

Field Generated by Division Points of Certain Formal Group Laws

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Abstract

In this article we study the Galois group of field generated by division points of special class of formal group laws and prove an equivalent condition for the group to be abelian. Further, we explore relations between the endomorphism ring of a formal group and the Galois group of field generated by division points. These results extend the results in [SS1], [SS2].

1 Introduction

Let p be an odd prime and let K be a finite extension of \mathbb{Q}_p . Put O_K to be the ring of integers of K , let \mathfrak{p}_K denote the unique maximal ideal of O_K and $v_{\mathfrak{p}_K}(\cdot)$ be the valuation associated to it. Fix an algebraic closure $\overline{\mathbb{Q}_p}$ and $|\cdot|_p$ be an fixed extension of the absolute value. Let \overline{O} be the ring of integers of $\overline{\mathbb{Q}_p}$ and $\overline{\mathfrak{p}}$ be the unique maximal ideal of \overline{O} . Clearly $\overline{\mathfrak{p}} \cap K = \mathfrak{p}_K$.

Let A be an integrally closed, complete subring of O_K . Assume that \mathfrak{p}_A is the maximal ideal, $K(A)$ is the field of fractions and $k(A)$ is the field of residues. Put $[k(A) : \mathbb{F}_p] = f_A$. Let \mathfrak{F} be a (one dimensional, commutative) formal group-law defined over O_K admitting an A module structure. π be a generator of \mathfrak{p}_A and say

$$[\pi](X) = \pi X + a_2 X^2 + a_3 X^3 + \cdots \in O_K[[X]] \quad (1.1)$$

with at least one $a_i \in O_K - \mathfrak{p}_K$. Then $\min \{i \mid |a_i|_p = 1\} = p^h$ for some positive integer h (see [Haz], 18.3.2). Now if π_1 is another generator of \mathfrak{p}_A and

$$[\pi_1](X) = \pi_1 X + b_2 X^2 + \cdots \in O_K[[X]]$$

then $b_{p^h} \in O_K - \mathfrak{p}_K$ and $\min \{i \mid |b_i|_p = 1\} = p^h$. This integer h is called the height of \mathfrak{F} as formal A module. If $a_i \in \mathfrak{p}_K$ for all $i \geq 2$ then we say, height of \mathfrak{F} is infinity. We shall only consider formal A modules of finite height.

Let A be as above and \mathfrak{F} be a formal A module over O_K . It defines a A module structure on \mathfrak{p}_K which naturally extends to a A -structure on $\overline{\mathfrak{p}}$. We shall denote the corresponding addition by $\oplus_{\mathfrak{F}}$ to distinguish it from usual addition.

Let π be a generator of \mathfrak{p}_A . For each $n \geq 1$, use $\mathfrak{F}[\pi^n]$ to denote the π^n -torsion submodule of $\overline{\mathfrak{p}}$. For any sub-field L of $\overline{\mathbb{Q}_p}$, let $L(\pi^n)$ be the subfield of $\overline{\mathbb{Q}_p}$ generated by $\mathfrak{F}[\pi^n]$ over L and put $L(\pi^\infty) = \bigcup_{i \geq 1} L(\pi^i)$. We shall adopt the convention $\mathfrak{F}[\pi^0] = \{0\}$.

Fix a generator π of \mathfrak{p}_K and put $K_\pi = \mathbb{Q}_p(\pi)$. Use A_π to denote the ring of integers of K_π . For simplicity we shall write \mathfrak{p}_{A_π} as \mathfrak{p}_π and f_{A_π} as f_π . Note that π is a generator of \mathfrak{p}_π . Let \mathfrak{F} be a formal A_π module of height h defined over O_K .

The next remark summarizes main results proved in appendix-A of [SS1]:

Remark 1.1 : A) i) With notation as above $f_\pi \mid h$.

Let $n \geq 1$. Put $q = p^h$ and $h_{r,\pi} = \frac{h}{f_\pi}$. Then the following statements are true :

ii) $\mathfrak{F}[\pi^n] \cong (A_\pi/\pi^n A_\pi)^{h_{r,\pi}}$ as A_π modules.

iii) If $z \in \mathfrak{F}[\pi^n] - \mathfrak{F}[\pi^{n-1}]$, then $K(z)/K$ is a totally ramified extension of degree $q^{n-1}(q-1)$.

B) Results comparable to (A.i) and (A.ii) already seem to exist in literature (for example, see [Lub2]). Their complete proof is included in section-2 of appendix-A ([SS1]).

C) In appendix-A of [SS1] we made an assumption :

$\forall n \geq 1, K(z) = K(\pi^{n+1})$ for any $z \in \mathfrak{F}[\pi^{n+1}] - \mathfrak{F}[\pi^n]$.

We shall discuss more on this assumption in present article. Modulo this assumption the following theorem was proved :

Let $n, q, h_{r,\pi}$ be as before. In the following statements $h_{r,\pi}$ will be abbreviated as h_r .

Let f be the degree of the extension of residue fields associated to the extension K/\mathbb{Q}_p . We shall assume that $h \mid f$.

