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LINEAR SYZYGY GRAPH AND LINEAR RESOLUTION

ERFAN MANOUCHEHRI AND ALI SOLEYMAN JAHAN

Abstract. For each squarefree monomial ideal $I \subset S = k[x_1, \ldots, x_n]$, we associate a simple graph G_I by using the first linear syzygies of I. In cases, where G_I is a cycle or a tree, we show the following are equivalent:
(a) I has a linear resolution;
(b) I has linear quotients;
(c) I is variable-decomposable.
In addition, with the same assumption on G_I, we characterize all monomial ideals with a linear resolution. Using our results, we characterize all Cohen-Macaulay codimension 2 monomial ideals with a linear resolution. As an other application of our results, we also characterize all Cohen-Macaulay simplicail complexes in cases that $G_\Delta \cong G_{I_{\Delta^\vee}}$ is a cycle or a tree.

Introduction

Let $S = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field k and I be a monomial ideal in S. We say that I has a d-linear resolution if the graded minimal free resolution of I is of the form:

$$0 \longrightarrow S(-d-p)^{\beta_p} \cdots \longrightarrow S(-d-1)^{\beta_1} \longrightarrow S(-d)^{\beta_0} \longrightarrow I \longrightarrow 0.$$

In general it is not easy to find ideals with linear resolution. Note that the free resolution of a monomial ideal and, hence, its linearity depends in general on the characteristic of the base field.

Let $I \subseteq S$ be a monomial ideal. We denote by $G(I)$ the unique minimal monomial set of generators of I. We say that I has linear quotients if there exists an order $\sigma = u_1, \ldots, u_m$ of $G(I)$ such that the colon ideal $< u_1, \ldots, u_{i-1} >: u_i$ is generated by a subset of the variables, for $i = 2, \ldots, m$. Any order of the generators for which, I has linear quotients, will be called an admissible order. Ideals with linear quotients were introduced by Herzog and Takayama [16]. Note that linear quotients is purely combinatorial property of an ideal I and, hence, does not depend on the characteristic of the base field. Suppose that I is a graded ideal generated in degree d. It is known that if I has linear quotients, then I has a d-linear resolution [13, Proposition 8.2.1].

The concept of variable-decomposable monomial ideal was first introduced by Rahmati and Yassemi [19] as a dual concept of vertex-decomposable simplicial complexes. In case that $I = I_{\Delta^\vee}$, they proved that I is variable-decomposable if and only if Δ is vertex-decomposable. Also they proved if a monomial ideal I is variable-decomposable, then it has linear quotients. Hence for monomial ideal generated in one degree, we have the following implications:
I is variable-decomposable $\Longrightarrow$ I has linear quotients $\Longrightarrow$ I has a linear resolution.
However, there are ideals with linear resolution but without linear quotients, see [5], and ideals with linear quotients which are not variable-decomposable, see [19, Example 2.24].

The problem of existing 2-linear resolution is completely solved by Fröberg [12] (See also [18]). Any ideal of S which is generated by squarefree monomials of degree 2 can be assumed as edge ideal of a simple graph. Fröberg proved that the edge ideal of a finite simple graph G has a linear resolution if and only if the complementary graph $\bar{G}$ of G is chordal. Trying to generalize the result of Fröberg for monomial ideals generated in degree d, $d \geq 3$, is an interesting problem on which several mathematicians including E. Emtander [7] and R.Woodroofe [23] have worked.

It is known that monomial ideals with 2-linear resolution have linear quotients [14]. Let $I = I_{\Delta^\vee}$ be a squarefree monomial ideal generated in degree d which has a linear resolution. By a result of Eagon-Reiner [6], we know Δ is a Cohen-Macaulay of dimension $n-d$. In [1] Soleyman Jahan and Ajdani proved if Δ is a Cohen-Macaulay simplicial complex of codimension 2, then Δ is vertex-decomposable. Hence, by [19, Theorem 2.10], $I_{\Delta^\vee}$ is a variable-decomposable monomial ideal generated in degree 2. Therefore if $I = I(G)$ is the edge ideal of a simple graph G, then the following are equivalent:

(a) I has a linear resolution;
(b) I has linear quotients;
(c) I is variable-decomposable ideal.

So it is natural to look for some other classes of monomial ideals with the same property.

The paper proceeds as follows. In Section 1, we associated a simple graph G_I to a squarefree monomial ideal I generated in degree $d \geq 2$. In Theorem 1.17, we show that if $G_I \cong C_m$, $m \geq 4$, then I has a linear resolution if and only if it has linear quotients and it is equivalent to I is a variable-decomposable. With the same assumption on G_I, we characterize all monomial ideals with a linear resolution.

In Section 2, we consider monomial ideal I where G_I is a tree. We prove that if I has linear relations, then G_I is a tree if and only if $\operatorname{proj}\dim(I) = 1$ (see Theorem 2.2). In Theorem 2.6 we show that if G_I is a tree, then the following are equivalent:

(a) I has a linear resolution;
(b) I has linear relations;
(c) $G_I^{(u,v)}$ is a connected graph for all u, and v in $G(I)$;
(d) If $u = u_1, u_2, \ldots, u_s = v$ is the unique path between u and v in G_I, then $F(u_j) \subset F(u_i) \cup F(u_k)$ for all $1 \leq i \leq j \leq k \leq s$;
(e) L has a linear resolution for all $L \subseteq I$, where $G(L) \subset G(I)$ and G_L is a line

In addition, it is shown that I has a linear resolution if and only if it has linear quotients and if and only if it is variable-decomposable, provided that G_I is a tree (see Theorem 2.7).

Let Δ_I be the Scarf complex of I. In Theorem 2.15 we prove that in the case that G_I is a tree, I has a linear resolution if and only if $G_I \cong \Delta_I$.

In Section 3, as applications of our results in Corollary 3.1, we characterize all Cohen-Macaulay monomial ideals of codimension 2 with a linear resolution. Let $t \geq 2$ and $I_t(C_n)$ ($I_t(L_n)$) be the path ideal of length t for n-cycle C_n (n-line L_n). We show that $I_t(C_n)$ ($I_t(L_n)$ has a linear resolution if and only if $t = n-2$ or $t = n-1$ ($t \geq n/2$), see Corollary 3.4 and Corollary 3.5.

Finally, we consider simplicial complex $\Delta = \langle F_1, \ldots, F_m \rangle$. It is shown that Δ is connected in codimension one if and only if $G_{I_{\Delta^\vee}}$ is a connected graph, see Lemma 4.1. In Corollary 4.2, we show that $I_{\Delta^\vee}$ has linear relations if and only if $\Delta^{(F,G)}$ is connected in codimension one for all facets F and G of Δ. Also, we introduce a simple graph G_Δ on vertex set $\{F_1, \ldots, F_m\}$ which is isomorphic to $G_{I_{\Delta^\vee}}$. As Corollaries of our results, we show that if G_Δ is a cycle or a tree, then the following are equivalent:

(a) Δ is Cohen-Macaulay;

(b) Δ is pure shellable;

(c) Δ is pure vertex-decomposable.

In addition, with the same assumption on G_Δ all Cohen-Macaulay simplicial complexes are characterized.

Note that for monomial ideal $I =< u_1, \dots, u_m >$ and monomial u in S, I has a linear resolution (has linear quotients, is variable-decomposable) if and only if uI has a linear resolution (has linear quotients, is variable-decomposable). Hence, without the loss of generality, we assume that $\gcd(u_i : u_i \in G(I)) = 1$. Also, one can see that a monomial ideal I has a linear resolution (has linear quotients, is variable-decomposable) if and only if its polarization has a linear resolution (has linear quotients, is variable-decomposable). Therefore in this paper we only consider squarefree monomial ideals.

1. MONOMIAL IDEALS WHOSE G_I IS A CYCLE

Let I be a monomial ideal which is generated in one degree. First, we recalling some definitions and known facts which will be useful later.

Proposition 1.1. [13, Proposition 8.2.1] Suppose $I \subseteq S$ is a monomial ideal generated in degree d. If I has linear quotients, then I has a d-linear resolution.

Let $u = x_1^{a_1} \dots x_n^{a_n}$ be a monomial in S. Set $F(u) := \{i : a_i > 0\} = \{i : x_i \mid u\}$. For another monomial v, we set $[u, v] = 1$ if $x_i^{a_i} \nmid v$ for all $i \in F(u)$. Otherwise, we set $[u, v] \neq 1$. For a a monomial ideal $I \subseteq S$, set $I_u =< u_i \in G(I) : [u, u_i] = 1 >$ and $I^u =< u_j \in G(I) : [u, u_j] \neq 1 >$.

