SOME INEQUALITIES FOR INTERPOLATIONAL OPERATOR MEANS

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ABSTRACT. Using the properties of geometric mean, we shall show for any $0 \le \alpha, \beta \le 1$,

$$f(A\nabla_{\alpha}B) \le f((A\nabla_{\alpha}B)\nabla_{\beta}A) \sharp_{\alpha}f((A\nabla_{\alpha}B)\nabla_{\beta}B) \le f(A) \sharp_{\alpha}f(B)$$

whenever f is a non-negative operator log-convex function, $A, B \in \mathcal{B}(\mathcal{H})$ are positive operators, and $0 \leq \alpha, \beta \leq 1$. As an application of this operator mean inequality, we present several refinements of the Aujla subadditive inequality for operator monotone decreasing functions.

Also, in a similar way, we consider some inequalities of Ando's type. Among other things, it is shown that if Φ is a positive linear map, then

$$\Phi\left(A\sharp_{\alpha}B\right) \leq \Phi\left(\left(A\sharp_{\alpha}B\right)\sharp_{\beta}A\right)\sharp_{\alpha}\Phi\left(\left(A\sharp_{\alpha}B\right)\sharp_{\beta}B\right) \leq \Phi\left(A\right)\sharp_{\alpha}\Phi\left(B\right).$$

1. INTRODUCTION AND PRELIMINARIES

We denote the set of all bounded linear operators on a Hilbert space \mathcal{H} by $\mathcal{B}(\mathcal{H})$. An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be positive (denoted by $A \ge 0$) if $\langle Ax, x \rangle \ge 0$ for all $x \in \mathcal{H}$. If a positive operator is invertible, it is said to be strictly positive and we write A > 0.

The axiomatic theory for connections and means for pairs of positive matrices have been studied by Kubo and Ando [10]. A binary operation σ defined on the cone of strictly positive operators is called an operator mean if for A, B > 0,

- (i) $I\sigma I = I$, where I is the identity operator;
- (ii) $C^*(A\sigma B) C \leq (C^*AC) \sigma (C^*BC), \forall C \in \mathcal{B}(\mathcal{H});$
- (iii) $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n \sigma B_n \downarrow A \sigma B$, where $A_n \downarrow A$ means that $A_1 \ge A_2 \ldots$ and $A_n \to A$ as $n \to \infty$ in the strong operator topology;
- (iv)

(1.1)
$$A \leq B \& C \leq D \Rightarrow A\sigma C \leq B\sigma D, \forall C, D > 0.$$

For a symmetric operator mean σ (in the sense that $A\sigma B = B\sigma A$), a parametrized operator mean σ_{α} ($\alpha \in [0, 1]$) is called an interpolational path for σ (or Uhlmann's interpolation for σ) if it satisfies

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- (c1) $A\sigma_0 B = A$ (here we recall the convention $T^0 = I$ for any positive operator T), $A\sigma_1 B = B$, and $A\sigma_{\frac{1}{2}}B = A\sigma B$;
- (c2) $(A\sigma_{\alpha}B)\sigma(A\sigma_{\beta}B) = A\sigma_{\frac{\alpha+\beta}{2}}B$ for all $\alpha, \beta \in [0,1];$
- (c3) the map $\alpha \in [0, 1] \mapsto A\sigma_{\alpha}B$ is norm continuous for each A and B.

It is straightforward to see that the set of all $\gamma \in [0, 1]$ satisfying

(1.2)
$$(A\sigma_{\alpha}B) \sigma_{\gamma} (A\sigma_{\beta}B) = A\sigma_{(1-\gamma)\alpha+\gamma\beta}B$$

for all α, β is a convex subset of [0, 1] including 0 and 1. Therefore (1.2) is valid for all $\alpha, \beta, \gamma \in [0, 1]$ (see [7, Lemma 1]).