Put $K_\pi^{h_r}$ to be the unique unramified extension of degree h_r of K_π in $\overline{\mathbb{Q}_p}$. Use $A_\pi^{h_r}$ to denote the ring of integers and $U_{i,K_\pi^{h_r}}$ ($i \geq 0$) to denote the i -th unit

group of this field. Then :

i) $K(z) = K(\pi^n)$ for any $z \in \mathfrak{F}[\pi^n] - \mathfrak{F}[\pi^{n-1}]$.

ii) $K(\pi^n)/K$ is a Galois extension with

$$\text{Gal}(K(\pi^n)|K) \cong U_{0, K^\pi} / U_{n, K^\pi}.$$

iii) Let k and i be integers such that $1 \leq k \leq n$ and $q^{k-1} \leq i \leq q^k - 1$. Then

$$G_i(K(\pi^n)|K) = \text{Gal}(K(\pi^n)|K(\pi^k)).$$

D) The results of appendix-A of [SS1] go through for $p = 2$ also.

Remark 1.2 : i) If K/\mathbb{Q}_p is an unramified extension then one can take $\pi = p$.

In this case $A_\pi = \mathbb{Z}_p$ and any formal group law over O_K is A_π module.

ii) Note that these results are similar to the results proved in classical Lubin-Tate theory (see [Neu], chapter 3, section 6, 7 and 8). Though the hypothesis looks general it will turn out the assumption is quite strong and we shall explore equivalent versions of it in section-2 of this article.

iii) Consider the π -adic Tate module $T_\pi(\mathfrak{F}) = \varprojlim \mathfrak{F}[\pi^n]$. A priori it has A_π module structure and the absolute Galois group $G_K = \text{Gal}(\overline{\mathbb{Q}_p}|K)$ acts on $T_\pi(\mathfrak{F})$ by A_π module morphism. From remark-1.1(A) it is easy to see that $T_\pi(\mathfrak{F})$ is a free A_π module of rank $h_{r,\pi}$. But as a consequence of remark-1.1(C) (modulo the assumption) we can put a $A_\pi^{h_{r,\pi}}$ module structure on $T_\pi(\mathfrak{F})$ consistent with A_π module structure. In this situation one can also see that G_K acts on $T_\pi(\mathfrak{F})$ via $A_\pi^{h_{r,\pi}}$ module morphisms (Lemma- 3.18, lemma-3.19, appendix-A, [SS1]).

iv) In appendix-A of [SS1] results in section-2 and lemma 3.1-3.8 are independent of the assumption.

v) Let A be a complete, integrally closed subring of O_K containing A_π . If \mathfrak{F} is a formal A module of height h defined over O_K then we can prove analogues of results in remark-1.1 (A) and (C) with $h_{r,\pi}$ replaced by $\frac{h}{f_A}$.

vi) In this article we shall frequently refer to appendix-A and appendix-B without explicitly mentioning [SS1].

vii) The author feels that the results from appendix-A and appendix-B needs to be studied more and currently the findings are being written up in parts ([SS1], [SS2]). Once he finds the picture in satisfactory shape, he intends to rewrite them together in more comprehensive form.

Notations and conventions :

We often use notations introduced in this section without explicitly defining them again.

Let π be a generator of \mathfrak{p}_K and let \mathfrak{F} be a formal A_π module over O_K . Such a formal module will be called a π -unramified group law over O_K .

We shall use notations

$$q_h = p^h,$$

$$h_{r,\pi} = \frac{h}{f_\pi}.$$

If h, π are clear from context then we shall use the abbreviations q, h_r .

A p -adic field is a finite extension of \mathbb{Q}_p .

For $n \geq 1$, μ_n be the group of n -th roots of unity in $\overline{\mathbb{Q}_p}$.

If L is a p -adic field then O_L is the ring of integers and \mathfrak{p}_L is the unique maximal ideal of O_L . Say residue degree of L is f . Then $\mu_{p^f-1} \subset O_L$ and $\mu_{p^f-1} \cup \{0\}$ forms a canonical set of representatives for the residue field. We shall always work with this set of representatives.

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2 An equivalence

In this section we study more about the assumption mentioned in introduction. For this we need to introduce the language of p -divisible groups. We follow the treatment in [Tate]. Note that our set-up will be bit more general and we shall consider π -torsion points rather than p -torsion subgroups (if K/\mathbb{Q}_p is unramified then one can choose $\pi = p$). The hypothesis and notation for present section is as in theorem-1.1 of introduction.

Put $R = O_K[[X]]$. Define a O_K algebra homomorphism $\phi : R \rightarrow R$ by defining $\phi(X) = [\pi](X)$. Note that ϕ makes R into a R module. Now we have the following lemma :

Lemma 2.1 : The module structure defined above makes R into a free R module of finite rank.

Proof : We shall show that $\{1, \dots, X^{q-1}\}$ is a base .

First we shall show that the set is linearly independent.

Let $f_0(X), \dots, f_{q-1}(X) \in R$ be such that

$$\sum_{i=0}^{q-1} \phi(f_i(X))X^i = 0. \quad (2.1)$$

We would like to show $f_i(X) = 0$ for each $0 \leq i \leq q-1$.