Definition 1.2. Let I be a monomial ideal with $G(I) = \{u_1, \dots, u_m\}$. A monomial $u = x_1^{a_1} \dots x_n^{a_n}$ is called shedding if $I_u \neq 0$ and for each $u_i \in G(I_u)$ and $l \in F(u)$, there exists $u_j \in G(I^u)$ such that $u_j : u_i = x_l$. Monomial ideal I is r-decomposable if $m = 1$ or else has a shedding monomial u with $\mid F(u) \mid \leq r + 1$ such that the ideals I_u and I^u are r-decomposable.

A monomial ideal is decomposable if it is r-decomposable for some $r \geq 0$. A 0-decomposable ideal is called variable-decomposable. In [19] the authors proved the following result:

Theorem 1.3. Let I be a monomial ideal with $G(I) = \{u_1, \dots, u_m\}$. Then I is decomposable if and only if it has linear quotients.

Let I be a squarefree monomial ideal and

$$F : 0 \longrightarrow F_p \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow I \longrightarrow 0$$

be the minimal graded free $S-$resolution of I, where $F_i = \bigoplus_j S(-j)^{\beta_{ij}}$ for all i. Set $\varphi : F_0 \longrightarrow I$ and $\psi : F_1 \longrightarrow F_0$, where φ maps a basis element e_i of F_0 to $u_i \in G(I)$ and ψ maps a basis element g_i of F_1 to an element of a minimal generating set of $\ker(\varphi)$. Monomial ideal I has *linear relations* if $\ker(\varphi)$ is generated minimally by a set of linear forms.

We associate to I a simple graph G_I whose vertices are labeled by the elements of $G(I)$. Two vertices u_i and u_j are adjacent if there exist variables x, y such that $xu_i = yu_j$. This graph was first introduced by Bigdeli, Herzog and Zaare-Nahandi [3].

Remark 1.4. If I is a squarefree monomial ideal, then two type of 3-cycle $u_{i_1}, u_{i_2}, u_{i_3}$ may appear in G_I.

(i): If $F(u_{i_1}) = A \cup \{j,k\}, F(u_{i_2}) = A \cup \{i,k\}$ and $F(u_{i_3}) = A \cup \{i,j\}$. Then $x_i e_{i_1} - x_k e_{i_3} = (x_i e_{i_1} - x_j e_{i_2}) + (x_j e_{i_2} - x_k e_{i_3})$. In this case one of the linear forms can be written as a linear combination of two other linear forms.

(ii): If $F(u_{i_1}) = A \cup \{i\}, F(u_{i_2}) = A \cup \{j\}$ and $F(u_{i_3}) = A \cup \{k\}$. In this case the three linear forms are independent.

The number of the minimal generating set of $\ker(\varphi)$ in degree $d+1$ is $\beta_{1(d+1)}$ and $\beta_{1(d+1)} \leq \mid E(G_I) \mid$. It is clear that equality holds if G_I has no C_3 of type (i). If G_I has a C_3 of type (i), then we remove one edge of this cycle. In this way, we obtain a graph G_I with no C_3 of type (i) and called it the first syzygies graph of I.

Our aim is to study minimal free resolution of I via some combinatorial properties of G_I. Set $x_F := \prod_{i \in F} x_i$ for each $F \subset [n] = \{1, \ldots, n\}$.

Remark 1.5. Let I be a squarefree monomial ideal. If $u_i = x_{F_i}$ and $u_j = x_{F_j}$ are two elements in $G(I)$ such that $w_i u_i = w_j u_j$, then there exists a monomial $w \in S$ such that $w_i = w x_{F_j \setminus F_i}$ and $w_j = w x_{F_i \setminus F_j}$.

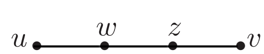
Lemma 1.6. Let I be squarefree monomial ideal. If there is a path of length t between u and v in G_I, then one can obtain monomials w_i and w_j from the given path such that $w_i u = w_j v$ and $\deg w_i$=$\deg w_j \leq t$.

Proof. We proceed by induction on t. The case $t = 1$ is obvious. Let $t = 2$ and u, w, v be a path of length 2 in G_I. Since u and w and w and v are adjacent, we have $x_{i_1} u = x_{i_2} w$ and $x_{i3} w = x_{i4} v$. Hence $x_{i_1} x_{i_3} u = x_{i_2} x_{i_4} v$.

Now assume that $t > 2$ and $u = u_{i_0}, u_{i_1}, \ldots, u_{i_{t-1}}, u_{i_t} = v$ is a path of length t. Hence $u = u_{i_0}, u_{i_1}, \ldots, u_{i_{t-1}}$ is a path of length $t-1$. Using induction hypothesis, we conclude that there are monomials $w_i^{'}$ and $w_j^{'}$ such that $w_i^{'} u = w_j^{'} u_{i_{t-1}}$, where $\deg w_i^{'} = \deg w_j^{'} \leq t-1$. Since v and $u_{i_{t-1}}$ are adjacent, there exist variable x, y such that $x u_{i_{t-1}} = y v$. Therefore $x w_i^{'} u = y w_j^{'} v$ and $\deg x w_i^{'} = \deg y w_j^{'} \leq t$. □

The following example shows that the inequality $\deg w_i$=$\deg w_j \leq k$ can be pretty strict.

Example 1.7. Consider monomial ideal $I =< u, v, w, z > \subset k[x_1, \ldots, x_5]$, where $u = x_1 x_2 x_3$, $w = x_1 x_2 x_4$, $z = x_1 x_4 x_5$ and $v = x_3 x_4 x_5$. We have a path of length 3 between u and v, but $x_4 x_5 u = x_1 x_2 v$.



Lemma 1.8. Let I be squarefree monomial ideal which has linear relations. Then G_I is a connected graph.

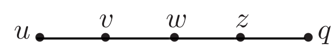
Proof. For any $u_i, u_j \in G(I)$, there exist monomials w_i and w_j such that $w_i u_i = w_j u_j$ and, hence, $w_i e_i - w_j e_j \in \ker(\varphi)$. Since $\ker(\varphi)$ is generated by linear forms one has :

$$w_i e_i - w_j e_j = f_{i_1}(x_{k_1} e_i - x_{k_2}{}' e_{i_2}) + f_{i_2}(x_{k_2} e_{i_2} - x_{k_3}{}' e_{i_3}) + \ldots + f_{i_t}(x_{k_t} e_{i_t} - x_{k_{t+1}}{}' e_j),$$

where $f_{ij} \in S$ for $j = 0, \ldots, t$. Therefore $u_i, u_{i_2}, \ldots, u_{i_t}, u_j$ is a path in G_I. □

The following example shows that the converse of Lemma 1.8 is not true in general.

Example 1.9. Consider monomial ideal $I =< u, v, w, z, q >\subset k[x_1, \ldots, x_5]$, where $u = x_1x_2x_3$, $v = x_1x_2x_4$, $w = x_1x_4x_5$, $z = x_4x_5x_6$ and $q = x_3x_5x_6$. It is easy to see that G_I is the following connected graph.



However I has not linear relations. It's minimal free S-resolutions is:

$$0 \longrightarrow S(-6) \longrightarrow S(-4)^4 + S(-5) \longrightarrow S(-3)^5 \longrightarrow I \longrightarrow 0.$$

Remark 1.10. Let I be a squarefree monomial ideal and $u = u_{i_1}, u_{i_2}, \ldots, u_{i_{t-1}}, u_{i_t} = v$ be a path in G_I. If $r \in F(v)$ and $r \notin F(u)$, then x_r is the coefficient of some e_{i_j} in the linear relations which comes from the given path.

Remark 1.11. Let $u = u_{i_1}, u_{i_2}, \ldots, u_{i_{t-1}}, u_{i_t} = v$ be a path in G_I. We know there exist minimal(with respect to divisibility) monomials w and $w^{'}$ such that $we_{i_1} - w^{'}e_{i_t} \in \ker(\varphi)$ and, hence,
$we_{i_1} - w^{'}e_{i_t} = f_{i_1}(x_{k_1}e_{i_1} - x_{k_2}{}^{'}e_{i_2}) + f_{i_2}(x_{k_2}e_{i_2} - x_{k_3}{}^{'}e_{i_3}) + \ldots +$
$f_{i_{t-1}}(x_{k_{t-1}}e_{i_{t-1}} - x_{k_t}{}^{'}e_{i_t})$.

