

# Selective survey on spaces of closed subgroups of topological groups

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**Abstract.** We survey different topologizations of the set  $\mathcal{S}(G)$  of all closed subgroups of a topological group  $G$  and demonstrate some applications in *Topological Groups, Model Theory, Geometric Group Theory, Topological Dynamics*.

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Some words in place of introduction. For a topological group  $G$ ,  $\mathcal{S}(G)$  denotes the set of all closed subgroup of  $G$ . There are many ways to endow  $\mathcal{S}(G)$  with a topology related to the topology of  $G$ . Among them, the most intensively studied are Chabauty topology rooted in *Geometry of Numbers* and the Vietoris topology went from *General Topology*; both coincide if  $G$  is compact. The spaces of closed subgroups are interesting by their own sake but also have some deep applications in *Topological Groups and Model Theory, Geometric Group Theory and Dynamical Systems*. The survey is my subjective look at this area.

Content: Chabauty spaces; Vietoris spaces; Other topologizations.

## 1 Chabauty spaces

**1.1. From Minkowski to Chabauty.** We recall that a *lattice*  $L$  in  $\mathbb{R}^n$  is a discrete subgroup of rank  $n$ . We denote  $\min L$  the length of a shortest non-zero vector from  $L$ ,  $\text{vol}(\mathbb{R}^n/L)$  is the volume of a basic parallelepiped of  $L$ .

A sequence  $(L_m)_{m \in \omega}$  of lattices in  $\mathbb{R}^n$  converges to the lattice  $L$  if, for each  $m \in \omega$ , one can choose a basis  $a_1(m), \dots, a_n(m)$  of  $L_m$  and a basis  $a_1, \dots, a_n$  of  $L$  such that the sequence  $(a_i(m))_{m \in \omega}$  converges to  $a_i$  for each  $i \in \{1, \dots, n\}$ . This convergence of lattices was introduced by H. Minkowski [1], and its usage in *Geometry of Numbers* (see [2]) is based on the following theorem of K. Mahler [3].

**Theorem 1.1.** *Let  $\mathcal{M}$  be a set of lattices in  $\mathbb{R}^n$ . Every sequence in  $\mathcal{M}$  has a convergent subsequence if and only if there exist two constants  $C > 0$ ,  $c > 0$  such that  $\min L > c$ ,  $\text{vol}(\mathbb{R}^n \setminus L) < C$  for each  $L \in \mathcal{M}$ .*

What we know now as Chabauty topology was invented by C. Chabauty in [4] in order to extend Theorem 1.1 to lattices in connected Lie groups. A discrete subgroup  $L$  of a connected Lie group  $G$  is called a *lattice* if the quotient space  $G/L$  is compact.

Let  $X$  be a Hausdorff locally compact space and let  $\exp X$  denotes the set of all closed subsets of  $X$ . The sets

$$\{F \in \exp X : F \cap K = \emptyset\}, \{F \in \exp X : F \cap U \neq \emptyset\},$$

where  $K$  runs over all compact subsets of  $X$  and  $U$  runs over all open subsets of  $X$ , form the subbase of the *Chabauty topology* on  $\exp X$ . The space  $\exp X$  is compact and Hausdorff. If  $X$  is discrete then  $\exp X$  is homeomorphic to the Cantor cube  $\{0, 1\}^{|X|}$ .

We note also that a net  $(F_\alpha)_{\alpha \in \mathcal{I}}$  converges in  $\exp X$  to  $F$  if and only if

- for every compact  $K$  of  $X$  such that  $K \cap F = \emptyset$ , there exists  $\beta \in \mathcal{I}$  such that  $F_\alpha \cap K = \emptyset$  for each  $\alpha > \beta$ ;
- for every  $x \in F$  and every neighbourhood  $U$  of  $x$ , there exists  $\gamma \in \mathcal{I}$  such that  $F_\alpha \cap U \neq \emptyset$  for each  $\alpha > \gamma$ .

If  $G$  is a locally compact group then  $\mathcal{S}(G)$  is a closed subspace of  $\exp G$  (so  $\mathcal{S}(G)$  is compact);  $\mathcal{S}(G)$  is called the *Chabauty space* of  $G$ .

**Theorem 1.2[4].** *Let  $G$  be a connected unimodular Lie group. A set  $\mathcal{M}$  of lattices in  $G$  is relatively compact in  $\mathcal{M}$  if and only if there exists constant  $C > 0$  and a neighbourhood  $U$  of the identity  $e$  of  $G$  such that  $L \cap U = \{e\}$  and  $\text{vol}(G/L) < C$  for each  $L \in \mathcal{M}$ .*

With some technical improvement made in [5], the paper [4] is included in [6, Chapter 8].

