An L^2-index formula of monopoles with Dirac-type singularities

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Abstract

We prove the Fredholmness of Dirac operators of monopoles with Dirac-type singularities on oriented complete Riemannian 3-folds, and we also calculate the L^2-indices of them.

1 Introduction

Let (X, g) be a complete oriented Riemannian 3-fold with the bounded scalar curvature. Let $Z \subset X$ be a finite subset. We fix a spin structure on X. Let (V, h) be a Hermitian vector bundle on $X \setminus Z$ and A a connection on (V, h). Let $\Phi \in \mathrm{End}(V)$ be a skew-Hermitian endomorphism of (V, h). The tuple (V, h, A, Φ) is said to be a monopole on $X \setminus Z$ if the tuple (V, h, A, Φ) satisfies the Bogomolny equation $F(A) = *\nabla_A(\Phi)$, where $F(A)$ is the curvature of A and $*$ is the Hodge operator. Moreover, A point $p \in Z$ is a Dirac-type singularity of (V, h, A, Φ) of weight $\vec{k}_p = (k_{p,i}) \in \mathbb{Z}^{\mathrm{rank}(V)}$ if the monopole (V, h, A, Φ) satisfies a certain asymptotic behavior around $p \in Z$ (See Definition 2.1 (ii).). We set the Dirac operators $\partial\!\!\!/^{\pm}_{(A,\Phi)} : \Gamma(X \setminus Z, V \otimes S_X) \to \Gamma(X \setminus Z, V \otimes S_X)$ of (V, h, A, Φ) to be $\partial\!\!\!/^{\pm}_{(A,\Phi)}(s) := \partial\!\!\!/_A(s) \pm \Phi \otimes \mathrm{Id}_{S_X}$, where S_X is the spinor bundle on X and $\partial\!\!\!/_A$ is the Dirac operator of (V, h, A). We regard $\partial\!\!\!/^{\pm}_{(A,\Phi)}$ as a closed operator $L^2(X \setminus Z, V \otimes S_X) \to L^2(X \setminus Z, V \otimes S_X)$ by considering derivation as a current. The main result is the following.

Theorem 1.1 (Theorem 4.4)**.** Let (V, h, A, Φ) be a monopole of rank r on $X \setminus Z$ such that each $p \in Z$ is a Dirac-type singularity of (V, h, A, Φ) with weight $\vec{k}_p = (k_{p,i}) \in \mathbb{Z}^r$. We assume that (V, h, A, Φ) satisfies the following conditions (*the Råde condition*).

- Both Φ and $F(A)$ are bounded.
- We have $\nabla_A(\Phi)|_x = o(1)$ as $x \to \infty$.
- There exists a compact religion $Y \supset Z$ such that Y has a smooth boundary ∂Y, and the inequality $\inf_{x \in X \setminus Y}\{|\lambda| \mid \lambda \text{ is an eigenvalue of } \Phi(x)\} > 0$ is satisfied.

Then the Dirac operators $\partial\!\!\!/^{\pm}_{(A,\Phi)}$ are Fredholm and adjoint to each other. Moreover, their indices $\mathrm{Ind}(\partial\!\!\!/^{\pm}_{(A,\Phi)})$ are given as follows:

$$\mathrm{Ind}(\partial\!\!\!/^{\pm}_{(A,\Phi)}) = \mp\left\{\sum_{p \in Z}\sum_{k_{p,i}>0} k_{p,i} + \int_{\partial Y} ch(V^+)\right\} = \pm\left\{\sum_{p \in Z}\sum_{k_{p,i}<0} k_{p,i} + \int_{\partial Y} ch(V^-)\right\},$$

where $V^{\pm}$ is a subbundle of $V|_{\partial Y}$ spanned by the eigenvectors of $\mp\sqrt{-1}\Phi$ with positive eigenvalues.

The celebrated index theorem proved by Atiyah and Singer have been applied in a wide range including gauge theory, differential topology and complex geometry. However, The indices of elliptic differential operators on odd-dimensional closed manifolds are always 0. Therefore we consider the index theorems of elliptic operators on odd-dimensional open manifolds. On one hand, Callias [3] proved the index theorem of the Dirac operators of $SU(2)$-bundles on $\mathbb{R}^{2n+1}$ that satisfies a certain asymptotic behavior at infinity. Callias's index theorem is generalized to the Dirac operators of vector bundles on odd-dimensional complete spin manifolds by Råde [10]. On the other hand, Kronheimer [6] defined the notion of Dirac-type singularities of monopoles on flat Riemannian 3-folds, and Pauly [9] generalize it to any Riemannian 3-folds. Moreover, Pauly proved the index theorem of the deformation complexes on $SU(2)$-monopoles with Dirac-type singularities on closed oriented 3-folds. However, Pauly's argument essentially needs the condition $\sum_i k_{p,i} = 0$ for any $p \in Z$, and it is difficult to apply the argument to calculate the indices of the Dirac operators of (V, h, A, Φ) even if X is a closed manifold.

The proof of the main result is divided into two parts. First we extend Pauly's argument and calculate the indices of $\eth^{\pm}_{(A,\Phi)}$ when X is a closed manifold (Theorem 3.4 and Corollary 3.12), by constructing a lift of (V, h, A, Φ) on a 4-dimensional closed manifold equipped with an S^1-action. Next we combine our result and Råde's index theorem in [10], and obtain the index formula on general complete Riemannian 3-folds (Theorem 4.4).

This result was obtained in the study of the inverse transform of the Nahm transform from L^2-finite instantons on the product of $\mathbb{R}$ and a 3-dimensional torus T^3 to Dirac-type singular monopoles on the dual torus $\hat{T}^3$ of T^3 in [11].

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2 Preliminary

2.1 Monopoles with Dirac-type singularities

We recall the definition of monopoles with Dirac-type singularities by following [5].

Definition 2.1. Let (X, g) be an oriented Riemannian 3-fold and $*_g$ be the Hodge operator on X. If there is no risk of confusion, then we abbreviate $*_g$ to just $*$.

(i) Let (V, h) be a Hermitian vector bundle with a unitary connection A on X. Let Φ be a skew-Hermitian endomorphism of V. The tuple (V, h, A, Φ) is said to be a monopole on X if it satisfies the Bogomolny equation $F(A) = *\nabla_A(\Phi)$.

(ii) Let $Z \subset X$ be a discrete subset. Let (V, h, A, Φ) be a monopole of rank $r \in \mathbb{N}$ on $X \setminus Z$. A point $p \in Z$ is called a Dirac-type singularity of the monopole (V, h, A, Φ) with weight $\vec{k}_p = (k_{p,i}) \in \mathbb{Z}^r$ if the following holds.

- There exists a small neighborhood B of p such that $(V, h)|_{B \setminus \{p\}}$ is decomposed into a sum of Hermitian line bundles $\bigoplus_{i=1}^r F_{p,i}$ with $\deg(F_{p,i}) = \int_{\partial B} c_1(F_{p,i}) = k_{p,i}$.

- In the above decomposition, we have the following estimates,

$$\begin{cases} \Phi = \dfrac{\sqrt{-1}}{2R_p} \displaystyle\sum_{i=1}^{r} k_{p,i} \cdot Id_{F_{p,i}} + O(1) \\ \nabla_A(R_p\Phi) = O(1), \end{cases}$$

  where R_p is the distance from p.

For a monopole (V, h, A, Φ) on $X \setminus Z$, if each point $p \in Z$ is a Dirac-type singularity, then we call (V, h, A, Φ) a Dirac-type singular monopole on (X, Z).

Remark 2.2. Let X be a compact manifold and $Z \subset X$ a finite subset. For a Dirac-type singular monopole (V, h, A, Φ) on (X, Z), we have $\sum_{p\in Z}\sum_i k_{p,i} = 0$ by the Stokes theorem, where $\vec{k} = (k_{p,i}) \in \mathbb{Z}^{\mathrm{rank}(V)}$ is the weight of (V, h, A, Φ) at $p \in Z$.

We also recall the notion of instantons.

Definition 2.3. Let (Y, g) be an oriented Riemannian 4-fold. For a Hermitian vector bundle (V, h) on Y and a connection A on (V, h), the tuple (V, h, A) is an instanton if the ASD equation $F(A) = - * F(A)$ is satisfied.

Remark 2.4. If (Y, g) is a Kähler surface with the Kähler form ω, the condition that a tuple (V, h, A) is an instanton on Y is equivalent to the one that $(V, \overline{\partial}_A, h)$ is a holomorphic Hermitian vector bundle satisfying the Hermite-Einstein condition $F(A) \wedge \omega = 0$, where $\overline{\partial}_A$ is the $(0, 1)$-part of ∇_A.

For example, we recall the flat Dirac monopole of weight $k \in \mathbb{Z}$. Let $g_{i,\mathrm{Euc}}$ denote the canonical metric on $\mathbb{R}^i$. For $i \in \mathbb{N}$, we denote by $r_i : \mathbb{R}^i \to \mathbb{R}$ the distance from $0 \in \mathbb{R}^i$. Let $p : \mathbb{R}^3 \setminus \{0\} \to S^2(\simeq \mathbb{P}^1)$ be the projection. Let $\mathcal{O}(k)$ be a holomorphic line bundle on $\mathbb{P}^1$ of degree k. Let $h_{\mathcal{O}(k)}$ be a Hermitian metric of $\mathcal{O}(k)$ that the Chern connection $A_{\mathcal{O}(k)}$ of $(\mathcal{O}_k, h_{\mathcal{O}(k)})$ has a constant mean curvature. Then $(p^*\mathcal{O}(k), p^*h_{\mathcal{O}(k)}, p^*A_{\mathcal{O}(k)}, \sqrt{-1}k/2r)$ is a Dirac-type singular monopole on $(\mathbb{R}^3, \{0\})$, where r is the distance from the origin. We call this monopole the flat Dirac monopole of weight k, and denote by (L_k, h_k, A_k, Φ_k).

We will recall the equivalent condition proved by Pauly [9]. Let $U \subset \mathbb{R}^3$ be a neighborhood of $0 \in \mathbb{R}^3$. Let g be a Riemannian metric on U. We assume that the canonical coordinate of $\mathbb{R}^3$ is a normal coordinate of g at 0. Set the Hopf map $\pi : \mathbb{R}^4 = \mathbb{C}^2 \to \mathbb{R}^3 = \mathbb{R} \times \mathbb{C}$ to be $\pi(z_1, z_2) = (|z_1|^2 - |z_2|^2, 2z_1z_2)$, where we set $z_i = x_i + \sqrt{-1}y_i$. We also set the $S^1(= \mathbb{R}/2\pi\mathbb{Z})$-action on $\mathbb{C}^2$ to be $\theta \cdot (z_1, z_2) := (e^{\sqrt{-1}\theta}z_1, e^{-\sqrt{-1}\theta}z_2)$. Then the restriction $\pi : \mathbb{R}^4 \setminus \{0\} \to \mathbb{R}^3 \setminus \{0\}$ forms a principal S^1-bundle. Then we have $\pi^*r_3 = r_4^2$.

Lemma 2.5. There exist a harmonic function $f : U \setminus \{0\} \to \mathbb{R}$ with respect to the metric g and a 1-form ξ on $\pi^{-1}(U)$ such that the following hold.

- The 1-form $\omega := \xi/\pi^*f$ is a connection of $\pi : \mathbb{R}^4 \setminus \{0\} \to \mathbb{R}^3 \setminus \{0\}$, *i.e.* ω is S^1-invariant, and we have $\omega(\partial_\theta) = 1$. Here ∂_θ is the generating vector field of the S^1-action on $\mathbb{R}^4 \setminus \{0\}$.
- We have $d\omega = \pi^*(* - df)$.
- We have the following estimates:

$$\begin{cases} f = 1/2r_3 + o(1) \\ \xi = 2(-y_1dx_1 + x_1dy_1 + y_2dx_2 - x_2dy_2) + O(r_4^2). \end{cases}$$

- The symmetric tensor $g_4 = \pi^* f(\pi^* g + \xi^2)$ is a Riemannian metric of $L^2_{5,\mathrm{loc}}$-class on $\pi^{-1}(U)$, and we have an estimate $|g_4 - 2g_{4,\mathrm{Euc}}|_{g_{4,\mathrm{Euc}}} = O(r_4)$. Here a function on $\pi^{-1}(U)$ is of $L^2_{k,\mathrm{loc}}$-class if every derivative of f up to order k has a finite L^2-norm on any compact subset of $\pi^{-1}(U)$.

Proposition 2.6 (Proposition 5 in [9])**.** Let (V, h, A) be a Hermitian vector bundle on $U \setminus \{0\}$ of rank r, and $\Phi \in \mathrm{End}(V)$ be a skew-Hermitian endomorphism. The tuple (V, h, A, Φ) is a monopole on $U \setminus \{0\}$ if and only if the tuple $(\pi^*V, \pi^*h, \pi^*A + \xi \otimes \pi^*\Phi)$ is an instanton on $\pi^{-1}(U) \setminus \{0\}$ with respect to the metric $g_4 = \pi^* f(\pi^* g + \xi^2)$. Moreover, 0 is a Dirac-type singularity of (V, h, A, Φ) of weight $\vec{k} = (k_i) \in \mathbb{Z}^r$ if and only if the following hold.

- The instanton $(\pi^*V, \pi^*h, \pi^*A - \pi^*\Phi \otimes \xi)$ can be prolonged over $\pi^{-1}(U)$, and the prolonged connection is represented by an $L^2_{6,\mathrm{loc}}$-valued connection matrix. We will denote by (V_4, h_4, A_4) the prolonged instanton.
- The weight of the S^1-action on the fiber $V_4|_0$ agrees with $\vec{k}$ up to a suitable permutation.

Remark 2.7.