Typical interpolational means are so-called power means

$$Am_{v}B = A^{\frac{1}{2}} \left(\frac{1}{2} \left(I + \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{v} \right) \right)^{\frac{1}{v}} A^{\frac{1}{2}}, \quad -1 \le v \le 1$$

and their interpolational paths are [8, Theorem 5.24],

$$Am_{\nu,\alpha}B = A^{\frac{1}{2}} \left((1-\alpha)I + \alpha \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\nu} \right)^{\frac{1}{\nu}} A^{\frac{1}{2}}, \quad 0 \le \alpha \le 1.$$

In particular, we have

$$Am_{1,\alpha}B = A\nabla_{\alpha}B = (1-\alpha)A + \alpha B,$$

$$Am_{0,\alpha}B = A\sharp_{\alpha}B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\alpha}A^{\frac{1}{2}},$$

$$Am_{-1,\alpha}B = A!_{\alpha}B = \left(A^{-1}\nabla_{\alpha}B\right)^{-1}.$$

They are called the weighted arithmetic, weighted geometric, and weighted harmonic interpolations respectively. It is well-known that

(1.3)
$$A!_{\alpha}B \le A\sharp_{\alpha}B \le A\nabla_{\alpha}B, \quad 0 \le \alpha \le 1$$

In [5], Aujla et al. introduced the notion of operator log-convex functions in the following way: A continuous real function $f: (0, \infty) \to (0, \infty)$ is called operator log-convex if

(1.4)
$$f(A\nabla_{\alpha}B) \le f(A) \,\sharp_{\alpha}f(B) \,, \quad 0 \le \alpha \le 1$$

for all positive operators A and B. After that, Ando and Hiai [2] gave the following characterization of operator monotone decreasing functions: Let f be a continuous non-negative function on $(0, \infty)$. Then the following conditions are equivalent:

- (a) f is operator monotone decreasing;
- (b) f is operator log-convex;
- (c) $f(A\nabla B) \leq f(A) \sigma f(B)$ for all positive operators A, B and for all symmetric operator means σ .

In Theorem 2.1 below, we provide a more precise estimate than (1.4) for operator log-convex functions. As a by-product, we improve both inequalities in (1.3). Additionally, we give refinement and two reverse inequalities for the triangle inequality.

Our main application of Theorem 2.1 is a subadditive behavior of operator monotone decreasing functions. Recall that a concave function (not necessarily operator concave) $f: (0, \infty) \rightarrow [0, \infty)$ enjoys the subadditive inequality

(1.5)
$$f(a+b) \le f(a) + f(b), a, b > 0.$$

Operator concave functions (equivalently, operator monotone) do not enjoy the same subadditive behavior. However, in [3] it was shown that an operator concave function $f: (0, \infty) \rightarrow (0, \infty)$ satisfies the norm version of (1.5) as follows

(1.6)
$$|||f(A+B)||| \le |||f(A) + f(B)|||,$$

for positive definite matrices A, B and any unitraily invariant norm ||| |||. Later, the authors in [6] showed that (1.6) is still valid for concave functions $f : (0, \infty) \to (0, \infty)$ (not necessarily operator concave).

We emphasize that (1.6) does not hold without the norm. In [4], it is shown that an operator monotone decreasing function $f: (0, \infty) \to (0, \infty)$ satisfies the subadditive inequality

(1.7)
$$f(A+B) \le f(A)\nabla f(B),$$

for the positive matrices A, B.

In Corollary 2.1, we present multiple refinements of (1.7).

The celebrated Ando's inequality asserts that if Φ is a positive linear map and $A, B \in \mathcal{B}(\mathcal{H})$ are positive operators, then

(1.8)
$$\Phi(A\sharp_{\alpha}B) \le \Phi(A) \sharp_{\alpha}\Phi(B), \quad 0 \le \alpha \le 1.$$

Recall that, a linear map Φ is positive if $\Phi(A)$ is positive whenever A is positive. We improve and extend this result to Uhlmann's interpolation $\sigma_{\alpha\beta}$ ($0 \le \alpha, \beta \le 1$). Precisely speaking, we prove that

$$\Phi (A\sigma_{\alpha\beta}B) \leq \Phi ((A\sigma_{\alpha}B)\sigma_{\beta} (A\sigma_{0}B))\sigma_{\alpha}\Phi ((A\sigma_{\alpha}B)\sigma_{\beta} (A\sigma_{1}B))$$
$$\leq \Phi (A)\sigma_{\alpha\beta}\Phi (B).$$

This result is included in Section 3.

2. On the operator log-convexity

Our first main result in this paper reads as follows.

Theorem 2.1. Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators and $0 \le \alpha \le 1$. If f is a non-negative operator monotone decreasing function, then

(2.1)
$$f(A\nabla_{\alpha}B) \leq f((A\nabla_{\alpha}B)\nabla_{\beta}A) \sharp_{\alpha}f((A\nabla_{\alpha}B)\nabla_{\beta}B) \leq f(A) \sharp_{\alpha}f(B)$$

for any $0 \leq \beta \leq 1$.

