \times S^{\ell-1}M$ by formula

$$w_h = \big((\otimes^{h-1} dx^1) \otimes_s (\otimes^{k-h} dx^2)\big) \otimes \big((\otimes^{\ell-(H-h+1)} dx^1) \otimes_s (\otimes^{H-h} dx^2)\big).$$

Then (24) yields

$$\lambda w_h = e^{2\alpha(x)}(f_{H,h} + f_{H,h-1}).$$

This implies the recursive formula

$$f_{H,h} = \lambda(e^{-2\alpha(x)} w_h) - f_{H,h-1}.$$

Thus for every $h \in \{0,\dots,H\}$ there exists $w'_h \in S^{k-1}M \times S^{\ell-1}M$ such that

$$f_{H,h} = \lambda w'_h + (-1)^h f_{H,0}. \tag{28}$$

Therefore there exists $w_H \in S^{k-1}M \times S^{\ell-1}M$ such that

$$f_H = \sum_{h=0}^{H} a_{H,h} f_{H,h} = \lambda w_H + f_{H,0} \sum_{h=0}^{H} (-1)^h a_{H,h}.$$

If $f \in \ker \mathrm{Sym}A$ it holds by the first part of this proof that

$$\mathrm{Sym}A f_H = (\mathrm{Sym}A f_{H,0}) \bigg(\sum_{i=0}^{H} (-1)^h a_{H,h} \bigg) = 0.$$

Since $\mathrm{Sym}A f_{H,0} \neq 0$ it follows that $\sum_{i=0}^{H} (-1)^h a_{H,h} = 0$ whence $f_H = \lambda w_H$. This implies $f = \lambda w$ for some $w \in S^{k-1}M \times S^{\ell-1}M$.

If $f \in \ker \mathrm{Sym}A$ and we cannot write $f = \sum_{m=1}^{M} u_m$, where each u_m satisfies (27), then there exists u_m that satisfies

$$(\otimes^h dx^1) \otimes_s (\otimes^{k-h} dx^2)\big) \otimes \big((\otimes^{\ell-a} dx^1) \otimes_s (\otimes^a dx^2), \quad \min\{k,\ell\} < h + a \leq \max\{k,\ell\}.$$

Therefore $f_H \neq 0$ for some $\min\{k,\ell\} < H \leq \max\{k,\ell\}$ and there exist two sub cases. If $k < H \leq \ell$, then

$$f_H = \sum_{h=0}^{k} a_{H,h} f_{H,h}, \quad f_{H,h} = \big((\otimes^h dx^1) \otimes_s (\otimes^{k-h} dx^2)\big) \otimes \big((\otimes^{\ell-(H-h)} dx^1) \otimes_s (\otimes^{H-h} dx^2)\big).$$

If $\ell < H \leq k$, then

$$f_H = \sum_{h=0}^{\ell} a_{H,h} f_{H,h}, \quad f_{H,h} = \big((\otimes^{H-\ell+h} dx^1) \otimes_s (\otimes^{k-h-H+\ell} dx^2)\big) \otimes \big((\otimes^h dx^1) \otimes_s (\otimes^{\ell-h} dx^2)\big).$$

By an analogous recursive argument as before, we find that $f = \lambda w$, for some $w \in S^{k-1}M \times S^{\ell-1}M$. This completes the proof. □

By the proof of the previous Lemma we can write any $f \in S^k M \times S^\ell M$ in the form

$$f = \lambda w + \sum_{H=0}^{k+\ell} r_H f_{H,0}, \quad r_H \in C^\infty(M), \tag{29}$$

for some $w \in S^{k-1}M \times S^{\ell-1}M$. Next, we prove that

$$A\mathrm{Sym}A f_{H,0} = (-1)^\ell f_{H,0} + \lambda w, \quad H \in \{1,\dots,k+\ell\}. \tag{30}$$

We assume first that $H \leq \min\{k,\ell\}$. Then

$$f_{H,0} = \big(\otimes^k dx^2\big) \otimes \big((\otimes^{\ell-H} dx^1) \otimes_s (\otimes^H dx^2)\big).$$