- If $g = g_{3,\mathrm{Euc}}$, we can choose $f = 1/2r_3$ and $\xi = 2(-y_1 dx_1 + x_1 dy_1 + y_2 dx_2 - x_2 dy_2)$. Then we have $g_4 = 2g_{4,\mathrm{Euc}}$.
- By the Sobolev embedding theorem, the connection matrix of A_4 is of C^3 class.

Let $h_{\mathbb{C}}$ be the canonical Hermitian metric on $\mathbb{C}$. We set the Hermitian line bundle $(\tilde{L}, \tilde{h}) := (\pi^{-1}(U) \setminus \{0\}) \times_{U(1)} (\mathbb{C}, h_{\mathbb{C}})$ on $U \setminus \{0\}$ and take the connection $\tilde{A}$ induced by ω. Then $(\tilde{L}, \tilde{h}, \tilde{A}, \sqrt{-1}f)$ is a monopole on U with respect to g, and 0 is the Dirac-type singularity of weight 1. We call the monopole $(\tilde{L}_k, \tilde{h}_k, \tilde{A}_k, \sqrt{-1}kf) := (\tilde{L}^{\otimes k}, \tilde{h}^{\otimes k}, \tilde{A}^{\otimes k}, \sqrt{-1}kf)$ a Dirac monopole of weight k with respect to g. The following proposition is a partial generalization of [7, Proposition 5.2].

Proposition 2.8. Let (V, h, A, Φ) be a monopole on $U \setminus \{0\}$, and assume that the point 0 is a Dirac-type singularity of weight $\vec{k} = (k_i) \in \mathbb{Z}^r$. Then there exist a neighborhood $U' \subset U$ and a unitary isomorphism $\varphi : V|_{U' \setminus \{0\}} \simeq (\bigoplus_{i=1}^r \tilde{L}_{k_i})|_{U' \setminus \{0\}}$ such that the following estimates hold.

$$|A - \varphi^*(\bigoplus \tilde{A}_{k_i})| = O(1).$$
$$|\Phi - \varphi^*(\sum \sqrt{-1}k_i f \,\mathrm{Id}_{L_{k_i}})| = O(1).$$

Proof. Let (V', h', A', Φ') be the monopole $\bigoplus_{i=1}^r (\tilde{L}_{k_i}, \tilde{h}_{k_i}, \tilde{A}_{k_i}, \sqrt{-1}k_i f)$. By Proposition 2.6, the instantons $(\pi^*V, \pi^*h, \pi^*A - \pi^*\Phi \otimes \xi)$ and$(\pi^*V', \pi^*h', \pi^*A' - \pi^*\Phi' \otimes \xi)$ can be prolonged over $\pi^{-1}(U)$, and denote by (V_4, h_4, A_4) and (V'_4, h'_4, A'_4) respectively. Then the weights of S^1-actions on the fiber of V_4 and V'_4 at the origin coincide with each other, and the connections A_4 and A'_4 are S^1-invariant. Hence there exist an S^1-invariant neighborhood $U'_4 \subset \pi^{-1}(U)$ of 0 and an S^1-equivariant unitary isomorphism $\varphi_4 : V_4|_{U'_4} \to V'_4|_{U'_4}$ such that $A_4 - \varphi_4^*(A'_4)$ vanishes at the origin. Hence we have $|A_4 - \varphi_4^*(A'_4)| = O(r_4)$. Since $f = 1/2r_3 + o(1)$ and ξ is orthogonal to $\pi^*(T^*\mathbb{R}^3)$ with the metric $g_4 = \pi^* f(\pi^* g + \xi^2)$, the unitary isomorphism $\varphi : V|_{U' \setminus \{0\}} \to V'|_{U' \setminus \{0\}}$ induced by φ_4 satisfies the desired estimates, where we put $U' := \pi(U'_4)$. ∎

By the estimates in Lemma 2.5, we also obtain the following approximation.

Corollary 2.9. Let (V, h, A, Φ) be a monopole on $U \setminus \{0\}$, and assume that the point 0 is a Dirac-type singularity of weight $\vec{k} = (k_i) \in \mathbb{Z}^r$. Then there exist a neighborhood $U' \subset U$ and a unitary isomorphism $\varphi : V|_{U'} \simeq (\bigoplus_{i=1}^r L_{k_i})|_{U'}$ such that the following estimates hold.

$$|A - \varphi^*(\bigoplus A_{k_i})| = O(1).$$
$$|\Phi - \varphi^*(\frac{\sqrt{-1}}{2r_3}\sum k_i \mathrm{Id}_{L_{k_i}})| = O(1).$$

2.2 Local properties of harmonic spinors of the flat Dirac monopoles

Let (X, g) be an n-dimensional oriented spin manifold with a fixed spin structure. We denote by S_X the spinor bundle on X, and by $\mathrm{clif} : T^*X \to \mathrm{End}(S_X)$ the Clifford product. If n is an odd number, then we assume $(\sqrt{-1})^{(n+1)/2}\mathrm{clif}(\mathrm{vol}_{(X,g)}) = -\mathrm{Id}_{S_X}$, where we use the canonical linear isomorphism between the exterior algebra and the Clifford algebra. The spinor bundle S_X has the induced connection A_{S_X} by the Levi-Civita connection on X, and we set the Dirac operator $\partial\!\!\!/_X : \Gamma(X, S_X) \to \Gamma(X, S_X)$ to be $\partial\!\!\!/_X(f) := \mathrm{clif} \circ \nabla_{A_{S_X}}(f)$. For a vector bundle (V, h) on X and a connection A on (V, h), we also set the Dirac operator $\partial\!\!\!/_A : \Gamma(X, S_X \otimes V) \to \Gamma(X, S_X \otimes V)$ to be $\partial\!\!\!/_A(s) := \mathrm{clif} \circ \nabla_{A_{S_X}\otimes A}(s)$. If n is even, then we have the decomposition $S_X = S_X^+ \oplus S_X^-$, and the Dirac operator $\partial\!\!\!/_A$ is also decomposed into sum of the positive and negative Dirac operators $\partial\!\!\!/_A^\pm : \Gamma(X, S_X^\pm \otimes V) \to \Gamma(X, S_X^\mp \otimes V)$. If $\dim(X) = 3$, then for a monopole (V, h, A, Φ) on X we also set the Dirac operators $\partial\!\!\!/_{(A,\Phi)}^\pm : \Gamma(X, V \otimes S_X) \to \Gamma(X, V \otimes S_X)$ to be $\partial\!\!\!/_{(A,\Phi)}^\pm(f) := \partial\!\!\!/_A(f) \pm (\Phi \otimes \mathrm{Id}_{S_X})(f)$.

For a differential operator $P : \Gamma(X, V_1) \to \Gamma(X, V_2)$ between Hermitian vector bundles (V_1, h_1) and (V_2, h_2) on X, we regard P as the closed operator $P : L^2(X, V_1) \to L^2(X, V_2)$ with the domain $\mathrm{Dom}(P) := \{s \in L^2(X, V_1) \mid P(s) \in L^2\}$, where $P(s)$ is the derivative as a current. We regard $\mathrm{Dom}(P)$ as a Banach space equipped with the graph norm $||s||_P := ||s||_{L^2} + ||P(s)||_{L^2}$.

Remark 2.10. Any 3-dimensional oriented manifolds are parallelizable, and hence they have spin structures.

Let $S_{\mathbb{R}^3}$ be the spinor bundle on $\mathbb{R}^3$ with respect to the trivial spin structure, and d be the trivial connection on $S_{\mathbb{R}^3}$. By using the projection $p : \mathbb{R}^3 \setminus \{0\} \to S^2$, We combine the Dirac operators of the Dirac monopole $(L_k, h_k, A_k, \Phi_k) = (p^*\mathcal{O}(k), p^*h_{\mathcal{O}(k)}, p^*A_{\mathcal{O}(k)}\sqrt{-1}k/2r)$ with the Dirac operators of $\mathcal{O}(k)$ on $\mathbb{P}^1 = S^2$. Let $S_{S^2} = S_{S^2}^+ \oplus S_{S^2}^-$ be the spinor bundle on (S^2, g_{S^2}), and $\partial\!\!\!/_{S^2}^\pm : \Gamma(S^2, S_{S^2}^\pm) \to \Gamma(S^2, S_{S^2}^\mp)$ the Dirac operators on S^2. By the isometry $\mathbb{R}^3 \simeq \left(\mathbb{R}_+ \times S^2, (dr_3)^2 + r_3^2 g_{S^2}\right)$ we obtain the unitary isomorphisms $S_{\mathbb{R}^3}|_{\mathbb{R}^3\setminus\{0\}} \simeq p^*S_{S^2}$. According to Nakajima [8], under the identification $S_{\mathbb{R}^3}|_{\mathbb{R}^3\setminus\{0\}} \simeq p^*S_{S^2}$ the Dirac operator $\partial\!\!\!/_{\mathbb{R}^3}$ on $\mathbb{R}^3 \setminus \{0\}$ is written as follows:

$$\partial\!\!\!/_{\mathbb{R}^3} = \frac{1}{r_3}\begin{pmatrix} \sqrt{-1}(r_3\frac{\partial}{\partial r_3} + 1) & \partial\!\!\!/_{S^2}^- \\ \partial\!\!\!/_{S^2}^+ & -\sqrt{-1}(r_3\frac{\partial}{\partial r_3} + 1) \end{pmatrix}.$$

Therefore we obtain the following equality.

$$\partial\!\!\!/^{\pm}_{(A_k,\Phi_k)} = \frac{1}{r_3}\left(\begin{array}{cc} \sqrt{-1}(r_3\dfrac{\partial}{\partial r_3}+\dfrac{2\pm k}{2}) & \partial\!\!\!/^{-}_{\mathcal{O}(k)} \\ \partial\!\!\!/^{+}_{\mathcal{O}(k)} & -\sqrt{-1}(r_3\dfrac{\partial}{\partial r_3}+\dfrac{2\mp k}{2}) \end{array}\right).$$

By the isomorphisms $S^+_{S^2} \simeq \Omega^{0,0}(\mathcal{O}(-1))$, $S^-_{S^2} \simeq \Omega^{0,1}(\mathcal{O}(-1))$ and $\partial\!\!\!/_{S^2} = \partial\!\!\!/^+_{S^2} + \partial\!\!\!/^-_{S^2} = \sqrt{2}(\overline{\partial}_{\mathcal{O}(-1)} + \overline{\partial}^{\star}_{\mathcal{O}(-1)})$, we obtain $\mathrm{Ker}(\partial\!\!\!/^+_{\mathcal{O}(k)}) \simeq H^0(\mathbb{P}^1, \mathcal{O}(k-1))$ and $\mathrm{Ker}(\partial\!\!\!/^-_{\mathcal{O}(k)}) \simeq H^1(\mathbb{P}^1, \mathcal{O}(k-1))$, where $\overline{\partial}^{\star}_{\mathcal{O}(-1)}$ is the formal adjoint of $\overline{\partial}_{\mathcal{O}(-1)}$. Let $f^{\pm}_{\nu} \in L^2(S^2, S^{\pm}_{S^2} \otimes \mathcal{O}(k))$ $(\nu \in \mathbb{N})$ be the all eigenvectors of the operators $\partial\!\!\!/^-_{\mathcal{O}(k)} \circ \partial\!\!\!/^+_{\mathcal{O}(k)}$ and $\partial\!\!\!/^+_{\mathcal{O}(k)} \circ \partial\!\!\!/^-_{\mathcal{O}(k)}$ with non-zero eigenvalues respectively. We set $n_\nu > 0$ to be the eigenvalue of $f^{\pm}_{\nu}$. Then, According to [1], we have $\{n_\nu\} = \{q^2 + |k|q \ ; \ q \in \mathbb{N}\}$. We set $q_\nu > 0$ to satisfy $n_\nu = q_\nu^2 + |k|q_\nu$. We may assume that $\{f^{\pm}_{\nu}\}$ forms an orthonormal system and satisfies the relations $\partial\!\!\!/^{\pm}_{\mathcal{O}(k)}(f^{\pm}_{\nu}) = \sqrt{n_\nu} f^{\mp}_{\nu}$ for any $\nu \in \mathbb{N}$. By the elliptic inequality and the Sobolev inequality, there exist $C', C'' > 0$ such that $||f^{\pm}_{\nu}||_{L^6} < C''||f^{\pm}_{\nu}||_{L^2_1} \leq C'(||f^{\pm}_{\nu}||_{L^2} + ||\partial\!\!\!/^{\pm}_{\mathcal{O}(k)}(f^{\mp}_{\nu})||_{L^2}) = C'(1 + \sqrt{n_\nu})$. Then by the interpolation inequality we obtain $||f^{\pm}_{\nu}||_{L^3} \leq (||f^{\pm}_{\nu}||_{L^2})^{1/2} \cdot (||f^{\pm}_{\nu}||_{L^6})^{1/2} = C\sqrt{1 + \sqrt{n_\nu}}$, where we put $C := \sqrt{C'}$. Hence we obtain the following lemma.

Lemma 2.11. We have the estimate $||f^{\pm}_{\nu}||_{L^3} = O(\sqrt{q_\nu})$.

Through the above arguments, we obtain the following proposition.

Proposition 2.12. Let s be a section of $L_k \otimes S_{\mathbb{R}^3}$ on a punctured ball $B(r)^* := \{x \in \mathbb{R}^3 \mid 0 < |x| < r\}$ for some $r > 0$.