**1.2. Pontryagin-Chabauty duality.** This duality was established in [7] and detailed in [8]. We use the standard abbreviation LCA for a locally compact Abelian group. Let  $G$  be a LCA-group,  $G^\wedge$  denotes its dual group,  $G^\wedge = \text{Hom}(G, \mathbb{R}/\mathbb{Z})$  and let  $\varphi$  denotes the canonical bijection  $\mathcal{S}(G) \longrightarrow \mathcal{S}(G^\wedge)$ ,  $\varphi(X) = \{f \in G^\wedge : X \subseteq \ker f\}$ .

**Theorem 1.3.** *For every LCA-group  $G$ , the bijection  $\varphi : \mathcal{S}(G) \longrightarrow \mathcal{S}(G^\wedge)$  is a homeomorphism.*

Typically, Theorem 1.3 applies to replace  $\mathcal{S}(G)$  by  $\mathcal{S}(G^\wedge)$  in the case of a compact Abelian group  $G$ .

In what follows we use the notations:  $\mathbb{C}_n$  is the cyclic group of order  $n$ ,  $\mathbb{C}_{p^\infty}$  is the quasi-cyclic (or Prüffer)  $p$ -group,  $\mathbb{Z}$  is the discrete group of integers,  $\mathbb{Z}_p$  is the group of  $p$ -adic integers,  $\mathbb{Q}_p$  is the additive group of the field of  $p$ -adic numbers.

**1.3.  $\mathcal{S}(G)$  for compact  $G$ .** The following two lemmas from [9] are the basic technical tools in this area.

**Lemma 1.1.** *If  $G, H$  are compact groups and  $\varphi : G \longrightarrow H$  is a continuous surjective homomorphism then the mapping  $\mathcal{S}(G) \longrightarrow \mathcal{S}(H)$ ,  $X \longmapsto \varphi(X)$  is continuous and open.*

The continuity is easy but to prove the openness we need

**Lemma 1.2.** *Let  $G$  be a compact group,  $X \in \mathcal{S}(G)$ . Then the following subsets from a base of neighbourhoods of  $X$  is  $\mathcal{S}(G)$ :*

$$\mathcal{N}_X(U, N, x_1, \dots, x_n) = \{u^{-1}Yu : Y \in \mathcal{S}(G), Y \subseteq XN, Y \cap x_1U \neq \emptyset, \dots, Y \cap x_nU \neq \emptyset, \}$$

where  $U$  is a neighbourhood of the identity of  $G$ ,  $N$  is closed normal subgroup such that  $G/N$  is a Lie group,  $x_1, \dots, x_n$  are arbitrary elements of  $X$ ,  $n \in \mathbb{N}$ .

In particular, if  $G$  is a compact Lie group then Lemma 1.2 states that there is a neighbourhood  $\mathcal{N}$  of  $X$  such that each subgroup  $Y \in \mathcal{N}$  is conjugated to some subgroup of  $X$ . The key part in the proof of Lemma 1.2 plays the Montgomery-Yang theorem on tubes [10], see also [11, Theorem 5.4 from Chapter 2].

We recall that the *cellularity* (or Souslin number)  $c(X)$  of a topological space  $X$  is the supremum of cardinalities of disjoint families of open subsets of  $X$ . A topological space  $X$  is called *dyadic* if  $X$  is a continuous image of some Cantor cube  $\{0, 1\}^\kappa$ .

The *weight*  $w(X)$  of a topological space  $X$  is the minimal cardinality of open bases of  $X$ .

**Theorem 1.4 [9].** *For every compact group  $G$ , we have  $c(\mathcal{S}(G)) \leq \aleph_0$ . In addition, if  $w(G) \leq \aleph_1$  then  $\mathcal{S}(G)$  is dyadic.*

**Theorem 1.5 [12].** *Let a group  $G$  be either profinite or compact and Abelian. If  $w(G) > \aleph_2$  then the space  $\mathcal{S}(G)$  is not dyadic.*

**Theorem 1.6 [12].** *Let  $G$  be an infinite compact Abelian group such that  $w(G) \leq \aleph_1$ . Then the space  $\mathcal{S}(G)$  is homeomorphic to the Cantor cube  $\{0, 1\}^{w(G)}$  if and only if  $\mathcal{S}(G)$  has no isolated points.*

An Abelian group  $G$  is called *Artinian* if every increasing chain of subgroups of  $G$  is finite; every such group is isomorphic to the direct sum  $\bigoplus_{p \in F} \mathbb{C}_{p^\infty} \oplus K$ , where  $F$  is a finite set of primes,  $K$  is a finite subgroup. An Abelian group  $G$  is called *minimax* if  $G$  has a finitely generated subgroup  $N$  such that  $G/N$  is Artinian.

