SOME INEQUALITIES FOR INTERPOLATIONAL OPERATOR MEANS

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ABSTRACT. Using the properties of geometric mean, we shall show for any $0 \leq \alpha, \beta \leq 1$,

$$
f(A\nabla_{\alpha}B) \le f((A\nabla_{\alpha}B)\nabla_{\beta}A)\sharp_{\alpha}f((A\nabla_{\alpha}B)\nabla_{\beta}B) \le f(A)\sharp_{\alpha}f(B)
$$

whenever f is a non-negative operator log-convex function, $A, B \in \mathcal{B}(\mathcal{H})$ are positive operators, and $0 \leq \alpha, \beta \leq 1$. As an application of this operator mean inequality, we present several refinements of the Aujla subadditive inequality for operator monotone decreasing functions.

Also, in a similar way, we consider some inequalities of Ando's type. Among other things, it is shown that if Φ is a positive linear map, then

$$
\Phi(A\sharp_{\alpha}B) \leq \Phi((A\sharp_{\alpha}B)\sharp_{\beta}A)\sharp_{\alpha}\Phi((A\sharp_{\alpha}B)\sharp_{\beta}B) \leq \Phi(A)\sharp_{\alpha}\Phi(B).
$$

1. Introduction and Preliminaries

We denote the set of all bounded linear operators on a Hilbert space $\mathcal H$ by $\mathcal B(\mathcal H)$. An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be positive (denoted by $A \geq 0$) if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$. If a positive operator is invertible, it is said to be strictly positive and we write $A > 0$.

The axiomatic theory for connections and means for pairs of positive matrices have been studied by Kubo and Ando [\[10\]](#page-8-0). A binary operation σ defined on the cone of strictly positive operators is called an operator mean if for $A, B > 0$,

- (i) $I \sigma I = I$, where I is the identity operator;
- (ii) $C^*(A \sigma B) C \leq (C^*AC) \sigma (C^*BC)$, $\forall C \in \mathcal{B}(\mathcal{H})$;
- (iii) $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n \sigma B_n \downarrow A \sigma B$, where $A_n \downarrow A$ means that $A_1 \geq A_2 \ldots$ and $A_n \to A$ as $n \to \infty$ in the strong operator topology;
- (iv)

$$
(1.1) \t\t A \le B \& C \le D \Rightarrow A \sigma C \le B \sigma D, \forall C, D > 0.
$$

For a symmetric operator mean σ (in the sense that $A\sigma B = B\sigma A$), a parametrized operator mean σ_{α} ($\alpha \in [0, 1]$) is called an interpolational path for σ (or Uhlmann's interpolation for σ) if it satisfies

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- (c1) $A\sigma_0B = A$ (here we recall the convention $T^0 = I$ for any positive operator T), $A\sigma_1B =$ B, and $A\sigma_{\frac{1}{2}}B = A\sigma B$;
- (c2) $(A\sigma_{\alpha}B)\sigma(A\sigma_{\beta}B) = A\sigma_{\frac{\alpha+\beta}{2}}B$ for all $\alpha, \beta \in [0,1]$;
- (c3) the map $\alpha \in [0,1] \mapsto A\sigma_{\alpha}B$ is norm continuous for each A and B.

It is straightforward to see that the set of all $\gamma \in [0, 1]$ satisfying

(1.2)
$$
(A\sigma_{\alpha}B)\,\sigma_{\gamma}\,(A\sigma_{\beta}B) = A\sigma_{(1-\gamma)\alpha+\gamma\beta}B
$$

for all α, β is a convex subset of [0,1] including 0 and 1. Therefore [\(1.2\)](#page-1-0) is valid for all $\alpha, \beta, \gamma \in [0, 1]$ (see [\[7,](#page-8-1) Lemma 1]).

