Convexity properties related to extremal functions

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Abstract. In 1962 Józef Siciak introduced in Transactions of the AMS [Si1] his famous polynomial extremal function, which was intensively investigated and applied in pluripotential theory and polynomial approximations related to the Chebyshev norm on the compacts in \mathbb{C}^N . In particular, starting from middle seventies the Siciak extremal function was one of the most important tool to investigate the behavior of derivatives of polynomials. The pioneer was Wiesław Pleśniak in his researches of quasianalytic functions in the sense of Bernstein. In the circle of papers (most important joint with Wiesław Pawłucki) there were shawn deep connections between behavior Siciak extremal function near compact K and bounds for derivatives of polynomials. In particular, in 1990 W. Pleśniak [Pl1] introduced condition (P) which is equivalent to Markov property of compact K. In the same paper there was stated a problem which property of Siciak's extremal function are necessary to Markov's property. In particular, thus Markov sets are non pluripolar that is Siciak's extremal function is finite at every point. Much more stronger question is on Hölder continuity of the logarithm of the Siciak extremal function, which plays a role of the pluricomplex Green function (see [K] for excelent presentation). This problem can be formulate in more general case of arbitrary norms q on the space of polynomials. In the present paper we, continuing our earlier researches, investigate the connection between behavior of generalizations of Siciak's function and the behavior of norms of derivatives of polynomials. In particular we get some deep properties of Markov factors $M_n(q,k)$ related to the main problems. One of the main result is the Kolmogorov-Landau type property of $M_n(q,k)^{1/k}$ which is a condition on the triangle sequence of family of derivatives of polynomials not for particular polynomials as for direct analogons of the Kolmogorov-Landau remarkable inequalities: $\log M_n(q,k)^{1/k} \leq \log const. + (1 - \frac{\log k}{\log n}) \log M_n(q,1) + \frac{\log k}{\log n} M_n(q,n)^{1/n}, \ 1 \leq k \leq n.$ It seems that this condition is satisfied for arbitrary norm q. Separately this condition (a weaker version is sufficient) gives nothing. But if we assume that q has A. Markov's property with respect to q and satisfies a condition C(q) > 0 then q possesses Vladimir Markov property. In the case $q(P) = ||P||_E$ this means that non pluripolar Markov sets possese Hölder continuous pluricomplex Green function (in the one dimensional case Markov sets are not polar [B-C]). This is presented in last section. Earlier we investigate a number of extremal functions, between them related to Pleśniak condition and to V. Markov's property. We shall consider mainly one dimensional case, but there is no problem to generalize for many variables.

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1. Introduction.

The vector space of polynomials of N variables with complex coefficients we shall denote by $\mathbb{P}(\mathbb{C}^N)$. Then $\mathbb{P}_n(\mathbb{C}^N) = \{P \in \mathbb{P}(\mathbb{C}^N) : \deg P \leq n\}$. If we consider a norm q(P) = ||P|| in $\mathbb{P}(\mathbb{C}^N)$ we shall get a normed space $X_q = (\mathbb{P}(\mathbb{C}^N), q)$ and finite dimensional spaces $X_{q,n} = (\mathbb{P}_n(\mathbb{C}^N), q)$ with the dual $X_{q,n}^*$. Thus, as it is well known, $q(P) = \sup\{|\Lambda(P)| : \Lambda \in X_{q,n}^*, ||\Lambda||^* = 1\}$.

A main motivation of this paper and a lot of earlier researches is to get bounds of partial derivatives of polynomials in spaces $X_{q,n}$ and to investigate them. We can consider a bound for $|\Lambda(D^{\alpha}P)|$, where $\Lambda \in X_{q,n}^*$ or a supremum $|\Lambda(D^{\alpha}P)|$, $\Lambda \in \mathcal{A} \subset X_{q,n}^*$, where \mathcal{A} is a bounded set. In particular we shall consider $||D^{\alpha}P||$.

A basic observation is an obvious fact

$$P(z+\zeta) = \sum_{|\alpha| < \deg P} \frac{1}{\alpha!} D^{\alpha} P(z) \zeta^{\alpha},$$

where as usually $\alpha! = \alpha_1! \cdots \alpha_N!, \ \zeta^{\alpha} = \zeta_1^{\alpha_1} \cdots \zeta_N^{\alpha_N}.$

Next step is a choice of a norm in \mathbb{C}^N , consider the unit ball \mathbb{B} with respect to this norm and next we can take a Borel probabilistic measure which is supported on $\partial_s \mathbb{B}$ the Shilov boundary of \mathbb{B} . Actually there is well known that the complex equilibrum measure $\mu_{\mathbb{B}} = (2\pi)^{-N} (dd^c V_{\mathbb{B}})^N$ has this property.

We shall present a few examples. To do this let us recall some standard notations. The unit disk in \mathbb{C} is \mathbb{D} , the unit ciricle is \mathbb{T} , \mathbb{D}_N is the polidisk in \mathbb{C}^N , while \mathbb{T}^N is the N dimensional tori, which is equal $\mathbb{T}^N = \text{extr}(\mathbb{D}_N)$. By \mathbb{B}_N is denoted the unit Euclidean ball (with respect to the standard inner product), $\mathbb{S}_N = \partial \mathbb{B}_N = \text{extr}(\mathbb{B}_N)$.

Example 1.1.
$$||z|| = ||z||_{\infty} = \max(|z_1|, \dots, |z_N|), \ \mathbb{B} = \mathbb{D}_N, \ \partial_s \mathbb{B} = \partial_s \mathbb{D}_N = \mathbb{T}^N,$$

 $\mu_{\partial_s \mathbb{B}} = d\sigma_1 \cdots d\sigma_N,$

where $d\sigma_i$ is the normalized arclength mesure on \mathbb{T} , that is

$$\int_{\mathbb{C}^N} \varphi(z) d\mu_{\partial_s \mathbb{B}}(z) = \left(\frac{1}{2\pi}\right)^N \int_{[0,2\pi]^N} \varphi(e^{i\theta_1}, \dots, e^{i\theta_N}) d\theta_1 \dots d\theta_N.$$

Now we have (equivalently Cauchy integral formula can be used)

$$\left(\frac{1}{2\pi}\right)^N \int_{[0,2\pi]^N} P(z + (r_1 e^{i\theta_1}, \dots r_N e^{i\theta_N})) e^{-i\theta \cdot \alpha} d\theta = r_1^{\alpha_1} \cdots r_N^{\alpha_N} \frac{1}{\alpha!} D^{\alpha} P(z).$$

Hence

$$D^{\alpha}P(z) = \alpha! r_1^{-\alpha_1} \cdots r_N^{-\alpha_N} \left(\frac{1}{2\pi}\right)^N \int_{[0,2\pi]^N} P(z + (r_1 e^{i\theta_1}, \dots r_N e^{i\theta_N})) e^{-i\theta \cdot \alpha} d\theta,$$

$$|\Lambda(D^{\alpha}P(z))| \le \alpha! r_1^{-\alpha_1} \cdots r_N^{-\alpha_N} \left(\frac{1}{2\pi}\right)^N \int_{[0,2\pi]^N} |\Lambda(P(z + (r_1 e^{i\theta_1}, \dots r_N e^{i\theta_N})))| d\theta,$$

$$||D^{\alpha}P(z)|| \leq \alpha! r_1^{-\alpha_1} \cdots r_N^{-\alpha_N} \left(\frac{1}{2\pi}\right)^N \int_{[0,2\pi]^N} ||P(z + (r_1 e^{i\theta_1}, \dots r_N e^{i\theta_N}))|| d\theta,$$

$$||D^{\alpha}P(z)|| \leq \alpha! r_1^{-\alpha_1} \cdots r_N^{-\alpha_N} \left(\left(\frac{1}{2\pi}\right)^N \int_{[0,2\pi]^N} ||P(z + (r_1 e^{i\theta_1}, \dots r_N e^{i\theta_N}))||^p d\theta\right)^{1/p}, \ p \geq 1.$$

In particular

$$||D^{\alpha}P(z)|| \leq \alpha |r_1^{-\alpha_1} \cdots r_N^{-\alpha_N} \max_{\theta \in [0,2\pi]^n} ||P(z + (r_1 e^{i\theta_1}, \dots r_N e^{i\theta_N}))||$$

$$\leq \alpha |r_1^{-\alpha_1} \cdots r_N^{-\alpha_N} \varphi_n(q, (r_1, \dots, r_N))||P(z)||,$$

where

$$\varphi_n(q, (r_1, \dots, r_N)) := \sup\{||P(z + \zeta)|| : ||\zeta_1| \le r_1, \dots, ||\zeta_N|| \le r_N, \deg P \le n, ||P(z)|| \le 1\}$$

$$= \sup\{|\Lambda(P(z + \zeta))| : \Lambda \in X_{q,n}^*, ||\Lambda||^* = 1, ||\zeta_1|| \le r_1, \dots, ||\zeta_N|| \le r_N, \deg P \le n, ||P(z)|| \le 1\}.$$
Simirally, if $\Lambda \in X_{q,n}^*, ||\Lambda||^* = 1$ then we put

$$\varphi_n(q, \Lambda, (r_1, \dots, r_n)) := \sup\{|\Lambda(P(z+\zeta))| : |\zeta_1| \le r_1, \dots, |\zeta_N| \le r_N, \deg P \le n, ||P(z)|| \le 1\}.$$

Thus

$$\varphi_n(q, (r_1, \dots, r_N)) = \sup \{ \varphi_n(q, \Lambda, (r_1, \dots, r_n)) : \Lambda \in X_{q,n}^*, ||\Lambda||^* = 1 \}.$$

Finally, if $1 \leq p \leq \infty$ then we define $\varphi_n(q, p, \Lambda, (r_1, \dots, r_n))$ by

$$\sup \left(\left(\frac{1}{2\pi} \right)^N \int_{[0,2\pi]^N} |\Lambda(P(z+(r_1e^{i\theta_1},\ldots,r_Ne^{i\theta_N}))|^p d\theta \right)^{1/p}$$

and $\varphi_n(q, p, (r_1, \ldots, r_n))$ to being equal to

$$\sup_{\||\Lambda|\|^*=1} \left(\left(\frac{1}{2\pi} \right)^N \int_{[0,2\pi]^N} |\Lambda(P(z + (r_1 e^{i\theta_1}, \dots, r_N e^{i\theta_N}))|^p d\theta \right)^{1/p}$$

$$\leq \sup \left(\left(\frac{1}{2\pi} \right)^N \int_{[0,2\pi]^N} ||P(z + (r_1 e^{i\theta_1}, \dots, r_N e^{i\theta_N}))|^p d\theta \right)^{1/p},$$

where in both cases the supremum is taken over all polynomials P with $1 \le \deg P \le n, ||P(z)|| \le 1$. A specially important is the case p = 2.