**Theorem 1.7 [12].** *For a compact Abelian group  $G$ , the space  $\mathcal{S}(G)$  has an isolated point if and only if the dual group  $G^\wedge$  is minimax.*

**1.4.  $\mathcal{S}(G)$  for LCA  $G$ .** The space  $\mathcal{S}(\mathbb{R})$  is homeomorphic to the segment  $[0, 1]$ . By [13],  $\mathcal{S}(\mathbb{R}^2)$  is homeomorphic to the sphere  $\mathbf{S}^4$ . For  $n \geq 3$ ,  $\mathcal{S}(\mathbb{R}^n)$  is not a topological manifold and its structure is far from understanding, see [14].

**Theorem 1.8 [15].** *The space  $\mathcal{S}(G)$  of a LCA-group  $G$  is connected if and only if  $G$  has a subgroup topologically isomorphic to  $\mathbb{R}$ .*

If  $F$  is a non-solvable finite group then  $\mathcal{S}(\mathbb{R} \times F)$  is not connected [8, Proposition 8.6].

**Theorem 1.9 [8].** *The space  $\mathcal{S}(G)$  of a LCA-group  $G$  is totally disconnected if and only if  $G$  is either totally disconnected or each elements of  $G$  belongs to a compact subgroup.*

Some more information on  $\mathcal{S}(G)$  for LCA  $G$  can be find in [8] and references there, in particular, on topological dimension of  $\mathcal{S}(G)$ .

By Theorems 1.4 and 1.3,  $c(\mathcal{S}(G)) \leq \aleph_0$  for every discrete Abelian group. We say that a topological space  $X$  has *Shanin number*  $\omega$  if any uncountable family  $\mathcal{F}$  of non-empty open subsets of  $X$  has an uncountable subfamily  $\mathcal{F}'$  such that  $\bigcap \mathcal{F}' \neq \emptyset$ . Evidently, if a space  $X$  has Shanin number  $\omega$  then  $c(X) \leq \aleph_0$ . By [16, Theorem 1], for every discrete Abelian group  $G$ , the space  $\mathcal{S}(G)$  has Shanin number  $\omega$ . By [16, Theorem 3], for every infinite cardinal  $\tau$ , there exists a solvable discrete group  $G$  such that  $c(\mathcal{S}(G)) = |G| = \tau$ .

**1.5.  $S(G)$  as a lattice.** The set  $S(G)$  has the natural structure of a lattice with the operations  $\vee$  and  $\wedge$ , where  $A \wedge B = A \cap B$  and  $A \vee B$  is the smallest closed subgroup of  $G$  containing  $A$  and  $B$ . In this subsection, we formulate some results from [17] on interrelations between the topological and lattice structures on  $S(G)$ .

For  $g \in G$ ,  $\overline{\langle g \rangle}$  denotes the subgroup of  $G$  topologically generated by  $g$ . A totally disconnected locally compact group  $G$  is called *periodic* if  $\overline{\langle g \rangle}$  is compact for each  $g \in G$ . In this case,  $\pi(G)$  denotes the set of all prime numbers such that  $p \in \pi(G)$  if and only if there is  $g \in G$  such that  $\overline{\langle g \rangle}$  is topologically isomorphic either to  $\mathbb{C}_{p^n}$  or to  $\mathbb{Z}_p$ ; this  $g$  is called a *topological  $p$ -element*.

**Theorem 1.10.** *For a compact group  $G$ , the following statements are equivalent*

(i)  $\wedge$  is continuous;

(ii)  $\wedge$  and  $\vee$  are continuous;

(iii)  $G$  is the semidirect product  $K \rtimes P$ , where  $K$  is profinite with finite Sylow  $p$ -subgroups,  $P$  is Abelian profinite and each Sylow  $p$ -subgroup of  $G$  is  $\mathbb{Z}_p$ ,  $\pi(K) \cap \pi(P) = \emptyset$  and the centralizer of each Sylow  $p$ -subgroup of  $G$  has finite index in  $G$ .

**Theorem 1.11.** *For a locally compact group  $G$ , the operation  $\wedge$  is continuous if and only if the followings conditions are satisfied*

(i)  $G$  is either discrete or periodic;

(ii)  $\wedge$  is continuous in  $\mathcal{S}(H)$  for each compact subgroup  $H$  of  $G$ ;

(iii) the centralizer of each topological  $p$  element of  $G$  is open.

We recall that a torsion group  $G$  is *layerly finite* if the set  $\{g \in G : g^n = e\}$  is finite for each  $n \in \mathbb{N}$ . A layerly finite group  $G$  is called *thin* if each Sylow  $p$ -subgroup of  $G$  is finite (equivalently,  $G$  has no subgroup isomorphic to  $\mathbb{C}_{p^\infty}$ ).