If for each j, $1 \leq j \leq t$, $F(u_{i_j}) \subseteq F(u) \cup F(v)$ and $x_l \mid w$, then $x_l \nmid w^{'}$. By Remark 1.10 x_l is the coefficient of some e_{i_r} which appear in the above equation. Hence, there exist u_{i_j} such that $l \in F(u_{i_j})$. Since $F(u_{i_j}) \subseteq F(u) \cup F(v)$ and $l \notin F(u)$, one has $l \in F(v)$. So $x_l \nmid w^{'}$. Similarly for arbitrary x_r where $x_r \mid w^{'}$, one has $x_r \nmid w$. Hence we conclude that $w = x_{F(v)\setminus F(u)}$ and $w^{'} = x_{F(u)\setminus F(v)}$.

Remark 1.12. Let w_{i_1} and w_{i_t} be two minimal monomials (with respect to divisibility) in S such that $w_{i_1}e_{i_1} - w_{i_t}e_{i_t} \in \ker(\varphi)$. Assume that
$w_{i_1}e_{i_1} - w_{i_t}e_{i_t} = f_{i_1}(x_{k_1}e_{i_1} - x_{k_2}{}^{'}e_{i_2}) + f_{i_2}(x_{k_2}e_{i_2} - x_{k_3}{}^{'}e_{i_3}) + \ldots + f_{i_{t-1}}(x_{k_{t-1}}e_{i_{t-1}} - x_{k_t}{}^{'}e_{i_t})$.
If $x_i \nmid u_{i_1}$ and there exist u_{i_r}, $2 \leq r \leq t$, such that $x_i \mid u_{i_r}$, then $x_i \mid w_{i_1}$. We may assume that r is the smallest number with the property that $x_i \mid u_{i_r}$. We know that $f_{i_{r-2}}(x_{k_{r-2}}e_{i_{r-2}} - x_{k_{r-1}}{}^{'}e_{i_{r-1}}) + f_{i_{r-1}}(x_ie_{i_{r-1}} - x_{k_r}{}^{'}e_{i_r})$ is a part of above equation. Since in the above equation $e_{i_{r-1}}$ must be eliminated, we have $f_{i_{r-1}}x_i = f_{i_{r-2}}x_{k_{r-1}}{}^{'}$. Hence, $x_i \mid f_{i_{r-2}}$. Also, $e_{i_{r-2}}$ must be eliminated and, hence, one has $f_{i_{r-2}}x_{k_{r-2}} = f_{i_{r-3}}x_{k_{r-2}}{}^{'}$. Therefore $x_i \mid f_{i_{r-3}}$. Continuing these procedures yields $x_i \mid f_{i_1}$, i.e $x_i \mid w_{i_1}$.

Similarly if $x_i \nmid u_{i_t}$ and there exist u_{i_r}, $1 \leq r \leq t-1$, such that $x_i \mid u_{i_r}$, then $x_i \mid w_{i_t}$.

For all $u, v \in G(I)$, let $G_I^{(u,v)}$ be the induced subgraph of G_I on vertex set

$$V(G_I^{(u,v)}) = \{w \in G(I) : F(w) \subseteq F(u) \cup F(v)\}.$$

The following fact was proved by Bigdeli, Herzog and Zaare-Nahandi [3]. Here we present a different proof of it.

Proposition 1.13. Let I be a squarefree monomial ideal which is generated in degree d. Then I has linear relations if and only if $G_I^{(u,v)}$ is connected for all $u, v \in G(I)$.

Proof. Assume that I has linear relations and $u, v \in G(I)$. We know that $x_{F(v)\setminus F(u)}e_u - x_{F(u)\setminus F(v)}e_v \in \ker(\varphi)$. Since $\ker(\varphi)$ is generated by linear forms

$$x_{F(v)\setminus F(u)}e_u - x_{F(u)\setminus F(v)}e_v =$$
$$f_{i_1}(x_{k_1}e_{i_1} - x_{k_2}{}'e_{i_2}) + f_{i_2}(x_{k_2}e_{i_2} - x_{k_3}{}'e_{i_3}) + \ldots + f_{i_{t-1}}(x_{k_{t-1}}e_{i_{t-1}} - x_{k_t}{}'e_t).$$

Hence $u = u_{i_1}, u_{i_2}, \ldots, u_{i_{t-1}}, u_{i_t} = v$ is a path in G_I. Now it is enough to show that $F(u_{i_j}) \subseteq F(u_{i_1}) \bigcup F(u_{i_t})$ for all i_j, $1 < j < t$. Assume to the contrary that there exist k, $1 < k < t$, such that $F(u_{i_k}) \nsubseteq F(u_{i_1}) \bigcup F(u_{i_t})$. Let $l \in F(u_{i_k})$ and $l \notin F(u_{i_1}) \bigcup F(u_{i_t})$. By Remark 1.12 $x_l \mid x_{F(v)\setminus F(u)}$ and $x_l \mid x_{F(u)\setminus F(v)}$. This is a contradiction.

For converse, we know that $\ker(\varphi)$ is generated by $x_{F_v\setminus F_u}e_u - x_{F_u\setminus F_v}e_v$, where $u, v \in G(I)$. By our assumption, $G_I^{(u,v)}$ is a connected graph for all $u, v \in G(I)$. Therefore there exist a path $u = u_{i_1}, u_{i_2}, \ldots, u_{i_{t-1}}, u_{i_t} = v$ between u and v in $G^{(u,v)}$. By Remark 1.11, one has

$$x_{F(v)\setminus F(u)}e_{i_1} - x_{F(u)\setminus F(v)}e_{i_t} = f_{i_1}(x_{k_1}e_{i_1} - x_{k_2}{}'e_{i_2}) + \ldots + f_{i_{t-1}}(x_{k_{t-1}}e_{i_{t-1}} - x_{k_t}{}'e_t).$$

Hence, $x_{F(v)\setminus F(u)}e_{i_1} - x_{F(v)\setminus F(u)}e_{i_t}$ is a linear combination of linear forms. □

Lemma 1.14. Let I be a squarefree monomial ideal. Then one can assign to each cycle of G_I an element in $\ker(\psi)$.

Proof. Let $u_{i_1}, u_{i_2}, \ldots, u_{i_{t-1}}, u_{i_t}, u_{i_1}$ be a cycle in G_I. Then we have two paths u_{i_1}, u_{i_2} and $u_{i_2}, \ldots, u_{i_t}, u_{i_1}$. Since $\{u_{i_1}, u_{i_2}\} \in E(G_I)$, there exist variables x and y such that $xe_{i_1} - ye_{i_2} \in \ker(\varphi) = \operatorname{Im}(\psi)$. This is an element in the minimal set of generators of $\ker(\varphi)$. Hence, there exist a basis element g of F_1 such that $\psi(g) = xe_{i_1} - ye_{i_2}$.

By Lemma 1.6, there exist monomials w_1 and w_2 in S such that $w_1e_{i_1} - w_2e_{i_2} = f_{i_2}(x_{k_2}e_{i_2} - x_{k_3}{}'e_{i_3}) + \ldots + f_{i_t}(x_{k_t}e_{i_t} - x_{k_{t+1}}{}'e_{i_1}) = \psi(\sum_{j=2}^{t} f_{i_j}g_{i_j})$. Remark 1.5 implies that $w_1 = hx_{F(u_{i_2})\setminus F(u_{i_1})} = hx$ and $w_2 = hx_{F(u_{i_1})\setminus F(u_{i_2})} = hy$. Therefore, we have

$$h(xe_{i_1} - ye_{i_2}) = w_1e_{i_1} - w_2e_{i_2}.$$

This implies that $h\psi(g) = \psi(\sum_{j=2}^{t} f_{i_j}g_{i_j})$ and, hence, $(hg - \sum_{j=2}^{t} f_{i_j}g_{i_j}) \in \ker\psi$. Since $g \neq g_{i_j}$ for all $1 \leq j \leq r$ one has $(hg - \sum_{j=2}^{t} f_{i_j}g_{i_j}) \neq 0$ □

Remark 1.15. Let w be an element of a minimal set of generators of $\ker(\psi)$. If $w = \sum h_ig_i$, where g_i is a basis element of F_1 and $0 \neq h_i \in S$, then h_i is a monomial. Without loss of generality, we may assume that $\psi(g_1) = t_1{}'e_1 - t_2e_2$. Let $u \in \operatorname{supp}(h_1)$ be a monomial. Since ut_2e_2 must be eliminated, there exist a basis element g_j of F_1 such that $\psi(g_j) = (t_2{}'e_2 - t_3e_l)$. Without loss of generality, we may assume $j = 2$ and $l = 3$. Hence, $t_2\frac{u}{t_2{}'} = u' \in \operatorname{supp}(h_2)$. Again since $u't_3e_3$ must be eliminated, without loss of generality, we may assume there exist a basis element g_3 of F_1 such that $\psi(g_3) = (t_3{}'e_3 - t_4e_4)$. Therefore $t_3\frac{u'}{t_3{}'} = u'' \in \operatorname{supp}(h_3)$. Continuing these procedures yields $\psi(g_l) = (t_l{}'e_l - t_1e_1)$ and $t_l\frac{u^{l-2}}{t_l{}'} = u^{l-1} \in \operatorname{supp}(h_l)$. Hence we obtain a cycle in G_I in this way. Now if there exist another monomial $v \in \operatorname{supp}(h_1)$ with $u \neq v$, then by the similar argument one can find a new cycle in G_I. Hence, Lemma 1.14 implies that w is a combination of some other elements of $\ker(\psi)$, a contradiction. So h_i is a monomial.