Assume that

$$f_i(X) = \sum_{j \geq 0} a_{ij}X^j$$

for each $0 \leq i \leq q-1$.

Put $m_i = \min\{v_{\mathfrak{p}_K}(a_{ij}) \mid j \geq 1\}$. for each $0 \leq i \leq q-1$ and $M = \min\{m_i \mid 0 \leq i \leq q-1\}$. If $M = \infty$ there is nothing to prove. Say, $M < \infty$. Define, $g_i(X) = \pi^{-M}f_i(X) \in R$ and $b_{ij} = \pi^{-M}a_{ij}$ for $0 \leq i \leq q-1$ and $j \geq 0$. From (2.1) it follows that :

$$\sum_{i=0}^{q-1} \phi(g_i(X))X^i = 0.$$

Put $M_i = \min\{j \mid |a_{ij}|_p = 1\}$ and $m = \min\{M_i \mid 0 \leq i \leq q-1\}$. Clearly $m < \infty$.

Now modulo \mathfrak{p}_K

$$\sum_{i=0}^{q-1} \phi(g_i(X))X^i = u(b_{0m} + b_{1m}X + \dots + b_{(q-1)m}X^{q-1})X^{mq} + \text{higher order terms}$$

where u is an unit and $b_{0m}, \dots, b_{(q-1)m} \in O_K$.

By choice of m at least one of b_{im} is non-zero modulo \mathfrak{p}_K . Thus the sum must be non-zero modulo \mathfrak{p}_K contrary to our assumption. Hence $M = \infty$ and each of $f_i(X)$ must be 0 as desired.

Now let $f(X) \in R$. We want to show that there are elements $a_0(X), \dots, a_{q-1}(X) \in R$ such that

$$f(X) = \sum_{i=0}^{q-1} \phi(a_i(X))X^i.$$

First we prove a claim :

Claim 2.2 : There are elements $a_0, \dots, a_{q-1} \in O_K$ and $g(X) \in R$ such that

$$f(X) = a_0 + \dots + a_{q-1}X^{q-1} + [\pi](X)f_1(X).$$

Proof : First notice that there are elements $a_0(1), \dots, a_{q-1}(1) \in O_K$ and

$g(1)(X), f(1)(X) \in R$ such that

$$f(X) = a_0(1) + \cdots + a_{q-1}(1)X^{q-1} + [\pi](X)g(1)(X) + \pi(f(1)(X))$$

since modulo \mathfrak{p}_K , $[\pi](X)$ is $X^q u_0(X)$ where $u_0(X) \in R$ is an unit. Now we can repeat the argument for $f(1)(X)$ and inductively construct elements $a_0, \cdots, a_{q-1} \in O_K, f_1(X) \in R$ as desired. \square

We can repeat the argument for $f_1(X)$ and inductively construct elements $a_0(X), \cdots, a_{q-1}(X)$ satisfying

$$f(X) = \phi(a_0(X)) + \cdots + \phi(a_{q-1}(X))X^{q-1}.$$

This completes the proof of lemma. \square

Let $S = \text{Spf}(O_K[[X]])$. ϕ induces a O_K -map $S \rightarrow S$. Denote this map by $[\pi]$. Note that \mathfrak{F} gives structure of a group-scheme on S and $[\pi] : S \rightarrow S$ is a homomorphism with respect to this group structure. For $\nu \geq 0$ put $G_\nu = \text{Ker}([\pi^\nu])$. Note that $G_\nu = \text{Spec}(A_\nu)$ where $A_\nu \cong R/[\pi^\nu](X)R$. So each G_ν is a connected finite group scheme over O_K of order $p^{\nu h}$. We have an exact sequence:

$$0 \rightarrow G_\nu \xrightarrow{i_\nu} G_{\nu+1} \xrightarrow{[\pi^\nu]} G_{\nu+1}$$

for each $\nu \geq 0$ where i_ν is the inclusion. Following Tate ([Tate, section-2.2]) we can construct a p -divisible group $\mathfrak{F}_\pi = \varinjlim G_i$.

Let $\text{Hom}_{O_K}(\mathfrak{F}, \mathfrak{F})$ be the set of all group homomorphisms of \mathfrak{F} defined over O_K and let $\text{Hom}_{O_K}^{A_\pi}(\mathfrak{F}, \mathfrak{F})$ denote the set of all A_π module morphisms of \mathfrak{F} defined over O_K . By a result of Hazewinkel any group endomorphism of \mathfrak{F} is automatically an A_π module morphism (see [Haz, 21.1.4]). So these two sets are same and we shall use the notation $\text{Hom}_{O_K}(\mathfrak{F}, \mathfrak{F})$.