Proof. Assume f is operator monotone decreasing. We start with the useful identity

(2.2)
$$A\nabla_{\alpha}B = ((A\nabla_{\alpha}B)\nabla_{\beta}A)\nabla_{\alpha}((A\nabla_{\alpha}B)\nabla_{\beta}B) + (A\nabla_{\alpha}B)\nabla_{\beta}B) + (A\nabla_{\alpha}B)\nabla_{\beta}B + (A\nabla_{\alpha}B)\nabla_{\beta}B + (A\nabla_{\alpha}B)\nabla_{\beta}B + (A\nabla_{\alpha}B)\nabla_{\beta}B) + (A\nabla_{\alpha}B)\nabla_{\beta}B + (A\nabla_$$

which follows from (1.2) with $A = A \nabla_0 B$ and $B = A \nabla_1 B$. Then we have

$$(A\nabla_{\alpha}B) = f\left(\left((A\nabla_{\alpha}B)\nabla_{\beta}A\right)\nabla_{\alpha}\left((A\nabla_{\alpha}B)\nabla_{\beta}B\right)\right)$$
$$\leq f\left((A\nabla_{\alpha}B)\nabla_{\beta}A\right)\sharp_{\alpha}f\left((A\nabla_{\alpha}B)\nabla_{\beta}B\right)$$

(2.4) $\leq (f(A\nabla_{\alpha}B) \sharp_{\beta}f(A)) \sharp_{\alpha} (f(A\nabla_{\alpha}B) \sharp_{\beta}f(B))$

(2.5)
$$\leq \left(\left(f\left(A\right)\sharp_{\alpha}f\left(B\right)\right)\sharp_{\beta}f(A)\right)\sharp_{\alpha}\left(\left(f\left(A\right)\sharp_{\alpha}f\left(B\right)\right)\sharp_{\beta}f(B)\right)\right)$$

$$(2.6) \qquad \qquad = \left(\left(f\left(A\right)\sharp_{\alpha}f\left(B\right)\right)\sharp_{\beta}\left(f(A)\sharp_{0}f(B)\right)\right)\sharp_{\alpha}\left(\left(f\left(A\right)\sharp_{\alpha}f\left(B\right)\right)\sharp_{\beta}\left(f(A)\sharp_{1}f(B)\right)\right)$$

(2.7)
$$= f(A) \sharp_{(1-\beta)\alpha+\beta\alpha} f(B)$$
$$= f(A) \sharp_{\alpha} f(B)$$

where the inequalities (2.3), (2.4) and (2.5) follow directly from the log-convexity assumption on f together with (1.1), the equalities (2.6) and (2.7) are obtained from the property (c1) and (1.2), respectively. This completes the proof.

As promised in the introduction, we present the following refinement of Aujla inequality (1.7), as a main application of Theorem 2.1.

Corollary 2.1. Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators. If f is a non-negative operator monotone decreasing function, then

$$f(A+B) \leq f(3A\nabla B) \sharp f(A\nabla 3B)$$
$$\leq f(2A) \sharp f(2B)$$
$$\leq f(2A) \nabla f(2B)$$
$$\leq f(A) \nabla f(B).$$

Proof. In Theorem 2.1, let $\alpha = \beta = \frac{1}{2}$ and replace (A, B) by (2A, 2B). This implies the first and second inequalities immediately. The third inequality follows from the second inequality in (1.3), while the last inequality follows properties of operator means and the fact that f is operator monotone decreasing.

Remark 2.1. Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators and $0 \le \alpha \le 1$. If f is a function satisfying

(2.8)
$$f(A\nabla_{\alpha}B) \leq f((A\nabla_{\alpha}B)\nabla_{\beta}A) \sharp_{\alpha}f((A\nabla_{\alpha}B)\nabla_{\beta}B),$$

for $0 \le \beta \le 1$, then f is operator monotone decreasing. This follows by taking $\beta = 1$ in (2.8) and equivalence between (a) and (b) above.

Corollary 2.2. Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators. If g is a non-negative operator monotone increasing, then

$$g(A\nabla_{\alpha}B) \ge g((A\nabla_{\alpha}B)\nabla_{\beta}A) \sharp_{\alpha}g((A\nabla_{\alpha}B)\nabla_{\beta}B) \ge g(A) \sharp_{\alpha}g(B)$$

for any $0 \leq \alpha, \beta \leq 1$.