Typical interpolational means are so-called power means

$$
Am_{\nu}B = A^{\frac{1}{2}} \left(\frac{1}{2} \left(I + \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{\nu} \right) \right)^{\frac{1}{\nu}} A^{\frac{1}{2}}, \quad -1 \le \nu \le 1
$$

and their interpolational paths are [\[8,](#page-8-2) Theorem 5.24],

$$
Am_{\nu,\alpha}B = A^{\frac{1}{2}} \Big((1 - \alpha) I + \alpha \Big(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \Big)^{\nu} \Big)^{\frac{1}{\nu}} A^{\frac{1}{2}}, \quad 0 \le \alpha \le 1.
$$

In particular, we have

$$
Am_{1,\alpha}B = A\nabla_{\alpha}B = (1 - \alpha) A + \alpha B,
$$

\n
$$
Am_{0,\alpha}B = A\sharp_{\alpha}B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\alpha} A^{\frac{1}{2}},
$$

\n
$$
Am_{-1,\alpha}B = A!_{\alpha}B = \left(A^{-1}\nabla_{\alpha}B\right)^{-1}.
$$

They are called the weighted arithmetic, weighted geometric, and weighted harmonic interpolations respectively. It is well-known that

(1.3)
$$
A!_{\alpha}B \le A \sharp_{\alpha} B \le A \nabla_{\alpha} B, \quad 0 \le \alpha \le 1
$$

In [\[5\]](#page-8-3), Aujla et al. introduced the notion of operator log-convex functions in the following way: A continuous real function $f:(0,\infty) \to (0,\infty)$ is called operator log-convex if

(1.4)
$$
f(A\nabla_{\alpha}B) \le f(A) \sharp_{\alpha} f(B), \quad 0 \le \alpha \le 1
$$

for all positive operators A and B. After that, Ando and Hiai $[2]$ gave the following characterization of operator monotone decreasing functions: Let f be a continuous non-negative function on $(0, \infty)$. Then the following conditions are equivalent:

- (a) f is operator monotone decreasing;
- (b) f is operator log-convex;
- (c) $f (A \nabla B) \leq f (A) \sigma f (B)$ for all positive operators A, B and for all symmetric operator means σ .

In Theorem [2.1](#page-3-0) below, we provide a more precise estimate than [\(1.4\)](#page-1-1) for operator log-convex functions. As a by-product, we improve both inequalities in [\(1.3\)](#page-1-2). Additionally, we give refinement and two reverse inequalities for the triangle inequality.

Our main application of Theorem [2.1](#page-3-0) is a subadditive behavior of operator monotone decreasing functions. Recall that a concave function (not necessarily operator concave) $f:(0,\infty) \to$ $(0, \infty)$ enjoys the subadditive inequality

(1.5)
$$
f(a+b) \le f(a) + f(b), a, b > 0.
$$

Operator concave functions (equivalently, operator monotone) do not enjoy the same subad-ditive behavior. However, in [\[3\]](#page-8-5) it was shown that an operator concave function $f:(0,\infty) \to$ $(0, \infty)$ satisfies the norm version of (1.5) as follows

$$
|||f(A+B)||| \le |||f(A)+f(B)|||,
$$

for positive definite matrices A, B and any unitraily invariant norm $||| |||$. Later, the authors in [\[6\]](#page-8-6) showed that [\(1.6\)](#page-2-1) is still valid for concave functions $f:(0,\infty) \to (0,\infty)$ (not necessarily operator concave).

We emphasize that (1.6) does not hold without the norm. In [\[4\]](#page-8-7), it is shown that an operator monotone decreasing function $f : (0, \infty) \to (0, \infty)$ satisfies the subadditive inequality

$$
(1.7) \t\t f(A+B) \le f(A)\nabla f(B),
$$

for the positive matrices A, B.

In Corollary [2.1,](#page-3-1) we present multiple refinements of [\(1.7\)](#page-2-2).

The celebrated Ando's inequality asserts that if Φ is a positive linear map and $A, B \in \mathcal{B}(\mathcal{H})$ are positive operators, then

(1.8)
$$
\Phi(A\sharp_\alpha B) \leq \Phi(A)\sharp_\alpha \Phi(B), \quad 0 \leq \alpha \leq 1.
$$

Recall that, a linear map Φ is positive if $\Phi(A)$ is positive whenever A is positive. We improve and extend this result to Uhlmann's interpolation $\sigma_{\alpha\beta}$ ($0 \leq \alpha, \beta \leq 1$). Precisely speaking, we prove that

$$
\Phi(A\sigma_{\alpha\beta}B) \leq \Phi((A\sigma_{\alpha}B)\sigma_{\beta}(A\sigma_{0}B))\sigma_{\alpha}\Phi((A\sigma_{\alpha}B)\sigma_{\beta}(A\sigma_{1}B))
$$

$$
\leq \Phi(A)\sigma_{\alpha\beta}\Phi(B).
$$

This result is included in Section [3.](#page-7-0)

2. On the operator log-convexity

Our first main result in this paper reads as follows.