(i) If we have $s \in L^2(B(r)^*, L_k \otimes S_{\mathbb{R}^3}) \cap \mathrm{Ker}(\partial\!\!\!/^+_{(A_k,\Phi_k)})$, then there exists a sequence $\{c_\nu\} \subset \mathbb{C}$ such that we have

$$s = \sum_{\nu \in \mathbb{N}} c_\nu \left(a^+_\nu(r) f^+_\nu + a^-_\nu(r) f^-_\nu\right).$$

Here the functions $a^{\pm}_{\nu}$ are given as follows:

$$a^+_\nu(r) = r^{-1+q_\nu+|k|/2}.$$

$$a^-_\nu(r) = \frac{q_\nu + \max(0,k)}{\sqrt{-1}\sqrt{q_\nu^2 + |k|q_\nu}}\, r^{-1+q_\nu+|k|/2}.$$

(ii) If we have $s \in L^2(B(r)^*, L_k \otimes S_{\mathbb{R}^3}) \cap \mathrm{Ker}(\partial\!\!\!/^-_{(A_k,\Phi_k)})$, then there exist a sequence $\{c_\nu\} \subset \mathbb{C}$, $\alpha^+ \in \mathrm{Ker}(\partial\!\!\!/^+_{\mathcal{O}(k)}) = H^0(\mathbb{P}^1, \mathcal{O}(k-1))$ and $\alpha^- \in \mathrm{Ker}(\partial\!\!\!/^-_{\mathcal{O}(k)}) = H^1(\mathbb{P}^1, \mathcal{O}(k-1))$ such that we have

$$s = \sum_{+,-} \alpha^{\pm}\rho^{\pm}(r) + \sum_{\nu \in \mathbb{N}} c_\nu \left(b^+_\nu(r) f^+_\nu + b^-_\nu(r) f^-_\nu\right).$$

Here the functions $\rho^{\pm}$ and $b^{\pm}_{\nu}$ are given as follows:

$$\rho^{\pm}(r) = r^{-1\pm k/2}.$$

$$b^+_\nu(r) = r^{-1+q_\nu+|k|/2}.$$

$$b^-_\nu(r) = \frac{q_\nu + \max(0,-k)}{\sqrt{-1}\sqrt{q_\nu^2 + |k|q_\nu}}\, r^{-1+q_\nu+|k|/2}.$$

By the above proposition, we obtain the following corollary.

Corollary 2.13. For arbitrary positive numbers $r > r' > 0$, the restriction map $L^2(B(r)^*, L_k \otimes S_{\mathbb{R}^3}) \cap \mathrm{Ker}(\partial\!\!\!/^{\pm}_{(A_k,\Phi_k)}) \to L^2(B(r')^*, L_k \otimes S_{\mathbb{R}^3})$ is a compact map.

As a preparation of Proposition 2.15, we prove the following lemma.

Lemma 2.14. Let $t_0 > 0$ be a positive number and α a real number. Set the constant C_α is given by

$$C_\alpha = \begin{cases} |2\alpha - 1|^{-1/2} & (\alpha \neq 1/2) \\ 1 & (\alpha = 1/2). \end{cases}$$

There exists a compact operator $K_\alpha : L^2(0, t_0) \to C^0([0, t_0])$ such that for any $f \in L^2(0, t_0)$, the function $g := K_\alpha(f)$ satisfies the estimate $|g(t)| \leq C_\alpha ||f||_{L^2} \cdot t^{1/2}(1 + \log(t_0/t)^{1/2}) \leq C_\alpha ||f||_{L^2} \cdot \sqrt{t_0}(1 + 1/\sqrt{e})$ and the differential equation $t\partial_t(g/t) + \alpha(g/t) = f$, where $C^0([0, t_0])$ is the Banach space consisting of bounded continuous functions on $[0, t_0]$.

Proof. We set $g = K_\alpha(f)$ to be

$$g(t) := \begin{cases} t^{-\alpha+1} \displaystyle\int_0^t f(x)x^{\alpha-1}dx & (\alpha > 1/2) \\ -t^{-\alpha+1} \displaystyle\int_t^{t_0} f(x)x^{\alpha-1}dx & (\alpha \leq 1/2). \end{cases}$$

Then, by a direct calculation we have $t\partial_t(g/t) + \alpha(g/t) = f$. If $\alpha \neq 1/2$, then we obtain $|g(t)| \leq t^{-\alpha+1}||f||_{L^2}\sqrt{t^{2\alpha-1}/|2\alpha - 1|} = |2\alpha - 1|^{-1/2}||g||_{L^2} \cdot t^{1/2}$. If $\alpha = 1/2$, then we have $|g(t)| \leq ||f||_{L^2} \cdot t^{1/2} \log(t_0/t)^{1/2}$. As a consequence of the above inequalities, we obtain the desired estimate. By this estimate, the compactness of K_α follows from the Ascoli-Arzelà theorem and the differential equation. □

Proposition 2.15. Let $r > 0$ be a positive number. There exists a compact map $G^{\pm} : L^2(B(r)^*, L_k \otimes S_{\mathbb{R}^3}) \to L^2(B(r)^*, L_k \otimes S_{\mathbb{R}^3})$ such that we have $R(G^{\pm}) \subset \mathrm{Dom}(\partial\!\!\!/^{\pm}_{(A_k,\Phi_k)})$ and $\partial\!\!\!/^{\pm}_{(A_k,\Phi_k)} \circ G^{\pm} = \mathrm{Id}$. Moreover, we have $R(G^{\pm}) \subset L^3(B(r)^*, L_k \otimes S_{\mathbb{R}^3})$, and hence $G^{\pm} : L^2 \to L^3$ is bounded.

Proof. The proof for $\partial\!\!\!/^{+}_{(A_k,\Phi_k)}$ remains valid for $\partial\!\!\!/^{-}_{(A_k,\Phi_k)}$. Hence we prove only for $\partial\!\!\!/^{+}_{(A_k,\Phi_k)}$. The subspace that is spanned by $\mathrm{Ker}(\partial\!\!\!/^{\pm}_{\mathcal{O}(k)})$ and $\{f^{\pm}_\nu\}$ is dense in $L^2(S^2, S_{S^2} \otimes \mathcal{O}(k))$. Hence for any $s \in L^2(B(r)^*, L_k \otimes S_{\mathbb{R}^3})$ there exist measurable maps $\alpha^{\pm} : (0, r) \to \mathrm{Ker}(\partial\!\!\!/^{\pm}_{\mathcal{O}(k)})$ and $s^{\pm}_\nu : (0, r) \to \mathbb{C}$ such that we have

$$s = \alpha^+ + \alpha^- + \sum_\nu \left(s^+_\nu f^+_\nu + s^-_\nu f^-_\nu\right)$$

and

$$||s||^2_{L^2} = ||r_3\alpha^+||^2_{L^2} + ||r_3\alpha^-||^2_{L^2} + \sum_\nu \left(||r_3 s^+_\nu||^2_{L^2} + ||r_3 s^-_\nu||^2_{L^2}\right).$$

By some linear-algebraic operations and Lemma 2.14, we can take an element $t=\beta^+ + \beta^- + \sum_\nu (t_\nu^+ f_\nu^+ + t_\nu^- f_\nu^-) \in L^2(B(r)^*, L_k \otimes S_{\mathbb{R}^3})$ such that we have $\partial\!\!\!/^+_{(A_k,\Phi_k)}(t) = s$ and

$$\begin{aligned}||t||_{L^2}^2 &= ||r_3\beta^+||_{L^2}^2 + ||r_3\beta^-||_{L^2}^2 + \sum_\nu \left(||r_3 t_\nu^+||_{L^2}^2 + ||r t_\nu^-||_{L^2}^2\right) \\ &\leq ||r_3\alpha^+||_{L^2}^2 + ||r_3\alpha^-||_{L^2}^2 \\ &\quad + \sum_\nu \left\{ \max(C_{1+(2q_\nu+k)/2}, C_{1-(2q_\nu+k)/2})^2 \left(||r_3 s_\nu^+||_{L^2}^2 + ||r_3 s_\nu^-||_{L^2}^2\right)\right\},\end{aligned}$$

where C_α is the constant in Lemma 2.14. Then We set $G^+(s) := t$, and G^+ is linear because all constructions of G^+ are linear. Since $C_{1\pm(2q_\nu+k)/2} = o(1)$ $(\nu \to \infty)$, the compactness of G^+ is deduced from the compactness of K_α in Lemma 2.14.

By the definition we have $2\sqrt{q_\nu}\cdot C_{1\pm(2q_\nu+k)/2} \to 1$ $(\nu \to \infty)$. Hence $||f_\nu^\pm||_{L^3}\cdot C_{1\pm(2q_\nu+k)/2} = O(1)$ by Lemma 2.11. Therefore we obtain $||t||_{L^3} < \infty$ and the proof is complete. □

Corollary 2.16. For any positive numbers $r > r' > 0$, the restriction map $L^2(B(r)^*, L_k \otimes S_{\mathbb{R}^3}) \cap \mathrm{Dom}(\partial\!\!\!/^\pm_{(A_k,\Phi_k)}) \to L^2(B(r')^*, L_k \otimes S_{\mathbb{R}^3})$ is a compact operator.

Proof. Let $\{f_n\}$ be a bounded sequence in $L^2(B(r)^*, L_k \otimes S_{\mathbb{R}^3}) \cap \mathrm{Dom}(\partial\!\!\!/^\pm_{(A_k,\Phi_k)})$. By using $G^\pm$ in Proposition 2.15, we set $\tilde{f}_n := G^\pm(\partial\!\!\!/^\pm_{(A_k,\Phi_k)}(f_n))$. Since $G^\pm$ is compact, there exists a subsequence $\{f_{n_k}\}$ such that $\{\tilde{f}_{n_k}\}$ is convergent. Hence we may assume that $\{\tilde{f}_n\}$ is convergent. Then we have $\partial\!\!\!/^\pm_{(A_k,\Phi_k)}(f_n - \tilde{f}_n) = 0$. By Corollary 2.13, $\{(f_n - \tilde{f}_n)|_{B(r')}\}$ has a convergent subsequence. Therefore $\{f_n|_{B(r')}\}$ also has a convergent subsequence. □

2.3 A local lift of the Dirac operators of the flat Dirac monopoles

Let $k \in \mathbb{Z}$ be an integer. For the flat Dirac monopole $(V, h, A, \Phi) := (L_k, h_k, A_k, \sqrt{-1}k/2r_3)$ on $(\mathbb{R}^3, \{0\})$, we denote by (V_4, h_4, A_4) the prolongation of the instanton $(\pi^*V, \pi^*h, \pi^*A - \xi \otimes \pi^*\Phi)$ over $\mathbb{R}^4$, where $\xi = 2\{(x_1dy_1 - y_1dx_1) - (x_2dy_2 - y_2dx_2)\}$. We compare the Dirac operators $\partial\!\!\!/^\pm_{(A,\Phi)}$ and $\partial\!\!\!/^\pm_{A_4}$.

We denote by X and P the punctured spaces $\mathbb{R}^3 \setminus \{0\}$ and $\mathbb{R}^4 \setminus \{0\}$ respectively. Set the function $f : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}_+$ to be $f(t, x, y) := 1/2r_3$. We also set $g_P := 2g_{4,\mathrm{Euc}}$. Since $g_P = 2g_{4,\mathrm{Euc}} = \pi^* f(\pi^* g + \xi^2)$, we have the orthogonal decomposition $TP \simeq \mathbb{R}\partial_\theta \oplus \pi^*TX$. Let $\mathscr{S}$ be the spin structure of $\mathbb{R}^3$ *i.e.* $\mathscr{S}$ is a principal $Spin(3)$-bundle on $\mathbb{R}^3$ that satisfies $\mathscr{S} \times_{Spin(3)} (\mathbb{R}^3, g_{3,Euc}) \simeq T\mathbb{R}^3$. Let $\rho : Spin(3) \to Spin(4)$ be the lift of the homomorphism $SO(3) \to SO(4)$ which is induced by $\mathbb{R}^3 \ni p \to (0, p) \in \mathbb{R}^4$. We set $\mathscr{S}_4 := \pi^*(\mathscr{S}) \times_\rho Spin(4)$. Then we have $\mathscr{S}_4 \times_{Spin(4)} (\mathbb{R}^4 \setminus \{0\}) \simeq (P \times \mathbb{R}) \oplus \pi^*TX$, and hence $\mathscr{S}_4$ is a spin structure on P. Under the isomorphisms $Spin(3) \simeq SU(2)$ and $Spin(4) \simeq SU(2)_+ \times SU(2)_-$, the homomorphism ρ is written as $\rho(g) = (g, g)$. Therefore we have the following proposition.

Proposition 2.17. The following claims are satisfied.

- We have the unitary isomorphisms $\pi^*S_X \simeq S_P^\pm$.
- Under the above isomorphisms, the Clifford product on P can be represented as follows:

$$\begin{aligned}\mathrm{clif}_P(\xi) &= (\pi^* f)^{-1/2} \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}. \\ \mathrm{clif}_P(\pi^*\alpha) &= (\pi^* f)^{-1/2} \begin{pmatrix} 0 & \mathrm{clif}_X(\alpha) \\ \mathrm{clif}_X(\alpha) & 0 \end{pmatrix} \qquad (\alpha \in \Gamma(X, \Omega^1(X))).\end{aligned}$$

Since the isomorphisms $\pi^*S_X \simeq S_P^{\pm}$ are unitary, we have $||\pi^*s||^2_{L^2(P)} = \int_P |\pi^*s|^2(-\pi^*f^2 \cdot \xi \wedge \pi^* d\mathrm{vol}_X) = 2\pi||f^{1/2}s||^2_{L^2(X)}$ for any $s \in \Gamma(X, V \otimes S_X)$. Hence the operator $\pi^\dagger(s) := \pi^*((2\pi f)^{-1/2}\, s)$ are isometric isomorphisms between $L^2(X, V \otimes S_X)$ and $L^2(P, V_4 \otimes S_P^{\pm})^{S^1}$.