Lemma 1.16. Let I, φ and ψ be as mention in above. If $\ker\varphi$ is generated by linear forms, then corresponding to every element in a minimal set of generators of $\ker(\psi)$ there is a cycle in G_I.

Proof. Let $\sum_{i=1}^{n} h_ig_i$ an element of a minimal generating set of $\ker(\psi)$. Then $\psi(\sum_{i=1}^{n} h_ig_i)$ $= \sum_{i=1}^{n} h_i\psi(g_i) = 0$, where g_i is a basis element of F_1 and h_i is monomial for $i = 1, \ldots .n$.

Then $-h_1\psi(g_1) = \sum_{i=2}^n h_i\psi(g_i)$. Assume that $\psi(g_1) = x_{i_1}e_{i_1} - x_{i_2}e_{i_2}$. So u_{i_1}, u_{i_2} is a path in G_I.

The left-hand side of above equation is of the form $w_{i_1}e_{i_1} - w_{i_2}e_{i_2}$. By proof of Lemma 1.8, the right-hand side of the above equation is of the form

$$f_{i_2}(x_{k_2}e_{i_2} - x_{k_3}{}'e_{i_3}) + f_{i_3}(x_{k_3}e_{i_3} - x_{k_4}{}'e_{i_4}) + \ldots + f_{i_t}(x_{k_t}e_{i_t} - x_{k_{t+1}}{}'e_{i_1}),$$

where $e_{i_t} \neq e_{i_2}$. If $e_{i_t} = e_{i_2}$, then $x_{k_{t+1}}{}' = x_{i_1}$ and $x_{k_t} = x_{i_2}$. Hence, g_1 appears in the right-hand side of equation , a contradiction. Thus $u_{i_2}, u_{i_3}, \ldots, u_{i_t}, u_{i_1}$ is a path which is different from path u_{i_1}, u_{i_2}. □

Theorem 1.17. Let $I \subset S$ be a squarefree monomial ideal such that $G_I \cong C_m$, $m \geq 4$. Then the following conditions are equivalent:

(a) I has a linear resolution;
(b) $m = n$ and with a suitable relabeling of variables for all j one has $x_i \mid u_j$ for all i, $i + 1 \neq j$ and $i \neq j$, where $n + 1 = 1$;
(c) I is variable-decomposable ideal;
(d) I has linear quotients.

Proof. $(a) \Rightarrow (b)$: Assume that I has a linear resolution. Since G_I is a cycle, by Lemma 1.14 and Lemma 1.16, $\ker(\psi) =< w >$. Let $w = \sum_{i=1}^m h_i g_i$ where g_i is a basis element of F_1 and h_i is a monomial in S for $i = 1, \ldots, m$. Without loss of generality, we may assume that $G_I = u_1, u_2, \ldots, u_m, u_1$. Then

$$\psi(w) = \sum_{i=1}^m h_i\psi(g_i) = h_1(x_{t_1}e_1 - x_{t_2}{}'e_2) + h_2(x_{t_2}e_2 - x_{t_3}{}'e_3) + \ldots + h_m(x_{t_m}e_m - x_{t_1}{}'e_1) = 0.$$

Therefore, $h_1x_{t_1}e_1 = h_m x_{t_1}{}'e_1$. Since, I has d-linear resolution and $\deg(e_i) = d$, we conclude that $\deg(h_i) = 1$ for all i. Consequently, $h_1 = x_{t_1}{}'$ and $h_m = x_{t_1}$. By similar argument $h_j = x_{t_j}{}'$ and $h_j = x_{t_{j+1}}$. Hence, $x_{t_{j+1}} = x_{t_j}{}'$ for all $1 \leq j \leq m-1$. So $\ker(\varphi)$ is minimally generated by the following linear forms.

$(x_{t_m}e_1 - x_{t_2}e_2), (x_{t_1}e_2 - x_{t_3}e_3), (x_{t_2}e_3 - x_{t_4}e_4), \ldots,$
$(x_{t_{m-2}}e_{m-1} - x_{t_m}e_m), (x_{t_{m-1}}e_m - x_{t_1}e_1)$.

For an arbitrary variable x_i in S there exits u_i and u_j in $G(I)$ such that $x_i \mid u_i$ and $x_i \nmid u_j$. Hence, by Remark 1.10 $x_i \in \{x_{t_1}, x_{t_2}, \ldots, x_{t_m}\}$. It is clear that the variables $x_{t_1}, x_{t_2}, \ldots, x_{t_m}$ are distinct and, hence, $n = m$.

Set $x_{t_{-1}} = x_{t_{m-1}}, x_{t_{m+1}} = x_{t_1}$, $e_0 = e_m$ and $e_{m+1} = e_1$. For $1 \leq i \leq m-1$, we have $\varphi(x_{t_{i-2}}e_{i-1} - x_{t_i}e_i) = 0$ and, hence, $x_{t_i} \mid u_{i-1}$ and $x_{t_i} \nmid u_i$. Also, from $\varphi(x_{t_i}e_{i+1} - x_{t_{i+2}}e_{i+2}) = 0$, we have $x_{t_i} \nmid u_{i+1}$ and $x_{t_i} \mid u_{i+2}$. By Remark 1.10 $x_{t_i} \mid u_j$ for $j \neq i, i+1$.

$(b) \Rightarrow (c)$: It is easy to see that $I_{x_1} =< u_1, u_2 >$ is variable-decomposable, $I^{x_1} =< u_3, \ldots, u_n >$ and $u = x_1$ is a shedding variable. Also, it is clear that x_2 is a shedding variable for I^{x_1} and $(I^{x_1})^{x_2} =< u_4, \ldots, u_n >$, $(I^{x_1})_{x_2} =< u_3 >$. Continuing these procedures yields that I^{x_1} is variable-decomposable. Hence, I is variable-decomposable ideal.

$(c) \Rightarrow (d)$ follows by Theorem 1.3.

$(d) \Rightarrow (a)$ follows by Proposition 1.1. □

Corollary 1.18. Let $I \subset S$ be a squarefree monomial ideal generated in degree 2 and assume that $G_I \cong C_m$, $m \geq 4$. Then I has a linear resolution if and only if $m = 4$.

Remark 1.19. Let I be a squarefree monomial ideal. If $G_I \cong C_3$, then I has linear quotients. Hence I has a linear resolution.

Let I be a squarefree monomial ideal generated in degree 2. We may assume that $I = I(G)$ is the edge ideal of a graph G. Hence, by Fröberg's result, $I(G)$ has a linear resolution if and only if $\bar{G}$ is a chordal graph. If $G \cong C_m$, then $\bar{G}$ is chordal if and only if $m = 3$ or $m = 4$. In this situation $G \cong C_m$ if and only if $G_I \cong C_m$. Hence, in this case our result is coincide to Fröberg's result.

Corollary 1.20. Let I be a squarefree monomial ideal generated in degree d where $G_I \cong C_m$. If $d+2 < n$ or $m \neq n$, then I can not has a d-linear resolution.

Example 1.21. Consider monomial ideal $I =< xy, zy, zq, qx >\subset k[x, y, z, q]$. The graph G_I is 4-cycle. Since $d = 2$, $n = 4$ and $d + 2 = n$, I has a 2-linear resolution.

$$0 \longrightarrow S(-4) \longrightarrow S(-3)^4 \longrightarrow S(-2)^4 \longrightarrow I \longrightarrow 0.$$

Example 1.22. For monomial ideal $I =< xyz, yzq, zqw, qwe, wex, xye >\subset k[x, y, z, q, e, w]$, we have $G_I \cong C_6$. Therefore I has not a 3-linear resolution, since $d = 3$, $n = 6$ and $d + 2 < n$. The resolution of I is:

$$0 \longrightarrow S(-6) \longrightarrow S(-4)^6 \longrightarrow S(-3)^6 \longrightarrow I \longrightarrow 0.$$

2. Linear resolution of monomial ideals whose G_I is a tree

Let I be a squarefree monomial ideal such that G_I is a tree. In this section we study linear resolution of such monomial ideals. We know that each line is a tree, therefore first we consider the following:

Proposition 2.1. Let $I =< u_1, \ldots, u_m >$ be a squarefree monomial ideal generated in degree d. If $G_I = u_1, u_2, \ldots, u_m$ is a line, then the following conditions are equivalent:

(a) I has a linear resolution;
(b) For any $1 \leq j \leq k \leq i \leq m$
$$F(u_k) \subseteq F(u_i) \cup F(u_j);$$
(c) I is variable-decomposable ideal;
(d) I has linear quotients.