Theorem 2.1. Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators and $0 \leq \alpha \leq 1$. If f is a non-negative operator monotone decreasing function, then

(2.1)
$$
f(A\nabla_{\alpha}B) \le f((A\nabla_{\alpha}B)\nabla_{\beta}A)\sharp_{\alpha}f((A\nabla_{\alpha}B)\nabla_{\beta}B) \le f(A)\sharp_{\alpha}f(B)
$$

for any $0 \leq \beta \leq 1$.

Proof. Assume f is operator monotone decreasing. We start with the useful identity

(2.2)
$$
A \nabla_{\alpha} B = ((A \nabla_{\alpha} B) \nabla_{\beta} A) \nabla_{\alpha} ((A \nabla_{\alpha} B) \nabla_{\beta} B),
$$

which follows from [\(1.2\)](#page-1-0) with $A = A\nabla_0B$ and $B = A\nabla_1B$. Then we have

$$
f(A\nabla_{\alpha}B) = f(((A\nabla_{\alpha}B)\nabla_{\beta}A)\nabla_{\alpha} ((A\nabla_{\alpha}B)\nabla_{\beta}B))
$$

(2.3) $\leq f((A\nabla_{\alpha}B)\nabla_{\beta}A)\sharp_{\alpha}f((A\nabla_{\alpha}B)\nabla_{\beta}B)$

(2.4) $\langle (f (A \nabla_{\alpha} B) \sharp_{\beta} f(A)) \sharp_{\alpha} (f (A \nabla_{\alpha} B) \sharp_{\beta} f(B)) \rangle$

(2.5)
$$
\leq ((f(A)\sharp_{\alpha}f(B))\sharp_{\beta}f(A))\sharp_{\alpha}((f(A)\sharp_{\alpha}f(B))\sharp_{\beta}f(B))
$$

$$
(2.6) \qquad \qquad = \left(\left(f(A) \sharp_{\alpha} f(B) \right) \sharp_{\beta} \left(f(A) \sharp_{0} f(B) \right) \right) \sharp_{\alpha} \left(\left(f(A) \sharp_{\alpha} f(B) \right) \sharp_{\beta} \left(f(A) \sharp_{1} f(B) \right) \right)
$$

(2.7)
$$
= f(A) \sharp_{(1-\beta)\alpha+\beta\alpha} f(B)
$$

$$
= f(A) \sharp_{\alpha} f(B)
$$

where the inequalities (2.3) , (2.4) and (2.5) follow directly from the log-convexity assumption on f together with (1.1) , the equalities (2.6) and (2.7) are obtained from the property (c1) and [\(1.2\)](#page-1-0), respectively. This completes the proof.

As promised in the introduction, we present the following refinement of Aujla inequality [\(1.7\)](#page-2-2), as a main application of Theorem [2.1.](#page-3-0)

 \Box

Corollary 2.1. Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators. If f is a non-negative operator monotone decreasing function, then

$$
f(A + B) \le f(3A \nabla B) \sharp f(A \nabla 3B)
$$

\n
$$
\le f(2A) \sharp f(2B)
$$

\n
$$
\le f(2A) \nabla f(2B)
$$

\n
$$
\le f(A) \nabla f(B).
$$

Proof. In Theorem [2.1,](#page-3-0) let $\alpha = \beta = \frac{1}{2}$ $\frac{1}{2}$ and replace (A, B) by $(2A, 2B)$. This implies the first and second inequalities immediately. The third inequality follows from the second inequality in (1.3) , while the last inequality follows properties of operator means and the fact that f is operator monotone decreasing. **Remark 2.1.** Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators and $0 \leq \alpha \leq 1$. If f is a function satisfying

(2.8)
$$
f(A\nabla_{\alpha}B) \le f((A\nabla_{\alpha}B)\nabla_{\beta}A)\sharp_{\alpha}f((A\nabla_{\alpha}B)\nabla_{\beta}B),
$$

for $0 \le \beta \le 1$, then f is operator monotone decreasing. This follows by taking $\beta = 1$ in [\(2.8\)](#page-4-0) and equivalence between (a) and (b) above.