Following the argument in [Tate, section-2.2] we obtain a bijection $\text{Hom}_{O_K}(\mathfrak{F}, \mathfrak{F}) \rightarrow \text{Hom}_{O_K}(\mathfrak{F}_\pi, \mathfrak{F}_\pi)$ induced by natural map. $T_\pi(\mathfrak{F})$ be the Tate module associated to \mathfrak{F}_π ([Tate, section-2.4]). Main result of the theory implies the canonical map $\text{Hom}_{O_K}(\mathfrak{F}_\pi, \mathfrak{F}_\pi) \rightarrow \text{Hom}_{G_K}(T_\pi(\mathfrak{F}), T_\pi(\mathfrak{F}))$ is bijective where $G_K = \text{Gal}(\overline{\mathbb{Q}}_p/K)$ and we are considering $T_\pi(\mathfrak{F})$ as G_K module. This proves that the natural map

$$\text{Hom}_{O_K}(\mathfrak{F}, \mathfrak{F}) \rightarrow \text{Hom}_{G_K}(T_\pi(\mathfrak{F}), T_\pi(\mathfrak{F})) \quad (2.2)$$

is bijective.

With this set-up we have :

Theorem 2.3 : Assume $h \mid f$ as in hypothesis of theorem-1.2. $A_\pi^{h_r}$ be as in remark-1.3(iii). The following are equivalent :

- i) $\text{Gal}(K(\pi^\infty)|K)$ is abelian.
- ii) Assumption (from remark-1.1(C)) holds for each $n \geq 1$.
- iii) \mathfrak{F} has a formal $A_\pi^{h_r}$ module structure over O_K which extends A_π module structure.

Proof : We shall show (i) \iff (ii) and (ii) \iff (iii).

(i) \implies (ii) We know that $K(\pi^{n+1})|K$ is Galois and given $z_1, z_2 \in \mathfrak{F}[\pi^{n+1}] - \mathfrak{F}[\pi^n]$ there is a $\tau \in G_K$ such that $\tau(z_1) = z_2$ (see lemma-3.1 and lemma-3.8 in appendix-A). It follows that if $z \in \mathfrak{F}[\pi^{n+1}] - \mathfrak{F}[\pi^n]$ then $K(\pi^{n+1})$ is Galois closure of $K(z)$ over K . Each sub-extension of an abelian extension is Galois. Hence the implication.

(ii) \implies (i) Remark-1.1(C) (in particular lemma-3.18, lemma-3.19 from appendix-A) implies

$$\text{Gal}(K(\pi^\infty)|K) \cong U_{0, K_\pi^{h_r}}.$$

So $\text{Gal}(K(\pi^\infty)|K)$ is abelian. Hence the implication.

(ii) \implies (iii) Consider $\tau_\infty \in \text{Hom}_{G_K}(T_\pi(\mathfrak{F}), T_\pi(\mathfrak{F}))$ (see remark-3.14 of appendix-A). Let $\tau_\infty(X) \in O_K[[X]]$ be the corresponding element in $\text{Hom}_{O_K}(\mathfrak{F}, \mathfrak{F})$. Since $\tau_\infty^{q-1} = \text{Id}$, $\tau_\infty^{q-1}(X) = X$ where τ^{q-1} denotes $(q-1)$ -fold composition. From construction of the element τ_∞ (see lemma-3.13 in appendix-A) and the ‘analytic interpretation lemma’ (lemma-3.5 in appendix-A) we conclude that there is a primitive $(q-1)$ -th root of unity ζ such that

$$\tau_\infty(X) = \zeta X + \text{higher degree terms.} \quad (2.3)$$

Note that $A_\pi^{h_r} = A_\pi[\zeta] = A_\pi \oplus A_\pi \zeta \oplus \dots \oplus A_\pi \zeta^{h_r-1}$ as A_π module.

Thus one can define a ring homomorphism

$$[\cdot] : A_\pi^{h_r} \rightarrow \text{Hom}_{O_K}(\mathfrak{F}, \mathfrak{F})$$

by $\zeta \rightarrow \tau_\infty(X)$ and extending A_π linearly. This is well defined due to the direct sum decomposition mentioned above. It is easy to check that for each $\alpha \in A_\pi^{h_r}$

$$[\alpha](X) = \alpha X + \text{higher degree terms.}$$

So we have the desired $A_\pi^{h_r}$ module structure.

(iii) \implies (ii) First observe that $[k(A_\pi^{h_r}) : \mathbb{F}_p] = h$. From remark-1.3(v) $\mathfrak{F}[\pi^{n+1}]$ is a free $A_\pi^{h_r}/\pi^{n+1}A_\pi^{h_r}$ module of rank 1 for all $n \geq 1$. Let $z \in \mathfrak{F}[\pi^{n+1}] - \mathfrak{F}[\pi^n]$. Then z generates $\mathfrak{F}[\pi^{n+1}]$ as $A_\pi^{h_r}$ module. So $\mathfrak{F}[\pi^{n+1}] \subseteq K(z)$. Hence assumption is verified.

This completes the proof. \square

Remark 2.4 : A) (iii) \implies (i) is a well-known result in literature ([Ser]).

B) Let L be an unramified extension of K (possibly infinite). Use \widehat{L} denote completion of L , $O_{\widehat{L}}$ for the ring of integers of \widehat{L} and $\mathfrak{p}_{\widehat{L}}$ be the maximal ideal of $O_{\widehat{L}}$. If π is a generator of \mathfrak{p}_K , then π also generates $\mathfrak{p}_{\widehat{L}}$. Let \mathfrak{F} be a A_π module of height h defined over $O_{\widehat{L}}$. Then one can prove analogues of results in appendix-A (see remark-1.1) for \mathfrak{F} (note that in this case we shall replace the fields $K(\pi^n)$ by $\widehat{L}(\pi^n)$ and K by \widehat{L} .) Proof is similar to proof in appendix-A. One can also prove an analogue of theorem-2.3.