Proof. $(a) \Rightarrow (b)$: Suppose, on the contrary, there exist $1 \leq j < k < i \leq m$ and $l \in F(u_k)$ such that $l \notin F(u_i) \cup F(u_j)$. Since I has a linear resolution, we have $x_{F(u_i)\setminus F(u_j)}e_j - x_{F(u_j)\setminus F(u_i)}e_i = f_i(x_{k_1}e_i - x_{k_2}{}' e_{i+1}) + f_{i+1}(x_{k_2}e_{i+1} - x_{k_3}{}' e_{i+2}) + \ldots + f_{j-1}(x_{k_{j-1}}e_{j-1} - x_{k_t}{}' e_j)$. By Remark 1.12, $x_l \mid x_{F(u_j)\setminus F(u_i)}$ and $x_l \mid x_{F(u_i)\setminus F(u_j)}$ which is a contradiction.

$(b) \Rightarrow (c)$: Let $F(u_2) \setminus F(u_1) = \{l\}$. From the facts that $F(u_2) \subseteq F(u_1) \cup F(u_i)$, $l \in F(u_2)$ and $u_2 : u_1 = x_l$, we conclude that $l \in F(u_i)$ for all $2 \leq i \leq m$, $I_{x_l} =< u_1 >$ and x_1 is a shedding. By induction on m, I^{x_l} is variable-decomposable, since I^{x_l} in a line of length $m - 1$.

$(c) \Rightarrow (d)$ follows by Theorem 1.3.

$(d) \Rightarrow (a)$ follows by Proposition 1.1.

Theorem 2.2. If I is a squarefree monomial ideal which has linear relations, then G_I is a tree if and only if $\operatorname{proj}\dim(I) = 1$.

Proof. If G_I is a tree, then G_I has no cycle. Therefore by Lemma 1.16, $\ker(\psi) = 0$. Hence the linear resolution of I is of the form

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow I \longrightarrow 0$$

and $\operatorname{proj}\dim(I) = 1$.

Conversely, assume that $\operatorname{proj}\dim(I) = 1$. Then $\ker(\psi) = 0$ and by Lemma 1.14 G_I has no cycle. Since I has linear relations, by Lemma 1.8, G_I is a connected graph. Therefore G_I is a tree. □

Proposition 2.3. Let I be a squarefree monomial ideal with $\operatorname{proj}\dim(I) = 1$. Then I has a linear resolution if and only if G_I is a connected graph.

Proof. Assume that G_I is a connected graph. Since $\operatorname{proj}\dim(I) = 1$, Lemma 1.14 implies that G_I has no cycle and, hence, it is a tree. So it is enough to show I has linear relations. For $u_i, u_j \in G(I)$ there exist a unique path between u_i and u_j in $G(I)$. Assume that

$$we_i - w^{'}e_j = f_{i_1}(x_{k_1}e_{i_1} - x_{k_2}{}^{'}e_{i_2}) + f_{i_2}(x_{k_2}e_{i_2} - x_{k_3}{}^{'}e_{i_3}) + \ldots + f_{i_{t-1}}(x_{k_{t-1}}e_{i_{t-1}} - x_{k_t}{}^{'}e_{i_t})$$

be an element of $\ker(\varphi)$ which is obtained from this path. If $we_i - w^{'}e_j = x_{F(u_j)\setminus F(u_i)}e_i - x_{F(u_i)\setminus F(u_j)}e_j$, we are done. So assume that the equality does not holds. Then $x_{F(u_j)\setminus F(u_i)}e_i - x_{F(u_i)\setminus F(u_j)}e_j$ is a minimal element in $\ker(\varphi)$. Hence, there exists $g \in F_1$ such that $\psi(g) = x_{F(u_j)\setminus F(u_i)}e_i - x_{F(u_i)\setminus F(u_j)}e_j$. Remark 1.5 implies that there exists a monomial $h \in S$ such that $h\psi(g) = we_i - w^{'}e_j = f_{i_1}\psi(g_{i_2}) + \ldots + f_{i_{t-1}}\psi(g_{i_{t-1}})$. Therefore $\psi(hg - f_{i_1}g_{i_2} - \cdots - f_{i_{t-1}}g_{i_{t-1}}) = 0$ and $hg - f_{i_1}g_{i_2} - \cdots - f_{i_{t-1}}g_{i_{t-1}} \neq 0$, a contradiction.

The converse follows from Lemma 1.8. □

Proposition 2.4. Let $I =< u_1, \ldots, u_m >$ be a squarefree monomial ideal generated in degree d which has linear quotients. Assume that G_I is a tree and v is a monomial in degree d which is a leaf in $G_{<I,v>}$. Then the following conditions are equivalent:

(a) $< I, v >$ has a linear resolution;
(b) Let u_i be the branch of v and $F(u_i) \setminus F(v) = \{l\}$, then $l \in \bigcap_{t=1}^{m} F(u_t)$;
(c) $< I, v >$ has linear quotients.

Proof. $(a) \Rightarrow (b)$: Suppose, on the contrary, that there exist a $1 \leq j \leq m$ such that $l \notin F(u_j)$. Let $v, u_i = u_{i_1}, u_{i_2}, \ldots, u_{i_{t-1}}, u_{i_t} = u_j$ be the unique path between v and u_j. Without loss of generality, we may assume that $l \in F(u_{i_r})$ for all r, $1 \leq r \leq t-1$. Since $< I, v >$ has a linear resolution, we have $x_{F(u_j)\setminus F(v)}e_v - x_{F(v)\setminus F(u_j)}e_j = f_0(x_{i_0}e_v - x_{i_1{}'}e_{i_1}) + f_1(x_{i_1}e_{i_1} - x_{i_2{}'}e_{i_2}) + \ldots + f_{t-1}(x_{i_{t-1}}e_{i_{t-1}} - x_{i_t{}'}e_t)$. By Remark 1.12, we know that $x_l \mid x_{F(u_j)\setminus F(v)}$ and $x_l \mid x_{F(v)\setminus F(u_j)}$, this is a contradiction .

$(b) \Rightarrow (c)$: We now that there is an admissible order $v_1, v_2, \ldots, v_m$ of $G(I)$. Since by our assumption $\{l\} = F(u_i) \setminus F(v)$ and $l \in F(u_j)$ for any $1 \leq j \leq m$, we conclude that the order $v_1, v_2, \ldots, v_m, v$ is an admissible order for $< I, v >$.

$(c) \Rightarrow (a)$ follows from Proposition 1.1. □

Proposition 2.5. Let $I =< u_1, \ldots, u_m >$ be a squarefree monomial ideal generated in degree d. If G_I is a tree, then I has a linear resolution if and only if L has a linear resolution for all $L \subseteq I$, where $G(L) \subset G(I)$ and G_L is a line.

Proof. Assume that I has a linear resolution. Since G_I is a tree, we have $\operatorname{proj}\dim(I) = 1$. So if $L \subset I$ with $G(L) \subset G(I)$ and G_L is a line, then L has linear relations and $\operatorname{proj}\dim(L) = 1$. Therefore L has a linear resolution.