Proof. It was shown in [2] that operator monotonicity of g is equivalent to operator log-concavity $(g(A\nabla_{\alpha}B) \ge g(A) \sharp_{\alpha}g(B))$. The proof goes in a similar way to the proof of Theorem 2.1. \Box

Remark 2.2. In [2, Remark 2.6], we have for non-negative operator monotone decreasing function f, any operator mean σ and A, B > 0,

(2.9)
$$f(A\nabla_{\alpha}B) \le f(A)!_{\alpha}f(B) \le f(A)\sigma f(B), \ 0 \le \alpha \le 1.$$

Better estimates than (2.9) may be obtained as follows, where $0 \le \alpha, \beta \le 1$,

$$\begin{split} f\left(A\nabla_{\alpha}B\right) &= f\left(\left((A\nabla_{\alpha}B\right)\nabla_{\beta}A\right)\nabla_{\alpha}\left((A\nabla_{\alpha}B\right)\nabla_{\beta}B\right)\right) \\ &\leq f\left((A\nabla_{\alpha}B)\nabla_{\beta}A\right)!_{\alpha}f\left((A\nabla_{\alpha}B)\nabla_{\beta}B\right) \\ &\leq \left(f\left(A\nabla_{\alpha}B\right)!_{\beta}f(A)\right)!_{\alpha}\left(f\left(A\nabla_{\alpha}B\right)!_{\beta}f(B)\right) \\ &\leq \left(\left(f\left(A\right)!_{\alpha}f\left(B\right)\right)!_{\beta}f(A)\right)!_{\alpha}\left(\left(f\left(A\right)!_{\alpha}f\left(B\right)\right)!_{\beta}f(B)\right) \\ &= \left(\left(f\left(A\right)!_{\alpha}f\left(B\right)\right)!_{\beta}\left(f(A)!_{0}f(B)\right)\right)!_{\alpha}\left(\left(f\left(A\right)!_{\alpha}f\left(B\right)\right)!_{\beta}\left(f(A)!_{1}f(B)\right)\right) \\ &= f\left(A\right)!_{(1-\beta)\alpha+\beta\alpha}f\left(B\right) \\ &= f\left(A\right)!_{\alpha}f\left(B\right) \\ &\leq f\left(A\right)\sigma f\left(B\right) \end{split}$$

In the following we improve the well-known weighted operator arithmetic-geometric-harmonic mean inequalities (1.3).

Theorem 2.2. Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators. Then

$$A!_{\alpha}B \leq \left(\left(A \sharp_{\alpha}B \right) \sharp_{\beta}A \right) !_{\alpha} \left(\left(A \sharp_{\alpha}B \right) \sharp_{\beta}B \right)$$
$$\leq A \sharp_{\alpha}B$$
$$\leq \left(\left(A \sharp_{\alpha}B \right) \sharp_{\beta}A \right) \nabla_{\alpha} \left(\left(A \sharp_{\alpha}B \right) \sharp_{\beta}B \right)$$
$$\leq A \nabla_{\alpha}B$$

for $0 \leq \alpha, \beta \leq 1$.

Proof. It follows from the proof of Theorem 2.1 that

(2.10)
$$A\sharp_{\alpha}B = \left(\left(A\sharp_{\alpha}B\right)\sharp_{\beta}A\right)\sharp_{\alpha}\left(\left(A\sharp_{\alpha}B\right)\sharp_{\beta}B\right), \quad 0 \le \alpha, \beta \le 1$$

Thus, we have

$$A\sharp_{\alpha}B = \left(\left(A\sharp_{\alpha}B \right) \sharp_{\beta}A \right) \sharp_{\alpha} \left(\left(A\sharp_{\alpha}B \right) \sharp_{\beta}B \right)$$

(2.11)
$$\leq \left(\left(A \sharp_{\alpha} B \right) \sharp_{\beta} A \right) \nabla_{\alpha} \left(\left(A \sharp_{\alpha} B \right) \sharp_{\beta} B \right)$$

(2.12)
$$\leq ((A\nabla_{\alpha}B)\nabla_{\beta}A)\nabla_{\alpha}((A\nabla_{\alpha}B)\nabla_{\beta}B)$$
$$= ((A\nabla_{\alpha}B)\nabla_{\beta}(A\nabla_{0}B))\nabla_{\alpha}((A\nabla_{\alpha}B)\nabla_{\beta}(A\nabla_{1}B))$$

 $(2.13) = A\nabla_{\alpha}B$

where in the inequalities (2.11) and (2.12) we used the weighted operator arithmetic-geometric mean inequality and the equality (2.13) follows from (1.2). This proves the third and fourth inequalities.