This implies

$$\begin{aligned}\mathrm{Sym} A f_{H,0} =& (-1)^\ell (\otimes^H dx^1 \otimes_s (\otimes^{k+\ell-H} dx^2))\\ =& (-1)^\ell \frac{1}{(k+\ell)!} \sum_{h=0}^{H} A_h (\otimes^h dx^1 \otimes_s (\otimes^{k-h} dx^2)) \otimes (\otimes^{H-h} dx^1 \otimes_s (\otimes^{\ell-H+h} dx^2)),\end{aligned}$$

where $\sum_{h=0}^H A_h = (k+\ell)!$. Using (28) we obtain

$$\begin{aligned}A\mathrm{Sym} A f_{H,0} =& (-1)^\ell \frac{1}{(k+\ell)!} \sum_{h=0}^{H} (-1)^h A_h f_{H,h} = (-1)^\ell \frac{1}{(k+\ell)!} \left(\sum_{h=0}^{H} A_h\right) f_{H,0} + \lambda w\\ =& (-1)^\ell f_{H,0} + \lambda w.\end{aligned}$$

If $\min\{k,\ell\} < H \leq \max\{k,\ell\}$ it follows by a similar argument that $A\mathrm{Sym}Af_{H,0} = (-1)^\ell f_{H,0} + \lambda w$. Therefore, we proved (30).

Equation (23) follows from Lemma 2 and (29)–(30).

4.2. **Analysis of operator** Ad^s**.** We note that $S^{k+\ell}M \subset S^k M \times S^\ell M$. Therefore, we can extend the inner derivative, d^s, to an operator $d^s : S^{k-1}M \times S^\ell M \to S^k M \times S^\ell M$ and evaluate $d^s - d'$. In this subsection, we show that for any $u \in S^{k-1}M \times S^\ell M$ the following equations hold,

$$A(d^s u - d'u) = \lambda w \quad \text{for some } w \in S^{k-1}M \times S^{\ell-1}M; \tag{31}$$

$$d'A = Ad'. \tag{32}$$

Since Ad^s and Ad' are linear it suffices to prove the claims for

$$u = r(x)\big((\otimes^{h-1} dx^1) \otimes_s (\otimes^{k-h} dx^2)\big) \otimes \big((\otimes^a dx^1) \otimes_s (\otimes^{\ell-a} dx^2)\big), \quad r \in C^\infty(M).$$

By (5) and (17) we have

$$\begin{aligned}Ad'u = (-1)^{\ell-a}\Bigg(&\left(\frac{\partial}{\partial x^1} r(x) - R_1\right)\big((\otimes^h dx^1) \otimes_s (\otimes^{k-h} dx^2)\big) \otimes \big((\otimes^{\ell-a} dx^1) \otimes_s (\otimes^a dx^2)\big)\\ &+ \left(\frac{\partial}{\partial x^2} r(x) - R_2\right)\big((\otimes^{h-1} dx^1) \otimes_s (\otimes^{k-h+1} dx^2)\big) \otimes \big((\otimes^{\ell-a} dx^1) \otimes_s (\otimes^a dx^2)\big)\Bigg),\end{aligned} \tag{33}$$

where $R_m = \sum_{s=1}^{k+\ell-1} r_{i_1,\dots,i_{s-1}p,i_{s+1},\dots,i_{k+\ell}} \Gamma^p_{mi_s}$, $m \in \{1,2\}$ and $r_{i_1,\dots,i_{s-1}p,i_{s+1},\dots,i_{k+\ell}} \in \{0,r\}$ depending on $(i_1,\dots,i_{k+\ell})$.

We write $H = h + a$, assume that $H \leq \min\{k,\ell\}$ and denote $\widetilde{R}_m = \frac{\partial}{\partial x^m} r(x) - R_m$. Then we obtain from (17) and (22),

$$\begin{aligned}d^s u =& \widetilde{R}_1 \frac{1}{(k+\ell)!} \sum_{j=0}^{H} A_j \big((\otimes^j dx^1) \otimes_s (\otimes^{k-j} dx^2)\big) \otimes \big((\otimes^{H-j} dx^1) \otimes_s (\otimes^{\ell+j-H} dx^2)\big)\\ &+ \widetilde{R}_2 \frac{1}{(k+\ell)!} \sum_{i=0}^{H-1} B_i \big((\otimes^i dx^1) \otimes_s (\otimes^{k-i} dx^2)\big) \otimes \big((\otimes^{H-i-1} dx^1) \otimes_s (\otimes^{\ell+i-H+1} dx^2)\big),\end{aligned}$$