3 Further results

Theorem-2.3 establishes an equivalence between properties of two different objects namely $\text{Gal}(K(\pi^\infty)|K)$ and $\text{Hom}_{O_K}(\mathfrak{F}, \mathfrak{F})$. In this section we study them separately.

For convenience we shall denote $\text{Hom}_{O_K}(\mathfrak{F}, \mathfrak{F})$ by $\text{End}_{O_K}(\mathfrak{F})$. This object has been extensively studied by Lubin (see [Lub1], [Lub2]). In first part of this section we recall some results from these two articles and rephrase some of them in context of A_π modules. Remember that any group endomorphism of \mathfrak{F} is an A_π module morphism.

3.1 The endomorphism ring

Let \mathfrak{F} be a A_π module of finite height defined over O_K for some generator π of \mathfrak{p}_K .

Consider the map

$$c : \text{End}_{O_K}(\mathfrak{F}) \rightarrow O_K$$

defined by $c(f(X)) = a_1$ where $f(X) = \sum_{i \geq 1} a_i X^i \in O_K[[X]]$.

By lemma-2.1.1 in [Lub1] c is an isomorphism onto a closed subring of O_K . From the hypothesis of A_π module structure on \mathfrak{F} , it follows that $A_\pi \subseteq c(\text{End}_{O_K}(\mathfrak{F}))$. Note that $\text{End}_{O_K}(\mathfrak{F})$ is noethrian, local ring. It contains a subring isomorphic to A_π which by abuse of notation will be identified with A_π .

Let $T_\pi(\mathfrak{F})$ be the π -adic Tate module. It is a free A_π module of rank h_r . Put,

$$\begin{aligned} V_\pi(\mathfrak{F}) &= T_\pi(\mathfrak{F}) \otimes_{A_\pi} K_\pi \\ \Lambda_\pi(\mathfrak{F}) &= \bigcup_{i \geq 1} \mathfrak{F}[\pi^i]. \end{aligned}$$

We have an isomorphism of A_π modules

$$\lambda : V_\pi(\mathfrak{F})/T_\pi(\mathfrak{F}) \rightarrow \Lambda_\pi(\mathfrak{F}).$$

Let L be a free A_π module of rank h_r contained in $V_\pi(\mathfrak{F})$ (will be called a ‘full lattice’) which is stable under the action of absolute Galois group G_K . Following the argument in [Lub2], with L one can associate an A_π module \mathfrak{G} defined over O_K which is isogenous to \mathfrak{F} and this association has certain functorial properties (see theorem-2.2, [Lub2]). Further if $L = T_\pi(\mathfrak{F})$, then one can take $\mathfrak{G} = \mathfrak{F}$ and and the isogeny to be identity.

The following lemma is a modified version of theorem-3.1 from [Lub2] :

Lemma 3.1.1 : Let \mathfrak{F} be a A_π module of finite height over O_K and assume that L is G_K stable full lattice in $V_\pi(\mathfrak{F})$ giving rise to a A_π module \mathfrak{G} over O_K as described above. Note that since \mathfrak{F} , \mathfrak{G} are isogenous, $c(\text{End}_{O_K}(\mathfrak{F}))$ and $c(\text{End}_{O_K}(\mathfrak{G}))$ have same fraction field in K (to be denoted F). Let $\zeta \in F$. Now $\zeta \in \text{End}_{O_K}(\mathfrak{G})$ if only if $\zeta L \subseteq L$.

Proof : Proof is similar to proof of theorem-3.1 from [Lub2]. \square

We shall use lemma-3.1.1 to prove the following result which is a generalised version of theorem-3.3 from [Lub2] and important for later development :

Theorem 3.1.2 : Let \mathfrak{F} be an A_π module of finite height over O_K for some

generator π of \mathfrak{p}_K . Then $\text{End}_{O_K}(\mathfrak{F})$ is integrally closed in its fraction field.

Proof : From lemma-3.8 in appendix-A we know that the absolute Galois group G_K acts transitively on $\mathfrak{F}[\pi] - \{0\}$. Hence only proper G_K submodule of $\mathfrak{F}[\pi]$ is $\{0\}$. We have an exact sequence of G_K modules :

$$0 \rightarrow [\pi]T_\pi(\mathfrak{F}) \rightarrow T_\pi(\mathfrak{F}) \rightarrow \mathfrak{F}[\pi] \rightarrow 0.$$