On one hand, we take a global flat unitary frame $\boldsymbol{e}^3 = (e^3_1, e^3_2)$ of S_X that satisfies the following.

$$\mathrm{clif}_X(dt)\boldsymbol{e}^3 = \boldsymbol{e}^3 \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}.$$
$$\mathrm{clif}_X(dx)\boldsymbol{e}^3 = \boldsymbol{e}^3 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$
$$\mathrm{clif}_X(dt)\boldsymbol{e}^3 = \boldsymbol{e}^3 \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}.$$

On the other hand, we have the isomorphisms $S_P^+ \simeq \Omega^{0,0}_{\mathbb{C}^2} \oplus \Omega^{0,2}_{\mathbb{C}^2}$ and $S_P^- \simeq \Omega^{0,1}_{\mathbb{C}^2}$. Moreover, under the isomorphisms we also have $\partial\!\!\!/_{A_4} = \sqrt{2}(\overline{\partial}_{A_4} + \overline{\partial}^{\bigstar}_{A_4})$ and $\mathrm{clif}_P(\alpha) = \sqrt{2}(\alpha^{(0,1)} \wedge - \lrcorner(\alpha^{(1,0)})^\flat)$ for a 1-form α on P, where $\lrcorner$ means the interior product and $(\alpha)^\flat$ is the image of α under the isomorphism $\Omega^{1,0}_{\mathbb{C}^2} \simeq T^{(0,1)}\mathbb{C}^2$ induced by the metric g_P. Here we set S^1-invariant global unitary frames $\boldsymbol{e}^{\pm} = (e_1^{\pm}, e_2^{\pm})$ of $S_P^{\pm}$ to be the following.

$$e_1^+ := 1.$$
$$e_2^+ := -\left(\pi^*(-\xi)^{0,1}/|\pi^*(-\xi)^{0,1}|\right) \wedge \left(\pi^*(d\bar{z})/|\pi^*(d\bar{z})|\right).$$
$$e_1^- := \pi^*(-\xi)^{0,1}/|\pi^*(-\xi)^{0,1}|.$$
$$e_2^+ := \pi^*(d\bar{z})/|\pi^*(d\bar{z})|.$$

Then, with respect to the frames $\boldsymbol{e}^{\pm}$ and $\boldsymbol{e}^3$, the representations of Clifford products of X and P coincide as in the sense of Proposition 2.17. Therefore we may assume $\pi^*\boldsymbol{e}^3 = \boldsymbol{e}^{\pm}$. Hence by a direct calculation we obtain the following proposition.

Proposition 2.18. For the flat Dirac monopole (V, h, A, Φ), the equalities

$$\pi^\dagger \circ \left(\partial\!\!\!/^+_{(A,\Phi)} f^{-1/2}\right)(s) = \partial\!\!\!/^+_{A_4} \circ \pi^\dagger(s)$$

and

$$\pi^\dagger \circ \left(f^{-1/2} \partial\!\!\!/^-_{(A,\Phi)}\right)(s) = \partial\!\!\!/^+_{A_4} \circ \pi^\dagger(s)$$

are satisfied for any $s \in \Gamma(X, V \otimes S_X)$.

3 An index formula of Dirac operators on compact 3-folds

Let (X, g) be a closed oriented spin 3-fold and Z a finite subset. Let $\mathscr{S}$ be a spin structure on (X, g) *i.e.* $\mathscr{S}$ is a principal $Spin(3)$-bundle on X such that $\mathscr{S} \times_{Spin(3)} (\mathbb{R}^3, g_{3,\mathrm{Euc}}) \simeq (TX, g)$. Let (V, h, A, Φ) be a Dirac-type singular monopole on (X, Z) of rank r, and we denote by $\vec{k}_p = (k_{p,i}) \in \mathbb{Z}^r$ the weight of (V, h, A, Φ) at each $p \in Z$.

3.1 Fredholmness of Dirac operators

For a sufficiently small $\varepsilon > 0$, we set $B(Z,\varepsilon) := \coprod_{p\in Z} B(p,\varepsilon) = \coprod_{p\in Z}\{x \in X \mid d_g(x,p) < \varepsilon\}$, where $d_g : X \times X \to \mathbb{R}$ is the distance function with respect to g. Let (x_p^1, x_p^2, x_p^3) be a normal coordinate at p on $B(p,\varepsilon)$, and set the flat metric g' on $B(Z,\varepsilon)$ to be $g'|_{B(p,\varepsilon)} := \sum_i (dx_p^i)^2$. We take a smooth bump function $\rho : X \to [0,1]$ satisfying $\rho(B(Z,\varepsilon/2)) = 1$ and $\rho(X \setminus B(Z, 3\varepsilon/4)) = 0$, and set a metric $\tilde{g} := (1-\rho)g + \rho \cdot g'$. Then we have $g|_{X\setminus B(Z,\varepsilon)} = \tilde{g}|_{X\setminus B(Z,\varepsilon)}$ and $|g - \tilde{g}|_g = O(R_p^2)$ on $B(p,\varepsilon)$ for any $p \in Z$, where R_p is the distance from p. Hence there exists an isometric isomorphism $\mu : (TX, g) \simeq (TM, \tilde{g})$ such that $\mu|_{X\setminus B(Z,\varepsilon)} = \mathrm{Id}_{TM}$ and $|\mu - \mathrm{Id}_{TM}|_g = O(R_p^2)$ on $B(p,\varepsilon)$ for any $p \in Z$. Therefore we obtain the following lemma.

Lemma 3.1. For a 1-form α, we have an equality $\mathrm{clif}_{(X,g)}(\alpha)|_{X\setminus B(Z,\varepsilon)} = \mathrm{clif}_{(X,\tilde{g})}(\alpha)|_{X\setminus B(Z,\varepsilon)}$ and an estimate $|\mathrm{clif}_{(X,g)}(\alpha) - \mathrm{clif}_{(X,\tilde{g})}(\alpha)| = |\alpha| \cdot O(R_p^2)$ on $B(p,\varepsilon)$ for any $p \in Z$, where $\mathrm{clif}_{(X,g)}$ and $\mathrm{clif}_{(X,\tilde{g})}$ denote the Clifford product with respect to g and $\tilde{g}$ respectively.

We also take a direct sum of the flat Dirac monopoles (V', h', A', Φ') on $(B(Z,\varepsilon) \setminus Z, g')$ to be $(V', h', A', \Phi')|_{B(p,\varepsilon)} = \bigoplus_{i=1}^r (L_{k_{p,i}}, h_{k_{p,i}}, A_{k_{p,i}}, \Phi_{k_{p,i}})$ for any $p \in Z$. By Corollary 2.9, there exists a unitary isomorphism $\varphi : V|_{B(Z,\varepsilon)\setminus Z} \simeq V'$ such that the estimates in Corollary 2.9 are satisfied. We set a connection $\tilde{A} := (1-\rho)A + \rho \cdot \varphi^* A'$ and an endomorphism $\tilde{\Phi} := (1-\rho)\Phi + \rho\cdot\varphi^*\Phi'$. Then for each $p \in Z$ the restriction $(V, h, \tilde{A}, \tilde{\Phi})|_{B(p,\varepsilon/2)\setminus\{p\}}$ is a direct sum of the flat Dirac monopoles, and $|A - \tilde{A}|$ and $|\Phi - \tilde{\Phi}|$ are bounded on $X \setminus Z$.

We denote by $\tilde{\eth}^{\pm}_{(A,\Phi)}$ and $\tilde{\eth}^{\pm}_{(\tilde{A},\tilde{\Phi})}$ the Dirac operators of the tuples (V, h, A, Φ) and $(V, h, \tilde{A}, \tilde{\Phi})$ with respect to the metrics $\tilde{g}$ respectively. In Proposition 3.3, we show the Fredholmness of $\tilde{\eth}^{\pm}_{(\tilde{A},\tilde{\Phi})}$. Consequently, we will prove the Fredholmness of $\eth^{\pm}_{(A,\Phi)}$ in Theorem 3.4.

Proposition 3.2. The injection maps $\mathrm{Dom}(\tilde{\eth}^{\pm}_{(\tilde{A},\tilde{\Phi})}) \to L^2(X, V \otimes S_X)$ are compact.

Proof. The norm $||s||_1 := ||s|_{X\setminus B(Z,\varepsilon/8)}||_{L^2} + ||s|_{B(Z,\varepsilon/4)^*}||_{L^2}$ on $L^2(X, V\otimes S_X)$ is equivalent to the ordinary L^2-norm on X. By the Rellich-Kondrachov theorem, the restriction maps $\mathrm{Dom}(\tilde{\eth}^{\pm}_{(\tilde{A},\tilde{\Phi})}) \ni s \to s|_{X\setminus B(Z,\varepsilon/8)} \in L^2(X \setminus B(Z,\varepsilon/8), S_X \otimes V)$ are compact. By Corollary 2.16, the restriction maps $\mathrm{Dom}(\tilde{\eth}^{\pm}_{(\tilde{A},\tilde{\Phi})}) \ni s \to s|_{B(Z,\varepsilon/4)^*} \in L^2(B(Z,\varepsilon/4), S_X \otimes V)$ are also compact. Hence the injection maps $\mathrm{Dom}(\tilde{\eth}^{\pm}_{(\tilde{A},\tilde{\Phi})}) \to L^2(X, V \otimes S_X)$ are compact. □

Proposition 3.3. The Dirac operators $\tilde{\eth}^{\pm}_{(\tilde{A},\tilde{\Phi})} : L^2(X\setminus Z, V\otimes S_X) \to L^2(X \setminus Z, V \otimes S_X)$ are closed Fredholm operators and adjoint to each other.

Proof. We show that $\tilde{\eth}^{\pm}_{(\tilde{A},\tilde{\Phi})}$ are adjoint to each other. For a densely defined closed operator F, we denote by F^* the adjoint of F. Take $\alpha \in \mathrm{Dom}\left((\tilde{\eth}^{\pm}_{(\tilde{A},\tilde{\Phi})})^*\right)$. Then we have $_1< (\tilde{\eth}^{\pm}_{(\tilde{A},\tilde{\Phi})})^*(\alpha), \varphi >_{L^2} = {}_1< \alpha, \tilde{\eth}^{\pm}_{(\tilde{A},\tilde{\Phi})}(\varphi) >_{L^2}$ for any $\varphi \in C_0^\infty(X\setminus Z, V\otimes S_X)$, where $C_0^\infty(X\setminus Z, V \otimes S_X)$ denotes the set of compact-supported smooth sections. Therefore $\alpha \in \mathrm{Dom}(\tilde{\eth}^{\mp}_{(\tilde{A},\tilde{\Phi})})$ and $(\tilde{\eth}^{\pm}_{(\tilde{A},\tilde{\Phi})})^*(\alpha) = \tilde{\eth}^{\mp}_{(\tilde{A},\tilde{\Phi})}(\alpha)$. We show the converse. Take $a \in \mathrm{Dom}(\tilde{\eth}^{\mp}_{(\tilde{A},\tilde{\Phi})})$ and $b \in \mathrm{Dom}(\tilde{\eth}^{\pm}_{(\tilde{A},\tilde{\Phi})})$. Because of the elliptic regularity, Proposition 2.12 and Proposition 2.15, we obtain $|a|, |b| \in L^3(X\setminus Z)$. Let $\kappa : \mathbb{R} \to [0,1]$ be a smooth function that satisfies the conditions $\kappa((-\infty,-1]) = \{0\}$, $\kappa([-1/2,\infty)) = \{1\}$. Set $\psi_n(x) = \kappa(n\cdot\log(d_{\tilde{g}}(x,Z)))$ for $n\in\mathbb{N}$, where we set $d_{\tilde{g}}(x,Z) := \min\{d_{\tilde{g}}(x,p) \mid p \in Z\}$. Since $\psi_n a$ has a compact support on $X\setminus Z$, we have $_1< \psi_n a, \tilde{\eth}^{\pm}_{(\tilde{A},\tilde{\Phi})}(b) >_{L^2} = {}_1< \tilde{\eth}^{\mp}_{(\tilde{A},\tilde{\Phi})}(\psi_n a), b >_{L^2} = {}_1< \psi_n \tilde{\eth}^{\mp}_{(\tilde{A},\tilde{\Phi})}(a), b>_{L^2} + {}_1<$

$\mathrm{clif}_X(d\psi_n)a, b >_{L^2}$. Since we have $|(\kappa(nx))'| \leq (x|\log(x)|)^{-1} \cdot ||\kappa'||_{L^\infty}$ for $0 < x < 1$, $|d\psi_n|$ is dominated by an L^3-function that is independent of n. Hence we obtain $\imath < a, \tilde{\not{\partial}}^{\pm}_{(\tilde{A},\tilde{\Phi})}(b) >_{L^2}=$ $\imath < \tilde{\not{\partial}}^{\mp}_{(\tilde{A},\tilde{\Phi})}(a), b >_{L^2}$ by the dominated convergence theorem. Therefore $a \in \mathrm{Dom}\left((\tilde{\not{\partial}}^{\pm}_{(\tilde{A},\tilde{\Phi})})^*\right)$ and $(\tilde{\not{\partial}}^{\pm}_{(\tilde{A},\tilde{\Phi})})^*(a) = \tilde{\not{\partial}}^{\mp}_{(\tilde{A},\tilde{\Phi})}(a)$.