**Theorem 1.12.** *Let  $G$  be a locally compact group. The operations  $\wedge$  and  $\vee$  are continuous if and only if  $G$  is periodic and topologically isomorphic to  $A \times B \times (C \rtimes D)$ , where  $C$  has a dense thin layerly finite subgroup,  $A, B, D$  are Abelian with Sylow  $p$ -subgroups  $\mathbb{C}_{p^\infty}, \mathbb{Q}_p$  or  $\mathbb{Z}_p$ , the sets  $\pi(A), \pi(B), \pi(G), \pi(D)$  are pairwise disjoint and the centralizer of each Sylow  $p$ -subgroup of  $G$  is open.*

**1.6. From Chabauty to local method.** A topological group  $G$  is called *topologically simple* if each closed normal subgroup of  $G$  is either  $G$  or  $\{e\}$ . Every topologically simple LCA-group is discrete and either  $G = \{e\}$  or  $G$  is isomorphic to  $\mathbb{C}_p$ .

Following the algebraic tradition, we say that a group  $G$  is *locally nilpotent (solvable)* if every finitely generated subgroup is nilpotent (solvable).

In [18, Problem 1.76], V. Platonov asked whether there exists a non-Abelian topologically simple locally compact locally nilpotent group. Now we sketch the negative answer to this question for locally solvable group obtained in [19].

Let  $G$  be a locally compact locally solvable group. We take  $g \in G \setminus \{e\}$ , choose a compact neighbourhood  $U$  of  $G$  and denote by  $\mathcal{F}$  the family of all topologically finitely generated subgroups of  $G$  containing  $g$ . We may assume that  $G$  is not topologically finitely generated so  $\mathcal{F}$  is directed by the inclusion  $\subset$ . For each  $F \in \mathcal{F}$ , we choose  $A_F, B_F \in \mathcal{S}(F)$  such that  $B_F \subset A_F$ ,  $A_F$  and  $B_F$  are normal in  $F$ ,  $A_F \cap U \neq \emptyset$ ,  $B_F \cap U = \emptyset$  and  $A_F/B_F$  is Abelian. Since  $\mathcal{S}(G)$  is compact, we can choose two subsets  $(A_\alpha)_{\alpha \in \mathcal{I}}$ ,  $(B_\alpha)_{\alpha \in \mathcal{I}}$  of the nets  $(A_F)_{F \in \mathcal{F}}$ ,  $(B_F)_{F \in \mathcal{F}}$  which converges to  $A, B \in \mathcal{S}(G)$ . Then  $A, B$  are normal in  $G$  and  $A/B$  is Abelian. Moreover,  $x \notin B$  and  $A \cap U \neq \emptyset$ . If  $A \neq \{G\}$  then  $A$  is a proper normal subgroup of  $G$ ; otherwise  $G/B$  is Abelian.

In [20], the Chabauty topology was defined on some systems of closed subgroups of locally compact group  $G$ . A system  $\mathfrak{A}$  of closed subgroups of  $G$  is called *subnormal* if

- $\mathfrak{A}$  contains  $\{e\}$  and  $G$ ;
- $\mathfrak{A}$  is linearly ordered by the inclusion  $\subset$ ;
- for any subset  $\mathfrak{M}$  of  $\mathfrak{A}$ , the closure of  $\bigcup_{F \in \mathfrak{M}} F \in \mathfrak{A}$  and  $\bigcap_{F \in \mathfrak{M}} F \in \mathfrak{A}$ ;
- whenever  $A$  and  $B$  comprise a jump in  $\mathfrak{A}$  (i.e  $B \subset A$  and no members of  $\mathfrak{A}$  lie between  $B$  and  $A$ ),  $B$  is a normal subgroup of  $A$ .

If the subgroup  $A, B$  form a jump then  $A/B$  is called a factor of  $G$ . The system is called *normal* if each  $A \in \mathfrak{A}$  is normal in  $G$ .

A group  $G$  is called an RN-group if  $G$  has a normal system with Abelian factors. Among the local theorems from [20], one can find the following: if every topologically finitely generated subgroup of a locally compact group  $G$  is an RN-group then  $G$  is an RN-group. In particular, every locally compact locally solvable group is an RN-group.

In 1941, see [21, pp. 78-83], A.I. Mal'tsev obtained local theorems for discrete groups as applications of the following general local theorem: if every finitely generated subsystem of an algebraic system  $A$  satisfies some property  $\mathcal{P}$ , which can be defined by some quasi universal second order formula, then  $A$  satisfies  $\mathcal{P}$ .