Example 1.2. Now let
$$||z|| = ||z||_2 = (|z_1|^2 + \cdots + |z_N|^N)^{1/2}$$
, $\mathbb{B} = \mathbb{B}_N$, $\partial_s \mathbb{B} = \mathbb{S}_N = S^{2N-1}$, $B_N = \{x \in \mathbb{R}^N : ||x||_2 \le 1\} = \mathbb{B}_N \cap \mathbb{R}^N$, $B_N^+ = \{z \in B_N : x_j \ge 0, j = 1, \dots, N\}$. $\mu_{\mathbb{B}_N} = d\sigma$,

where $d\sigma$ is the normalized surfaces mesure on \mathbb{S}_N ($|\mathbb{S}_N| = |S^{2N-1}| = 2\pi^N/(N-1)!$), that is

$$\int_{\mathbb{C}^N} \varphi(z) d\mu_{\mathbb{B}_N}(z) = \frac{(N-1)!}{2\pi^N} \int_{[0,2\pi]^N} \int_{S_+^{N-1}} \varphi(\rho_1 e^{i\theta_1}, \dots, \rho_N e^{i\theta_N}) \rho_1 \cdots \rho_N d\rho d\theta_1 \dots d\theta_N.$$

Here $S_+^{N-1} = \{x \in S^{N-1} : x_j \ge 0, j = 1, ..., N\}, d\rho$ is the standard surface measure on S^{N-1} . Let us recall that

$$\int_{S_{+}^{N-1}} f(\rho_{1}, \dots, \rho_{N}) d\rho = \int_{B_{N}^{+}} \frac{f(\rho_{1}, \dots, \rho_{N-1}, \sqrt{1 - \rho_{1}^{2} - \dots - \rho_{N-1}^{2}})}{\sqrt{1 - \rho_{1}^{2} - \dots - \rho_{N-1}^{2}}} d\rho_{1} \dots d\rho_{N-1}.$$

Now we can write

$$\frac{1}{|S^{2N-1}|} \int_{S^{2N-1}} P(z+r\eta)(\eta_1/|\eta_1|)^{-\alpha_1} \cdots (\eta_N/|\eta_N|)^{-\alpha_N} d\sigma(\eta) =
\frac{1}{|S^{2N-1}|} \int_{[0,2\pi]^N} \int_{B_{N-1}^+} P(z+r(\rho_1 e^{i\theta_1}, \dots, \rho_{N-1} e^{i\theta_{N-1}}, (1-\rho_1^2 - \dots \rho_{N-1}^2)^{1/2} e^{i\theta_N}))
\rho_1 \cdots \rho_{N-1} e^{-i\theta \cdot \alpha} d\rho_1 \cdots \rho_{N-1} d\theta_1 \cdots d\theta_N
= \frac{1}{2} r^{|\alpha|} \frac{1}{\binom{N-1+|\alpha|/2}{N-1}} \frac{\Gamma(\alpha_1/2) \cdots \Gamma(\alpha_N/2)}{\Gamma(|\alpha|/2)} D^{\alpha} P(z),$$

where

$$\chi_{\alpha}(\rho_1, \dots, \rho_{N-1}) = \rho_1^{\alpha_1+1} \cdots \rho_{N-1}^{\alpha_{N-1}+1} (1 - \rho_1^2 - \dots - \rho_{N-1}^2)^{\alpha_N/2}.$$

Hence

$$D^{\alpha}P(z) = 2r^{-|\alpha|} \binom{N-1+|\alpha|/2}{N-1} \frac{\Gamma(|\alpha|/2)}{\Gamma(\alpha_1/2)\cdots\Gamma(\alpha_N/2)} \cdot \frac{1}{|S^{2N-1}|} \int_{S^{2N-1}} P(z+r\eta)(\eta_1/|\eta_1|)^{-\alpha_1} \cdots (\eta_N/|\eta_N|)^{-\alpha_N} d\sigma(\eta)$$

and

$$\begin{split} |\Lambda(D^{\alpha}P(z))| &\leq 2r^{-|\alpha|} \binom{N-1+|\alpha|/2}{N-1} \frac{\Gamma(|\alpha|/2)}{\Gamma(\alpha_{1}/2)\cdots\Gamma(\alpha_{N}/2)} \frac{1}{|S^{2N-1}|} \int_{S^{2N-1}} |\Lambda(P(z+r\eta))| d\sigma(\eta) \\ &\leq 2r^{-|\alpha|} \binom{N-1+|\alpha|/2}{N-1} \frac{\Gamma(|\alpha|/2)}{\Gamma(\alpha_{1}/2)\cdots\Gamma(\alpha_{N}/2)} \varphi_{n}(\mathbb{B}_{N},q,\Lambda,r), \\ \|D^{\alpha}P(z)\| &\leq 2r^{-|\alpha|} \binom{N-1+|\alpha|/2}{N-1} \frac{\Gamma(|\alpha|/2)}{\Gamma(\alpha_{1}/2)\cdots\Gamma(\alpha_{N}/2)} \frac{1}{|S^{2N-1}|} \int_{S^{2N-1}} ||P(z+r\eta)|| d\sigma(\eta), \\ &\leq 2r^{-|\alpha|} \binom{N-1+|\alpha|/2}{N-1} \frac{\Gamma(|\alpha|/2)}{\Gamma(\alpha_{1}/2)\cdots\Gamma(\alpha_{N}/2)} \varphi_{n}(\mathbb{B}_{N},q,r), \end{split}$$

where

$$\varphi_n(\mathbb{B}_N, q, r) := \sup\{||P(z + \zeta)|| : \zeta \in r\mathbb{B}_N, \deg P \le n, ||P(z)|| \le 1\},$$

$$\varphi_n(\mathbb{B}_N, q, \Lambda, r) := \sup\{|\Lambda(P(z + \zeta))| : \zeta \in r\mathbb{B}_N, \deg P \le n, ||P(z)|| \le 1\}.$$

Remark 1.3. If q is the supremum norm with respect to a compact $K \subset \mathbb{C}^N$ then in definitions of φ_n we shall replace q by K and if $z_0 \in K$, $\Lambda(P(z)) = P(z_0)$, then we shall replace Λ by z_0 .

Example 1.4.

Let
$$K = \mathbb{D}, z_0 \in \mathbb{T}, \ \Lambda(P(z)) = P(z_0), \ ||P||_{\mathbb{D}} = 1, \ 1 \leq \deg P \leq n.$$
 Then

$$\int_{0}^{2\pi} |P(z+re^{i\theta})|^{2} \frac{d\theta}{2\pi} = \sum_{k=1}^{n} \left(\frac{1}{k!}\right)^{2} |P^{(k)}(z_{0})|^{2} r^{2k} \le \sum_{k=0}^{n} \left(\frac{1}{k!}\right)^{2} (n(n-1)\cdots(n-k+1))^{2} r^{2k}$$

$$= \sum_{k=0}^{n} \binom{n}{k}^{2} r^{2k}$$

with equality if $P(z) = z^n$. (Here we use Bernstein inequality for derivative of polynomials on the unit circle). Hence

$$\varphi_n(\mathbb{D}, z_0, 2, r) = \left(\sum_{k=0}^n \binom{n}{k}^2 r^{2k}\right)^{1/2}$$

and

$$\varphi_n(\mathbb{D}, \mathbb{D}, 2, r) = \left(\sum_{k=0}^n \binom{n}{k}^2 r^{2k}\right)^{1/2} \le (1+r)^n.$$

Moreover,

$$\inf_{r>0} r^{-l} \varphi_n(\mathbb{D}, \mathbb{D}, 2, r) \le \left(1/\binom{n}{l}^{1/l} \right)^{-l} \left(\sum_{k=0}^n \binom{n}{k}^2 (1/\binom{n}{l}^{1/l})^{2k} \right)^{1/2}$$

$$\le \binom{n}{l} \left(1 + 1/\binom{n}{l}^{1/l} \right)^n \le \binom{n}{l} (1 + l/n)^n \le e^l \binom{n}{l}.$$

Therefore

$$\binom{n}{l} \le \inf_{r>0} r^{-l} \varphi_n(\mathbb{D}, \mathbb{D}, 2, r) \le e^l \binom{n}{l}.$$

Example 1.5. Let $K = [-1, 1], z_0 \in [-1, 1], \Lambda(P(z)) = P(z_0), ||P||_{[-1, 1]} = 1, 1 \le \deg P \le n.$

$$\int_0^{2\pi} |P(z_0 + re^{i\theta})|^2 \frac{d\theta}{2\pi} = \sum_{k=1}^n \left(\frac{1}{k!}\right)^2 |P^k(z_0)|^2 r^{2k} \le \sum_{k=0}^n \left(\frac{1}{k!}\right)^2 T_n^{(k)}(1)^2 r^{2k} = \sum_{k=0}^n (T_n^{(k)}(1)/k!)^2 r^{2k}.$$

Hence

$$\sup \left\{ \int_0^{2\pi} |P(z_0 + re^{i\theta})|^2 \frac{d\theta}{2\pi}, \ z_0 \in [-1, 1] \right\} \le \sum_{k=0}^n (T_n^{(k)}(1)/k!)^2 r^{2k}$$

with equality for $P(z) = T_n(z)$. This gives equality

$$\varphi_n(\mathbb{D}, [-1, 1], 2, r) = \left(\sum_{k=0}^n (T_n^{(k)}(1)/k!)^2 r^{2k}\right)^{1/2} \le T_n(1+r).$$

Moreover,

$$\inf_{r>0} r^{-l} \varphi_n(\mathbb{D}, [-1, 1], 2, r)
\leq \left(1/(T_n^{(l)}(1)/l!)^{1/l}\right)^{-l} \left(\sum_{k=0}^n (T_n^{(k)}(1)/k!)^2 (1/(T_n^{(l)}(1)/l!)^{1/l})^{2k}\right)^{1/2}
\leq \frac{T_n^{(l)}(1)}{n} T_n \left(1 + (1/(T_n^{(l)}(1)/l!)^{1/l}\right) = \frac{T_n^{(l)}(1)}{n} g\left(h\left(1 + (1/(T_n^{(l)}(1)/l!)^{1/l}\right)^n\right)$$

$$= \frac{T_n^{(l)}(1)}{l!} g\left(h^n \left(1 + \left(1/n2^l \binom{n+l-1}{2l}\right)^{1/l}\right)\right)$$

$$\leq \frac{T_n^{(l)}(1)}{l!} \left(1 + (\sqrt{2} + \sqrt{6})\frac{l}{n+2l-1}\right)^n \leq e^{(\sqrt{2} + \sqrt{6})l} \frac{T_n^{(l)}(1)}{l!}.$$