From the observation above it follows that if L is a G_K stable full lattice containing $[\pi]T_\pi(\mathfrak{F})$ and contained in $T_\pi(\mathfrak{F})$ then either $L = [\pi]T_\pi(\mathfrak{F})$ or $L = T_\pi(\mathfrak{F})$. Let \mathfrak{m} be the maximal ideal of $\text{End}_{O_K}(\mathfrak{F})$. Clearly $[\pi] \in \mathfrak{m}$. Put $L = \mathfrak{m}T_\pi(\mathfrak{F})$. Clearly L is a G_K stable A_π submodule of $T_\pi(\mathfrak{F})$ containing $[\pi]T_\pi(\mathfrak{F})$. From structure theorem of finitely generated modules over PID it follows that L is free over A_π of rank h_r . By the observation above $L = [\pi]T_\pi(\mathfrak{F})$ or $L = T_\pi(\mathfrak{F})$. Since $\text{End}_{O_K}(\mathfrak{F})$ is noetherian and $T_\pi(\mathfrak{F})$ is a finitely generated non-zero module it is clear that $L \neq T_\pi(\mathfrak{F})$. So $L = [\pi]T_\pi(\mathfrak{F})$.

Let $\phi \in \mathfrak{m}$. We shall show that there is a $\phi_1 \in \text{End}_{O_K}(\mathfrak{F})$ such that $\phi = [\pi]\phi_1$. Consider the element $\zeta = c(\phi)\pi^{-1}$ in fraction field of $c(\text{End}_{O_K}(\mathfrak{F}))$. From the equality above it follows that $\zeta T_\pi(\mathfrak{F}) \subseteq T_\pi(\mathfrak{F})$. By lemma-3.1.1 there is a $\phi_1 \in \text{End}_{O_K}(\mathfrak{F})$ such that $c(\phi_1) = \zeta$. This ϕ_1 satisfies $\phi = \pi\phi_1$.

The argument above proves that \mathfrak{m} is a principal ideal generated by π . Since $\text{End}_{O_K}(\mathfrak{F})$ is a local, noetherian domain we conclude that it is a DVR and hence the theorem. \square

Remark 3.1.3 : $\text{End}_{O_K}(\mathfrak{F})$ is isomorphic to a closed subring of O_K containing A_π . Let F be the fraction field of $c(\text{End}_{O_K}(\mathfrak{F}))$. Then we have a tower of fields $K_\pi \subseteq F \subseteq K$. Since K/K_π is unramified, F/K_π is also unramified. The theorem above shows that $c(\text{End}_{O_K}(\mathfrak{F})) = O_F$, the ring of integers of F . Thus \mathfrak{F} has a O_F module structure over O_K extending A_π module structure. Let f_{O_F} be the degree of residue extension of O_F . Since F/K_π is unramified $[F : K_\pi] = \frac{f_{O_F}}{f_\pi}$ and \mathfrak{F} has same height h with respect to O_F . From theorem-1.1 we have $f_{O_F} \mid h$. Hence $F \subseteq K_\pi^{h_r}$. Thus $K_\pi \subseteq F \subseteq K_\pi^{h_r}$. If $F = K_\pi^{h_r}$, we say the endomorphism ring of \mathfrak{F} has full height.

3.2 The Galois Group

Let \mathfrak{F} be a π -unramified group law of height h defined over O_K . In this subsection we study the group $\text{Gal}(K(\pi^\infty)|K)$. For simplicity we shall use the

notation G_K^∞ .

Note that G_K acts $T_\pi(\mathfrak{F})$ by A_π linear maps. $T_\pi(\mathfrak{F})$ is a free module A_π of rank h_r . Let $\{z_1, \dots, z_{h_r}\}$ be a fixed A_π base for $T_\pi(\mathfrak{F})$. This choice identifies $\text{End}_{A_\pi}(T_\pi(\mathfrak{F}))$ with $M_{h_r}(A_\pi)$ and invertible endomorphisms with $\text{Gl}_{h_r}(A_\pi)$. Thus we have a group homomorphism

$$\rho(\mathfrak{F}) : G_K \rightarrow \text{Gl}_{h_r}(A_\pi).$$

Put $H = \rho(\mathfrak{F})(G_K)$.

Note that kernel of $\rho(\mathfrak{F})$ is $\text{Gal}(\overline{\mathbb{Q}_p} | K(\pi^\infty))$ and $H \cong G_K^\infty$ as an abstract group. Further it is easy to see that $\rho(\mathfrak{F})$ is continuous with respect to usual profinite topology. Since G_K is compact with respect to pro-finite topology, H is closed subgroup of $\text{Gl}_{h_r}(A_\pi)$ and the isomorphism of abstract groups mentioned above is actually an isomorphism of topological groups.

Let R be the A_π submodule of $M_{h_r}(A_\pi)$ generated by H . Note that, we have a canonical surjective ring homomorphism

$$\rho(\mathfrak{F}) : A_\pi[G_K] \rightarrow R$$

induced by $\rho(\mathfrak{F})$ satisfying $a.z = \rho(\mathfrak{F})(a).z$ for all $a \in A_\pi[G_K]$ and $z \in T_\pi(\mathfrak{F})$.

Let

$$\phi : \text{End}_{O_K}(A_\pi) \rightarrow M_{h_r}(A_\pi)$$

be the canonical ring homomorphism. Put $\text{Im}(\phi) = E$.

Let R' denote the commutant (ie. centralizer) of R in $M_{h_r}(A_\pi)$. Clearly $E \subseteq R'$.