In [22], Mal'tsev's local theorem was generalized on topological algebraic system. The part of the model-theoretical Compactness Theorem in Mal'tsev arguments plays some convergents of closed subsets. A net  $(F_\alpha)_{\alpha \in \mathcal{I}}$  of closed subsets of a topological space  $X$   $S$ -converges to a closed subset  $F$  if

- for every  $x \in F$  and every neighbourhood  $U$  of  $x$ , there exists  $\beta \in \mathcal{I}$  such that  $F_\alpha \cap U \neq \emptyset$  for each  $\alpha > \beta$ ;
- for every  $y \in X \setminus F$ , there exist a neighbourhood  $\mathcal{V}$  of  $y$  and  $\gamma \in \mathcal{I}$  such that  $F_\alpha \cap \mathcal{V} = \emptyset$  for each  $\alpha > \gamma$ .

Every net of closed subsets of an arbitrary (!) topological space has a convergent subnet. If  $X$  is a Hausdorff locally compact space then  $S$ -convergence coincides with convergence in the Chabauty topology.

**1.7 Spaces of marked groups.** Let  $F_k$  be the free group of rank  $k$  with the free generators  $x_1, \dots, x_k$  and let  $\mathcal{G}_k$  denotes the set of all normal subgroups of  $F_k$ . In the metric form, the Chabauty topology on  $\mathcal{G}_k$  was introduced in [23] as a reply on the Gromov's idea of topologizations of some sets of groups [24].

Let  $G$  be a group generated by  $g_1, \dots, g_k$ . The bejection  $x_i \mapsto g_i, g_1, \dots, g_n$  can be extended to the homomorphism  $f : F_k \rightarrow G$ . With the correspondence  $G \mapsto \ker f$ ,  $\mathcal{G}_k$  is called the *space marked  $k$ -generated groups*.

A couple of papers in development of [23] is directed to understand how large in topological sense are well-known classes of finitely generated groups, or how a given  $k$ -generated group is placed in  $\mathcal{G}_k$ , see [25]. Among applications of  $\mathcal{G}_k$ , we mention the construction of topologizable Tarski Monsters in [26].

**1.8 Dynamical development.** Every locally compact group  $G$  acts on the Chabauty space  $\mathcal{S}(G)$  by the rule:  $(g, H) \mapsto g^{-1}Hg$ . Under this action, every minimal closed invariant subset of  $\mathcal{S}(G)$  is called a *uniformly recurrent subgroup*, URS for short. The study of URSs was initiated by Glasner and Weiss [27] with the following observation.

Let a locally compact group  $G$  acts on a compact  $X$  so that is  $G$  minimal, i.e. the orbit of each point  $x \in X$  is dense. We consider the mapping  $Stab : X \rightarrow \mathcal{S}(G)$  defined by  $Stab(x) = \{g \in G : gx = x\}$ . Then there is the unique URS contained in the closure of  $Stab(X)$ . This URS is called the *stabilizer URS*. Glasner and Weiss asked whether every URS of a locally compact group  $G$  arises as the stabilizer URS of a minimal action of  $G$  on a compact space. This question was answered in the affirmative in [28].

## 2 Vietoris spaces

For a topological space  $X$ , the Vietoris topology on the set  $exp X$  of all closed subsets of  $X$  is defined by the subbase of open sets

$$\{F \in exp X : F \subseteq U\}, \{F \in exp X : F \cap V \neq \emptyset\},$$

where  $U, \mathcal{V}$  run over all open subsets of  $X$ .

A net  $(F_\alpha)_{\alpha \in \mathcal{I}}$  converges to  $F$  in  $exp X$  if and only if

- for each open subset  $U$  of  $X$  such that  $F \subseteq U$ , there exists  $\beta \in \mathcal{I}$  such that  $F_\alpha \subseteq U$  for each  $\alpha > \beta$ ;
- for each  $x \in F$  and each neighbourhood  $\mathcal{V}$  of  $x$ , there exists  $\gamma \in \mathcal{I}$  such that  $F_\alpha \cap \mathcal{V} \neq \emptyset$  for each  $\alpha > \gamma$ .

If  $X$  is regular then  $\mathcal{S}(G)$  is closed in  $\exp G$ . As to my knowledge, the spaces  $\mathcal{S}(G)$ , where  $G$  needs not to be compact, endowed with the Vietoris topologies appeared in [29] with characterization of LCA-groups  $G$  such that the canonical mapping  $\varphi : \mathcal{S}(G) \rightarrow \mathcal{S}(G^\wedge)$  is a homeomorphism.

**2.1. Compactness.** It is naively to ask a constructive description of arbitrary topological groups  $G$  with compact space  $\mathcal{S}(G)$  because we know nothing even about  $G$  with  $S(G) = 2$ .