Therefore

$$\frac{T_n^{(l)}(1)}{l!} \le \inf_{r>0} r^{-l} \varphi_n(\mathbb{D}, [-1, 1], 2, r) \le e^{(\sqrt{2} + \sqrt{3})l} \frac{T_n^{(l)}(1)}{l!}.$$

Remark 1.6. In two above examples we have obtained the following. Let $M_n(K, l) := \sup\{||P^{(l)}||_K : \deg P \le n, ||P||_K = 1\}$. Then

$$\varphi_n(\mathbb{D}, K, 2, r) \le \left(\sum_{k=0}^n \left(\frac{M_n(K, k)}{k!}\right)^2 r^{2l}\right)^{1/2},$$

$$\frac{M_n(K, l)}{l!} \le \inf_{r>0} r^{-l} \varphi_n(\mathbb{D}, K, 2, r) \le e(K)^l \frac{M_n(K, l)}{l!}$$

with

$$e(K) \leq \sup_{n \geq 1} \sup_{1 \leq l \leq n} \left(\sum_{k=0}^{n} \left(\frac{M_n(K,k)^{1/k}}{M_n(K,l)^{1/l}} \frac{(l!)^{1/l}}{(k!)^{1/k}} \right)^{2k} \right)^{1/2l}$$

$$\leq \sup_{n \geq 1} \sup_{1 \leq l \leq n} \left(\sum_{k=0}^{n} \left(\frac{M_n(K,k)^{1/k}}{M_n(K,l)^{1/l}} \frac{(l!)^{1/l}}{(k!)^{1/k}} \right)^k \right)^{1/l}$$

Remark 1.7. We can repeat constructions from the above examples by considering another norms in \mathbb{C}^N , for example $||z||_p = (|z_1|^p + \cdots + |z_N|^p)^{1/p}$. We shall consider below the general case.

Definition 1.8. Fix a norm q(P) = ||P|| in $\mathbb{P}(\mathbb{C}^N)$, a circular and absorbing set $\mathbb{B} \subset \mathbb{C}^N$ (for any compact C there exists an r>0 such that $K\subset [0,r]\mathbb{B}$) and a linear functional $\Lambda\in X_{q,n}^*$ with $||\Lambda||^* = 1$. Then for any $r \geq 0$ define

$$\varphi_n(\mathbb{B},q,\Lambda,r) := \sup\{|\Lambda(P(z+\zeta))| : \ \zeta \in r\mathbb{B}, \deg P \leq n, ||P(z)|| \leq 1\},$$

$$\varphi(\mathbb{B},q,\Lambda,r) := \sup_{n\geq 1} \varphi_n(\mathbb{B},q,\Lambda,r)^{1/n},$$

$$v(\mathbb{B},q,\Lambda,r) := \log \varphi(\mathbb{B},q,\Lambda,r), \ u(\mathbb{B},q,\Lambda,t) := v(\mathbb{B},q,\Lambda,e^t), t \in \mathbb{R},$$

$$\varphi_n(\mathbb{B},q,r) := \sup\{||P(z+\zeta)|| : \ \zeta \in r\mathbb{B}, \deg P \leq n, ||P(z)|| \leq 1\},$$

$$\varphi(\mathbb{B},q,r) := \sup_{n\geq 1} \varphi_n(\mathbb{B},q,r)^{1/n},$$

$$v(\mathbb{B},q,r) := \log \varphi(\mathbb{B},q,r), \ u(\mathbb{B},q,t) := v(\mathbb{B},q,e^t), t \in \mathbb{R}.$$

Definition 1.9. If a norm q(P) = ||P|| in $\mathbb{P}(\mathbb{C}^N)$ is fixed then define

$$M_n(q,\alpha) := \sup\{||D^{\alpha}P|| : \deg P \le n, ||P|| = 1\},$$

$$\int M_n(q,\beta)^{1/|\beta|} (\alpha!)^{1/|\alpha|} \rangle^{|\beta|} \rangle^{1/|\beta|}$$

$$e(q) := \sup_{n \ge 1} \sup_{1 \le |\alpha| \le n} \left(\sum_{|\beta| \le n} \left(\frac{M_n(q,\beta)^{1/|\beta|}}{M_n(q,\alpha)^{1/|\alpha|}} \frac{(\alpha!)^{1/|\alpha|}}{(\beta!)^{1/|\beta|}} \right)^{|\beta|} \right)^{1/|\alpha|}.$$

Remark 1.10.

- If we define $M_n^{\bullet}(q,k) := \sup\{\|D^{\alpha}P\|^{1/|\alpha|} : |\alpha| \leq k, \deg P \leq n, \|P\| = 1\}$ then one cane easily check that $M_n^{\bullet}(q,k) = \max_{j=1,\dots,N} M_n(q,e_j)$.
- If $e(q) < \infty$ then

$$\max_{|\alpha|=k} \left(\frac{M_n(q,\alpha)}{\alpha!} \right)^{1/k} \le \inf_{r>0} r^{-k} \varphi_n(\mathbb{B}_N, q, r) \le e(q)^k \max_{|\alpha|=k} \left(\frac{M_n(q,\alpha)}{\alpha!} \right)^{1/k}.$$

Let us note that in the case $q(P) = ||P||_{\mathbb{D} \cup \{2\}}$ we have $e(q) = \infty$. Thus condition $e(q) < \infty$ gives a some restriction. Here we can ask that condition $e(q) < \infty$ is equivalent to exits a constant C such that $\max_{|\alpha|=k} \left(\frac{M_n(q,\alpha)}{\alpha!}\right)^{1/k} \leq \inf_{r>0} r^{-k} \varphi_n(\mathbb{B}_N, q, r) \leq C^k \max_{|\alpha|=k} \left(\frac{M_n(q,\alpha)}{\alpha!}\right)^{1/k}$.

The second question is that condition $\sup_{n\geq 2}\max_{1\leq j\leq N}\log\frac{M_n(q,e_j)}{\log n}<\infty$ is necessary to satisfy the condition $e(q)<\infty$.

Proposition 1.11. Let \mathbb{B}_1 and \mathbb{B}_2 be two unit closed balls in \mathbb{C}^N such that $A_1\mathbb{B}_1 \subset \mathbb{B}_2 \subset A_2\mathbb{B}_1$. Then

$$A_1^k \inf_{r>0} r^{-k} \varphi_n(\mathbb{B}_1, q, r) \le \inf_{r>0} r^{-k} \varphi_n(\mathbb{B}_2, q, r) \le A_2^k \inf_{r>0} r^{-k} \varphi_n(\mathbb{B}_1, q, r).$$

Applying arguments from [BB-C1] one can prove the following important facts.

Theorem 1.12. The following functions are convex functions on \mathbb{R} (possibly some of them are equal to $+\infty$):

$$\log \varphi_n(\mathbb{B}, q, \Lambda, e^t), \ \log \varphi(\mathbb{B}, q, \Lambda, e^t), \ \log \varphi_n(\mathbb{B}, q, e^t), \ \log \varphi(\mathbb{B}, q, e^t).$$

Remark 1.13. Since $\log \varphi_n(\mathbb{B}, q, e^t)$ is a convex function, we get inequality

$$\varphi_n(\mathbb{B}, q, rs) \le \varphi_n(\mathbb{B}, q, r^p)^{1/p} \varphi_n(\mathbb{B}, q, s^q)^{1/q}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

As a direct consequence of this theorem and known properties of convex functions we get an important properties (c.f. [BB-C1]).

Corollary 1.14. The following functions if are finite then are continuous and increasing on $(0,\infty)$:

$$\log \varphi_n(\mathbb{B}, q, \Lambda, r), \ \log \varphi(\mathbb{B}, q, \Lambda, r), \ \log \varphi_n(\mathbb{B}, q, r), \ \log \varphi(\mathbb{B}, q, r).$$

Applying known (but still dificult to prove) we get one of reasons that introduced notions can be helpful.

Corollary 1.15. If $\log \varphi(\mathbb{B}, q, \Lambda, r)$ or $\log \varphi(\mathbb{B}, q, r)$ is finite then this function is differentiable except possibly countable set of points and is twice differentiable almost everywhere (Alexandrov's theorem).

2. Radial modifications of Siciak's extremal function.

If $E \subset \mathbb{C}^N$ is a compact set then Siciak's extremal function $\Phi_E(z) = \Phi(E,z)$ is usually defined as

$$\Phi(E, z) := \sup\{|P(z)|^{1/\deg P} : \deg P \ge 1, \ ||P||_E \le 1\}, \ z \in \mathbb{C}^N.$$

In connection with $\Phi(E,z)$ there are also considered functions $\Phi_n(E,z)$, where

$$\Phi_n(E, z) = \sup\{|P(z)| : \deg P \le n, ||P||_E \le 1\}, z \in \mathbb{C}^N.$$

There is known that (c.f. 3.2 in [Si3]) for all $z \in \mathbb{C}^N$

$$\Phi(E, z) = \sup_{n \ge 1} \Phi_n(E, z)^{1/n} = \lim_{n \to \infty} \Phi_n(E, z)^{1/n}.$$

The L-capacity is defined as $C(E):=\liminf_{z\to\infty}||z||_2/\Phi(E,z)$ (cf. [K],[Si2],[Si3]), which is Choquet capacity [Ko] and has product property $C(E\times F)=\min(C(E),C(F))$ [BB-C1].

Analogously we can define $C_{\nu}(E) := \liminf_{z \to \infty} \nu(z)/\Phi(E, z)$, where ν is a norm in \mathbb{C}^{N} . We refer to [BB-C1] for examples, where $C_{\nu}(E)$ is explicitly computed. In the case $\nu(z) = ||z||_{p}$ we also have product property: $C_{\nu}(E \times F) = \min(C_{\nu_{1}}(E), C_{\nu_{2}}(F))$, where $\nu_{j}(z_{j}) = ||z_{j}||_{p}$.