We show that the kernel of $\tilde{\not{\partial}}^{\pm}_{(\tilde{A},\tilde{\Phi})}$ is finite-dimensional. By Proposition 3.2, the identity map of $\mathrm{Ker}(\tilde{\not{\partial}}^{\pm}_{(\tilde{A},\tilde{\Phi})})$ is a compact operator. Hence we obtain $\dim(\mathrm{Ker}(\tilde{\not{\partial}}^{\pm}_{(\tilde{A},\tilde{\Phi})})) < \infty$. Since the Dirac operators $\tilde{\not{\partial}}^{\pm}_{(\tilde{A},\tilde{\Phi})}$ are adjoint to each other, the claim $\dim(R(\tilde{\not{\partial}}^{\pm}_{(\tilde{A},\tilde{\Phi})})^{\perp}) < \infty$ can be deduced from $\dim(\mathrm{Ker}(\tilde{\not{\partial}}^{\mp}_{(\tilde{A},\tilde{\Phi})})) < \infty$, where $R(\cdot)$ means the range of the operator and $\perp$ means the orthogonal complement in L^2.

To prove that $R(\tilde{\not{\partial}}^{\pm}_{(\tilde{A},\tilde{\Phi})})$ is closed, it suffices to show that there exists a constant $C > 0$ such that the condition $||s||_{L^2} < C||\tilde{\not{\partial}}^{\pm}_{(\tilde{A},\tilde{\Phi})}(s)||_{L^2}$ holds for any $s \in \mathrm{Dom}(\tilde{\not{\partial}}^{\pm}_{(\tilde{A},\tilde{\Phi})}) \cap \left(\mathrm{Ker}(\tilde{\not{\partial}}^{\pm}_{(\tilde{A},\tilde{\Phi})})\right)^{\perp}$. Suppose that there is no such a constant $C > 0$, then we can take a sequence $\{s_n\} \subset \mathrm{Dom}(\tilde{\not{\partial}}^{\pm}_{(\tilde{A},\tilde{\Phi})}) \cap \left(\mathrm{Ker}(\tilde{\not{\partial}}^{\pm}_{(\tilde{A},\tilde{\Phi})})\right)^{\perp}$ such that the conditions $||s_n|| = 1$ and $||\tilde{\not{\partial}}^{\pm}_{(\tilde{A},\tilde{\Phi})}(s_n)||_{L^2} < 1/n$ are satisfied. By Proposition 3.2, we may assume that $\{s_n\}$ converges to some $s_\infty \in L^2$. Since $||s_n||_{L^2} = 1$ for any $n \in \mathbb{N}$, we have $s_\infty \in \left(\mathrm{Ker}(\tilde{\not{\partial}}^{\pm}_{(\tilde{A},\tilde{\Phi})})\right)^{\perp} \setminus \{0\}$. However, we also have $\tilde{\not{\partial}}^{\pm}_{(\tilde{A},\tilde{\Phi})}(s_n) \to 0$, and hence $\tilde{\not{\partial}}^{\pm}_{(\tilde{A},\tilde{\Phi})}(s_\infty) = 0$, which is impossible. Therefore the condition holds for some $C > 0$ and $R(\tilde{\not{\partial}}^{\pm}_{(\tilde{A},\tilde{\Phi})})$ is closed. □

Theorem 3.4. The Dirac operators $\not{\partial}^{\pm}_{(A,\Phi)}$ are closed Fredholm operators and adjoint to each other, and they have the same indices of $\tilde{\not{\partial}}^{\pm}_{(\tilde{A},\tilde{\Phi})}$.

Proof. Since $|A-\tilde{A}|$ and $|\Phi-\tilde{\Phi}|$ are bounded on $X\setminus Z$, by Proposition 3.2 the operators $\tilde{\not{\partial}}^{\pm}_{(A,\Phi)}$ are closed Fredholm operators and adjoint to each other, and they have the same indices of $\tilde{\not{\partial}}^{\pm}_{(\tilde{A},\tilde{\Phi})}$.

We will prove that $\tilde{\not{\partial}}^{\pm}_{(A,\Phi)}$ and $\not{\partial}^{\pm}_{(A,\Phi)}$ have the same domains and indices. First we show $\mathrm{Dom}(\not{\partial}^{\pm}_{(A,\Phi)}) \subset \mathrm{Dom}(\tilde{\not{\partial}}^{\pm}_{(A,\Phi)})$. By Lemma 3.1, there exists $C_0 > 0$ such that the estimate

$$|\tilde{\not{\partial}}^{\pm}_{(A,\Phi)}(s) - \not{\partial}^{\pm}_{(A,\Phi)}(s)| < C_0\, R_p^2 \cdot |\nabla_{A_X \otimes A}(s)| \tag{1}$$

holds on a neighborhood of $p \in Z$ for any $s \in \Gamma(X \setminus Z, S_X \otimes V)$. Let $\kappa : [0,\infty] :\to [0,1]$ be a smooth bump function satisfying

$$\kappa(x) = \begin{cases} 0 & (1 \leq x) \\ 1 & (3/8 \leq x \leq 3/4) \\ 0 & (x \leq 1/3). \end{cases}$$

For $\delta > 0$, we set a function $\varphi_\delta : X \to [0,1]$ to be $\varphi_\delta(x) = \kappa(\delta^{-1}d_g(x,Z))$. By the Weitzenböck formula $\nabla^{\star_g}_{A_X\otimes A}\nabla_{A_X\otimes A} = \not{\partial}^{-}_{(A,\Phi)}\not{\partial}^{+}_{(A,\Phi)} - \Phi^2 + Sc(g)$, we have $||\nabla_{A_{S_X}\otimes A}(\varphi_\delta s)||^2_{L^2} = ||\not{\partial}^{+}_{(A,\Phi)}(\varphi_\delta s)||^2_{L^2} + ||\Phi(\varphi_\delta s)||^2_{L^2} + \int_X Sc(g)|\varphi_\delta s|^2 d\mathrm{vol}_M$ for any $s \in \Gamma(X \setminus Z, V \otimes S_X)$, where $\nabla^{\star_g}_{A_X\otimes A}$ is the formal adjoint of $\nabla_{A_X\otimes A}$ with respect to g and $Sc(g)$ is the scalar curvature of g. Therefore, there exists $C_1 > 0$ such that for any sufficiently small $\delta > 0$ we have $||\nabla_{A_{S_X}\otimes A}(s)||_{L^2(U_p(3\delta/8,3\delta/4))} \leq$

$C_1(||\partial\!\!\!/^{\pm}_{(A,\Phi)}(s)||_{L^2(U_p(\delta/3,\delta))} + \delta^{-1}||s||_{L^2(U_p(\delta/3,\delta))})$ holds for any $s \in \Gamma(X \setminus Z, V \otimes S_X)$, where we put $U_p(\delta_1, \delta_2) := \{x \in X \mid \delta_1 < d_g(p,x) < \delta_2\}$. We set $\delta_i := 4\varepsilon/(3 \cdot 2^i)$ for $i \in \mathbb{Z}_{\geq 0}$. Then we have

$$\begin{aligned}
&||\tilde{\partial\!\!\!/}^{\pm}_{(A,\Phi)}(s) - \partial\!\!\!/^{\pm}_{(A,\Phi)}(s)||_{L^2(B(p,\varepsilon))} \\
\leq& C_0||R_p^2 \cdot \nabla_{A_X \otimes A}(s)||_{L^2(B(p,\varepsilon))} \\
\leq& C_0 \sum_{i=0}^{\infty} ||R_p^2 \cdot \nabla_{A_X \otimes A}(s)||_{L^2(U_p(3\delta_i/8, 3\delta_i/4))} \\
\leq& C_0 C_1 \sum_{i=0}^{\infty} \left\{ \delta_i^2 ||\partial\!\!\!/^{\pm}_{(A,\Phi)}(s)||_{L^2(U_p(\delta_i/3,\delta_i))} + \delta_i ||s||_{L^2(U_p(\delta_i/3,\delta_i))} \right\} \\
\leq& C_0 C_1 \Big(\delta_0^2 ||\partial\!\!\!/^{\pm}_{(A,\Phi)}(s)||_{L^2(B(p,\delta_0))} + \delta_0 ||s||_{L^2(B(p,\delta_0))} \Big)
\end{aligned}$$

Hence there exists $C_2 = C_2(\varepsilon) > 0$ such that $||\tilde{\partial\!\!\!/}^{\pm}_{(A,\Phi)}(s) - \partial\!\!\!/^{\pm}_{(A,\Phi)}(s)||_{L^2} \leq C_2(||s||_{L^2} + ||\partial\!\!\!/^{\pm}_{(A,\Phi)}(s)||_{L^2})$, and we have $C_2 = O(\varepsilon)$ ($\varepsilon \to 0$). Hence we obtain $\mathrm{Dom}(\partial\!\!\!/^{\pm}_{(A,\Phi)}) \subset \mathrm{Dom}(\tilde{\partial\!\!\!/}^{\pm}_{(A,\Phi)})$. We show the converse. Let $\tilde{A}_X$ denote the connection on S_X induced by the Levi-Civita connection of $(X, \tilde{g})$. By the definition of Dirac-type singularity, we have $|\nabla_A(\Phi)| = |F(A)| = O(R_p^{-2})$ around $p \in Z$. Therefore from the Weitzenböck formula $\nabla^{\star_{\tilde{g}}}_{\tilde{A}_X \otimes A} \nabla_{\tilde{A}_X \otimes A} = \partial\!\!\!/^{-}_{(A,\Phi)} \partial\!\!\!/^{+}_{(A,\Phi)} - \Phi^2 + \mathrm{clif}(\nabla_A(\Phi) - *_{\tilde{g}} F(A))$ and a similar argument as above, it follows that there exists $C_3 = C_3(\varepsilon) > 0$ such that $||\tilde{\partial\!\!\!/}^{\pm}_{(A,\Phi)}(s) - \partial\!\!\!/^{\pm}_{(A,\Phi)}(s)||_{L^2} \leq C_3(||s||_{L^2} + ||\tilde{\partial\!\!\!/}^{\pm}_{(A,\Phi)}(s)||_{L^2})$, and $C_3 = O(\varepsilon)$ ($\varepsilon \to 0$). Therefore $\mathrm{Dom}(\partial\!\!\!/^{\pm}_{(A,\Phi)}) = \mathrm{Dom}(\tilde{\partial\!\!\!/}^{\pm}_{(A,\Phi)})$. Moreover, Their graph norms are also equivalent. It is a well-known fact that sufficiently small deformations of Fredholm operators remain Fredholm. Hence the operators $\partial\!\!\!/^{\pm}_{(A,\Phi)}$ are closed Fredholm operators, and they have the same indices as ones of $\tilde{\partial\!\!\!/}^{\pm}_{(\tilde{A},\tilde{\Phi})}$. □

3.2 An index calculation on a compact 3-folds

3.2.1 A lift of singular monopoles to closed 4-folds

For an arbitrary 3-fold N and a principal S^1-bundle P defined on outside of a point $x \in N$, we set $\deg_x(P) := \int_{\partial B} c_1(P)$, where B is a small neighborhood of x.

We take a finite subset $Z' \subset X$ satisfying the conditions $|Z'| = |Z|$ and $Z \cap Z' = \emptyset$, and set $\tilde{Z} = Z \cup Z'$. By the Mayer-Vietoris exact sequence induced by the open covering $X = B_\varepsilon(\tilde{Z}) \cup (X \setminus \tilde{Z})$, we can prove that there exists a principal S^1-bundle $\pi : P \to X \setminus \tilde{Z}$ such that we have $\deg_p(P) = -1$ for $p \in Z$ and $\deg_{p'}(P) = 1$ for $p' \in Z'$. We take a metric $\hat{g}$ on X that is flat on $B(\tilde{Z}, \varepsilon/2)$. Let $f : X \setminus \tilde{Z} \to \mathbb{R}_+$ be a smooth function. Let $\omega \in \Omega^1(P, \mathbb{R})$ be a connection of P. We assume that for any $p \in Z$ (resp. Z') the tuple $((P,\omega) \times_{S^1} (\mathbb{C}, h_{\mathbb{C}}), -\sqrt{-1}f)|_{B(p,\varepsilon/2)}$ (resp. $((P,\omega) \times_{S^1} (\mathbb{C}, h_{\mathbb{C}}), \sqrt{-1}f)|_{B(p,\varepsilon/2)}$) is the flat Dirac monopole of weight -1 (resp. 1) with respect to $\hat{g}$. Set a one-form $\xi := \omega/\pi^* f$ and a metric $g_P := \pi^*\hat{g} + \xi^2$ on P. We choose the global 4-form $-\xi \wedge \pi^* \mathrm{vol}_{(X,\hat{g})}$ as the orientation of P.

Proposition 3.5. The following claims are satisfied.

- The 4-fold P has the spin structure induced by the one of X.
- Let v be a vector field on X. By the isomorphism $TP = \mathbb{R}\partial_\theta \oplus \pi^* TX$ induced by ω, we regard $\pi^* v$ as a vector field on P. Then for $F \in C^\infty(X)$ we have $\pi^*(v \cdot F) = \pi^* v \cdot \pi^* F$.

- For the spinor bundles $S^{\pm}$, we have the unitary isomorphism $S^{\pm}_P \simeq \pi^*(S_X)$.
- Under the above isomorphisms, the Clifford product on P can be represented as follows:

$$\begin{aligned}
\mathrm{clif}_P(\xi) &= \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix} \\
\mathrm{clif}_P(\pi^*\alpha) &= \begin{pmatrix} 0 & \mathrm{clif}_X(\alpha) \\ \mathrm{clif}_X(\alpha) & 0 \end{pmatrix} \qquad (\alpha \in \Gamma(X, \Omega^1(X))).
\end{aligned}$$

Proof. Let $i : SO(3) \to SO(4)$ be the injection induced by $\mathbb{R}^3 \ni x \to (0, x) \in \mathbb{R}^4$, and take the homomorphism $\rho : Spin(3) \to Spin(4)$ to be the lift of i. Set $\mathscr{S}_P := \pi^*\mathscr{S} \times_\rho Spin(4)$. Then we have $\mathscr{S}_P \times_{Spin(4)} (\mathbb{R}^4, g_{4,\mathrm{Euc}}) \simeq (P \times (\mathbb{R}, g_{1,\mathrm{Euc}})) \oplus (\pi^*TX, \pi^*g) \simeq TP$. Hence $\mathscr{S}_P$ is a spin structure on P. The second claim is trivial from some direct calculations.