As for the first and second inequalities, replace A and B by A^{-1} and B^{-1} , respectively, in the third and fourth inequalities

$$A\sharp_{\alpha}B \le \left(\left(A\sharp_{\alpha}B\right)\sharp_{\beta}A\right) \nabla_{\alpha} \left(\left(A\sharp_{\alpha}B\right)\sharp_{\beta}B\right) \le A\nabla_{\alpha}B$$

which we have just shown. Then take the inverse to obtain the required results (thanks to the identity $A^{-1}\sharp_{\alpha}B^{-1} = (A\sharp_{\alpha}B)^{-1}$). This completes the proof.

Remark 2.3. We notice that similar inequalities maybe obtained for any symmetric mean σ , as follows. First, observe that if σ, τ are two symmetric means such that $\sigma \leq \tau$, then the set $T = \{t : 0 \leq t \leq 1 \text{ and } \sigma_t \leq \tau_t\}$ is convex. Indeed, assume $t_1, t_2 \in T$. Then for the positive operators A, B, we have

$$A\sigma_{\frac{t_1+t_2}{2}}B = (A\sigma_{t_1}B)\sigma(A\sigma_{t_2}B)$$
$$\leq (A\tau_{t_1}B)\tau(A\tau_{t_2}B)$$
$$= A\tau_{\frac{t_1+t_2}{2}}B,$$

where we have used the assumptions $\sigma \leq \tau$ and $t_1, t_2 \in T$. This proves that T is convex, and hence T = [0, 1] since $0, 1 \in T$, trivially. Thus, we have shown that if $\sigma \leq \tau$ then $\sigma_{\alpha} \leq \tau_{\alpha}$, for all $0 \leq \alpha \leq 1$. Now noting that

$$A\sigma_{\alpha}B = \left((A\sigma_{\alpha}B)\sigma_{\beta}A \right)\sigma_{\alpha} \left((A\sigma_{\alpha}B)\sigma_{\beta}B \right),$$

and proceeding as in Theorem 2.1, we obtain

$$(2.14) f(A\nabla_{\alpha}B) \leq f((A\nabla_{\alpha}B)\nabla_{\beta}A)\sigma_{\alpha}f((A\nabla_{\alpha}B)\nabla_{\beta}B) \leq f(A)\sigma_{\alpha}f(B)$$

for any $0 \le \beta \le 1$ and the operator log-convex function f. This provides a more precise estimate than (c) above.

On the other hand, proceeding as in Theorem 2.2, we obtain

(2.15)
$$A\sigma_{\alpha}B \leq ((A\sigma_{\alpha}B)\sigma_{\beta}A)\nabla_{\alpha}((A\sigma_{\alpha}B)\sigma_{\beta}B) \leq A\nabla_{\alpha}B,$$

observing that $\sigma_{\alpha} \leq \nabla_{\alpha}$. This provides a refinement of the latter basic inequality.

Taking into account (2.2), it follows that

$$A + B = \alpha A + (1 - \alpha) (A\nabla B) + \alpha B + (1 - \alpha) (A\nabla B).$$

As a consequence of this inequality, we have the following refinement of the well-known triangle inequality

 $||A + B|| \le ||A|| + ||B||.$

Corollary 2.3. Let $A, B \in \mathcal{B}(\mathcal{H})$. Then, for $\alpha \in \mathbb{R}$,

$$||A + B|| \le ||\alpha A + (1 - \alpha) (A\nabla B)|| + ||\alpha B + (1 - \alpha) (A\nabla B)|| \le ||A|| + ||B||.$$

Remark 2.4. Using Corollary 2.3, we obtain the reverse triangle inequalities

$$||A|| - ||B|| \le \frac{1}{2} \left(||A\nabla_{-\alpha}(2B)|| + ||A\nabla_{\alpha}(2B)|| - 2 ||B|| \right) \le ||A - B||$$

and

$$||B|| - ||A|| \le \frac{1}{2} \left(||B\nabla_{-\alpha}(2A)|| + ||B\nabla_{\alpha}(2A)|| - 2 ||A|| \right) \le ||A - B||,$$

where $\alpha \in \mathbb{R}$.