Now for $r \geq 0$ define

$$\varphi_n(r) = \varphi_n(E, r) := \sup\{|P(z + \zeta)| : z \in E, ||\zeta||_2 \le r, \deg P \le n, ||P||_E \le 1\}$$

$$= \varphi_n(\mathbb{B}_N, E, r) = \sup\{\Phi_n(E, z + \zeta) : z \in E, ||\zeta||_2 \le r\},$$

$$\varphi(r) = \varphi(E, r) := \sup_{n \ge 1} \varphi_n(E, r)^{1/n} = \sup\{\Phi(E, z + \zeta) : z \in E, ||\zeta||_2 \le r\},$$

and

$$v(r) = v(E, r) = \log \varphi(E, r), \ v_n(r) = \log \varphi_n(E, r), \ r \ge 1,$$

$$u(t) = u(E, t) = v(e^t), \ u_n(t) = v_n(e^t), \ t \in \mathbb{R}.$$

An important tool in polynomial approximation theory plays the homogeneous capacity $\sigma(E)$ related to the homogeneous Siciak extremal function $\Psi(E,z)$:

$$\Psi(E, z) = \sup_{n>1} \Psi_n(E, z)^{1/n} = \lim_{n \to \infty} \Psi_n(E, z)^{1/n},$$

where

 $\Psi_n(E,z) = \sup\{|P(z)|: P \text{ homogeneous of degree } n, ||P||_E \le 1\}, z \in \mathbb{C}^N.$

$$\sigma_{\nu}(E) := \liminf_{z \to \infty} \frac{\nu(z)}{\Psi(E, z)} = \inf_{\nu(z) = 1} \frac{1}{\Psi(E, z)} = \frac{1}{\sup_{\nu(z) = 1} \Psi(E, z)}.$$

There is known(c.f. [Kor1] - [Kor4]) the following description of homogeneous capacity in the case $\nu(z) = ||z||_{\infty}$:

$$\sigma_{\nu}(E) = \inf_{n \ge 1} \beta_n(E) = \lim_{n \to \infty} \beta_n(E),$$

where

$$\beta_n(E) = \inf_{|\alpha|=n} \inf_{b_{\beta} \in \mathbb{C}} \|z^{\alpha} - \sum_{|\beta|=n, \beta \neq \alpha} b_{\beta} z^{\beta}\|_E^{1/n}.$$

The constants $\beta_n(E)$ are optimal in the following deep result, which was proved by Koreavaar refining earlier joint lemma with Wiegerinck (cf. [Kor1] - [Kor4]).

Proposition 2.1. If $E \subset S^{N-1} \subset \mathbb{R}^N$ satisfy $\beta_n(E) > 0$ then for any $f \subset \mathbb{R}^\infty$ function on a neighborhood of some point $a \in \mathbb{R}^N$ one has inequality:

$$\max_{|\alpha|=n} \binom{n}{\alpha} |D^{\alpha} f(a)| \le \sup_{y \in E} \left| \left(\frac{d}{dt} \right)^n f(a+ty)|_{t=0} \right| / \beta_n(E)^n \le \sup_{y \in E} \left| \left(\frac{d}{dt} \right)^n f(a+ty)|_{t=0} \right| / \sigma(E)^n.$$

Remark 2.2. The proposition fails in the most interesting case $E = \{e_1, \ldots, e_N\}$, especially in the case of polynomials. However if we consider family of constants $M_n(q, \alpha)$ then probably the following is true:

if q is an arbitrary norm in $\mathbb{P}(\mathbb{C}^N)$ then there exists a positive constant a=a(q) such that

$$\max_{|\alpha|=k} M_n(q,\alpha) \le a(q)^k \max_{1 \le j \le N} M_n(q,ke_j).$$

Let us note, as an example, that for $q(P) = ||P||_{\mathbb{D}_N}$ one can take $a(q) = e^N$. Similarly, in the case $q(P) = ||P||_{[-1,1]^N}$ we can put $a(q) = e^{2N}$.

Example 2.3.

result [B1] (cf. [B2]):

(1) If
$$E = \{z \in \mathbb{C}^N : ||z|| \le 1\}$$
 (||z|| is a norm in \mathbb{C}^N) then $\varphi(E, r) = 1 + r/C(E)$. By [Mo]
$$(dd^c \log \varphi(E, ||z||_2)^N = \frac{1}{4} \left(\frac{1}{2}\right)^{N-1} \left(\frac{1}{||z||_2}\right)^N \left(\frac{1}{||z||_2 + C(E)}\right)^{N+1} C(E),$$

$$\lim_{z \to \infty} 2^{N+1} ||z||_2^{2N+1} (dd^c \log \varphi(E, ||z||_2)^N = C(E).$$

(2) If $E = \{z \in \mathbb{R}^N \subset \mathbb{C}^N = \mathbb{R}^N + i\mathbb{R}^N + i\mathbb{R}^N : \nu(z) \leq 1\}$ then $\varphi(E, r) = h(1 + r/(2C(E)))$, where $h(t) = t + \sqrt{t^2 - 1}$, $t \geq 1$. (We also have $h(t) = g|_{[1, +\infty)}^{-1}(t)$, $g(t) = \frac{1}{2}(t + 1/t) = t - \hat{g}(t)$, $\hat{g}(t) = \frac{1}{2}(t - 1/t)$.) In this case we can calculate

$$(dd^c \log \varphi(E, ||z||_2)^N = \frac{1}{4} \left(\frac{1}{2}\right)^{N-1} \left(\frac{1}{||z||_2}\right)^{3N/2} \left(\frac{1}{||z||_2 + 4C(E)}\right)^{N/2+1} 2C(E),$$

$$\lim_{z \to \infty} 2^{N+1} ||z||_2^{2N+1} (dd^c \log \varphi(E, ||z||_2)^N = 2C(E).$$

(3) If E is the closed unit ball in \mathbb{C}^N with respect to a norm n(z) = ||z|| then there is known (cf. [Si2],[Si3]) that $\Psi(E,z) = ||z||$ (while $\Phi(E,z) = \max(1,||z||)$), whence

$$C_{\nu}(E) = \sigma_{\nu}(E) = \frac{1}{\sup_{\nu(z)=1} ||z||}.$$

(4) A situation is much more complicated if E is a convex symmetric body in \mathbb{R}^N . There was known in the case E is the unit Euclidean ball in \mathbb{R}^N than $\Psi(E,z) = L(z) = \left(\frac{||z||_2^2 + |z_1^2 + ... z_N^2|}{2}\right)^{1/2}$ is the Lie norm (which gives $\sigma(E) = \frac{1}{\sqrt{2}} > \frac{1}{2} = C(E)$).

If N > 2 a situation is quite unclear. But in the case N = 2 there is known the following

Let S be the unit ball with respect to a norm N in \mathbb{R}^2 . If $u(t) = \log N(1,t)$ then

$$\Psi(S,(z_1,z_2)) = |z_1| \exp \mathcal{P}u(z_2/z_1),$$

with

$$\mathcal{P}u(\zeta) = (\Im \zeta) \frac{1}{\pi} \int_{-\infty}^{\infty} |\zeta - t|^{-2} u(t) dt = \frac{1}{\pi} \int_{-\infty}^{\infty} u(ty + x) \frac{dt}{1 + t^2},$$

where $\zeta = x + iy$, $y \ge 0$.

In particular, ff $N_m(x) = (x_1^{2m} + x_2^{2m})^{1/(2m)}$ and $S_m = \{x \in \mathbb{R}^2 : N_m(x) = 1\}$, then for all $z \in \mathbb{C}^2$,

$$\Psi(S_m, z) = \left[\prod_{j=1}^m \left(|z_1|^2 - 2\alpha_j \Re(z_1 \overline{z_2}) + |z_2|^2 + 2|\beta_j| \Im(z_1 \overline{z_2})| \right)^{1/2} \right]^{1/m},$$

where $\zeta_j = \alpha_j + i\beta_j \in \sqrt[2m]{-1}$, $j = 1, \dots, m$, with $\zeta_j \neq \overline{\zeta_k}$ for $j \neq k$. If $N_{\infty}(x) = \max(|x_1|, |x_2|)$ and $S_{\infty} = \{x \in \mathbb{R}^2 : N_{\infty}(x) = 1\}$, then for all $z \in \mathbb{C}^2$,

$$\Psi(S_{\infty}, z) = \exp\left[\int_{0}^{2\pi} \log\left(|z_{1}|^{2} - 2\cos\theta\Re(z_{1}\overline{z_{2}}) + |z_{2}|^{2} + 2|\sin\theta\Im(z_{1}\overline{z_{2}})|\right)^{1/2} \frac{d\theta}{2\pi}\right].$$

Since $S_1 = \{x \in \mathbb{R}^2 : |x_1| + |x_2| = 1\} = L^{-1}(S_\infty)$, where $L(z_1, z_2) = (z_1 - z_2, z_1 + z_2)$, we get

$$\Psi(S_1, z) = \Psi(S_{\infty}, L(z))$$

$$= \exp\left[\int_0^{2\pi} \log\left(2|z_1|^2 + 2|z_2|^2 - 2\cos\theta(|z_1|^2 - |z_2|^2) + 4|\sin\theta\Im(z_1\overline{z_2})|\right) \frac{d\theta}{4\pi}\right].$$

Example 2.4.

(1) $E = \{z \in \mathbb{C} : |z| \le R\}, \ u(t) = \log \varphi(E, e^t) = \log(1 + ce^t), \ c = \frac{1}{C(E)} = \frac{1}{R}, \ \frac{u''(t)}{u'(t)} = \frac{1}{1 + ce^t}, \lim_{t \to \infty} e^t \frac{u''(t)}{u'(t)} = \frac{1}{c}.$

$$\Delta \log \varphi(E,|z|) = \frac{c}{|z|(1+c|z|)^2}, \lim_{z \to \infty} |z|^3 \Delta \log \varphi(E,|z|) = \frac{1}{c}.$$

- (2) $E = [a, b], u(t) = \log h(1 + ce^t), c = \frac{1}{2C(E)} = \frac{2}{b-a}, \frac{u''(t)}{u'(t)} = \frac{1}{2+ce^t}. \lim_{t \to \infty} e^t \frac{u''(t)}{u'(t)} = \frac{1}{c}.$ $\Delta \log \varphi(E, |z|) = \frac{c^2}{((1+c|z|)^2 - 1)^{3/2}}, \lim_{z \to \infty} |z|^3 \Delta \log \varphi(E, |z|) = \frac{1}{c}.$
- (3) $E = \{z \in \mathbb{C} : \Phi([-1,1],z) \leq R\}, R \geq 1, u(t) = \log h(g(R) + e^t) \log R, \frac{u''(t)}{u'(t)} = \frac{\frac{g(R)+1}{2}}{g(R)+1+e^t} + \frac{\frac{g(R)-1}{2}}{g(R)-1+e^t}. \lim_{t\to\infty} e^t \frac{u''(t)}{u'(t)} = g(R).$

$$\Delta \log \varphi(E, |z|) = \frac{\hat{g}(R)^2 + g(R)|z|}{|z|((g(R) + |z|)^2 - 1)^{3/2}}, \lim_{z \to \infty} |z|^3 \Delta \log \varphi(E, |z|) = g(R).$$