Corollary 2.2. Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators. If g is a non-negative operator monotone increasing, then

$$
g(A\nabla_{\alpha}B) \ge g((A\nabla_{\alpha}B)\nabla_{\beta}A)\sharp_{\alpha}g((A\nabla_{\alpha}B)\nabla_{\beta}B) \ge g(A)\sharp_{\alpha}g(B)
$$

for any $0 \le \alpha, \beta \le 1$.

Proof. It was shown in [\[2\]](#page-8-4) that operator monotonicity of g is equivalent to operator log-concavity $(g (A\nabla_{\alpha}B) \ge g(A)\sharp_{\alpha}g(B))$. The proof goes in a similar way to the proof of Theorem [2.1.](#page-3-0) \square

Remark 2.2. In [\[2,](#page-8-4) Remark 2.6], we have for non-negative operator monotone decreasing function f, any operator mean σ and $A, B > 0$,

(2.9)
$$
f(A\nabla_{\alpha}B) \le f(A)!_{\alpha}f(B) \le f(A)\sigma f(B), \ 0 \le \alpha \le 1.
$$

Better estimates than [\(2.9\)](#page-4-1) may be obtained as follows, where $0 \le \alpha, \beta \le 1$,

$$
f(A\nabla_{\alpha}B) = f(((A\nabla_{\alpha}B)\nabla_{\beta}A)\nabla_{\alpha}((A\nabla_{\alpha}B)\nabla_{\beta}B))
$$

\n
$$
\leq f((A\nabla_{\alpha}B)\nabla_{\beta}A) \cdot \Delta f((A\nabla_{\alpha}B)\nabla_{\beta}B)
$$

\n
$$
\leq (f(A\nabla_{\alpha}B) \cdot \Delta f(A)) \cdot \Delta (f(A\nabla_{\alpha}B) \cdot \Delta f(B))
$$

\n
$$
\leq ((f(A) \cdot \Delta f(B)) \cdot \Delta f(A)) \cdot \Delta (f(A) \cdot \Delta f(B)) \cdot \Delta f(B))
$$

\n
$$
= ((f(A) \cdot \Delta f(B)) \cdot \Delta f(A) \cdot \Delta f(B))) \cdot \Delta (f(A) \cdot \Delta f(B)) \cdot \Delta f(A) \cdot \Delta f(B))
$$

\n
$$
= f(A) \cdot \Delta f(B)
$$

\n
$$
= f(A) \cdot \Delta f(B)
$$

\n
$$
\leq f(A) \cdot \sigma f(B)
$$

In the following we improve the well-known weighted operator arithmetic-geometric-harmonic mean inequalities [\(1.3\)](#page-1-2).

Theorem 2.2. Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators. Then

$$
A!_{\alpha}B \le ((A \sharp_{\alpha} B) \sharp_{\beta} A) !_{\alpha} ((A \sharp_{\alpha} B) \sharp_{\beta} B)
$$

\n
$$
\le A \sharp_{\alpha} B
$$

\n
$$
\le ((A \sharp_{\alpha} B) \sharp_{\beta} A) \nabla_{\alpha} ((A \sharp_{\alpha} B) \sharp_{\beta} B)
$$

\n
$$
\le A \nabla_{\alpha} B
$$

for $0 \leq \alpha, \beta \leq 1$.

Proof. It follows from the proof of Theorem [2.1](#page-3-0) that

(2.10)
$$
A\sharp_{\alpha}B = ((A\sharp_{\alpha}B)\sharp_{\beta}A)\sharp_{\alpha}((A\sharp_{\alpha}B)\sharp_{\beta}B), \quad 0 \le \alpha, \beta \le 1.
$$

Thus, we have

$$
A\sharp_{\alpha}B = ((A\sharp_{\alpha}B)\sharp_{\beta}A)\sharp_{\alpha}((A\sharp_{\alpha}B)\sharp_{\beta}B)
$$

(2.11)
$$
\leq ((A\sharp_{\alpha}B)\sharp_{\beta}A)\nabla_{\alpha}((A\sharp_{\alpha}B)\sharp_{\beta}B)
$$

(2.12)
\n
$$
\leq ((A\nabla_{\alpha}B)\nabla_{\beta}A)\nabla_{\alpha}((A\nabla_{\alpha}B)\nabla_{\beta}B)
$$
\n
$$
= ((A\nabla_{\alpha}B)\nabla_{\beta}(A\nabla_{0}B))\nabla_{\alpha}((A\nabla_{\alpha}B)\nabla_{\beta}(A\nabla_{1}B))
$$
\n(2.13)
\n
$$
= A\nabla_{\alpha}B
$$

where in the inequalities (2.11) and (2.12) we used the weighted operator arithmetic-geometric mean inequality and the equality (2.13) follows from (1.2) . This proves the third and fourth inequalities.