**Theorem 2.1.** [30]. *Let  $G$  be a locally compact group. The space  $\mathcal{S}(G)$  is compact if and only if  $G$  is one of the following groups*

- (i)  $G$  is compact;
- (2)  $\mathbb{C}_{p_1^\infty} \times \dots \times \mathbb{C}_{p_n^\infty} \times K$ , where  $p_1, \dots, p_n$  are distinct prime numbers,  $K$  is finite and each  $p_i$  is not a divisor of  $|K|$ ;
- (3)  $Q_p \times K$ , where  $K$  is finite and  $p$  does not divide  $|K|$ .

Similar characterization of groups with compact  $\mathcal{S}(G)$  is given in [31] provided that  $G$  has a base at the identity consisting of subgroups.

**Theorem 2.2.** [32]. *Let  $G$  be a locally compact group. A closed subset  $\mathcal{F}$  of  $\mathcal{S}(G)$  is compact if and only if the following conditions are satisfied*

- (i) every descending chain of non-compact subgroups from  $\mathcal{F}$  is finite;
- (ii) every closed subset  $\mathcal{F}'$  of  $\mathcal{F}$  has only finite number of non-compact subgroups maximal in  $\mathcal{F}$ ;
- (iii) if a closed subset  $\mathcal{F}'$  of  $\mathcal{F}$  has no non-compact subgroups then  $\cup \mathcal{F}'$  is compact.

Two corollaries: every compact in  $\mathcal{L}(G)$  consisting of non-compact subgroups is scattered; a subset  $\mathcal{F}$  is compact if and only if  $\mathcal{F}$  is countably compact.

For locally compact groups with  $\sigma$ -compact space  $\mathcal{S}(G)$  see [33], a description of LCA-groups with locally compact space  $\mathcal{S}(G)$  is obtained in [34].

A topological group  $G$  is called *inductively compact* if every finite subset of  $G$  is contained in compact subgroup. For a group  $G$ ,  $K(G)$  and  $IK(G)$  denote the sets of all compact and closed inductively compact subgroups.

**Theorem 2.3.** [35]. *For every locally compact group  $G$ ,  $IK(G)$  is the closure of  $K(G)$ .*

Two corollaries: if  $G$  is a connected Lie group then  $K(G)$  is closed;  $\mathcal{S}(G)$  is a  $k$ -space for each locally compact group  $G$  of countable weight, i.e. the topology of  $\mathcal{S}(G)$  is uniquely determined by the family of all compact subsets of  $\mathcal{S}(G)$ .

**2.2. Metrizable and normality.** LCA-groups  $G$  with metrizable and normal space  $\mathcal{S}(G)$  are characterized by S. Panasyuk in the candidate thesis *Normality and metrizability of the space of closed subgroups*, Kyiv University, 1989. These lists are completely constructive but too cumbersome so we formulate only

**Theorem 2.4.** *For a discrete Abelian group  $G$ , the following statements are equivalent*

- (i)  $\mathcal{S}(G)$  is metrizable;
- (ii)  $\mathcal{S}(G)$  is normal;
- (iii)  $G$  has a finitely generated subgroup  $H$  such that  $G/H = \mathbb{C}_{p_1^\infty} \times \dots \times \mathbb{C}_{p_n^\infty}$ , where  $p_1, \dots, p_n$  are distinct primes.

In general case, metrizability and normality of  $\mathcal{S}(G)$  are not equivalent but if  $G$  a connected semisimple Lie group then  $\mathcal{S}(G)$  is metrizable iff  $\mathcal{S}(G)$  is normal iff  $G$  is compact, see [36], [37]. The space  $\mathcal{S}(G)$  of every connected solvable Lie group is metrizable [36].

**2.3. Some cardinal invariants.** We remind that  $c(X)$  denotes the cellularity of  $X$ .

**Theorem 2.5.**[9] *For every infinite locally compact group  $G$ , we have  $c(\mathcal{S}(G)) \leq c(G)$ .*

**Theorem 2.6.** [38]. *For every locally compact group  $G$ , the following conditions are equivalent*

- (i)  $\mathcal{S}(G)$  is of countable pseudocharacter;
- (ii)  $\mathcal{S}(G)$  is of countable tightness;
- (iii)  $\mathcal{S}(G)$  is sequential;

(iv)  $w(G) \leq \aleph_0$ .

### 3 Other topologizations

**3.1. Bourbaki uniformities.** Let  $(X, \mathcal{U})$  be a uniform space. The uniformity  $\mathcal{U}$  induces the uniformity  $\tilde{\mathcal{U}}$  on the set  $\mathcal{F}(X)$  all non-empty closed subsets of  $X$  which has as a base the family of sets of the form

$$\{(A, B) \in \mathcal{F}(X) \times \mathcal{F}(X) : B \subseteq U(A), A \subseteq U(B)\},$$

whenever  $U \in \mathcal{U}$ . The uniformity  $\tilde{\mathcal{U}}$  was introduced in [39, Chapter 2, § 1] and  $\tilde{\mathcal{U}}$  is called *the Bourbaki* (sometimes, Hausdorff-Bourbaki) *uniformity*.