(4) $E = [-1, 1] \times \overline{\mathbb{D}}_R$

$$u(t) = \max\left(\log h(1+e^t), \log(1+e^t/R)\right) = \begin{cases} \log h(1+e^t), & R \ge \frac{1}{2} \\ \log h(1+e^t), & t < \log\left(\frac{2}{(1/R-1)^2-1}\right), & 0 < R < \frac{1}{2}, \\ \log(1+e^t/R), & t > \log\left(\frac{2}{(1/R-1)^2-1}\right) \end{cases}$$

$$\frac{u''(t)}{u'(t)} = \begin{cases} \frac{1}{2+e^t}, \ t < \log\left(\frac{2}{(1/R-1)^2-1}\right), \ 0 < R < \frac{1}{2}, \\ \frac{1}{1+e^t/R}, \ t > \log\left(\frac{2}{(1/R-1)^2-1}\right) \end{cases}$$

(5) $E = \mathbb{D} \cup \{z_0\}, \ z_0 \notin \mathbb{D}$. Then $\varphi(E, r) = |z_0| + r, \ r > 0$ and $\varphi(E, 0) = 1$. Further, $\log \varphi(E, r) = \log(1 + r/|z_0|) + \log|z_0| = \log \varphi(|z_0|\mathbb{D}, r) + \log|z_0|$, whence $\frac{u''(t)}{u'(t)} = \frac{|z_0|}{|z_0| + r}$,

$$\Delta \log \varphi(E, |z|) = \frac{1/|z_0|}{|z|(1+|z|/|z_0|)^2}, \lim_{z \to \infty} |z|^3 \Delta \log \varphi(E, |z|) = |z_0| = \lim_{t \to \infty} e^t \frac{u''(t)}{u'(t)}.$$

Let us note (for $z_0 = 2$) that applying the formula $\varphi(E, r) = 2 + r$ and considering polynomials $P_n(z) = (z - a_n)z^{n-1}$, where $a_n = 3\frac{2^{n-1}}{2^{n-1}+1} - 1$ we get bounds

$$(n-1)\cdots(n-k+1)2^{-k}(n+\frac{k}{3}(2^n-1)) \le M_n(E,k) \le e^k 2^{n-k}n^k.$$

Let us recall mentioned above David Monn result from [Mo] (it is only one paper published by this mathematician).

Proposition 2.5. If U is the C^2 plurisubharmonic function on \mathbb{C}^N that is radial $(U(z) = u(||z||_2)$ with $u \in C^2(\mathbb{R}_+)$ then

$$(dd^{c}U)^{n}(z) = \frac{1}{4} \left(\frac{u'(||z||_{2})}{2||z||_{2}} \right)^{N-1} \left(u''(||z||_{2}) + \frac{1}{||z||_{2}} u'(||z||_{2}) \right).$$

Corollary 2.6. If $\lim_{r\to\infty} ru'(r) = 1$ then

$$\lim_{z \to \infty} 2^{N+1} ||z||_2^{2N+1} (dd^c U)^n(z) = \lim_{t \to \infty} e^t \frac{v''(t)}{v'(t)}, \ v(t) = u(e^t).$$

Remark 2.7. Let use notice some observations in the one dimensional case.

• We have $\varphi_1(E,r) = 1 + M_1(E)r$, where

$$M_1(E) = \sup\{||P'||_E : P \in \mathbb{P}_1(\mathbb{C}), ||P||_E = 1\}.$$

Hence

$$(1+M_1(E)r)^n \le \varphi_n(E,r), \ n \ge 1,$$

or equivalently $(1+r)^n \leq \varphi_n(E, r/M_1(E))$. As an application we get the following: For all $r, s \geq 0$

$$\varphi_n(E, rs) \le \varphi_n(E, r)\varphi_n(E, s/M_1(E)).$$

Moreover, if $r \geq 1/M_1(E)$ then

$$\varphi_n(E, r+s) \le \varphi_n(E, r)\varphi_n(E, s).$$

Proof. As a consequence of Bernstein's inequality we get

$$\varphi_n(E, rs) \le \varphi_n(E, r) \max(1, s)^n \le \varphi_n(E, r)(1 + s)^n \le \varphi_n(E, r)\varphi_n(E, s/M_1(E)).$$

Analogously

$$\varphi_n(E, r+s) \le \varphi_n(E, r)(1+s/r)^n \le \varphi_n(E, r)(1+M_1(E)s)^n \le \varphi_n(E, r)\varphi_n(E, s).$$

• Let E be a Bernstein set, that is

$$M(E,1) = \sup \left\{ \frac{||P'||_E}{\deg P}, \deg P \ge 1, ||P||_E = 1 \right\} < +\infty.$$

Then

$$(1 + M_1(E)r)^n \le \varphi_n(E, r) \le (1 + M(E, 1)r)^n$$

and

$$\varphi_n(E, r+s) \le \varphi_n(E, (M(E, 1)/M_1(E))r)\varphi_n(E, (M(E, 1)/M_1(E))s).$$

• Define (cf. [BKMO])

$$||P||_r = \sum_{k=0}^{\infty} \frac{1}{k!} ||P^{(k)}||_E r^k, \ r \ge 0, \ ||P||_0 = ||P||_E.$$

Then

$$\sup_{|\zeta| \le r} ||P(x+\zeta)||_E \le ||P||_r \le (\deg P + 1) \sup_{|\zeta| \le r} ||P(x+\zeta)||_E.$$

Hence

$$\varphi_n(E, r) \le \sup\{||P||_r : P \in \mathbb{P}_n(\mathbb{C}), ||P||_0 \le 1\} \le (n+1)\varphi_n(E, r)$$

and therefore

$$\varphi(E,r) = \lim_{n \to \infty} \sup\{||P||_r : P \in \mathbb{P}_n(\mathbb{C}), ||P||_0 \le 1\}^{1/n}.$$

If E is a compact subset of \mathbb{C}^N with C(E) > then there is known (c.f. [BB-C1]) that $u(t) = \log \varphi(E, e^t)$ is a convex increasing function and $\Lambda(t) = u(t) - t$ is a (convex) decreasing one with $\Lambda(t) \searrow -\log C(E)$. In particular

$$\lim_{r \to \infty} \frac{\varphi(E,r)}{r} = \frac{1}{C(E)} \text{ and } \frac{\varphi(E,r)}{r} \searrow \frac{1}{C(E)}.$$

Proposition 2.8. Assume that $v_n(E,r)$ is finite for r > 0. Then there is a positive constant $C_n(E)$ such that

$$v_n(E, r) - \log r \searrow - \log C_n(E)$$

and thus
$$C_n(E) = \lim_{r \to +\infty} \frac{r}{\varphi_n(E,r)} = \sup_{r>0} \frac{r}{\varphi_n(E,r)}$$
, which implies $\varphi_n(E,r) \ge \frac{r}{C_n(E)}$.

3. A RADIAL EXTREMAL FUNCTION RELATED TO A NORM IN $\mathbb{P}(\mathbb{C})$

Proposition 3.1. Assume that $v_n(q, r)$ is finite for r > 0. Then there is a positive constant $C_n(q)$ such that

$$v_n(q,r) - \log r \searrow - \log C_n(q)$$

and thus $C_n(q) = \lim_{r \to +\infty} \frac{r}{\varphi_n(q,r)} = \sup_{r>0} \frac{r}{\varphi_n(q,r)}$, which implies $\varphi_n(q,r) \geq \frac{r}{C_n(q)}$. Moreover

$$t_n(q)^{1/n} \ge C_n(q), \ n \ge 1, \ t(q) \ge \min_{n \ge 1} C_n(q)$$
 and $t(q) \ge C(q)$.

Here $C(q) := \lim_{r \to \infty} \frac{r}{\varphi(q,r)}$ or equivalently $-\log C(q) = \lim_{r \to \infty} (v(q,r) - \log r)$ and more precisely $v(q,r) - \log r \searrow -\log C(q)$.

Proof. Fix $n \geq 1$ and a polynomial $P \in \mathbb{P}_n(\mathbb{C})$, ||P|| = 1, and a continuous functional l with $||l||^* = 1$. Consider the function

$$g(\zeta) = \frac{1}{n} \log |l(P(x+\zeta)) - \log |\zeta| \in SH(\mathbb{C} \setminus \mathbb{D}_{r_0}), \ r_0 > 0.$$

Since g is bounded from above, we have, by the maximum principle for subharmonic functions, the inequality $g(\zeta) \leq \max_{|\zeta| = r_0} g(\zeta)$. Taking the supremum we get the bound

$$v_n(q,r) \le v_n(q,r_0) + \log r - \log r_0, \ r \ge r_0.$$

Now consider the function $\psi(t) = v_n(q, e^t) - t$. It is a convex function that is bounded from above which implies $\liminf_{t \to +\infty} \frac{1}{t} \psi(t) \le 0$ and by Lemma ψ is a decreasing function. In particular the limit $\lim_{r \to \infty} (v_n(q, r) - \log r) =: -\log C_n(q)$ exists and $-\log C_n(q) = \inf_{r>0} (v_n(q, r) - \log r)$.

Similarly, assuming $v(q, r_0)$ is finite for an $r_0 > 0$ and applying analogous arguments we get existence of the limit $\lim_{r \to \infty} (v(q, r) - \log r) =: -\log C(q)$ and $-\log C(q) = \inf_{r > 0} (v(q, r) - \log r)$.

Now let $T_n(q) = T_n(q, \cdot)$ be n - th Chebyshev polynomial for $q: T_n(q)$ is a monic polynomial of degree n such that

$$t_n(q) = ||T_n(q)|| =: \inf\{||P_n||: P_n \text{ is a monic polynomial of degree } n\}.$$

Then

$$\sup_{|\zeta|=r} \log \left(\frac{||T_n(q, x + \zeta)||}{||T_n(q)||} \right)^{1/n} \le v_n(q, r)$$

which easily gives $-\log C_n(q) \ge -\log ||T_n(q)||^{1/n}$. Analogously we get inequality $-\log C(q) \ge -\log ||T_n(q)||^{1/n}$ and therefore $t(q) \ge C(q)$.

4. Markov's inequality in \mathbb{C}

Let E be a compact subset of \mathbb{C} . Applying Cauchy's integral formula one can easily prove the following.