For the converse, by our assumption there exists a monomial ideal $J_0 \subset I$ such that $G(J_0) = \{u_{i_1}, \ldots, u_{i_t}\} \subset G(I)$, G_{J_0} is a line and has linear resolution. Therefore J_0 has linear quotients. Take $v \in V(G_I) \setminus V(G_{J_0})$ such that v and u_{i_j} are adjacent in G_I for some $1 \leq j \leq t$. Set $F(u_{i_j}) \setminus F(v) = \{l\}$. Since J_0 has linear quotients there exist a path between u_{i_r} and u_{i_j} for all $1 \leq r \leq t$. Therefore we have line $u_{i_r}, \dots, u_{i_j}, v$ in G_I. By our hypothesis $L =< u_{i_r}, \dots, u_{i_j}, v >$ has a linear resolution and Proposition 2.1 implies that $F(u_{i_j}) \subseteq F(v) \cup F(u_{i_r})$. Therefor $\{l\} \in F(u_{i_r})$ and Proposition 2.4 implies that $J_1 =< J_0, v >$ has linear quotients. Now replace J_0 by J_1 and do the same procedure until we obtain I. □

Theorem 2.6. Let I be a squarefree monomial ideal which is generated in one degree. If G_I is a tree, then the following conditions are equivalent:

(a) I has a linear resolution;
(b) I has linear relations;
(c) $G_I^{(u,v)}$ is a connected graph for all u, and v in $G(I)$;
(d) If $u = u_1, u_2, \dots, u_s = v$ is the unique path between u and v in G_I, then $F(u_j) \subset F(u_i) \cup F(u_k)$ for all $1 \leq i \leq j \leq k \leq s$;
(e) L has a linear resolution for all $L \subseteq I$, where $G(L) \subset G(I)$ and G_L is a line

Proof. $(a) \Rightarrow (b)$ is trivial. $(b) \Rightarrow (c)$ follows from Proposition 1.13. $(c) \Rightarrow (d)$: for all $1 \leq i \leq j \leq k \leq s$, $G_I^{(u_i,u_k)}$ is connected and u_j is a vertex of this graph. Therefore $F(u_j) \subset F(u_i) \cup F(u_k)$. $(d) \Rightarrow (e)$ follows from Proposition 2.1. $(e) \Rightarrow (a)$ follows from Proposition 2.5. □

Theorem 2.7. Let I be a squarefree monomial ideal generated in degree d. If G_I is a tree, then the following are equivalent:

(a) I has a linear resolution;
(b) I is variable-decomposable ideal;
(c) I has linear quotients.

Proof. $(a) \Rightarrow (b)$: We know that $\operatorname{proj}\dim(I) = 1$, since G_I is a tree and I has a linear resolution. With out loss of generality we may assume that u_1 is a vertex of degree one in G_I and u_2 be the unique neighborhood of u_1 in G_I. Set $F(u_2) \setminus F(u_1) = \{l\}$. Proposition 1.13 implies that $G_I^{(u_1,u_i)}$ is a connected graph for all u_i. If $l \notin F(u_i)$ for some $i > 2$, then $F(u_2) \nsubseteq F(u_1) \cup F(u_i)$ and $u_2 \notin V(G_I^{(u_1,u_i)})$. Therefore $G_I^{(u_1,u_i)}$ is not connected, a contradictions. Hence $I_{x_l} = \{u_1\}$ and $G(I^{x_l}) = G(I) \setminus \{u_1\}$. It is easy to see that x_l is a shedding variable. Since $G_{I^{x_l}}$ is a tree and has linear relations, by induction on $|G(I)|$, we conclude that I^{x_l} is variable-decomposable. Therefore I is variable-decomposable ideal. $(b) \Rightarrow (c)$ follows by Theorem 1.3. $(c) \Rightarrow (a)$ follows by Proposition 1.1. □

Remark 2.8. In Theorem 2.7, we show that if G_I is a tree and I has a linear resolution, then I has linear quotients. In the following we present an admissible order for I in this case. we choose order $u_{r_1}, \dots, u_{r_m}$ for the elements of $G(I)$ such that the subgraph on vertices $\{u_{r_1}, \dots, u_{r_t}\}$ is a connected graph for $1 \leq t \leq m$. We show tat this order is an admissible order. If this order is not an admissible order, then there exists a $j < i$ such that for all $k < i$ with $F(u_{r_k}) \setminus F(u_{r_i}) = \{l\}$, we have $l \notin F(u_{r_j})$. Since there is a path $u_{r_j}, \dots, u_{r_k}, u_{r_i}$ Remark 1.12 implies that $x_l \mid x_{F(u_{r_i}) \setminus F(u_{r_j})}$ and $x_l \mid x_{F(u_{r_j}) \setminus F(u_{r_i})}$, a contradiction.

A simplicial complex Δ over a set of vertices $[n] = \{1, \dots, n\}$ is a collection of subsets of $[n]$ with the property that $\{i\} \in \Delta$ for all i and if $F \in \Delta$, then all subsets of F are also in Δ. An element of Δ is called a face and the dimension of a face F is defined as $|F| - 1$, where $|F|$ is the number of vertices of F. The maximal faces of Δ under inclusion are called facets and the set of all facets denoted by $\mathcal{F}(\Delta)$. The dimension of the simplicial complex Δ is the maximal dimension of its facets. A subcollection of Δ is a simplicial complex whose facets are also facets of Δ. In other words a simplicial complex generated by a subset of the set of facets of Δ. Let Δ be a simplicial complex on $[n]$ of dimension $d-1$. For each $0 \leq i \leq d-1$ the ith skeleton of Δ is the simplicial complex $\Delta^{(i)}$ on $[n]$ whose faces are those faces F of Δ with $\mid F \mid \leq i+1$. We say that a simplicial complex Δ is connected if for facets F and G of Δ there exists a sequence of facets $F = F_0, F_1, \ldots, F_{q-1}, F_q = G$ such that $F_i \cap F_{i+1} \neq \emptyset$ for $i = 0, \ldots, q-1$. Observe that Δ is connected if and only if $\Delta^{(1)}$ is connected.

Let Δ be a simplicial complex on $[n]$. The Stanley-Reisner ideal of Δ is a squarefree monomial ideal $I_\Delta =< x_{i_1} \ldots x_{i_p} \mid \{x_{i_1}, \ldots, x_{i_p}\} \notin \Delta >$. Conversely, let $I \subseteq k[x_1, \ldots, x_n]$ be a squarefree monomial ideal. The Stanley-Reisner complex of I is the simplicial complex Δ on $[n]$ such that $I_\Delta = I$. The Alexander dual of Δ is the simplicial complex $\Delta^\vee =< \{x_1, \ldots, x_n\} \setminus F \mid F \notin \Delta >$.

Definition 2.9. [9] Let Δ be a simplicial complex. A facet $F \in \mathcal{F}(\Delta)$ is said to be a leaf of Δ if either F is the only facet of Δ or there exists a facet $G \in \mathcal{F}(\Delta)$ with $G \neq F$, called a branch F, such that $H \cap F \subseteq G \cap F$ for all $H \in \mathcal{F}(\Delta)$ with $H \neq F$. A connected simplicial complex Δ is a tree if every nonempty subcollection of Δ has a leaf. If Δ is not necessarily connected, but every subcollection has a leaf, then Δ is called a forest.

If Δ is a simplicial tree, then we can always order the facets $F_1, \ldots, F_q$ of Δ such that F_i is a leaf of the induced subcomplex $< F_1, \ldots, F_{i-1} >$. Such an ordering on the facets of Δ is called a leaf order. A simplicial complex Δ is a quasi-forest if Δ has a leaf order. A connected quasi-forest is called a quasi-tree.

Consider an arbitrary monomial ideal $I =< u_1, \ldots, u_m >$. For any subset σ of $\{1, \ldots, m\}$, we write u_σ for the least common multiple of $\{u_i \mid i \in \sigma\}$ and set $\mathrm{a}_\sigma = \deg u_\sigma$. Let $G(I) = \{u_1, \ldots, u_m\}$. The Scarf complex Δ_I is the collection of all subsets of $\sigma \subset \{1, \ldots, m\}$ such that u_σ is unique. As first noted by Diane Taylor [21], given a monomial ideal I in a polynomial ring S minimally generated by monomials $u_1, \ldots, u_m$, a free resolution of I can be given by the simplicial chain complex of a simplex with m vertices. Most often Taylors resolution is not minimal. The Taylor complex F_{Δ_I} supported on the Scarf complex Δ_I is called the algebraic Scarf complex of the monomial ideal I. For more information about Taylor complex we refer to [17].

The following results will be used later.

Lemma 2.10. [17]If I is a monomial ideal in S, then every free resolution of S/I contains the algebraic Scarf complex F_{Δ_I} as a subcomplex.

Proposition 2.11. [10, Corollary 4.7] Every simplicial tree is the Scarf complex of a monomial ideal I and supports a minimal resolution of I.

In [11] Faridi and Hersey studied minimal free resolution of squarefree monomial ideals with projective dimension 1. They prove the following.

Theorem 2.12. Let I be a squarefree monomial ideal in a polynomial ring S and Δ be a simplicial complex such that $I = I_\Delta$. Then the following statements are equivalent:

(a) $\operatorname{proj\,dim} I \leq 1$;
(b) $\Delta^\vee$ is a quasi-forest;
(c) S/I has a minimal free resolution supported on a graph-tree.

If Δ is a simplicial complex and $\dim \Delta = 1$, then the geometric realization of Δ is a graph. In this situation we say Δ is a graph.

Lemma 2.13. Let I be a monomial ideal. Set $G = \Delta_I^{(1)}$. If G_I is a c_3-free graph, then G_I is a subgraph of G.