The bijection in (2.2) proves that ϕ is injective and $E = R'$.

F be the fraction field of $c(\text{End}_{O_K}(A_\pi))$. Note that we have a tower of fields $K_\pi \subseteq F \subseteq K_\pi^{h_r}$ and $[F : K_\pi^{h_r}] = \frac{f_F}{f_\pi}$ where we are using the abbreviation f_F for f_{O_F} . So, $O_F = A_\pi[\zeta]$ where ζ is a primitive $(p^{f_F} - 1)$ -th root of unity and

$$O_F = A_\pi \oplus \dots \oplus A_\pi \zeta^{\frac{f_F}{f_\pi} - 1}$$

as A_π modules.

Note that $T_\pi(A_\pi)$ is a free O_F module of rank h/f_F . Let $\{z_1, \dots, z_{h/f_F}\}$ be a O_F base. Then $\{[\zeta]^i z_j \mid 0 \leq i \leq f_F/f_\pi - 1, 1 \leq j \leq h/f_F\}$ is a A_π base for $T_\pi(\mathfrak{F})$. We shall use such base now on, order it using reverse dictionary order (ie $(i, j) \leq (i_1, j_1)$ iff $(j, i) \leq (j_1, i_1)$ in dictionary order) and assume that $\rho(\mathfrak{F})$

and ϕ are defined with respect to this ordered base.

In the following calculation we shall partition a $h_r \times h_r$ matrix into square blocks of f_F/f_π matrices. For convenience put $m = f_F/f_\pi$ and $n = h/f_F$.

Now $\phi([\zeta]) = (A_{ij})_{n \times n}$ where $A_{ij} = 0_{m \times m}$ whenever $i \neq j$ and $A_{ii} = A$ for all $1 \leq i \leq n$ where

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}_{m \times m}$$

Let $Y = (y_{ij})_{m \times m}$ be a matrix whose entries are indeterminates. Note that the relation $AY = YA$ holds if and only if relations $\{R(i) \mid 0 \leq i \leq m-1\}$ hold, where $R(i)$ is defined as

$$R(i) : y_{1,1+i} = y_{2,2+i} = \dots = y_{m,m+i}$$

with the convention that if some index exceeds m we shall read it modulo m .

Now $X = (X_{ij})_{n \times n}$ be a matrix in h_r^2 indeterminates where X_{ij} is a $m \times m$ matrix with $X_{ij} = (x_{pq,ij})_{m \times m}$. Note that $X\phi([\zeta]) = \phi([\zeta])X$ if and only if $AX_{ij} = X_{ij}A$ for all $1 \leq i, j \leq n$.

Let V be the sub-variety of $M_{h_r}(A_\pi)$ defined by

$$V = \{X \in M_{h_r}(A_\pi) \mid X\phi([\zeta]) = \phi([\zeta])X\}.$$

Clearly $H \subseteq R \subseteq V$.

From the relations above it is easy to see $V \otimes_{A_\pi} K_\pi$ has dimension $\leq n^2 m = \frac{h_r h}{f_F}$. Hence H (thought as a subset of $M_{h_r}(K_\pi)$) is contained in a variety of dimension $\leq \frac{h_r h}{f_F}$.

Remark 3.2.1 : i) Consider the case $f_F = h$. From theorem-2.3 and remark-1.1(C) it follows that $R \cong A_\pi^{h_r}$ and $H \cong U_{0, K_\pi^{h_r}}$ the unit group of this ring. Since $R = A_\pi[\zeta]$, it is a free A_π module of rank h_r . It follows that H is an algebraic subgroup of $\text{Gl}_{h_r}(A_\pi)$ and $H \otimes_{A_\pi} K_\pi$ has dimension exactly h_r . This leads to the following questions about general situation :

- a) Is H is always an algebraic subgroup ?
- b) Can H (thought as subset of $G_{h_r}(K_\pi)$) can be put inside a sub-variety of $\text{Gl}_{h_r}(K_\pi)$ of dimension $< \frac{h_r h}{f_F}$?

Further if $f_F = f_\pi$, is H an open subgroup of $G_{h_r}(A_\pi)$ (in usual profinite topology) ?

ii) Consider the reduction modulo π map $\mathrm{Gl}_{h_r}(A_\pi) \rightarrow \mathrm{Gl}_{h_r}(\mathbb{F}_{p^{f_\pi}})$ and assume that \tilde{H} is the image of H . The kernel of $H \rightarrow \tilde{H}$ is $\mathrm{Gal}(K(\pi^\infty)|K(\pi))$ and hence $\tilde{H} \cong \mathrm{Gal}(K(\pi)|K) \cong \mathbb{Z}/(q-1)\mathbb{Z}$.

iii) If G_K^∞ is abelian then H contains all the scalar matrices. Conversely, one can ask what can be said about G_K^∞ or in particular about f_F if H contains all scalar matrices.

iv) Note that one can construct A_π modules with distinct f_F (subject to the conditions $f_\pi | f_F | h$). For example a Lubin-Tate module has $f_F = h$. By theorem-5.1.2 in [Lub1] one can construct for any such f_F if K/\mathbb{Q}_p is unramified (with $\pi = p$).