Proposition 4.1. If $P \in \mathbb{P}_n(\mathbb{C})$, $n \geq 1$ with $||P||_E = 1$ then

$$||P'||_E \le \inf_{r>0} \frac{1}{r} \exp(nv_n(E,r)) \le \inf_{r>0} \frac{1}{r} \exp(nv(E,r)).$$

Proposition 4.2. Assume that $E \in \mathcal{AM}(M,m)$, which means that for an arbitrary $P \in \mathbb{P}(\mathbb{C})$ the following A. Markov type inequality is satisfied: $||P'||_E \leq M(\deg P)^m||P||_E$ (here $M > 0, m \geq 1$ are constants). Then we have the following bounds

$$v_n(E,r) \le M n^{m-1} r, \ r \ge 0.$$

Proposition 4.3. Assume that $E \in \mathcal{AM}(M, m)$. Then

$$v_n(E,1) \le M + (m-1)\log n.$$

Proof. Fix an $x \in E$ and $P \in \mathbb{P}_n(\mathbb{C})$. Consider the function

$$g(\zeta) = \frac{1}{n} \log |P(x+\zeta) - \log |\zeta| \in SH(\mathbb{C} \setminus \mathbb{D}_{r_0}), \ r_0 > 0.$$

Since g is bounded from above, we have by the maximum principle for subharmonic functions, the inequality $g(\zeta) \leq \max_{|\zeta|=r_0} g(\zeta)$. Taking the supremum we get the bound

$$v_n(E, r) \le v_n(E, r_0) - \log r_0 + \log r, \ r \ge r_0$$

and for $r_0 = \left(\frac{1}{n}\right)^{m-1}$ we obtain $v_n(E,1) \leq M + (m-1)\log n$.

Proposition 4.4. Assume that $q \in \mathcal{AM}(M,m)$, which means that for an arbitrary $P \in \mathbb{P}(\mathbb{C})$ the following A. Markov type inequality is satisfied: $||P'|| \leq M(\deg P)^m ||P||$ (here $M > 0, m \geq 1$ are constants). Then we have the following bounds

$$v_n(q,r) \le M n^{m-1} r, \ r \ge 0.$$

Proposition 4.5. Assume that $q \in \mathcal{VM}(M, m)$, which means that for an arbitrary $P \in \mathbb{P}(\mathbb{C})$ the following V. Markov type inequality is satisfied: $||P'^{(k)}|| \leq M^k \left(\frac{1}{k!}\right)^{m-1} (\deg P)^{km} ||P||$ (here $M > 0, m \geq 1$ are constants). Then we have the following bounds

$$v_n(q,r) \le mM^{1/m}r^{1/m}, \ r \ge 0.$$

Proof. Applying Taylor's expansion to $P \in \mathbb{P}_n(\mathbb{C})$ with ||P|| = 1 we can write

$$||P(x+\zeta)|| \le \sum_{k \le n} \frac{1}{k!} ||P^{(k)}(x)|| |\zeta|^k \le \sum_{k \le n} \frac{1}{k!} M^k \left(\frac{1}{k!}\right)^{m-1} n^{km} |\zeta|^k$$

$$\le \left(\sum_{k \le n} \frac{1}{k!} (M^{1/m} n |\zeta|^{1/m})^k\right)^m \le \left[\exp(M^{1/m} n |\zeta|^{1/m})\right]^m = \exp(m M^{1/m} n |\zeta|^{1/m}),$$

which implies

$$v_n(q,r) \le mM^{1/m}r^{1/m}.$$

We shall write $q \in HCP(\gamma, B)$ (γ, B) positive constants) if inequality $v(q, r) \leq Br^{\gamma}$ holds for an arbitrary r > 0.

Theorem 4.6. Let q be a fixed norm in $\mathbb{P}(\mathbb{C})$. Then we have implications

$$q \in VM(m, M) \Rightarrow q \in HCP(\frac{1}{m}, mM^{1/m}),$$

$$q \in HCP(\gamma, B) \Rightarrow q \in VM(\frac{1}{\gamma}, (\gamma eB)^{1/\gamma}).$$

Moreover, if $q \in HCP(\gamma, B)$ then

$$\gamma eB \cdot C(q)^{\gamma} \ge 1.$$

Hence, if $q \in VM(m, M)$ then

$$C(q) \ge \frac{1}{M}e^{-m}$$
.

Proof. The proof of the first implication was done. Assume $q \in HCP(\gamma, B)$ and take $P \in \mathbb{P}_n(\mathbb{C}), ||P|| = 1$. Then

$$||P^{(k)}|| \le k! \inf_{r>0} \frac{1}{r^k} \exp(nBr^{\gamma}) = k! \inf_{r>0} \exp(nBr^{\gamma} - k \log r).$$

The minimum is attained for $r = (k/nB\gamma)^{1/\gamma}$ which gives inequality

$$||P^{(k)}|| \le k! \left(\frac{n}{k}\right)^{k/\gamma} (B\gamma e)^{k/\gamma} \le (B\gamma e)^{k/\gamma} \left(\frac{1}{k!}\right)^{1/\gamma - 1} n^{k/\gamma}.$$

Assume again $q \in HCP(\gamma, B)$. Since v(q, r) is continuous then $v(q, [0, +\infty)) = [0, +\infty)$ and we can take a positive r such that $v(q, r) = \frac{1}{\gamma}$. Now

$$C(q) \ge \frac{r}{\exp v(q, r)} = \frac{r}{e^{1/\gamma}} = \frac{r}{(\gamma e v(q, r))^{1/\gamma}}$$
$$\ge \frac{r}{(\gamma e B r^{\gamma})^{1/\gamma}} = \frac{1}{(\gamma e B)^{1/\gamma}}.$$

5. Extremal functions related to Pleśniak's properties.

Definition 5.1. Fix a norm q in $\mathbb{P}(\mathbb{C}^N)$ we define a family of extremal radial functions

$$\mathcal{R}_k(q,r) := \sup_{n>1} \varphi_n \left(q, r(k!/M_n(q,k))^{1/k} \right)^{1/k}.$$

and

$$\mathcal{R}(q,r) := \sup_{n \ge 1} \sup_{1 \le k \le n} \varphi_n \left(q, r(k!/M_n(q,k))^{1/k} \right)^{1/k}.$$

As an example consider $E = \mathbb{D}$. Since $\varphi_n(E, r) = (1 + r)^n$ we get

$$\mathcal{R}_1(E,r) = \sup_{n>1} (1+r/n)^n = e^r,$$

and since $\varphi_n(E, r/\binom{n}{k}^{1/k})^{1/k} \leq (1 + rk/n)^{n/k} \leq e^r$, we obtain

$$\mathcal{R}(\mathbb{D}, r) = \mathcal{R}_1(\mathbb{D}, r) = e^r, \ r \ge 0.$$

In the case E = [-1, 1] we can estimate

$$\cosh \sqrt{2r} \le \mathcal{R}_1(E, r) \le e^{\sqrt{2r}}, \ \mathcal{R}(E, r) = e^{\sqrt{2r}}.$$

Let $E_R = \{z \in \mathbb{C} : |h(z)| \le R\} = \{z \in \mathbb{C} : \Phi([-1, 1], z) \le R\}, R > 1.$

Then $\varphi(E_R, r) = h(g(R) + r)/R$. One can check that

$$\sup_{n\geq 1} \varphi(E_R, r/n)^n = \lim_{n\to\infty} \varphi(E_R, r/n)^n = e^{2r/(R-1/R)} = e^{r/\sqrt{g^2(R)-1}}.$$

Hence

$$\mathcal{P}_1(E_R, r) \le e^{r/\sqrt{g^2(R) - 1}}.$$

Remark 5.2. We can define

$$\widetilde{\varphi}_n(q,r) = \sum_{k=0}^n \frac{1}{k!} M_n(E,k) r^k.$$

We have $\varphi_n(q,r) \leq \widetilde{\varphi}_n(q,r)$ and thus

$$\varphi_n \left(q, r(k!/M_n(q,k))^{1/k} \right)^{1/k} \le \left(\sum_{l=0}^n \frac{(k!)^{l/k}}{l!} \left(M_n(q,l)^{1/l}/M_n(q,k)^{1/k} \right)^l r^l \right)^{1/k}.$$

Hence, if $\sup_{n\geq 1} \sup_{1\leq k,l\leq n} M_n(q,l)^{1/l}/M_n(q,k)^{1/k} := a(q) < \infty$ then $\mathcal{R}(q,r) \leq e^{a(q)r}$.

Definition 5.3. Let us recall that $q \in \mathcal{AM}(m, M)$ iff $M_n(q, 1) \leq Mn^m$ and $q \in \mathcal{VM}(m, M)$ iff $M_n(q, k) \leq M^k n^{km}/(k!)^{m-1}$. This is a motivation to consider *Pleśniak's extremal functions*

$$\mathcal{P}_m(q,r) := \sup_{n>1} \varphi_n\left(q, \frac{r}{n^m}\right),$$

$$\mathcal{B}_{\alpha}(q,r) := \sup_{n \ge 1} \sup_{k \ge 1} \varphi_n \left(q, r \left(\frac{k}{n} \right)^{\alpha} \right)^{1/k}.$$

Let us observe that

$$\mathcal{P}_m(q,r) \le \mathcal{B}_m(q,r) = \sup_{k \ge 1} \mathcal{P}_m(q,rk^m)^{1/k}.$$

Remark 5.4. If $E \in \mathcal{M}(E, m, M)$ then

$$\widetilde{\varphi}_n(E,r) \le \sum_{k=0}^n \frac{1}{k!} (n \cdots (n-k+1))^m M^k r^k \le \sum_{k=0}^n \frac{1}{k!} n^{mk} M^k r^k \le \exp(Mrn^m).$$

Theorem 5.5. If $E \subset \mathbb{C}$ then for an arbitrary $P \in \mathbb{P}_n(\mathbb{C})$ and $m \geq 1$

$$||P'||_E \le n^m \inf_{t>0} \left(\frac{\mathcal{P}_m(E,t)}{t}\right) ||P||_E$$

and

$$||P^{(k)}||_E \le k! \left(\frac{n}{k}\right)^{km} \left(\inf_{t>0} \frac{\mathcal{B}_m(E,t)}{t}\right)^k ||P||_E.$$

If there exist constants $M > 0, m \ge 1$ such that for all $P \in \mathbb{P}_n(\mathbb{C})$

$$||P'||_E \le Mn^m ||P||_E,$$

then

$$\mathcal{P}_m(E,t) \le e^{Mt}$$
.