Let  $G$  be a topological group. We endow  $G$  with the left uniformity  $L$  and  $F(G)$  with the Bourbaki uniformity  $\tilde{L}$ . We denote by  $\mathcal{L}(G)$  and  $\mathcal{B}(G)$  the subspaces of  $\mathcal{F}(G)$  consisting of all subgroups and all totally bounded subsets of  $G$ .

**Theorem 3.1.[40]** *Let a group  $G$  has a base at the identity consisting of subgroups. The space  $\mathcal{L}(G)$  is compact if and only if  $G$  is totally bounded and  $K \cap G$  is dense in  $K$  for each closed subgroup  $K$  from the completion of  $G$ .*

In particular, if  $\mathcal{L}(G)$  is compact then  $G$  is totally minimal.

**Theorem 3.2.[40]** *If a group  $G$  is complete in the left uniformity then  $\mathcal{B}(G)$  is complete.*

We recall that a topological group  $G$  is *almost metrizable* if each neighbourhood of  $e$  contains a compact subgroup  $K$  such that the set of all open subsets containing  $K$  has a countable base. Every metrizable and every locally compact topological group are almost metrizable.

**Theorem 3.3.[40]** *If an almost metrizable group  $G$  is complete in the left uniformity then  $\mathcal{F}(G)$  is complete.*

In [41], Theorem 3.3 is proved with the bilateral uniformity on  $G$  (and so on  $\mathcal{F}(G)$ ) in place of the left uniformity.

**3.2. Functionally balanced groups.** For a topological group  $G$ , the set  $\mathcal{F}(G)$  has the natural structure of a semigroup with the operation  $(A, B) \mapsto cl AB$ .

**Theorem 3.4.[42]** *For a topological group  $G$ , the following statements are equivalent*

(i)  $\mathcal{F}(G)$  is a topological semigroup;

(ii) for every subset  $X$  of  $G$  and every neighbourhood  $U$  of  $e$ , there exists a neighbourhood  $V$  of  $e$  such that  $VX \subseteq XU$ ;

(iii) every bounded left uniformly continuous function on  $G$  is right uniformly continuous.

A topological group  $G$  is called *balanced* (or a SIN-group) if left and right uniformities of  $G$  coincide. A group  $G$  is called *functionally balanced* if  $G$  satisfies (iii) of Theorem 3.4. The study of functionally balanced groups was initiated by G. Itzkowitz [43].

The equivalence of (ii) and (iii) in Theorem 3.4 is a criterion for a topological group to be functionally balanced. In [44], this criterion was used to show that each almost metrizable functionally balanced group is balanced.

**3.3. Lattice topologies.** These topologies on a complete lattice  $\mathcal{L}(G)$  of closed subgroup are algebraically defined by the lattice structure of  $\mathcal{L}(G)$ .

For example, a net  $(A_\alpha)_{\alpha \in \mathcal{I}}$  in  $\mathcal{L}(G)$  *order-converges* to  $A \in \mathcal{L}(G)$  if there exist two nets  $(B_\alpha)_{\alpha \in \mathcal{I}}$ ,  $(C_\alpha)_{\alpha \in \mathcal{I}}$  in  $\mathcal{L}(G)$  such that, for each  $\alpha \in \mathcal{I}$ ,  $B_\alpha \subseteq A_\alpha \subseteq C_\alpha$  and  $\bigvee_{\alpha \in \mathcal{I}} B_\alpha = \bigwedge_{\alpha \in \mathcal{I}} C_\alpha = A$ . By [45], for a compact group  $G$ , every net in  $\mathcal{L}(G)$  has an order-convergent subset if and only if  $\mathcal{L}(G)$  endowed with the Shabauty topology is a topological lattice, see Theorem 1.10.

More on the lattices topologies on  $\mathcal{L}(G)$  in the case of a compact  $G$  can be find in [46].

**3.4. Segment topologies.** Let  $G$  be a topological group,  $\mathcal{P}_G$  is the family of all subsets of  $G$ ,  $[G]^{<\omega}$  is the family of all finite subsets of  $G$ . Each pair  $\mathcal{A}, \mathcal{B}$  of subsets of  $\mathcal{P}_G$  closed under finite unions define the segment topology on  $\mathcal{L}(G)$  with a base consisting of the segments.

$$[A, G \setminus B] = \{X \in \mathcal{L}(G) : A \subseteq X \subseteq G \setminus B\}, \quad A \in \mathcal{A}, \quad B \in \mathcal{B}.$$

