On Adjacency and e-Adjacency in General Hypergraphs: Towards a New e-Adjacency Tensor

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Abstract

In graphs, the concept of adjacency is clearly defined: it is a pairwise relationship between vertices. Adjacency in hypergraphs has to integrate hyperedge multi-adicity: the concept of adjacency needs to be defined properly by introducing two new concepts: k-adjacency - k vertices are in the same hyperedge - and e-adjacency - vertices of a given hyperedge are e-adjacent. In order to build a new e-adjacency tensor that is interpretable in terms of hypergraph uniformisation, we designed two processes: the first is a hypergraph uniformisation process (HUP) and the second is a polynomial homogeneisation process (PHP). The PHP allows the construction of the e-adjacency tensor while the HUP ensures that the PHP keeps interpretability. This tensor is symmetric and can be fully described by the number of hyperedges; its order is the range of the hypergraph, while extra dimensions allow to capture additional hypergraph structural information including the maximum level of k-adjacency of each hyperedge. Some results on spectral analysis are discussed.

Keywords: hypergraph, e-adjacency tensor, uniformisation, homogeneisation

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1 Adjacency in hypergraphs

A **hypergraph** $\mathcal{H} = (V, E)$ is a hyperedge family $E = \{e_i : e_i \subseteq V \wedge i \in [\![p]\!]\}$ [3] over the vertex set $V = \{v_i : i \in [\![n]\!]\}$ [1]. A **hypergraph with no repeated hyperedge** is a hypergraph where the hyperedges are distinct pairwise.

We write $k_{\max} = \max\{|e| : e \in E\}$ the **range of the hypergraph**.

Hyperedge multi-adicity calls for additional adjacency concepts.

Definition 1.1 k vertices are said k**-adjacent** if it exists an hyperedge that contains them. Vertices of a given hyperedge are said **e-adjacent**. The $\overline{k}$-adjacency of an hypergraph is the maximal value of k such that it exists vertices of the hypergraph that are k-adjacent.

Hypermatrices - abusively designated as tensors [2] - are used to store the adjacency multi-adic relationships. In k-uniform hypergraphs, where all hyperedges have the same cardinality $\overline{k} = k$, $\overline{k}$-adjacency and e-adjacency are equivalent; we use here the degree normalized $\overline{k}$-adjacency hypermatrix [3].

For general hypergraphs with no-repeated hyperedge, a first e-adjacency hypermatrix is defined in [4]. The value and the number of elements that are required to store this hypermatrix vary depending on the hyperedge cardinality; due to index repetition, tensor elements can not be interpreted directly in term of a hypergraph uniformisation process (HUP). To address this issue, we propose a new e-adjacency tensor [4].

2 A new e-adjacency tensor for general hypergraphs

We give here only the main steps. [5]

2.1 Decomposition in layers

The family $(E_k)_{1 \leqslant k \leqslant k_{\max}}$ where $E_k = \{e \in E : |e| = k\}$ constitutes a partition of E. $\mathcal{H}$ is decomposable uniquely into a k-uniform hypergraph direct sum $\mathcal{H} = \bigoplus_{k=1}^{k_{\max}} \mathcal{H}_k$ of increasing $k \in [\![k_{\max}]\!]$. The $\mathcal{H}_k = (V, E_k)$ - $k \in [\![k_{\max}]\!]$ - are called the **layers** of $\mathcal{H}$. Any of these $\mathcal{H}_k$ is representable by a degree-normalised $\overline{k}$-adjacency hypermatrix $\boldsymbol{A}_k = \left(a_{(k)\, i_1 \ldots i_k}\right)$.

[3] $[\![k; n]\!]$ is $\{i : i \in \mathbb{N} \wedge k \leqslant i \leqslant n\}$ and $[\![n]\!]$ is $[\![1, n]\!]$. $\mathcal{S}_k$ is the permutation set on $[\![k]\!]$.

[4] Details and proofs can be found in [5].

[5] Exponents into parenthesis refer to the order of the corresponding tensor; indices into parenthesis refer to a sequence of objects.

Symmetric cubical hypermatrices are bijectively mapped to homogeneous polynomials [6] through the hypermatrix multilinear matrix multiplication [7].