We have the isomorphisms $Spin(3) \simeq SU(2)$ and $Spin(4) \simeq SU(2)_+ \times SU(2)_-$. Under this isomorphism, we have $\rho(g) = (g, g)$. Hence we obtain the unitary isomorphism $S^{\pm}_P \simeq \pi^*(S_X)$. The last claim easily follows from the third one. □

We take another metric $\tilde{g}_P := \pi^*f \cdot g_P$. For $p \in Z$, the restriction $\pi : \pi^*(B(p, \varepsilon/2) \setminus \{p\}) \to B(p, \varepsilon/2) \setminus \{p\}$ can be identified with the Hopf fibration $(\mathbb{R}^4 \setminus \{0\}) \to (\mathbb{R}^3 \setminus \{0\})$. For $p' \in Z'$, we can also identify $\pi : \pi^*(B(p', \varepsilon/2) \setminus \{p'\}) \to B(p', \varepsilon/2) \setminus \{p'\}$ with the inverse-oriented Hopf fibration $(-\mathbb{R}^4 \setminus \{0\}) \to (\mathbb{R}^3 \setminus \{0\})$, where $-\mathbb{R}^4$ is the differentiable manifold $\mathbb{R}^4$ with the inverse orientation of the standard one of $\mathbb{R}^4$. Hence by taking the one-point compactification on the closure of each $\pi^*(B(p, \varepsilon/2) \setminus \{p\})$, we obtain a closed 4-fold $\tilde{P}$ equipped with an S^1-action. Then $\tilde{g}_P$ can be prolonged to a metric on $\tilde{P}$ as in Lemma 2.5. We extend the projection $\pi : P \to X \setminus \tilde{Z}$ to the smooth map $\tilde{P} \to X$, and we denote this map by the same letter π by abuse of notation. Set $Z_4 := \pi^{-1}(Z)$, $Z'_4 := \pi^{-1}(Z')$ and $\tilde{Z}_4 := \pi^{-1}(\tilde{Z})$. Then $\pi|_{\tilde{Z}_4} : \tilde{Z}_4 \to Z_4$ is a bijection. We have $\tilde{P} = P \sqcup \tilde{Z}_4$ and $\mathrm{codim}(\tilde{P}, \tilde{Z}) = 4$. Hence we obtain isomorphisms $\pi_1(P) \simeq \pi_1(\tilde{P})$ and $H^2(P, \mathbb{Z}/2\mathbb{Z}) \simeq H^2(\tilde{P}, \mathbb{Z}/2\mathbb{Z})$. Therefore the orientation and the spin structure of P induce the unique ones of $\tilde{P}$. Hence we obtain the following lemma.

Lemma 3.6. We have the unitary isomorphisms $S^{\pm}_{\tilde{P}}|_P \simeq (\pi^*S_X)|_P$. Under these isomorphisms, we have $\mathrm{clif}_{\tilde{P}}(v)|_P = \pi^*f^{-1/2} \cdot \mathrm{clif}_P(v)$ for $v \in \Omega^1(\tilde{P})$.

For the Dirac-type singular monopole (V, h, A, Φ) on (X, Z), we take a connection $\hat{A}$ and a skew-Hermitian endomorphism $\hat{\Phi}$ that satisfy the following conditions.

- For any $p \in Z$, $(V, h, \hat{A}, \hat{\Phi})|_{B(p,\varepsilon/2)\setminus\{p\}}$ is a direct sum of the flat Dirac monopoles with respect to the metric $\hat{g}$.
- For any $p' \in Z'$, $(V, h, \hat{A})|_{B(p',\varepsilon/2)}$ is a flat unitary bundle and $\hat{\Phi}|_{B(p',\varepsilon/2)} = 0$.
- The differences $|A - \hat{A}|$, $|\Phi - \hat{\Phi}|$ are bounded on $X \setminus \tilde{Z}$.

We denote by $\hat{\partial\!\!\!/}^{\pm}_{(\hat{A},\hat{\Phi})}$ the Dirac operators of $(V, h, \hat{A}, \hat{\Phi})$ with respect to the metric $\hat{g}$. By the same argument as Proposition 3.3 and Theorem 3.4, $\hat{\partial\!\!\!/}^{\pm}_{(\hat{A},\hat{\Phi})}$ are Fredholm and adjoint to each other, and the indices of $\hat{\partial\!\!\!/}^{\pm}_{(\hat{A},\hat{\Phi})}$ are the same as the ones of $\partial\!\!\!/^{\pm}_{(A,\Phi)}$.

We set $(V_4, h_4, A_4) := (\pi^*V, \pi^*h, \pi^*\hat{A} - \xi \otimes \pi^*\hat{\Phi})$ on $P \sqcup Z'_4$. By Proposition 2.6, (V_4, h_4, A_4) can be prolonged over $\tilde{P}$, and we denote it by the same symbols. Let $\partial\!\!\!/^{\pm}_{A_4} : \Gamma(\tilde{P}, S^{\pm}_{\tilde{P}} \otimes V_4) \to$

$\Gamma(\tilde{P}, S^{\mp}_{\tilde{P}} \otimes V_4)$ be the Dirac operators of (V_4, h_4, A_4). For a section $s \in \Gamma(X \setminus Z, V \otimes S_X)$, we have $||\pi^* s||^2_{L^2(\tilde{P}, \tilde{g}_P)} = 2\pi ||\sqrt{f} s||^2_{L^2(X, \hat{g})}$. Hence the operator $\pi^\dagger(s) := \pi^*(\sqrt{2\pi f^{-1}}\, s)$ preserves the L^2-norms. Since P is a principal S^1-bundle on X, $\pi^\dagger$ is an isometric isomorphism from $L^2(X \setminus \tilde{Z}, V \otimes S_X)$ to $L^2(\tilde{P}, V_4 \otimes S^{\pm}_{\tilde{P}})^{S^1}$, where $L^2(\tilde{P}, V_4 \otimes S_{\tilde{P}})^{S^1}$ is the closed subspace of $L^2(\tilde{P}, V_4 \otimes S_{\tilde{P}})$ consisting of S^1-invariant sections. For $i = 1, 2$, take smooth functions $\lambda^{\pm}_i : X \setminus \tilde{Z} \to \mathbb{R}_+$ satisfying the following conditions.

- The equality $\lambda^{\pm}_1 \lambda^{\pm}_2 = f^{-1/2}$ holds.
- The equality $\lambda^{\pm}_1 = \lambda^{\mp}_2$ holds.
- For any $p \in Z$, $\lambda^{+}_1|_{B(p,\varepsilon)\setminus\{p\}} = 1$.
- For any $p' \in Z'$, $\lambda^{+}_2|_{B(p',\varepsilon)\setminus\{p'\}} = 1$.

By Lemma 3.6 and Proposition 2.18, there exist compact-supported smooth endomorphisms $\epsilon^{\pm} \in \Gamma(X \setminus \tilde{Z}, \mathrm{End}(S_X \otimes V))$ such that we have $\pi^\dagger(\epsilon^{\pm})(s) = \partial\!\!\!/^{\pm}_{A_4} \circ \pi^\dagger(s) - \pi^\dagger \circ (\lambda^{\pm}_1 \hat{\partial\!\!\!/}^{\pm}_{(\hat{A}, \hat{\Phi})} \lambda^{\pm}_2)(s)$ for any $s \in \Gamma(X \setminus Z, S_X \otimes V)$. Let $D^{\pm}$ be the differential operator $\lambda^{\pm}_1 \hat{\partial\!\!\!/}^{\pm}_{(\hat{A}, \hat{\Phi})} \lambda^{\pm}_2 + \epsilon^{\pm}$ on $X \setminus \tilde{Z}$. We denote by $\mathrm{Ind}(\partial\!\!\!/^{\pm}_{A_4})^{S^1}$ the S^1-equivariant index of the closed operator $\partial\!\!\!/^{\pm}_{A_4} : L^2(\tilde{P}, V_4 \otimes S^{\pm}_{\tilde{P}})^{S^1} \to L^2(\tilde{P}, V_4 \otimes S^{\mp}_{\tilde{P}})^{S^1}$.

Proposition 3.7. Under the isometric isomorphism $\pi^\dagger$, the operators $D^{\pm}$ and $\partial\!\!\!/^{\pm}_{A_4}$ determine the same closed operators respectively. In particular, the operators $D^{\pm}$ are closed Fredholm operator adjoint to each other, and satisfy $\mathrm{Ind}(D^{\pm}) = \mathrm{Ind}(\partial\!\!\!/^{\pm}_{A_4})^{S^1}$.

Proof. We take an arbitrary $a \in \mathrm{Dom}(\partial\!\!\!/^{\pm}_{A_4})^{S^1}$, and set $b := \partial\!\!\!/^{\pm}_{A_4}(a)$. We will show $(\pi^\dagger)^{-1}(a) \in \mathrm{Dom}(D^{\pm})$ and $D^{\pm}((\pi^\dagger)^{-1}(a)) = (\pi^\dagger)^{-1}(b)$. Let φ be a compact-supported smooth section of $V \otimes S_X$ on $X \setminus \tilde{Z}$. Then $\pi^\dagger(\varphi)$ also has a compact support. Hence we have $\imath < a, (\partial\!\!\!/^{\pm}_{A_4})^{\star}(\pi^\dagger(\varphi)) >_{L^2} = \imath < b, \pi^\dagger(\varphi) >_{L^2}$. Since $(\pi^\dagger)^{-1}$ is isometric, we obtain $\imath < (\pi^\dagger)^{-1}(a), (D^{\pm})^{\star}(\varphi) > = \imath < (\pi^\dagger)^{-1}(b), \varphi >$. Therefore we have $(\pi^\dagger)^{-1}(a) \in \mathrm{Dom}(D^{\pm})$ and $D^{\pm}((\pi^\dagger)^{-1}(a)) = (\pi^\dagger)^{-1}(b)$. We prove the converse. We take an arbitrary $c \in \mathrm{Dom}(D^{\pm})$, and set $d := D^{\pm}(c)$. Let χ be a compact-supported smooth section of $V_4 \otimes S_{\tilde{P}}$ on $\tilde{P} \setminus \tilde{Z}_4$. We take the orthogonal decomposition $\chi = \chi^{S^1} + \chi^{\perp} \in L^2(\tilde{P}, V_4 \otimes S^{\mp}_{\tilde{P}})^{S^1} \oplus (L^2(\tilde{P}, V_4 \otimes S^{\mp}_{\tilde{P}})^{S^1})^{\perp}$. Then χ^{S^1} and $\chi^{\perp}$ are also compact-supported smooth sections on $\tilde{P} \setminus \tilde{Z}_4$, and we have $(\partial\!\!\!/^{\pm}_{A_4})^{\star}(\chi^{S^1}) \in L^2(\tilde{P}, V_4 \otimes S^{\mp}_{\tilde{P}})^{S^1}$ and $(\partial\!\!\!/^{\pm}_{A_4})^{\star}(\chi^{\perp}) \in (L^2(\tilde{P}, V_4 \otimes S^{\mp}_{\tilde{P}})^{S^1})^{\perp}$. Hence we obtain $\imath < \pi^\dagger(c), (\partial\!\!\!/^{\pm}_{A_4})^{\star}(\chi) >_{L^2} = \imath < \pi^\dagger(c), (\partial\!\!\!/^{\pm}_{A_4})^{\star}(\chi^{S^1}) >_{L^2} = \imath < c, (\pi^\dagger)^{-1}((\partial\!\!\!/^{\pm}_{A_4})^{\star}(\chi^{S^1})) >_{L^2} = \imath < c, (D^{\pm})^{\star}((\pi^\dagger)^{-1}(\chi^{S^1})) >_{L^2} = \imath < d, (\pi^\dagger)^{-1}(\chi^{S^1}) >_{L^2} = \imath < \pi^\dagger(d), \chi^{S^1} >_{L^2} = \imath < \pi^\dagger(d), \chi >_{L^2}$. Therefore $\partial\!\!\!/^{\pm}_{A_4}(\pi^\dagger(c)) = \pi^\dagger(d)$ holds on $P = \tilde{P} \setminus \tilde{Z}_4$. Here we prepare the following lemma.

Lemma 3.8. Take arbitrary $u \in L^2(\tilde{P}, V_4 \otimes S^{\pm}_{\tilde{P}})$ and $v \in L^2(\tilde{P}, V_4 \otimes S^{\mp}_{\tilde{P}})$. If u and v satisfy $\partial\!\!\!/^{\pm}_{A_4}(u) = v$ on P, then we have $\partial\!\!\!/^{\pm}_{A_4}(u) = v$ on whole $\tilde{P}$.