3. A GLIMPSE AT THE ANDO'S INEQUALITY

In this section, we present some versions and improvements of Ando's inequality (1.8).

Theorem 3.1. Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators and Φ be a positive linear map. Then for any $0 \le \alpha, \beta \le 1$,

(3.1)
$$\Phi\left(A\sharp_{\alpha}B\right) \leq \Phi\left(\left(A\sharp_{\alpha}B\right)\sharp_{\beta}A\right)\sharp_{\alpha}\Phi\left(\left(A\sharp_{\alpha}B\right)\sharp_{\beta}B\right) \leq \Phi\left(A\right)\sharp_{\alpha}\Phi\left(B\right).$$

In particular,

(3.2)
$$\sum_{j=1}^{m} A_{j} \sharp_{\alpha} B_{j} \leq \left(\sum_{j=1}^{m} \left(A_{j} \sharp_{\alpha} B_{j} \right) \sharp_{\beta} A_{j} \right) \sharp_{\alpha} \left(\sum_{j=1}^{m} \left(A_{j} \sharp_{\alpha} B_{j} \right) \sharp_{\beta} B_{j} \right) \\ \leq \left(\sum_{j=1}^{m} A_{j} \right) \sharp_{\alpha} \left(\sum_{j=1}^{m} B_{j} \right).$$

Proof. We omit the proof of (3.1) because it is proved in a way similar to that of (2.1) in Theorem 2.1. Now, if in (3.1) we take $\Phi: M_{nk}(\mathbb{C}) \to M_k(\mathbb{C})$ defined by

$$\Phi\left(\begin{pmatrix} X_{1,1} & & \\ & \ddots & \\ & & X_{n,n} \end{pmatrix}\right) = X_{1,1} + \ldots + X_{n,n}$$

and apply Φ to $A = \text{diag}(A_1, \ldots, A_n)$ and $B = \text{diag}(B_1, \ldots, B_n)$, we get (3.2).

In the following, we present a more general form of (3.1) will be shown.

Theorem 3.2. Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators and Φ be any positive linear map. Then we have the following inequalities for Uhlmann's interpolation $\sigma_{\alpha\beta}$ and $0 \le \alpha, \beta \le 1$,

$$\Phi (A\sigma_{\alpha\beta}B) \leq \Phi ((A\sigma_{\alpha}B)\sigma_{\beta} (A\sigma_{0}B))\sigma_{\alpha}\Phi ((A\sigma_{\alpha}B)\sigma_{\beta} (A\sigma_{1}B))$$
$$\leq \Phi (A)\sigma_{\alpha\beta}\Phi (B).$$

Proof. Thanks to (1.2), we obviously have

$$((A\sigma_{\alpha}B)\sigma_{\beta}(A\sigma_{0}B))\sigma_{\alpha}((A\sigma_{\alpha}B)\sigma_{\beta}(A\sigma_{1}B))$$
$$= (A\sigma_{\alpha(1-\beta)}B)\sigma_{\alpha}(A\sigma_{\alpha(1-\beta)+\beta}B)$$
$$= A\sigma_{\alpha\beta}B.$$

Now, the desired result follows directly from the above identities.

Remark 3.1. From simple calculations, we have the following inequalities for positive operators $A, B \in \mathcal{B}(\mathcal{H})$, any positive linear map Φ and $0 \le \alpha, \beta, \gamma, \delta \le 1$,

(3.3)
$$\Phi\left(A\sigma_{\alpha(1-\beta)+\beta((1-\alpha)\gamma+\alpha\delta)}B\right) \leq \Phi\left((A\sigma_{\alpha}B)\sigma_{\beta}\left((A\sigma_{\gamma}B)\right)\right)\sigma_{\alpha}\Phi\left((A\sigma_{\alpha}B)\sigma_{\beta}\left((A\sigma_{\delta}B)\right)\right) \\
\leq \Phi\left(A\right)\sigma_{\alpha(1-\beta)+\beta((1-\alpha)\gamma+\alpha\delta)}\Phi\left(B\right).$$

Apparently, (3.3) reduces to (3.2) when $\gamma = 0$ and $\delta = 1$.

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