If there exist constants $M > 0, m \ge 1$ such that for all $P \in \mathbb{P}_n(\mathbb{C})$

$$||P^{(k)}||_E \le M^k \left(\frac{1}{k!}\right)^{m-1} n^{km} ||P||_E,$$

then

$$\mathcal{B}_m(E,t) \le e^{mM^{1/m}t^{1/m}}.$$

Definition 5.6. Define

$$C_{\mathcal{P}}(E,m) := \sup_{t>0} \frac{t}{\mathcal{P}_m(E,t)}$$

and

$$C_{\mathcal{B}}(E,m) := \sup_{t>0} \frac{t}{\mathcal{B}_m(E,t)}.$$

Since $\mathcal{P}_m(E,t) \leq \mathcal{B}_m(E,t)$ we get inequality

$$C_{\mathcal{B}}(E,m) \leq C_{\mathcal{P}}(E,m).$$

Let us note that

$$||P'||_E \le n^m \frac{1}{C_P(E, m)} ||P||_E$$

and

$$||P^{(k)}||_E \le k! \left(\frac{n}{k}\right)^{km} \left(\frac{1}{C_{\mathcal{B}}(E,m)}\right)^k ||P||_E.$$

As a corollary (to Theorem) we get

$$C_{\mathcal{B}}(E,m) \ge e^{-m} \frac{1}{M}$$

and

$$C_{\mathcal{P}}(E,m) \ge e^{-1} \frac{1}{M}.$$

If
$$\varphi(E,r) \leq e^{Ar\alpha}$$
 then $\mathcal{B}_{1/\alpha}(E,t) \leq e^{At^{\alpha}}$.

If $\mathcal{B}_m(E,t) \leq e^{At^{1/m}}$ then $\varphi(E,r) \leq \inf_{s \geq 1} \mathcal{B}_s(E,r) \leq \mathcal{B}_m(E,r) \leq e^{Ar^{1/m}}$. We have

$$C(E) \ge \sup_{m>1} C_{\mathcal{B}}(E, m).$$

Now define

$$\mathcal{B}_{m}^{*}(E,r) = \sup_{k,n \geq 1} \varphi \left(r \left(\frac{k}{n} \right)^{m} \right)^{n/k} = \sup_{\sigma > 0} \varphi(r\sigma^{m})^{1/\sigma} = \sup_{\sigma > 0} \varphi(r\sigma)^{1/\sigma^{1/m}} = e^{A_{m}r^{1/m}},$$

$$A_{m} = \sup_{\sigma > 0} \frac{\log \varphi(E,\sigma)}{\sigma^{1/m}} = \sup_{\sigma > 0} \frac{\rho(E,\sigma)}{\sigma^{1/m}}.$$

$$C_{\mathcal{B}^*}(E, m) = \sup_{t>0} \frac{t}{\mathcal{B}_m^*(E, t)}.$$

Let us observe that

$$\mathcal{B}_{m}^{*}(E,r) = \sup_{n>1} \mathcal{P}_{m}^{*}(E,rk^{m})^{1/k},$$

where

$$\mathcal{P}_m^*(E,r) := \sup_{k>1} \varphi(E,r/n^m)^n.$$

We have $\mathcal{P}_m(E,r) \leq \mathcal{P}_m^*(E,r)$. In the case of E = [-1,1] one can check that $\mathcal{P}_2^*([-1,1],r) = \lim_{n \to \infty} \varphi([-1,1],r/n^2)^n = e^{\sqrt{2r}} = \mathcal{B}_2([-1,1],r)$. We shall see that it is a consequence of a little more general facts.

Proposition 5.7. We have $\mathcal{B}_m^*(E,r) = \mathcal{B}_m(E,r)$ and

$$C_{\mathcal{B}}(E, m) = C_{\mathcal{B}^*}(E, m) = H_{1/m}(E),$$

where $H_{\gamma}(E)$ was defined in [BB-C3]

Proof. It is clear that $\mathcal{B}_m(E,r) \leq \mathcal{B}_m^*(E,r)$. To prove opposite inequality let us observe that by Zaharjuta-Siciak theorem (cf. [Si2] or Proposition 1.3 in [Si3]) $\varphi(E,r) = \sup_{l>1} \varphi_l(E,r)^{1/l}$. Hence

$$\mathcal{B}_{m}^{*}(E,r) = \sup_{k,n,l \geq 1} \varphi_{l} \left(E, r \left(\frac{k}{n} \right)^{m} \right)^{n/kl} \leq \sup_{k,n,l \geq 1} \varphi_{ln} \left(E, r \left(\frac{k}{n} \right)^{m} \right)^{1/kl}$$
$$= \sup_{k,n,l \geq 1} \varphi_{ln} \left(E, r \left(\frac{kl}{ln} \right)^{m} \right)^{1/kl} \leq \mathcal{B}_{m}(E,r)$$

(we apply inequality $\varphi_l(E,r) \leq \varphi_{ln}(E,r)^{1/n}$.)

Let us recall (cf. Definition 16 in [BB-C3]) that for $\gamma \in (0,1]$

$$H_{\gamma}(E) = 1/(B(\gamma)\gamma e)^{1/\gamma}, \ B(\gamma) = \sup_{r>0} \frac{\log \varphi(E,r)}{r^{\gamma}} = \sup_{r>0} \frac{\rho(E,r)}{r^{\gamma}}.$$

We see that $A_m = B(1/m)$.

Now calculate

$$\frac{1}{C_{\mathcal{B}^*}(E,m)} = \inf_{t>0} \exp(A_m t^{1/m} - \log t) = \exp(A_m t_0^{1/m} - \log t_0),$$

where $t_0 = (m/A_m)^m$. Hence

$$C_{\mathcal{B}^*}(E, m) = \left(eA_m \frac{1}{m}\right)^{-m} = H_{1/m}(E).$$

Proposition 5.8. If $\mathcal{P}_m^*(E,r) = \lim_{n \to \infty} \varphi(E,r/n^m)^{1/n}$ then

$$\mathcal{P}_{m}^{*}(E, rk^{m})^{1/k} = \mathcal{P}_{m}^{*}(E, r), \ k \ge 1$$

and thus $\mathcal{B}_m(E,r) = \mathcal{P}_m^*(E,r)$.

Proof.

$$\mathcal{P}_m^*(E, rk^m)^{1/k} = \lim_{n \to \infty} \varphi(E, rk^m/(kn)^m)^{(kn)/k} = \mathcal{P}_m^*(E, r).$$

Remark 5.9. We know that assumption of the above proposition is satisfied if $E = \overline{\mathbb{D}}$ (m = 1) or E = [-1, 1] (m = 2). It is also true (with m = 1) in the case of $E_R = \{z \in \mathbb{C} : |h(z)| \leq R\}$. Here

$$\mathcal{P}_1^*(E_R, r) = \mathcal{B}_1(E_R, r) = e^{r/\sqrt{g^2(R)-1}}$$

In the general case we prove the following.

Theorem 5.10.

$$\mathcal{P}_m^*(E, rk^m)^{1/k} \le \max(\sup_{\sigma > 1} \varphi(E, r\sigma)^{1/\sigma^{1/m}}, \varphi(E, r)\mathcal{P}_m^*(E, r)).$$

Proof.

$$\begin{split} \mathcal{P}_m^*(E,rk^m)^{1/k} &= \max(\max_{1 \leq n \leq k} \varphi\left(E,r(k/n)^m\right)^{n/k}, \max_{0 \leq s \leq k-1} \sup_{l \geq 1} \varphi\left(E,rk^m/(kl+s)^m\right)^{(kl+s)/k} \\ &\leq \max(\sup_{\sigma \geq 1} \varphi(E,r\sigma)^{1/\sigma^{1/m}}, \max_{0 \leq s \leq k-1} \sup_{l \geq 1} \varphi\left(E,rk^m/(kl+s)^m\right)^{l+s/k} \\ &\leq \max(\sup_{\sigma \geq 1} \varphi(E,r\sigma)^{1/\sigma^{1/m}}, \varphi(E,r)\mathcal{P}_m^*(E,r)). \end{split}$$

6. Kolmogorov-Landau type conditions.

6.1. **Kolmogorov-Landau type theorems.** A problem related to the name Kolmogorov and Landau is the following (c.f. [MPF]).

Let $M_k(p,I) = M_k(p,I,f) = ||f^{(k)}||_p$, $0 \le k \le n$, where f is a real function on the real interval I, $||g||_p = (\int_I |g(x)|^p dx)^{1/p}$, $1 \le p \le \infty$. Find optimal constants $C_{nk}(p,I)$ such that for all $f \in \mathcal{C}^n(int(I))$

$$M_k(p, I, f) \le C_{nk}(p, I) M_0(p, I, f)^{1 - \frac{k}{n}} M_n(p, I, f)^{\frac{k}{n}}, \ 0 \le k \le n.$$

If we replace f by f' we get inequalities

$$M_k(p, I, f) \le C_{n-1, k-1}(p, I) M_1(p, I, f)^{1 - \frac{k-1}{n-1}} M_n(p, I, f)^{\frac{k-1}{n-1}}, \ 0 \le k \le n$$

or equivalently

$$\log M_k(p, I, f) \le \log C_{n-1, k-1}(p, I) + \left(1 - \frac{k-1}{n-1}\right) \log M_1(p, I, f) + \left(\frac{k-1}{n-1}\right) \log M_n(p, I, f),$$

 $1 \le k \le n$.

All results in this direction are rather hard to prove. Let us present an example (Neder inequality, c.f. (5.1) in [MPF]):

$$M_k(+\infty, [a, a+L]) \le (2n)^{2n} L^{-k} M_0(+\infty, [a, a+L]) + L^{n-k} M_n(+\infty, [a, a+L]).$$

Another result connected to bounded subset of \mathbb{R}^N is contained in R. Redheffer and W. Walter theorem (c.f. [MPF] and references given there):

If G is a bounded domain belonging to a class $K(\theta, H)$ (that contains a family of N dimesional intervals), if we put for all $u \in C^n(G)$, $U_k = \sup\{|D^{\alpha}u(x)| : |\alpha| = k, x \in G\}$, then there exists a constant $A = A(n, \theta)$ such that

$$\log U_k \le \log A + \left(1 - \frac{k}{n}\right) \log U_0 + \frac{k}{n} \log U_n^*,$$

where $U_n^* = \max(U_n, h^{-n}U_0)$.

It is rather difficult to say something about behavior of constants $A(n, \theta)$, even for special class of functions. Our goal will be give a modification of Kolmogorov-Landau type inequalities to polynomials, more precisely to factors $M_n(q, \alpha)$.