As for the first and second inequalities, replace A and B by A^{-1} and B^{-1} , respectively, in the third and fourth inequalities

$$
A\sharp_{\alpha}B \le ((A\sharp_{\alpha}B)\sharp_{\beta}A) \nabla_{\alpha} ((A\sharp_{\alpha}B)\sharp_{\beta}B) \le A \nabla_{\alpha}B
$$

which we have just shown. Then take the inverse to obtain the required results (thanks to the identity $A^{-1} \sharp_{\alpha} B^{-1} = (A \sharp_{\alpha} B)^{-1}$. This completes the proof.

Remark 2.3. We notice that similar inequalities maybe obtained for any symmetric mean σ , as follows. First, observe that if σ, τ are two symmetric means such that $\sigma \leq \tau$, then the set $T = \{t : 0 \le t \le 1 \text{ and } \sigma_t \le \tau_t\}$ is convex. Indeed, assume $t_1, t_2 \in T$. Then for the positive operators A, B, we have

$$
A\sigma_{\frac{t_1+t_2}{2}}B = (A\sigma_{t_1}B)\sigma(A\sigma_{t_2}B)
$$

$$
\leq (A\tau_{t_1}B)\tau(A\tau_{t_2}B)
$$

$$
= A\tau_{\frac{t_1+t_2}{2}}B,
$$

where we have used the assumptions $\sigma \leq \tau$ and $t_1, t_2 \in T$. This proves that T is convex, and hence $T = [0, 1]$ since $0, 1 \in T$, trivially. Thus, we have shown that if $\sigma \leq \tau$ then $\sigma_{\alpha} \leq \tau_{\alpha}$, for all $0 \leq \alpha \leq 1$. Now noting that

$$
A\sigma_{\alpha}B = ((A\sigma_{\alpha}B)\sigma_{\beta}A)\,\sigma_{\alpha}\left((A\sigma_{\alpha}B)\sigma_{\beta}B\right),\,
$$

and proceeding as in Theorem [2.1,](#page-3-0) we obtain

(2.14)
$$
f(A\nabla_{\alpha}B) \le f((A\nabla_{\alpha}B)\nabla_{\beta}A)\sigma_{\alpha}f((A\nabla_{\alpha}B)\nabla_{\beta}B) \le f(A)\sigma_{\alpha}f(B)
$$

for any $0 \le \beta \le 1$ and the operator log-convex function f. This provides a more precise estimate than (c) above.

On the other hand, proceeding as in Theorem [2.2,](#page-5-3) we obtain

(2.15)
$$
A\sigma_{\alpha}B \le ((A\sigma_{\alpha}B)\sigma_{\beta}A) \nabla_{\alpha} ((A\sigma_{\alpha}B)\sigma_{\beta}B) \le A \nabla_{\alpha}B,
$$

observing that $\sigma_{\alpha} \leq \nabla_{\alpha}$. This provides a refinement of the latter basic inequality.

Taking into account [\(2.2\)](#page-3-7), it follows that

$$
A + B = \alpha A + (1 - \alpha) (A \nabla B) + \alpha B + (1 - \alpha) (A \nabla B).
$$

As a consequence of this inequality, we have the following refinement of the well-known triangle inequality

 $||A + B|| \le ||A|| + ||B||$.

Corollary 2.3. Let $A, B \in \mathcal{B}(\mathcal{H})$. Then, for $\alpha \in \mathbb{R}$,

$$
||A + B|| \le ||\alpha A + (1 - \alpha) (A \nabla B)|| + ||\alpha B + (1 - \alpha) (A \nabla B)|| \le ||A|| + ||B||.
$$

Remark 2.4. Using Corollary [2.3,](#page-6-0) we obtain the reverse triangle inequalities

$$
||A|| - ||B|| \le \frac{1}{2} (||A\nabla_{-\alpha}(2B)|| + ||A\nabla_{\alpha}(2B)|| - 2||B||) \le ||A - B||
$$

and

$$
||B|| - ||A|| \le \frac{1}{2} (||B\nabla_{-\alpha}(2A)|| + ||B\nabla_{\alpha}(2A)|| - 2||A||) \le ||A - B||,
$$

where $\alpha \in \mathbb{R}$.