These topologies are studied in [47] in the following three cases:  $\mathcal{A} = \mathcal{B} = [G]^{<\omega}$ ;  $\mathcal{A} = \mathcal{P}_G$  and  $\mathcal{B} = [G]^{<\omega}$ ;  $\mathcal{A} = [G]^{<\omega}$ ,  $\mathcal{B} = \mathcal{P}_G$

**3.5.  $(\Sigma, \Theta)$ -topologies.** This general construction for topologizations of the set  $\mathcal{L}(G)$  of closed subgroups of a topological group  $G$  from [48] produces Chabauty, Vietoris, Bourbaki topologies and a plenty of other topologies

We assume that, for each  $H \in \mathcal{L}(G)$ ,  $\Sigma(H)$  is some family of open subsets of  $G$ ,  $\Sigma = \bigcup_{H \in \mathcal{L}(G)} \Sigma(H)$  and the following conditions are satisfied

- if  $U, \mathcal{V} \in \Sigma(H)$  then  $U \cap \mathcal{V}$  contains some  $W \in \Sigma(H)$ ;
- for every  $U \in \Sigma(H)$ , there exists  $\mathcal{V} \in \Sigma(H)$  such that  $U \in \Sigma(K)$  for each  $K \in \mathcal{L}(G)$ ,  $K \subseteq \mathcal{V}$ ;
- $\bigcap_{U \in \Sigma(H)} \overline{U} = H$  for each  $H \in \mathcal{L}(G)$ .

Then the family  $\{X \in \mathcal{L}(G) : X \subseteq U\}$ ,  $U \in \Sigma$  is a base for the  $\Sigma$ -topology on  $\mathcal{L}(G)$ .

Let  $\tau$  denotes the topology of  $G$ ,  $\mathcal{P}_\tau$  is the family of all subsets of  $\tau$ . We assume that, for each  $H \in \mathcal{L}(G)$ ,  $\Theta(H)$  is some subset of  $\mathcal{P}_\tau$  such that the following conditions are satisfied

- for every  $\alpha, \beta \in \Theta(H)$ , there is  $\gamma \in \Theta(H)$  such that  $\alpha < \gamma$ ,  $\beta < \gamma$  ( $\alpha < \beta$  means that, for every  $U \in \alpha$ , there exists  $V \in \beta$  such that  $V \subseteq U$ );
- for every  $\alpha \in \Theta(H)$ , there exists  $\beta \in \Theta(H)$  such that if  $K \in \mathcal{L}(G)$  and  $K \cap V \neq \emptyset$  for each  $V \in \beta$ , then  $\alpha < \gamma$  for some  $\gamma \in \Theta(K)$ ;
- for each  $H \in \mathcal{L}(G)$  and every neighbourhood  $V$  of  $x$ , there exists  $\alpha \in \Theta(H)$  such that  $x \in U$ ,  $U \subseteq V$  for some  $U \in \alpha$ .

Then the family  $\{X \in \mathcal{L}(G) : X \cap U \neq \emptyset \text{ for each } U \in \alpha\}$ , where  $\alpha \in \Theta(H)$ ,  $H \in \mathcal{L}(G)$ , is a base for the  $\Theta$ -topology on  $\mathcal{L}(G)$ .

The upper bound of  $\Sigma$ - and  $\Theta$ -topologies is called the  $(\Sigma, \Theta)$ -topology.

A net  $(H_\alpha)_{\alpha \in \mathcal{I}}$  converges in  $(\Sigma, \Theta)$ -topology to  $H \in \mathcal{L}(G)$  if and only if

- for any  $U \in \Sigma(H)$ , there exists  $\beta \in \mathcal{I}$  such that  $H_\alpha \subseteq U$  for each  $\alpha > \beta$ ;
- for any  $\alpha \in \Theta(H)$ , there exists  $\gamma \in \mathcal{I}$  such that  $H_\alpha \cap \mathcal{V} \neq \emptyset$  for each  $\alpha > \gamma$ .

In [48], one can find characterizations of  $G$  with compact and discrete  $\mathcal{L}(G)$  in some concrete  $(\Sigma, \Theta)$ -topologies.

**3.6. Hyperballeans of groups.** Let  $G$  be a discrete group. The set  $\{Fg : g \in G, F \in [G]^{<\omega}\}$  is a family of balls in the finitary coarse structure on  $G$ . For coarse structures and balleans see [49] and [50]. The finitary coarse structure on  $G$  induces the coarse structure on  $\mathcal{L}(G)$  in which  $\{X \in \mathcal{L}(G) : X \subseteq FA, A \in FX\}$ ,  $F \in [G]^{<\omega}$  is the family of balls centered at  $A \in \mathcal{L}(G)$ . The set  $\mathcal{L}(G)$  endowed with structure is called a hyperballean of  $G$ . Hyperballeans of groups carefully studied in [51] can be considered as asymptotic counterparts of Bourbaki uniformities.

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