Proof. One has $V(G) = V(G_I) = G(I)$. Let $\{u_i, u_j\} \in E(G_I)$. Assume that $\{u_i, u_j\}$ is not an edge in G. Then there exits $\sigma \subset \{1, \ldots, m\}$ such that $\{i, j\} \neq \sigma$ and $u_{\{i,j\}} = u_\sigma$. Let $r \in \sigma \setminus \{i, j\}$, then $F(u_r) \subseteq F(u_i) \cup F(u_j)$. We may assume that $F(u_i) = A \cup \{i\}$ and $F(u_j) = A \cup \{j\}$. Hence $F(u_r) \subseteq A \cup \{i, j\}$. Therefore $\{u_i, u_r\}$ and $\{u_j, u_r\}$ are in $E(G_I)$, which is a contradiction. □

Remark 2.14. Let I be a monomial ideal which has a linear resolution and $u_i, u_j \in G(I)$. Assume that $u_i = u_{i_1}, .., u_{i_{t-1}}, u_{i_t} = u_j$ is a path between u_i and u_j. Then $x_{F(u_j)\setminus F(u_i)}e_i - x_{F(u_i)\setminus F(u_j)}e_j \in \ker(\varphi)$ and is a linear combination of linear forms which comes from the given path. By Remark 1.10 and Remark 1.12, $F(u_{i_r}) \subseteq F(u_i) \bigcup F(u_j)$, for all $i \leq r \leq j$.

Theorem 2.15. Assume that I is a squarefree monomial ideal generated in degree d. If G_I is a tree, then I has a linear resolution if and only if $G_I \cong \Delta_I$.

Proof. Assume that I has a linear resolution. By Theorem 2.2 $\operatorname{proj\,dim} I \leq 1$ and by Lemma 2.10 $\dim \Delta_I = 1$. By Lemma 2.13 G_I is a subgraph of Δ_I. Now let $\{u_r, u_t\} \in \Delta_I$. Suppose that $\{u_r, u_t\} \notin E(G_I)$. Since $x_{F(u_t)\setminus F(u_r)}e_r - x_{F(u_r)\setminus F(u_t)}e_t \in \ker(\varphi)$ and I ha a linear resolution, we have

$$x_{F(u_t)\setminus F(u_r)}e_r - x_{F(u_r)\setminus F(u_t)}e_t = f_r(x_{k_r}e_r - x_{{k_{r+1}}'}e_{i_{r+1}}) + \ldots + f_j(x_{k_j}e_{i_j} - x_{{k_t}'}e_t).$$

Set $\sigma = \{r, i_{r+1}, \ldots, i_j, t\}$. By Remark 2.14 $m_\sigma = m_{\{r,t\}}$, which is a contradiction.

Conversely assume that $G_I \cong \Delta_I$. Therefore Δ_I ia a tree. By Proposition 2.11 Δ_I supports a minimal free resolution of I. Therefore by Theorem 2.12 $\operatorname{proj\,dim} I \leq 1$. If $\operatorname{proj\,dim} I = 0$, then I is principal monomial ideal and, hence, I has a linear resolution.

Let $\operatorname{proj\,dim} I = 1$. If I has not a linear resolution, then there exists $x_{F(u_j)\setminus F(u_i)}e_i - x_{F(u_i)\setminus F(u_j)}e_j \in \ker(\varphi)$ such that this element belong to a minimal set of generators of $\ker(\varphi)$ and $\deg(x_{F(u_i)\setminus F(u_j)}) \geq 2$. There exists a unique path $u_i = u_{i_1}, u_{i_2}, \ldots, u_{i_{t-1}}, u_{i_t} = u_j$ between u_i and u_j in G_I. By Lemma 1.6 and Remark 1.5 there exists a monomial w such that
$w(x_{F(u_j)\setminus F(u_i)}e_i - x_{F(u_i)\setminus F(u_j)}e_j) = f_{i_1}(x_{k_1}e_{i_1} - {x_{k_2}}'e_{i_2}) + f_{i_2}(x_{k_2}e_{i_2} - {x_{k_3}}'e_{i_3}) + \ldots + f_{i_{t-1}}(x_{k_{t-1}}e_{i_{t-1}} - {x_{k_t}}'e_{i_t})$.
Set $\psi(g) = x_{F(u_j)\setminus F(u_i)}e_i - x_{F(u_i)\setminus F(u_j)}e_j,\ \psi(g_1) = x_{k_1}e_{i_1} - {x_{k_2}}'e_{i_2}, \ldots,\ \psi(g_{t-1})$
$= x_{k_{t-1}}e_{i_{t-1}} - {x_{k_t}}'e_{i_t}$. Then $\psi(\Sigma_{r=1}^{t-1} f_{i_r}g_r - wg) = 0$. Since $\Sigma_{r=1}^{t-1} f_{i_r}g_r - wg \neq 0$, one has $\ker(\psi) \neq 0$, which is a contradiction. □

3. Linear resolution of some classes of monomial ideals

In this section as applications of our results, we determine linearity of resolution for some classes of monomial ideals.

Let I be a squarefree Cohen-Macaulay monomial ideal of codimension 2 and Δ be a simplicial complex such that $I = I_\Delta$. In [15] the authors showed that Δ is shellable. Moreover one can see that Δ is vertex decomposable, see [1]. Now assume that I is generated in one degree. Since $\operatorname{proj\,dim}(I) = 1$, as a corollary of Proposition 2.3 and Theorem 2.7, we have:

Corollary 3.1. Let I be a squarefree Cohen-Macaulay monomial ideal of codimension 2. Then I has a linear resolution if and only if G_I is a connected graph. Indeed in this case G_I is a tree and the following conditions are equivalent:

(i) I has a linear resolutions;
(ii) I has linear quotients;
(iii) I is variable decomposable.

The following example shows that there are Cohen-Macaulay monomial ideal of codimenstion 2 with and without a linear resolution.

Example 3.2. (i)- let $I = (xy, yz, zt) \subset K[x, y, z, t]$. Then I is Cohen-Macaulay monomial ideal of codimenstion 2 with a linear resolutions

(ii)- let $I =< xy, zt >\subset K[x, y, z, t]$. It is easy to see that I is a Cohen-Macaulay monomial ideal of codimension 2 which has not a linear resolution.

Remark 3.3. Let I be a squarefree monomial ideal. If G_I is a complete graph, then the following statements hold.

(a) I has a linear resolution;
(b) I is variable-decomposable ideal;
(c) I has linear quotients.

In [4] Conca and De Negri introduced path ideal of a graph. Let G be a directed graph on vertex set $\{1, \dots, n\}$. For integer $2 \leq t \leq n$, a sequence $i_1, \dots, i_t$ of distinct vertices of G is called a path of length t, if there are $t - 1$ distinct directed edges $e_1, \dots, e_{t-1}$, where e_j is an edge from i_j to i_{j+1}. The path ideal of G of length t is the monomial ideal $I_t(G) =< \prod_{j=1}^{t} x_{i_j} >$, where $i_1, \dots, i_t$ is a path of length t in G. Let C_n denote the n-cycle on vertex set $V = \{1, \dots, n\}$. In [8, proposition 4.1] it is shown that $S/I_2(C_n)$ is vertex decomposable/ shellable/ Cohen-Macaulay if and only if $n = 3$ or 5. Saeedi, Kiani and Terai in [20] showed that if $2 < t \leq n$, then $S/I_t(C_n)$ is sequentially Cohen-Macaulay if and only if $t = n$, $t = n-1$ or $t = (n-1)/2$. In [1] it is shown that $S/I_t(C_n)$ is Cohen-Macaulay if and only if it is shellable and if and only if $I_t(C_n)$ is vertex decomposable.

It is easy to see that if $t < n-1$, then $G_{I_t(C_n)} \cong C_n$. Hence, by Theorem 1.17 $I_t(C_n)$ has a linear resolution if and only if $t = n-2$. For $t = n-1$, since $G_{I_t(G)}$ is a complete graph, $I_t(G)$ has a linear resolution. Also, in these cases having a linear resolution is equivalent to have linear quotients and it is equivalent to variable decomposability of $I_t(C_n)$.

Corollary 3.4. $I_t(C_n)$ has a linear resolution if and only if $t = n - 2$ or $t = n - 1$. Moreover the following conditions are equivalent:

(a) $I_t(G)$ has a linear resolution;
(b) $I_t(G)$ is variable-decomposable ideal;
(c) $I_t(G)$ has linear quotients.

Corollary 3.5. Let L_n be a line on vertex set $\{1, \dots, n\}$ and $I_t(L_n)$ be the path ideal of L_n. Then $I_t(L_n)$ has a linear resolution if and only if $t \geq \frac{n}{2}$.