3. A glimpse at the Ando's inequality

In this section, we present some versions and improvements of Ando's inequality [\(1.8\)](#page-2-3).

Theorem 3.1. Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators and Φ be a positive linear map. Then for any $0 \le \alpha, \beta \le 1$,

(3.1)
$$
\Phi(A\sharp_{\alpha}B) \leq \Phi((A\sharp_{\alpha}B)\sharp_{\beta}A)\sharp_{\alpha}\Phi((A\sharp_{\alpha}B)\sharp_{\beta}B) \leq \Phi(A)\sharp_{\alpha}\Phi(B).
$$

In particular,

(3.2)
$$
\sum_{j=1}^{m} A_j \sharp_{\alpha} B_j \leq \left(\sum_{j=1}^{m} \left(A_j \sharp_{\alpha} B_j \right) \sharp_{\beta} A_j \right) \sharp_{\alpha} \left(\sum_{j=1}^{m} \left(A_j \sharp_{\alpha} B_j \right) \sharp_{\beta} B_j \right) \leq \left(\sum_{j=1}^{m} A_j \right) \sharp_{\alpha} \left(\sum_{j=1}^{m} B_j \right).
$$

Proof. We omit the proof of (3.1) because it is proved in a way similar to that of (2.1) in Theorem [2.1.](#page-3-0) Now, if in [\(3.1\)](#page-7-1) we take $\Phi : M_{nk}(\mathbb{C}) \to M_k(\mathbb{C})$ defined by

$$
\Phi\left(\begin{pmatrix} X_{1,1} & & \\ & \ddots & \\ & & X_{n,n} \end{pmatrix}\right) = X_{1,1} + \ldots + X_{n,n}
$$

and apply Φ to $A = \text{diag}(A_1, \ldots, A_n)$ and $B = \text{diag}(B_1, \ldots, B_n)$, we get [\(3.2\)](#page-7-2).

In the following, we present a more general form of (3.1) will be shown.

Theorem 3.2. Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators and Φ be any positive linear map. Then we have the following inequalities for Uhlmann's interpolation $\sigma_{\alpha\beta}$ and $0 \le \alpha, \beta \le 1$,

$$
\Phi(A\sigma_{\alpha\beta}B) \leq \Phi((A\sigma_{\alpha}B)\sigma_{\beta}(A\sigma_{0}B))\sigma_{\alpha}\Phi((A\sigma_{\alpha}B)\sigma_{\beta}(A\sigma_{1}B))
$$

$$
\leq \Phi(A)\sigma_{\alpha\beta}\Phi(B).
$$

Proof. Thanks to (1.2) , we obviously have

$$
((A\sigma_{\alpha}B)\sigma_{\beta}(A\sigma_{0}B))\sigma_{\alpha}((A\sigma_{\alpha}B)\sigma_{\beta}(A\sigma_{1}B))
$$

$$
= (A\sigma_{\alpha(1-\beta)}B)\sigma_{\alpha}(A\sigma_{\alpha(1-\beta)+\beta}B)
$$

$$
= A\sigma_{\alpha\beta}B.
$$

Now, the desired result follows directly from the above identities.

Remark 3.1. From simple calculations, we have the following inequalities for positive operators $A, B \in \mathcal{B}(\mathcal{H})$, any positive linear map Φ and $0 \leq \alpha, \beta, \gamma, \delta \leq 1$,

$$
\Phi\left(A\sigma_{\alpha(1-\beta)+\beta((1-\alpha)\gamma+\alpha\delta)}B\right) \leq \Phi\left(\left(A\sigma_{\alpha}B\right)\sigma_{\beta}\left(\left(A\sigma_{\gamma}B\right)\right)\right)\sigma_{\alpha}\Phi\left(\left(A\sigma_{\alpha}B\right)\sigma_{\beta}\left(\left(A\sigma_{\delta}B\right)\right)\right)
$$
\n
$$
\leq \Phi\left(A\right)\sigma_{\alpha(1-\beta)+\beta((1-\alpha)\gamma+\alpha\delta)}\Phi\left(B\right).
$$

Apparently, [\(3.3\)](#page-7-3) reduces to [\(3.2\)](#page-7-4) when $\gamma = 0$ and $\delta = 1$.

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