If we admit this lemma, then we obtain $\partial\!\!\!/^{\pm}_{A_4}(\pi^\dagger(c)) = \pi^\dagger(d)$ on $\tilde{P}$. Hence the proof is complete. □

proof of Lemma 3.8. Take $\varphi \in \Gamma(\tilde{P}, V_4 \otimes S^{\mp}_{\tilde{P}})$. Let $\kappa : \mathbb{R} \to [0,1]$ be a smooth function that satisfies $\kappa((-\infty, -1)) = \{0\}$ and $\kappa((-1/2, \infty)) = \{1\}$. Set $\psi_n : \tilde{P} \to [0,1]$ to be $\psi_n(x) := \kappa(n \log(d_{\tilde{g}_P}(x, \tilde{Z}_4)))$ for $n \in \mathbb{N}$. Then $\psi_n \cdot \varphi$ has a compact support on $\tilde{P} \setminus \tilde{Z}_4$. Hence we obtain $\imath < u, (\not{\partial}^{\pm}_{A_4})^{\star}(\psi_n \cdot \varphi) >_{L^2} = \imath < u, \psi_n \cdot (\not{\partial}^{\pm}_{A_4})^{\star}(\varphi) >_{L^2} + \imath < u, \mathrm{clif}_{\tilde{P}}(d\psi_n)\varphi >_{L^2} = \imath < v, \psi_n \cdot \varphi >_{L^2}$. Since we have an estimate $|\kappa'(nx)| \leq (x|\log(x)|)^{-1}||\kappa'||_{L^\infty}$ for $0 < x < 1$, $|d\psi_n|$ is dominated by an L^2-function that is independent of n. Therefore we obtain $\imath < u, (\not{\partial}^{\pm}_{A_4})^{\star}(\varphi) >_{L^2} = \imath < v, \varphi >$ by the dominated convergence theorem. $\square$

We will associate the S^1-invariant indices of $\not{\partial}^{\pm}_{A_4}$ and the indices of $\hat{\not{\partial}}^{\pm}_{(\hat{A},\hat{\Phi})}$.

Proposition 3.9. We have $\mathrm{Ind}(\not{\partial}^{\pm}_{A_4})^{S^1} = \mathrm{Ind}(\hat{\not{\partial}}^{\pm}_{(\hat{A},\hat{\Phi})})$.

Proof. If we prove $\mathrm{Ind}(\not{\partial}^{+}_{A_4})^{S^1} = \mathrm{Ind}(\hat{\not{\partial}}^{+}_{(\hat{A},\hat{\Phi})})$, then we obtain $\mathrm{Ind}(\not{\partial}^{-}_{A_4})^{S^1} = -\mathrm{Ind}(\not{\partial}^{+}_{A_4})^{S^1} = -\mathrm{Ind}(\hat{\not{\partial}}^{+}_{(\hat{A},\hat{\Phi})}) = \mathrm{Ind}(\hat{\not{\partial}}^{-}_{(\hat{A},\hat{\Phi})})$ because $\hat{\not{\partial}}^{\pm}_{(\hat{A},\hat{\Phi})}$ are adjoint to each other. Hence we only need to prove $\mathrm{Ind}(\not{\partial}^{+}_{A_4})^{S^1} = \mathrm{Ind}(\hat{\not{\partial}}^{+}_{(\hat{A},\hat{\Phi})})$. By Proposition 3.7, it suffices to show $\mathrm{Ind}(\hat{\not{\partial}}^{+}_{(\hat{A},\hat{\Phi})}) = \mathrm{Ind}(D^+)$. Since the support of ϵ^+ is compact in $X \setminus Z$, $\lambda_1^+ \hat{\not{\partial}}^{+}_{(\hat{A},\hat{\Phi})} \lambda_2^+$ is a closed Fredholm operator and it has the same index as D^+. By the same asymptotic analysis in Proposition 2.12, for any solutions $s \in \Gamma(X \setminus \tilde{Z}, S_X \otimes V)$ of the equation $\hat{\not{\partial}}^{+}_{(\hat{A},\hat{\Phi})}(s) = 0$, we have $s \in L^2$ if and only if $(\lambda_2^+)^{-1} s \in L^2$. Hence we have the natural equality $\mathrm{Ker}(\lambda_1^+ \hat{\not{\partial}}^{+}_{(\hat{A},\hat{\Phi})} \lambda_2^+) \cap L^2 = (\lambda_2^+)^{-1} \cdot (\mathrm{Ker}(\hat{\not{\partial}}^{+}_{(\hat{A},\hat{\Phi})}) \cap L^2)$, where $(\lambda_2^+)^{-1} \cdot (\mathrm{Ker}(\hat{\not{\partial}}^{+}_{(\hat{A},\hat{\Phi})}) \cap L^2)$ means the set $\{(\lambda_2^+)^{-1} \cdot s \mid s \in \mathrm{Ker}(\hat{\not{\partial}}^{+}_{(\hat{A},\hat{\Phi})}) \cap L^2\}$. By a similar way, we also have $\mathrm{Cok}(\hat{\not{\partial}}^{+}_{(\hat{A},\hat{\Phi})}) \cap L^2 = \mathrm{Ker}(\hat{\not{\partial}}^{-}_{(\hat{A},\hat{\Phi})}) \cap L^2$ and $\mathrm{Cok}(\lambda_1^+ \hat{\not{\partial}}^{+}_{(\hat{A},\hat{\Phi})} \lambda_2^+) \cap L^2 = \mathrm{Ker}(\lambda_1^- \hat{\not{\partial}}^{-}_{(\hat{A},\hat{\Phi})} \lambda_2^-) \cap L^2 = (\lambda_2^-)^{-1} \cdot (\mathrm{Ker}(\hat{\not{\partial}}^{-}_{(\hat{A},\hat{\Phi})}) \cap L^2)$. Therefore we obtain $\mathrm{Ind}(\hat{\not{\partial}}^{+}_{(\hat{A},\hat{\Phi})}) = \mathrm{Ind}(\lambda_1^+ \hat{\not{\partial}}^{+}_{(\hat{A},\hat{\Phi})} \lambda_2^+) = \mathrm{Ind}(D^+)$, which completes the proof. $\square$

By following [2], we calculate the S^1-equivariant index $\mathrm{Ind}(\not{\partial}^{\pm}_{A_4})^{S^1}$.

Lemma 3.10. For $p \in Z_4$ (resp. Z'), the weights of the fiber $S^+_{\tilde{P}}|_p$ and $S^-_{\tilde{P}}|_p$ are $(0,0)$ and $(-1,1)$ (resp. $(-1,1)$ and $(0,0)$) respectively.

Proof. For $p \in Z_4$, the projection $\pi|_{B(p,\varepsilon)} : B(p,\varepsilon) \to \pi(B(p,\varepsilon))$ can be identified with the Hopf fibration $\mathbb{R}^4 = \mathbb{C}^2 \to \mathbb{R}^3$ in Section 1. By the natural isomorphisms $S^+_{\mathbb{C}^2} \simeq \Omega^{0,0}_{\mathbb{C}^2} \oplus \Omega^{0,2}_{\mathbb{C}^2}$ and $S^-_{\mathbb{C}^2} \simeq \Omega^{0,1}_{\mathbb{C}^2}$, the weights of $S^+_{\tilde{P}}|_p$ and $S^-_{\tilde{P}}|_p$ are $(0,0)$ and $(-1,1)$ respectively. As a similar way, for $p' \in Z'_4$, the projection $\pi|_{B(p',\varepsilon)} : B(p',\varepsilon) \to \pi(B(p',\varepsilon))$ can be identified with the inverse-oriented Hopf fibration $-\mathbb{R}^4 \to \mathbb{R}^3$. Therefore the weights of $S^+_{\tilde{P}}|_{p'}$ and $S^-_{\tilde{P}}|_{p'}$ are $(-1,1)$ and $(0,0)$ respectively. $\square$

Proposition 3.11. The S^1-invariant index $\mathrm{Ind}(\not{\partial}^{\pm}_{A_4})^{S^1}$ is given as

$$\mathrm{Ind}(\not{\partial}^{\pm}_{A_4})^{S^1} = \mp \sum_{p \in Z} \sum_{k_{p,i} > 0} k_{p,i},$$

where $\vec{k}_p = (k_{p,i}) \in \mathbb{Z}^r$ is the weight of the monopole (V, h, A, Φ) at $p \in Z$.

Proof. According to [2], The S^1-invariant index $\mathrm{Ind}(\not{\partial}^{\pm}_{A_4})^{S^1}$ is given as

$$\mathrm{Ind}(\not{\partial}^{\pm}_{A_4})^{S^1} = (2\pi)^{-1} \int_{S^1} \sum_{p \in \tilde{Z}_4} \frac{\mathrm{tr}_\theta((S^{\pm}_{\tilde{P}} \otimes V_4)|_p) - \mathrm{tr}_\theta((S^{\mp}_{\tilde{P}} \otimes V_4)|_p)}{\mathrm{tr}_\theta(\bigwedge^{-1} T_p \tilde{P})} d\theta,$$

where tr_θ is trace of the action of $\theta \in S^1$ and $\bigwedge^{-1} T_p\tilde{P}$ means the virtual vector space $\bigoplus_{i=0}^{\infty}(-1)^i \bigwedge^i T_p\tilde{P}$. Then by Lemma 3.10 we have

$$\begin{aligned}
&\mathrm{tr}_\theta((S^{\pm}_{\tilde{P}} \otimes V_4)|_p) - \mathrm{tr}_\theta((S^{\mp}_{\tilde{P}} \otimes V_4)|_p) = \pm 2(1+\cos\theta)\sum_i \exp(2\pi\sqrt{-1}k_{p,i}\theta) && (p \in Z) \\
&\mathrm{tr}_\theta((S^{\pm}_{\tilde{P}} \otimes V_4)|_{p'}) - \mathrm{tr}_\theta((S^{\mp}_{\tilde{P}} \otimes V_4)|_{p'}) = \mp 2r(1-\cos\theta) && (p' \in Z') \\
&\mathrm{tr}_\theta(\textstyle\bigwedge^{-1} T_{\tilde{p}}\tilde{P}) = 4(1-\cos\theta)^2 && (\tilde{p} \in \tilde{Z}).
\end{aligned}$$

Hence by straightforward computation we obtain the conclusion. □

Hence we obtain the following corollary.

Corollary 3.12. The indices of the Dirac operators $\partial\!\!\!/^{\pm}_{(A,\Phi)}$ are given as follows:

$$\mathrm{Ind}(\partial\!\!\!/^{\pm}_{(A,\Phi)}) = \mp \sum_{p\in Z}\sum_{k_{p,i}>0} k_{p,i} = \pm \sum_{p\in Z}\sum_{k_{p,i}<0} k_{p,i},$$

where $\vec{k}_p = (k_{p,i}) \in \mathbb{Z}^r$ is the weight of the monopole (V, h, A, Φ) at $p \in Z$.

4 An index formula of Dirac operators on complete 3-folds

Let (X, g) be a complete oriented Riemannian 3-fold such that the scalar curvature $Sc(g)$ is bounded. We fix a spin structure on X. Let $i : Y \hookrightarrow X$ be a relative compact region with a smooth boundary ∂Y. We take the orientation of ∂Y to satisfy that $\nu \wedge \mathrm{vol}_{\partial Y}$ is positive for the inward normal unit 1-form $\nu \in i^*\Omega^1(X)$.

4.1 The non-singular case

Following [10], we recall the non-singular case. Let (V, h, A) be a Hermitian bundle with a connection on X and Φ be a skew-Hermitian endomorphism on V. We assume the following conditions.

- Both Φ and $F(A)$ are bounded.
- We have $\nabla_A(\Phi)|_x = o(1)$ as $x \to \infty$.
- The inequality $\inf_{x\in X\setminus Y}\{|\lambda| \mid \lambda \text{ is an eigenvalue of } \Phi(x)\} > 0$ is satisfied.

We call this conditions *the Råde condition*. In [10], Råde proved the following theorem.

Theorem 4.1. The differential operators $\partial\!\!\!/^{\pm}_{(A,\Phi)} = \partial\!\!\!/_A \pm \Phi : L^2(X, V \otimes S_X) \to L^2(X, V \otimes S_X)$ are closed Fredholm, and their indices are given as follows:

$$\mathrm{Ind}(\partial\!\!\!/^{\pm}_{(A,\Phi)}) = \mp \int_{\partial Y} \mathrm{ch}(V^+) = \pm \int_{\partial Y} \mathrm{ch}(V^-),$$

where $V^{\pm}$ is a subbundle of $V|_{\partial Y}$ spanned by the eigenvectors of $\mp\sqrt{-1}\Phi$ with positive eigenvalues.

4.2 The indices of twisted flat Dirac monopoles

Let (L_k, h_k, A_k, Φ_k) be the flat Dirac monopole of weight $k \in \mathbb{Z}$. For $a \in \mathbb{R} \setminus \{0\}$, we set $\Phi_{a,k} := \sqrt{-1}\,(a + (k/2r_3))$. Then $(L_k, h_k, A_k, \Phi_{a,k})$ is also a Dirac-type monopole on $(\mathbb{R}^3, \{0\})$.

Proposition 4.2. The operators $\partial\!\!\!/^{\pm}_{(A_k,\Phi_{a,k})}$ are Fredholm. Moreover, we have $\mathrm{Ind}(\partial\!\!\!/^{\pm}_{(A_k,\Phi_{a,k})}) = 0$ if $ak > 0$.