We build a family $P_{\mathcal{H}} = (P_k)$ of homogenous polynomials that are one-to-one mapped to the layers of the hypergraph. Considering $\boldsymbol{z} = (\boldsymbol{z}_0)^{\top}$ [6] - for all $i \in [\![n]\!]$: z^i represents $v_i \in V$ - and $(\boldsymbol{z})_{[k]} = (\boldsymbol{z}, ..., \boldsymbol{z}) \in (\mathbb{R}^n)^k$, $(\boldsymbol{z})_{[k]}.\boldsymbol{A_k}$ contains only one element: $P_k(\boldsymbol{z}_0) = \sum\limits_{1 \leqslant i_1, ..., i_k \leqslant n} a_{(k)\,i_1 ... i_k} z^{i_1} ... z^{i_k}$. As $\boldsymbol{A_k}$ is symmetric: $P_k(\boldsymbol{z}_0) = \sum\limits_{1 \leqslant i_1 \leqslant ... \leqslant i_k \leqslant n} \alpha_{(k)\,i_1 ... i_k} z^{i_1} ... z^{i_k}$ with $\alpha_{(k)\,i_1 ... i_k} = k! a_{(k)\,i_1 ... i_k}$.

2.2 Uniformisation and homogeneisation process

The hypergraph uniformisation process involves two elementary operations on weighted hypergraphs.

Operation 1: Let $\mathcal{H}_w = (V, E, w)$ be a weighted hypergraph. Let $y \notin V$. The **y-vertex-augmented hypergraph** of $\mathcal{H}_w$ is the weighted hypergraph $\overline{\mathcal{H}_{\overline{w}}} = \left(\overline{V}, \overline{E}, \overline{w}\right)$ where $\overline{V} = V \cup \{y\}$, $\overline{E} = \{\phi(e) : e \in E\}$ - with the map $\phi : \mathcal{P}(V) \to \mathcal{P}\left(\overline{V}\right)$ such that: $\forall A \in \mathcal{P}(V) : \phi(A) = A \cup \{y\}$ - and, $\overline{w}$ such that $\forall e \in E$: $\overline{w}(\phi(e)) = w(e)$.

Operation 2: The **merged hypergraph** $\widehat{\mathcal{H}_{\widehat{w}}} = \left(\widehat{V}, \widehat{E}, \widehat{w}\right)$ of two weighted hypergraphs $\mathcal{H}_a = (V_a, E_a, w_a)$ and $\mathcal{H}_b = (V_b, E_b, w_b)$ is the weighted hypergraph with vertex set $\widehat{V} = V_a \cup V_b$, with hyperedge family $\widehat{E} = E_a + E_b$ - constituted of all elements of E_a and all elements of E_b - such that $\forall e \in E_a$, $\widehat{w}(e) = w_a(e)$ and $\forall e \in E_b$, $\widehat{w}(e) = w_b(e)$.

The hypergraph uniformisation process starts by mapping each $\mathcal{H}_k$ to a weighted hypergraph $\mathcal{H}_{w_k,k} = (V, E_k, w_k)$ with: $\forall e \in E_k : w_k(e) = c_k$ with $c_k \in \mathbb{R}^{+*}$ and $k \in [\![k_{\max}]\!]$. c_k are dilatation coefficients introduced to guarantee that the generalized hand-shake lemma holds in the e-adjacency tensor. A set of pairwise distinct vertices $V_s = \{y_k : k \in [\![k_{\max} - 1]\!]\}$ is generated and such that no vertex of V_s is in V.

The HUP iterates over a two-phase step: the **inflation phase** (IP) and the **merging phase** (MP). At step $k > 1$ the input is the $(k-1)$-uniform weigthed hypergraph $\mathcal{K}_w$ obtained from the previous iteration; at step 1, $\mathcal{K}_w = \mathcal{H}_{w_1,1}$. In the IP, $\mathcal{K}_w$ is transformed into $\overline{\mathcal{K}_{\overline{w}}}$ the $k+1$-uniform y_k-vertex-augmented hypergraph of $\mathcal{K}_w$.

The MP elaborates the merged hypergraph $\widehat{\mathcal{K}_{\widehat{w}}}$ from $\overline{\mathcal{K}_{\overline{w}}}$ and $\mathcal{H}_{w_{k+1},k+1}$.

[6] We write $\boldsymbol{z}_0$ the variable list $z^1, ..., z^n$ and $\boldsymbol{z}_k$ the variable list $\boldsymbol{z}_0, y^1, ..., y^k$.