4 Applications to theory of local fields

This is the concluding section and its goal is to apply the theory developed in [SS1], [SS2] and in this article to construct extensions of local fields. This aspect requires more careful study and the author hopes to do so in some future article.

We begin by some simple observations :

Remark 4.1 : i) Lubin-Tate theory is used to generate totally ramified abelian extensions. If \mathfrak{F} is an unramified group law of height h over O_K such that the endomorphism ring has full height, then $K(\pi^\infty)/K$ is also a totally ramified abelian extension whose Galois group is isomorphic to the Galois group of Lubin-Tate extension of height h (remark-1.1, theorem-2.3). Now let $\mathfrak{F}_1, \mathfrak{F}_2$ be two unramified group laws (wrt π_1 and π_2 respectively) of height h over O_K having endomorphism rings of full height. Clearly $K(\Lambda_{\pi_1}(\mathfrak{F}_1), \Lambda_{\pi_2}(\mathfrak{F}_2))$ is an abelian extension of K . It is interesting to ask what is the Galois group of this extension. Put $L = K^{\mathrm{ur}}$. Let \hat{L} be the completion of L . Clearly $\mathrm{Gal}(\hat{L}(\Lambda_{\pi_1}(\mathfrak{F}_1))|\hat{L}) \cong \mathrm{Gal}(K(\Lambda_{\pi_1}(\mathfrak{F}_1))|K)$. One can consider the extension $\hat{L}(\Lambda_{\pi_1}(\mathfrak{F}_1), \Lambda_{\pi_2}(\mathfrak{F}_2))|\hat{L}$ and ask for its Galois group. Note that if \mathfrak{F}_1 and \mathfrak{F}_2 are Lubin-Tate extensions then $\hat{L}(\Lambda_{\pi_1}(\mathfrak{F}_1), \Lambda_{\pi_2}(\mathfrak{F}_2)) = \hat{L}(\Lambda_{\pi_1}(\mathfrak{F}_1))$. Further, for any two unramified group laws of same height $\mathfrak{F}_1, \mathfrak{F}_2$ we have $\hat{L}(\mathfrak{F}_1[\pi_1], \mathfrak{F}_2[\pi_2]) = \hat{L}(\mathfrak{F}_1[\pi_1])$ (see [SS2], theorem-1.1).

ii) Same question can be asked if we have two unramified group laws of distinct

height having endomorphism rings of full height.

iii) Now let \mathfrak{F} be an unramified group law of height h defined over O_K such that the endomorphism ring does not have full height. Then $K(\pi^\infty)|K$ is a non-abelian extension. This gives a method of constructing non-abelian extensions explicitly. One can ask for a family of unramified formal group laws so that any non-abelian extension of K is contained in one such of extension or inside compositum of several ones of them.

We end the section with a result on roots of unity.

4.1 Roots of unity

Let \mathfrak{F} be a unramified group law defined over O_K corresponding to a generator π . It is natural to ask the question if $\zeta_{p^n} \in K(\pi^n)$ for some (all) $n \geq 1$, where ζ_{p^n} is a primitive p^n -th root of unity. We may relax the condition a bit and ask an easier question if $\zeta_{p^n} \in K^{\text{ur}}(\pi^n)$ where K^{ur} is the maximal unramified extension of K inside $\overline{\mathbb{Q}_p}$.

This question is easier to answer for some special group laws. Let K be an unramified extension of \mathbb{Q}_p . If \mathfrak{F} is the formal group law associated to a supersingular elliptic curve E defined over K , then by Weil pairing $\zeta_{p^n} \in K(p^n)$ for all $n \geq 1$ (Note that in this case $E[p^n] = \mathfrak{F}[p^n]$). Further if \mathfrak{F} is the formal group law associated to an elliptic curve with good, ordinary reduction then $E[p^n] \subseteq K^{\text{ur}}(p^n)$ for all $n \geq 1$ and by Weil pairing we have $\zeta_{p^n} \in K^{\text{ur}}(p^n)$.

In this section we shall try to answer this question for an arbitrary formal group for $n = 1$ ie we would like to know if $\zeta_p \in K^{\text{ur}}(\pi)$.

First note that there is a formal group law \mathbb{G}_m defined over O_K given by $F(X, Y) = X + Y + XY$. This is a \mathbb{Z}_p module and as \mathbb{Z}_p module its height is 1. This implies \mathbb{G}_m can not have structure of an unramified group law unless K/\mathbb{Q}_p is unramified. It is easy to see, $\zeta_{p^n} - 1 \in \mathbb{G}_m[p^n]$ for all $n \geq 1$.

We have the following lemma :

Lemma 4.1.1 : Let K be an unramified extension of \mathbb{Q}_p . Let \mathfrak{F} be a formal group law defined over O_K . Then $\zeta_p \in K^{\text{ur}}(\mathfrak{F}[p])$.

Proof : The \mathbb{Z}_p modules \mathbb{G}_m and \mathfrak{F} both define unramified group laws over

O_K . So the result follows from corollary-2.1.2 in [SS2]. \square

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