Proof. The Fredholmness of $\partial\!\!\!/^{\pm}_{(A_k,\Phi_{a,k})}$ follows from Corollary 2.16 and some standard arguments. The proof for the case $a > 0$ and $k > 0$ works for the case $a < 0$ and $k < 0$ mutatis mutandis. Hence we may assume $a > 0$ and $k > 0$. We take $f^{\pm}_{\nu} \in L^2(S^2, S^{\pm}_{S^2} \otimes \mathcal{O}(k))$ $(\nu \in \mathbb{N})$ and $n_\nu > 0$ as in subsection 2.2. We set vector subspaces $W_0 := H^0(\mathbb{P}^1, \mathcal{O}(k-1)) \times \{0\}$ and $W_\nu := (f^+_\nu, 0)\mathbb{C} \oplus (0, f^-_\nu)\mathbb{C}$ of $L^2(S^2, \mathcal{O}(k) \otimes S_{S^2}) = L^2(S^2, \mathcal{O}(k) \otimes S^+_{S^2}) \times L^2(S^2, \mathcal{O}(k) \otimes S^-_{S^2})$. Then we have a decomposition $L^2(S^2, \mathcal{O}(k) \otimes S_{S^2}) = \widehat{\bigoplus}_{\nu \geq 0} W_\nu$, where $\widehat{\bigoplus}$ means L^2-completion of the direct sum. Hence we obtain the decomposition $L^2(\mathbb{R}^3 \setminus \{0\}, L_k \otimes S_{\mathbb{R}^3}) = \widehat{\bigoplus}_{\nu \geq 0} W_\nu \otimes L^2(\mathbb{R}_{>0}, r^2 dr)$, where $L^2(\mathbb{R}_{>0}, r^2 dr)$ is the weighted L^2-space on $\mathbb{R}_{>0}$ with the norm $||f||^2 = \int_{\mathbb{R}_{>0}} r^2 |f(r)|^2 dr$. We denote by E_ν the space $W_\nu \otimes L^2(\mathbb{R}_{>0}, r^2 dr)$. The Dirac operators $\partial\!\!\!/^{\pm}_{(A_k,\Phi_{a,k})}$ preserves this decomposition, and hence we obtain $\mathrm{Ind}(\partial\!\!\!/^{\pm}_{(A_k,\Phi_{a,k})}) = \sum_\nu \mathrm{Ind}\left(\partial\!\!\!/^{\pm}_{(A_k,\Phi_{a,k})}|_{E_\nu} : E_\nu \to E_\nu\right)$. Here we prepare the following lemma.

Lemma 4.3. We take Hermitian matrices

$$A_\nu = \begin{pmatrix} -(2+k)/2 & \sqrt{-1}n_\nu \\ -\sqrt{-1}n_\nu & -(2-k)/2 \end{pmatrix}, \quad B = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}.$$

We set the closed operator $P_\nu : \mathbb{C}^2 \otimes L^2(\mathbb{R}_{>0}, r^2 dr) \to \mathbb{C}^2 \otimes L^2(\mathbb{R}_{>0}, r^2 dr)$ to be $P_\nu(v) := \partial_r v - (A_\nu v/r + Bv)$. Then P_ν is closed Fredholm and $\mathrm{Ind}(P_\nu) = 0$.

By this lemma we have $\mathrm{Ind}\left(\partial\!\!\!/^{\pm}_{(A_k,\Phi_{a,k})}|_{E_\nu}\right) = 0$ unless $i = 0$. Hence $\mathrm{Ind}(\partial\!\!\!/^{\pm}_{(A_k,\Phi_{a,k})}) = \mathrm{Ind}\left(\partial\!\!\!/^{\pm}_{(A_k,\Phi_{a,k})}|_{E_0}\right)$. Moreover, we obtain $\mathrm{Ind}\left(\partial\!\!\!/^{\pm}_{(A_k,\Phi_{a,k})}|_{E_0}\right) = 0$ by a straight calculation. □

(proof of Lemma 4.3). The Fredholmness can be easily seen. We take a function $C_\nu : \mathbb{R}_{>0} \to \mathrm{Mat}(2, \mathbb{C})$ as

$$C_\nu(r) := \begin{cases} A_\nu/r & (r \leq 1) \\ B & (r > 1) \end{cases}$$

and set a differential operator $\tilde{P}_\nu$ to be $\tilde{P}(v) := \partial_r v - C_\nu(r)v$. Since a compact perturbation does not change the index, P_ν and $\tilde{P}_\nu$ have the same indices. We can write any elements of the kernels of $\tilde{P}_\nu$ and the adjoint operator $\tilde{P}^{\bigstar}_\nu$ explicitly, and there are no non-zero L^2-solutions of $\tilde{P}_\nu(v) = 0$ and $\tilde{P}^{\bigstar}_\nu(v) = 0$. Hence we obtain $\mathrm{Ind}(P_\nu) = 0$, which is the desired equality. □

4.3 The general case

Let $Z \subset Y \setminus \partial Y$ be a finite subset. Let (V, h, A, Φ) be a Dirac-type singular monopole on (X, Z) of rank r which satisfies the Råde condition. We denote by $\vec{k}_p = (k_{p,i}) \in \mathbb{Z}^r$ the weight of (V, h, A, Φ) at $p \in Z$.

Theorem 4.4. The Dirac operators $\partial\!\!\!/^{\pm}_{(A,\Phi)}$ are Fredholm and adjoint each other. The indices of $\partial\!\!\!/^{\pm}_{(A,\Phi)}$ are given as follows:

$$\mathrm{Ind}(\partial\!\!\!/^{\pm}_{(A,\Phi)}) = \mp\left\{\sum_{p\in Z}\sum_{k_{p,i}>0} k_{p,i} + \int_{\partial Y} ch(V^+)\right\} = \pm\left\{\sum_{p\in Z}\sum_{k_{p,i}<0} k_{p,i} + \int_{\partial Y} ch(V^-)\right\}.$$

Proof. We may assume that X is connected. The former claims are easy consequences of Corollary 2.16 and results in [10]. We calculate the indices of $\partial\!\!\!/^{\pm}_{(A,\Phi)}$ by using the excision formula in [4, Appendix B]. We set $k := \sum_{p\in Z}\sum_i k_{p,i}$.

First we consider the case $k = 0$. Let (V_0, h_0, A_0) be a trivial Hermitian bundle of rank r with the trivial connection on S^3 and Φ_0 the zero endomorphism on V_0. Let U_N be the northern closed hall ball of S^3. We take a compact neighborhood U of Z that is diffeomorphic to a closed ball. We replace $(V, h, A, \Phi)|_U$ and $(V_0, h_0, A_0, \Phi_0)|_{U_N}$, and obtain $(\tilde{V}^0, \tilde{h}^0, \tilde{A}^0, \tilde{\Phi}^0)$ on X and $(\tilde{V}_0, \tilde{h}_0, \tilde{A}_0, \tilde{\Phi}_0)$ on S^3. Then by the excision formula we have $\mathrm{ind}(\partial\!\!\!/^{\pm}_{(A,\Phi)}) + \mathrm{ind}(\partial\!\!\!/^{\pm}_{(A_0,\Phi_0)}) = \mathrm{ind}(\partial\!\!\!/^{\pm}_{(\tilde{A}^0,\tilde{\Phi}^0)}) + \mathrm{ind}(\partial\!\!\!/^{\pm}_{(\tilde{A}_0,\tilde{\Phi}_0)})$. Hence we obtain $\mathrm{ind}(\partial\!\!\!/^{\pm}_{(A,\Phi)}) = \mp\left(\sum_{k_{p,i}>0} k_{p,i} + \int_{\partial Y}\mathrm{ch}(V^+)\right)$ by Corollary 3.12 and Theorem 4.1.

Next we consider the case $k \neq 0$. The proof for the case $k > 0$ remain valid for $k < 0$ mutatis mutandis. Hence we may assume $k > 0$. Let (V_1, h_1, A_1) be a Hermitian bundle of rank r with a connection on S^3 outside the north pole p_N and the south pole p_S, and Φ_1 be a skew-Hermitian endomorphism of V_1. We assume that (V_1, h_1, A_1, Φ_1) is a Dirac-type singular monopole of weight $(k, 0, \ldots, 0)$ (resp. $(-k, 0, \ldots, 0)$) on a neighborhood of p_N (resp. p_S). We replace $(V, h, A, \Phi)|_U$ and $(V_1, h_1, A_1, \Phi_1)|_{U_N}$, and obtain $(\tilde{V}^1, \tilde{h}^1, \tilde{A}^1, \tilde{\Phi}^1)$ on X and $(\tilde{V}_1, \tilde{h}_1, \tilde{A}_1, \tilde{\Phi})$ on S^3. Then the excision formula shows $\mathrm{ind}(\partial\!\!\!/^{\pm}_{(A,\Phi)}) + \mathrm{ind}(\partial\!\!\!/^{\pm}_{(A_1,\Phi_1)}) = \mathrm{ind}(\partial\!\!\!/^{\pm}_{(\tilde{A}^1,\tilde{\Phi}^1)}) + \mathrm{ind}(\partial\!\!\!/^{\pm}_{(\tilde{A}_1,\tilde{\Phi}_1)})$. Hence $\mathrm{ind}(\partial\!\!\!/^{\pm}_{(A,\Phi)}) \mp k = \mathrm{ind}(\partial\!\!\!/^{\pm}_{(\tilde{A}^1,\tilde{\Phi}^1)}) \mp \sum_{k_{p,i}>0} k_{p,i}$. We set $(V_2, h_2, A_2, \Phi_2) := (L_{-k}, h_{-k}, A_{-k}, \Phi_{-k,-1}) \oplus (\underline{\mathbb{C}^{r-1}}, \underline{h_{\mathbb{C}^{r-1}}}, \underline{d}, 0)$ on $\mathbb{R}^3$, where $(\underline{\mathbb{C}^{r-1}}, \underline{h_{\mathbb{C}^{r-1}}}, \underline{d})$ be a trivial Hermitian bundle with the trivial connection on $\mathbb{R}^3$. We denote by $p \in X$ the singular point of $(\tilde{V}^1, \tilde{h}^1, \tilde{A}^1, \tilde{\Phi}^1)$. We glue $(\tilde{V}^1, \tilde{h}^1, \tilde{A}^1, \tilde{\Phi}^1)|_{X\setminus B(p,\varepsilon)}$ and $(V_2, h_2, A_2, \Phi_2)|_{\mathbb{R}^3\setminus B(0,\varepsilon)}$, and obtain $(\tilde{V}^2, \tilde{h}^2, \tilde{A}^2, \tilde{\Phi}^2)$ on $\tilde{X} = \big((X \setminus B(p,\varepsilon)) \sqcup \mathbb{R}^3 \setminus B(0,\varepsilon)\big) / \sim$, where $\sim$ is an identification of their boundaries. We also glue $(\tilde{V}^1, \tilde{h}^1, \tilde{A}^1, \tilde{\Phi}^1)|_{\overline{B(p,\varepsilon)}}$ and $(V_2, h_2, A_2, \Phi_2)|_{\overline{B(0,\varepsilon)}}$ and obtain $(\tilde{V}_2, \tilde{h}_2, \tilde{A}_2, \tilde{\Phi}_2)$ on $S^3_\varepsilon := (\overline{B(p,\varepsilon)} \sqcup \overline{B(0,\varepsilon)})/\sim$, where over-line means the closure. Then by the excision formula we have $\mathrm{ind}(\partial\!\!\!/^{\pm}_{(\tilde{A}^1,\tilde{\Phi}^1)}) = \mathrm{ind}(\partial\!\!\!/^{\pm}_{(\tilde{A}^1,\tilde{\Phi}^1)}) + \mathrm{ind}(\partial\!\!\!/^{\pm}_{(A_2,\Phi_2)}) = \mathrm{ind}(\partial\!\!\!/^{\pm}_{(\tilde{A}^2,\tilde{\Phi}^2)}) + \mathrm{ind}(\partial\!\!\!/^{\pm}_{(\tilde{A}_2,\tilde{\Phi}_2)}) = \mathrm{ind}(\partial\!\!\!/^{\pm}_{(\tilde{A}^2,\tilde{\Phi}^2)}) \mp k$. Since the tuple $(\tilde{V}^2, \tilde{h}^2, \tilde{A}^2, \tilde{\Phi}^2)$ satisfies the Råde condition, we obtain $\mathrm{ind}(\partial\!\!\!/^{\pm}_{(\tilde{A}^2,\tilde{\Phi}^2)}) = \mp\int_{\partial Y} ch(V^+)$. As a consequence of the above arguments, we obtain $\mathrm{ind}(\partial\!\!\!/^{\pm}_{(A,\Phi)}) = \mp(\int_{\partial Y} ch(V^+) + \sum_{k_{p,i}>0} k_{p,i})$, which is the desired equation. □

References

[1] Antonio López Almorox and Carlos Tejero Prieto, "Holomorphic spectrum of twisted Dirac operators on compact Riemann surfaces", Journal of Geometry and Physics (2006), Vol.56, 2069–2091.

[2] Michael Francis Atiyah and Isadore Manuel Singer, "The Index of Elliptic operators (III)", Annals of Mathematics (1968), Vol.87, no.3, 546–604.

[3] Constantine Callias, “Axial anomalies and index theorems on open spaces”, Communications in Mathematical Physics (1978), vol. 62, 213–234.

[4] Benoit Charbonneau, “Analytic aspects of Periodic Instantons”, Ph.D Theis, Massachusetts Institute of Technology (2004)

[5] Benoit Charbonneau and Jacques Hurtubise, “Singular Hermitian-Einstein Monopoles on the Product of a Circle and a Riemann Surface”, International mathematics research notices (2011), no. 1, 175–216.

[6] Peter Kronheimer, “Monopoles and Taub-NUT metrics”, Master Thesis, University of Oxford (1985)

[7] Takuro Mochizuki and Masaki Yoshino, “Some Characterizations of Dirac Type Singularity of Monopoles”, Communications in mathematical physics (2017), vol. 356, 613–625.

[8] Hiraku Nakajima, “Monopoles and Nahm’s equations”, Lecture Notes in Pure and Appl. Math, vol. 145, (1993)

[9] Marc Pauly, “Monopole moduli spaces for compact 3-manifolds”, Mathematische Annalen (1998), vol. 311, 125–146.

[10] Johan Råde, “Callias’ index theorem, elliptic boundary conditions, and cutting and gluing”, Communications in mathematical physics (1994), vol. 161, 51–61.

[11] Masaki Yoshino, “The Nahm transform of spatially periodic instantons”, arXiv:1804.05565 (2018).