where $\sum_{j=0}^{H} A_j = \sum B_{i=0}^{H-1} = (k+\ell)!$. This yields

(34)

$$\begin{aligned} Ad^s u = \widetilde{R}_1 \frac{1}{(k+\ell)!} \sum_{j=0}^{H} (-1)^{\ell-H+j} A_j \big((\otimes^j dx^1) \otimes_s (\otimes^{k-j} dx^2)\big) \otimes \big((\otimes^{\ell+j-H} dx^1) \otimes_s (\otimes^{H-j} dx^2)\big) \\ - \widetilde{R}_2 \frac{1}{(k+\ell)!} \sum_{i=0}^{H-1} (-1)^{\ell-H+i} B_i \big((\otimes^i dx^1) \otimes_s (\otimes^{k-i} dx^2)\big) \otimes \big((\otimes^{\ell+i-H+1} dx^1) \otimes_s (\otimes^{H-i-1} dx^2)\big). \end{aligned}$$

We define

$$g_{H,j} = \big((\otimes^j dx^1) \otimes_s (\otimes^{k-j} dx^2)\big) \otimes \big((\otimes^{\ell+j-H} dx^1) \otimes_s (\otimes^{H-j} dx^2)\big), \quad j \in \{0, \dots, H\},$$

and

$$v_{H,j} = \big((\otimes^j dx^1) \otimes_s (\otimes^{k-j-1} dx^2)\big) \otimes \big((\otimes^{\ell+j-H} dx^1) \otimes_s (\otimes^{H-j-1} dx^2)\big), \quad j \in \{1, \dots, H\}.$$

Then (24) implies that $\lambda v_{H,j} = e^{2\alpha(x)}(g_{H,j} + g_{H,j+1})$. We obtain

$$g_{H,j} = \lambda w_{H,j} + (-1)^{H-j} g_{H,H}, \quad \text{for some } w_j \in S^{k-1}M \times S^{\ell-1}M.$$

Thus

$$\begin{aligned} dA'u =& (-1)^{\ell-a}\Big(\widetilde{R}_1 g_{H,h} + \widetilde{R}_2 g_{H-1,h-1}\Big) \\ =& (-1)^{\ell}\Big(\widetilde{R}_1 g_{H,H} + \widetilde{R}_2 g_{H-1,H-1}\Big) + \lambda w', \quad \text{for some } w' \in S^{k-1}M \times S^{\ell-1}M \end{aligned}$$

and

$$\begin{aligned} Ad^s u =& \widetilde{R}_1 \frac{1}{(k+\ell)!} \sum_{j=0}^{H} (-1)^{\ell-H+j} A_j (\lambda w_{j,H} + (-1)^{H-j} g_{H,H}) \\ &+ \widetilde{R}_2 \frac{1}{(k+\ell)!} \sum_{i=0}^{H-1} (-1)^{\ell-H+i+1} B_i (\lambda w_{i,H-1} + (-1)^{H-1-i} g_{H-1,H-1}) \\ =& (-1)^{\ell}\Big(\widetilde{R}_1 g_{H,H} + \widetilde{R}_2 g_{H-1,H-1}\Big) + \lambda w'', \quad \text{for some } w'' \in S^{k-1}M \times S^{\ell-1}M. \end{aligned}$$

These identities imply

$$A(d^s u - d'u) = \lambda w, \quad w \in S^{k-1}M \times S^{\ell-1}M.$$

For the case $\min\{k,\ell\} < H \le \max\{k,\ell\}$, the proof is similar and is omitted. Therefore have proved (31).

Finally we prove equation (32). We note that

$$\begin{aligned} d'Au = \ (-1)^{\ell-a}\Big(& \widetilde{R}_1 \big((\otimes^h dx^1) \otimes_s (\otimes^{k-h} dx^2)\big) \otimes \big((\otimes^{\ell-a} dx^1) \otimes_s (\otimes^a dx^2)\big) \\ &+ \widetilde{R}_2 \big((\otimes^{h-1} dx^1) \otimes_s (\otimes^{k-h+1} dx^2)\big) \otimes \big((\otimes^{\ell-a} dx^1) \otimes_s (\otimes^a dx^2)\big)\Big). \end{aligned}$$

Thus (32) holds since the previous equation coincides with (33).

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