6.2. Kolmogorov-Landau triangle sequences.

Definition 6.1. Consider a triangle sequence of positive numbers

$$1 \le k \le n, \ n \in \mathbb{Z}_+ \ \varphi(n,k) > 0$$

and put $\varphi(n,k)^{1/k} =: \psi(n,k)$. We shall say sequence $\varphi(n,k)$ belongs to Kolmogorov-Landau class \mathcal{KL}^* iff for an arbitrary $n \in \mathbb{Z}_+, n > 1$ and every $1 \le k \le n$

$$\log \psi(n, k) \le \left(1 - \frac{\log k}{\log n}\right) \log \psi(n, 1) + \frac{\log k}{\log n} \log \psi(n, n).$$

In such a situation we shall write $\varphi(n,k) \in \mathcal{KL}^*$. Similarly, we shall say sequence $\varphi(n,k)$ belongs to Kolmogorov-Landau class \mathcal{KL} iff there exists a positive constant C such that for an arbitrary n > 1 and every $1 \le k \le n$

$$\log \psi(n,k) \le \log C + \left(1 - \frac{\log k}{\log n}\right) \log \psi(n,1) + \frac{\log k}{\log n} \log \psi(n,n).$$

We shall write $\varphi(n,k) \in \mathcal{KL}$.

Obviously $\varphi(n,k) \in \mathcal{KL}^* \Rightarrow \varphi(n,k) \in \mathcal{KL}^*$. We also see that \mathcal{KL}^* is a kind of convexity property and thus \mathcal{KL} is a kind of weak convexity condition.

Example 6.2. It is easy to check that the following sequences belong to \mathcal{KL}^* .

- (1) $\varphi(n,k) = k^k$
- (2) $\varphi(n,k) = e^{\sigma k}$
- (3) $\varphi(n,k) = \left(\frac{n}{k}\right)^{km}$
- $(4) \varphi(n,k) = k^{k} \cdot \left(\frac{n}{k}\right)^{km}$
- (5) $\varphi(n,k) = e^{k\sigma} \cdot k^k \cdot \left(\frac{n}{k}\right)^{km}$ (6) $\varphi(n,k) = 2^{n-k}n^k$.

In the examples below we used the following simple observations.

Proposition 6.3. If $\varphi_1(n,k), \varphi_2(k,n) \in \mathcal{KL}, m > 0$ then

- (a) $\varphi_1(n,k)\varphi_2(n,k) \in \mathcal{KL}$,
- (b) $\varphi_1(n,k)^m \in \mathcal{KL}$.
- (c) $\max(\varphi_1(n,k), \varphi_2(n,k)) \in \mathcal{KL}$.
- (d) If there exist positive constant A_1, A_2 such that $A_1^k \leq \frac{\varphi_1(n,k)}{\varphi_2(n,k)} \leq A_2^k$ then

$$\varphi_1(k,n) \in \mathcal{KL} \iff \varphi_2(k,n) \in \mathcal{KL}.$$

Example 6.4. In the following cases we can check that a sequence belongs to \mathcal{KL} with a given constant C (usually not optimal). We refer to Mitrinović book or to Wikipedia for needed inequalities for factorials n! and Newton symbols $\binom{n}{k}$ and left calculations to the reader.

- (1) $\varphi(n,k) = k!$, $\log C = \frac{11}{12} + \frac{1}{2e} + \frac{1}{2}\log(2\pi)$. (2) $\varphi(n,k) = \binom{n}{k}^m$, $C = e^m$. (3) $\varphi(n,k) = k! \cdot \binom{n}{k}^m$, $\log C = m + \frac{11}{12} + \frac{1}{2e} + \frac{1}{2}\log(2\pi)$.
- (4) $\varphi(n,k) = k!e^{-k} \exp((1+1/s)k^{\frac{s}{1+s}}n^{\frac{1}{1+s}}), \log C = \frac{11}{12} + \frac{1}{2e} + \frac{1}{2}\log(2\pi) + 1 + 1/s, \text{ where}$ $0 < s \le 1$.
- (5) $\varphi(n,k) = \left(\frac{n}{k}\right)^{km} (1 + \log(n/k))^{mk}$.

6.3. Kolmogorov-Landau norms. Let q(P) = ||P|| be a norm in $\mathbb{P}(\mathbb{C}^N)$.

Put

$$M(n,k) = M_q(n,k) := \sup\{||D^{\alpha}P||: |\alpha| = k, k \le \deg P \le n, ||P|| = 1\}.$$

Definition 6.5.

- (1) $q \in \mathcal{KL}$ if $M_q(n,k) \in \mathcal{KL}$.
- (2) $q \in \mathcal{KL}_*$ if there exists $\varphi(n,k) \in \mathcal{KL}$ such that $M_q(n,k) \leq \varphi(n,k)$ ($\varphi(n,k)$ will be called \mathcal{KL} majorant).

Example 6.6.

(1) Let $E = \overline{\mathbb{D}}$, $q(P) = ||P||_E$. There is well known that

$$M_q(n,k) = n(n-1)\cdots(n-k+1) = k!\cdot \binom{n}{k}.$$

By the Example 1.3 (3) we get $q \in \mathcal{KL}$.

(2) Consider $E = [-1, 1], q(P) = ||P||_E$. The famous Vladimir Markov inequality gives

$$M_q(n,k) = \frac{(n(n-1)\cdot(n-k+1))^2}{1\cdot 3\cdot (2k-1)} \le k! \binom{n}{k}^2.$$

Hence, by the Example 1.3 (3) we get $q \in \mathcal{KL}_*$. One can also check that $\varphi(n,k) = \frac{(k!)^3}{(2k)!} \in \mathcal{KL}$, which gives $q \in \mathcal{KL}$. Applying recent result by G. Sroka [Sr] one can check that $q_p \in \mathcal{KL}_*$, where $q_p(P) = \left(\frac{1}{2} \int_E |P(x)|^p dx\right)^{1/p}$, $p \geq 1$.

(3) Let $q(P) = ||P|| = \sum_{j=0}^{\infty} \left(\frac{1}{j!}\right)^m |P^{(j)}(0)|\tau^j, \ \tau > 0, \ m \ge 0.$

If
$$P(x) = a_0 + a_1 x + \dots + a_n x^n$$
 then $||P|| = \sum_{j=0}^{\infty} \left(\frac{1}{j!}\right)^{m-1} |a_j| \tau^j$.

Put $S_{q,n} := \{ P \in \mathbb{P}_n(\mathbb{C}) : ||P|| \leq 1 \}$ - this is a convex symmetric body in finite dimensional vector space $\mathbb{P}_n(\mathbb{C})$. Then one can calculate that

$$\operatorname{extr}(S_{q,n}) = \{ P_j(x) = \zeta_j(j!)^{m-1} \tau^{-j} x^j : |\zeta_j| = 1, \ j = 0, \dots, n \}.$$

Hence $\sup\{||P^{(k)}||: P \in S_{q,n}\} = \max\{||P_j^{(k)}||: j = 1, ..., n\}.$

Since $||x^l|| = (1/l!)^{m-1} \tau^l$, we get

$$||P_j^{(k)}|| = \tau^{-k} (j(j-1)\cdots(j-k+1))^m.$$

Hence

$$M_q(n,k) = \max_{k \le j \le n} ||P_j^{(k)}|| = \tau^{-k} (n(n-1)\cdots(n-k+1))^m.$$

Consequently $q \in \mathcal{KL}$.

- (4) If $E \subset \mathbb{C}^N$ is a Bernstein set $(||D^{\alpha}P||_E \leq B^{|\alpha|}(\deg P)^{|\alpha|}||P||_E)$ then $q = ||\cdot||_E \in \mathcal{KL}$.
- (5) If E is a compact subset of \mathbb{C} then $q(P) = \sum_{j=1}^{\infty} \frac{1}{j!} ||P^{(j)}||_{E} \tau^{j} \in \mathcal{KL}$.
- (6) If $q \in \mathcal{AM}$ $(M_q(n,k) \leq B^k n^{km})$ or $q \in \mathcal{VM}$ $(M_q(n,k) \leq B^k n^{km}/(k!)^{m-1})$ then $q \in \mathcal{KL}_*$.

Let us formulate the main results of this paper.

Theorem 6.7. If $q \in \mathcal{KL}$ then two conditions are equivalent

- (1) $q \in \mathcal{VM}$;
- (2) $q \in \mathcal{AM}, \ M(n,n) \leq A^n n!.$

Theorem 6.8. Two conditions are equivalent

- (1) $q \in \mathcal{VM}$;
- (2) there exists $\varphi(n,k)$ a \mathcal{KL} majorant such that $\varphi(n,1) \leq An^{\alpha}$, $\varphi(n,n) \leq B^n n!$.

Remark 6.9. If q is a norm in \mathbb{C}^N then we conjecture $q \in \mathcal{KL}$.

If the above conjecture is true, then applying [B-C] we get the following.

Corollary 6.10. If $E \subset \mathbb{C}$ then $E \in \mathcal{AM} \Leftrightarrow E \in \mathcal{VM}$.

Remark 6.11. Let q be the norm in Example 6.6 (3) with m > 1. Then $q \in \mathcal{KL}$, $q \in \mathcal{AM}$ but $q \notin \mathcal{VM}$ as it was proved in [BKMO].

Remark 6.12. Let us consider the following condition: $q \in \mathcal{M}_*(a, m)$ if and only if

$$\left(\frac{M_n(q,l)}{l!}\right)^{1/l} / \left(\frac{M_n(q,k)}{k!}\right)^{1/k} \le a \left(\frac{k}{l}\right)^m, \ \forall n \ge 1, \ 1 \le l \le k \le n,$$

where $m \geq 1$ is a constant. In particular, if k = n, we obtain a condition

$$M_n(q, l) \le a^l l! \left(\frac{n}{l}\right)^{lm} \left(M_n(q, n)^{1/n} / n!^{1/n}\right)^l.$$

Hence, if $M_n(q, n) \leq b^n n!$, we obtain V. Markov's inequality

$$M_n(q, l) \le (a(q)b(q))^l n^{ml}/l!^{m-1}.$$

Let us note that considered condition $\mathcal{M}_*(a, m)$ is not satisfied if $q(P) = ||P||_{\mathbb{D} \cup \{z_0\}}$, with $|z_0| > 1$. On the other hand this condition is satisfied if q(P) is a norm from Example 6.6(3).

Now we can formulate the following question: thus

$$q \in \mathcal{AM}(M,m) \Rightarrow q \in \mathcal{M}_*(a',m')$$

or (a weaker condition)

$$q \in \mathcal{AM}(M,m) \Rightarrow M_n(q,l) \leq a'^l l! \left(\frac{n}{l}\right)^{lm'} \left(M_n(q,n)^{1/n}/n!^{1/n}\right)^l$$
?

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