Proof. Let $L_n = 1, \dots, n$ be a line. It is easy to see that $G_{I_t(L_n)} \cong L_{n-t+1}$ and $I_t(L_n) =< \prod_{i=1}^{t} x_i, \dots, \prod_{i=t+1}^{2t} x_i, \dots, \prod_{i=n-t+1}^{n} x_i >$. If $n-t+1 > t+1$, then $F(u_2) \nsubseteq F(u_1) \bigcup F(u_n)$. Hence Theorem **??** implies that $I_t(G)$ has not a linear resolution. If $n - t + 1 \leq t + 1$, i.e $t \geq \frac{n}{2}$, then it is clear that for any $1 \leq j \leq k \leq i \leq m$ one has:

$$F(u_k) \subseteq F(u_i) \bigcup F(u_j).$$

Therefore, by Proposition 2.1, $I_t(G)$ has a linear resolution and the equivalent conditions hold. □

4. Cohen-Macaulay simplicial complex

Let $\Delta =< F_1, \ldots, F_m >$ be a simplicial complex on vertex set $[n]$ and $I_\Delta \subset k[x_1, \ldots, x_n]$ be its Stanley-Reisner ideal. For each $F \subset [n]$, we set $\bar{F}_i = [n] \setminus F_i$ and $P_F = (x_j : j \in F)$. It is well known that $I_\Delta = \bigcap_{i=1}^m P_{\bar{F}_i}$ and $I_{\Delta^\vee} = (x_{\bar{F}_i} : i = 1, \ldots, m)$, see [13]. The simplicial complex Δ is called pure if all facets of it have the same dimension. It is easy to see that Δ is pure if and only if $I_{\Delta^\vee}$ is generated in one degree. The k-algebra $k[\Delta] = S/I_\Delta$ is called the Stanley-Reisner ring of Δ. We say that Δ is Cohen-Macaulay over k if $k[\Delta]$ is Cohen-Macaulay. It is known Δ is a Cohen-Macaulay over k if and only if $I_{\Delta^\vee}$ has a linear resolution, see [6]. Since every Cohen-Macaulay simplicial complex is pure, in this section, we consider only pure simplicial complexes.

The simplicial complex Δ is called shellable if its facets can be ordered $F_1, F_2, \ldots, F_m$ such that, for all $2 \leq i \leq m$, the subcomplex $\langle F_1, \ldots, F_{i-1}\rangle \cap \langle F_i \rangle$ is pure of dimension $\dim(F_i) - 1$.

For the simplicial complexes Δ_1 and Δ_2 defined on disjoint vertex sets, the join of Δ_1 and Δ_2 is $\Delta_1 * \Delta_2 = \{F \cup G : F \in \Delta_1, G \in \Delta_2\}$. For a face F in Δ, the link, deletion and star of F in Δ are respectively, denoted by $\mathrm{link}_\Delta F$, $\Delta \setminus F$ and $\mathrm{star}_\Delta F$ and are defined by $\mathrm{link}_\Delta F = \{G \in \Delta : F \cap G = \varnothing, F \cup G \in \Delta\}$, $\Delta \setminus F = \{G \in \Delta : F \nsubseteq G\}$ and $\mathrm{star}_\Delta F = \langle F \rangle * \mathrm{link}_\Delta F$.

A face F in Δ is called a shedding face if every face G of $\mathrm{star}_\Delta F$ satisfies the following exchange property: for every $i \in F$ there is a $j \in [n] \setminus G$ such that $(G \cup \{j\}) \setminus \{i\}$ is a face of Δ. A simplicial complex Δ is recursively defined to be k-decomposable if either Δ is a simplex or else has a shedding face F with $\dim(F) \leq k$ such that both $\Delta \setminus F$ and $\mathrm{link}_\Delta F$ are k-decomposable. 0-decomposable simplicial complexes are called vertex decomposable.

It is clear that $x_{\bar{F}_i}$ and $x_{\bar{F}_j}$ are adjacent in $G_{I_{\Delta^\vee}}$ if and only if F_i and F_j are connected in codimension one, i.e, $|F_i \cap F_j| = |F_i| - 1$. A simplicial complex Δ is called connected in codimension one or strongly connected if for any two facets F and G of Δ there exists a sequence of facets $F = F_0, F_1, \ldots, F_{q-1}, F_q = G$ such that F_i and F_{i+1} is connected in codimension one for each $i = 1, \ldots, q-1$. Hence we have the following:

Lemma 4.1. A simplicial complex Δ is connected in codimension one if and only if $G_{I_{\Delta^\vee}}$ is a connected graph.

For facets F and G of Δ, we introduced a subcomplex $\Delta^{(F,G)} = \langle L \in \mathcal{F}(\Delta) : F \cap G \subset L\rangle$. It is easy to see that $\Delta^{(F,G)}$ is connected in codimension one if and only if $G_{I_{\Delta^\vee}}^{(x_{\bar{F}}, x_{\bar{G}})}$ is a connected graph. Hence by Proposition 1.13 we have:

Corollary 4.2. Let Δ be a simplicial complex on vertex set $[n]$. Then $I_{\Delta^\vee}$ has linear relations if and only if $\Delta^{(F,G)}$ is connected in codimension one for all facets F and G of Δ.

Suppose that Δ is a simplicial complex of dimension d, i.e, $|F_i| = d + 1$ for all i. We associate to Δ a simple graph G_Δ whose vertices are labeled by the facets of Δ. Two vertices F_i and F_j are adjacent if F_i and F_j are connected in codimension one. If F_i and F_j are adjacent in G_Δ, then $| F_i \cap F_{i+1} |= d$. It is easy to see that $| \bar{F}_i \cap \bar{F}_j |= n - d - 2$. Therefor $x_{\bar{F}_i}$ and $x_{\bar{F}_j}$ is adjacent in $G_{I_{\Delta^\vee}}$ and, hence, $G_\Delta \cong G_{I_{\Delta^\vee}}$.

Now assume that $G_\Delta \cong G_{I_{\Delta^\vee}}$ is a line. Proposition 2.1, implies that $I_{\Delta^\vee}$ has a linear resolution if and only if for any $1 \leq j \leq k \leq i \leq m$, $\bar{F}_k \subseteq \bar{F}_i \bigcup \bar{F}_j$. By Eagon-Reiner [6], we have the following:

Corollary 4.3. Let $\Delta =< F_1, \dots, F_m >$ be a pure simplicial complex. If $G_\Delta = F_1, F_2, \dots, F_m$ is a line, then Δ is a Cohen-Macaulay if and only if $F_i \cap F_j \subseteq F_k$ for any $1 \leq j \leq k \leq i \leq m$. Moreover in this case the following conditions are equivalent:

(a) $\Delta^{(F,G)}$ is connected in codimension one for all facets F and G in Δ.
(b) Δ is Cohen-Macaulay.
(c) Δ is shellabe
(d) Δ is vertex decomposable simplicial complex.

Also a consequence of Theorem 1.17, we have:

Corollary 4.4. Let $\Delta =< F_1, \dots, F_m >$ be a pure simplicial complex. If $G_\Delta \cong C_m$, then Δ is a Cohen-Macaulay if and only if $m = n$ and with a suitable relabeling of facets F_i, we have $i \in F_i \cap F_{i+1}$ and $i \notin F_j$ for all $j \neq i, i+1$ ($F_{m+1} = F_1$). Moreover in this case Δ is shellabe and vertex decomposable simplicial complex.

Again a corollary of Theorem 2.7, Theorem 2.15 and Theorem 2.6 we have:

Corollary 4.5. Let $\Delta =< F_1, \dots, F_m >$ be a pure simplicial complex. If G_Δ is a tree, then the following conditions are equivalent:

(a) $\Delta^{(F,G)}$ is connected in codimension one for all facets F and G in Δ.
(b) if $F = F_1, F_2, \dots, F_s = G$ is a unique path in G_Δ, then $F_i \cap F_k \subset F_j$ for all $1 \leq i \leq j \leq k \leq s$.
(c) Δ is Cohen-Macaulay.
(d) Δ is shellabe
(e) Δ is vertex decomposable.
(f) $G_\Delta \cong \Delta_{I_{\Delta^\vee}}$.

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Erfan manouchehri, Department of Mathematics, University of Kurdistan, P.O.Box:66177-15175, Sanadaj, Iran
E-mail address: erfanm6790@yahoo.com

Ali Soleyman Jahan, Ali Soleyman Jahan, Department of Mathematics, University of Kurdistan, P.O.Box:66177-15175, Sanadaj, Iran
E-mail address: A.solaimanjahan@uok.ac.ir