At the end of each step k is increased until it reaches $k_{\max}$: the last $\widehat{\mathcal{H}_{\widehat{w}}}$ obtained is called the V_s**-layered uniform hypergraph** of $\mathcal{H}$.

Proposition 2.1 *$\widehat{\mathcal{H}_{\widehat{w}}}$ captures exactly the e-adjacency of $\mathcal{H}$.*

In the polynomial homogeneisation process, $R_{\mathcal{H}} = (R_k)_{k\in[\![k_{\max}]\!]}$ the family of homogeneous polynomials of degree k is obtained iteratively from the family $(c_kP_k)_{k\in[\![k_{\max}]\!]}$: for all $k \in [\![k_{\max}]\!]$, c_kP_k maps one to one to $\mathcal{H}_{w_k,k}$.

We set $R_1(\boldsymbol{z}_o) = c_1P_1(\boldsymbol{z}_o) = c_1 \sum\limits_{i=1}^{n} a_{(1)\,i}z^i$. We generate $k_{\max}-1$ new pairwise distinct variables y^j, $j \in [\![k_{\max}-1]\!]$.

At step k, we suppose that: $R_k(\boldsymbol{z}_{k-1}) = \sum\limits_{j=1}^{k} c_j \sum\limits_{i_1,\dots,i_j=1}^{n} a_{(j)\,i_1\dots i_j} z^{i_1}\dots z^{i_j} \prod\limits_{l=j}^{k-1} y^l$, with the convention that: $\prod\limits_{l=j}^{k-1} y^l = 1$ if $j > k-1$. Then for $y^{k-1} \neq 0$:

$$\begin{aligned} R_{k+1}(\boldsymbol{z}_k) &= y^{k\,(k+1)}\left(R_k\left(\frac{\boldsymbol{z}_{k-1}}{y^{k\,(k)}}\right) + c_{k+1}P_{k+1}\left(\frac{\boldsymbol{z}_o}{y^{k\,(k+1)}}\right)\right)\\ &= R_k(\boldsymbol{z}_{k-1})\,y^k + c_{k+1}\sum_{i_1,\dots,i_{k+1}=1}^{n} a_{(k+1)\,i_1\,\dots\,i_{k+1}} z^{i_1}\dots z^{i_{k+1}} \end{aligned}$$

and for $y^k = 0$: $R_{k+1}(\boldsymbol{z}_{k-1}, 0) = c_{k+1} \sum\limits_{i_1,\dots,i_{k+1}=1}^{n} a_{(k+1)\,i_1\,\dots\,i_{k+1}} z^{i_1}\dots z^{i_{k+1}}$.

Even if $P_{k+1}(z_0) = 0$ the step above is performed: the degree of R_k will increase by 1.

2.3 Construction of the e-adjacency tensor

From $R_{\mathcal{H}} = (R_k)$ we build a symmetric tensor. R_k is an homogeneous polynomial with $n+k-1$ variables of order k. With $w_{(k)}$ for $w^1_{(k)}, \dots, w^n_{(k)}$, we have: $R_k\left(\boldsymbol{w}_{(k)}\right) = \sum\limits_{i_1,\dots,i_k=1}^{n+k-1} r_{(k)\,i_1\,\dots\,i_k} w^{i_1}_{(k)}\dots w^{i_k}_{(k)}$ where:

- ⋆ for $i \in [\![n]\!]$: $w^i_{(k)} = z^i$ and for $i \in [\![n+1; n+k-1]\!]$: $w^i_{(k)} = y^{i-n}$
- ⋆ for all $\forall j \in [\![k]\!]$, for $1 \leqslant i_1 < \dots < i_j \leqslant n$, for all $l \in [\![j+1;k]\!]$ [7] : $i_l = n+l-1$ and, for all $\sigma \in \mathcal{S}_k$:

$$r_{(k)\,\sigma(i_1)\dots\sigma(i_k)} = \frac{c_j\alpha_{(j)\,i_1\dots i_j}}{k!} = \frac{j!}{k!}c_j a_{(j)\,i_1\dots i_j}$$

[7] With the convention $[\![p,q]\!] = \emptyset$ if $p > q$

$\star$ otherwise $r_{(k)\,i_1\,\dots\,i_k}$ is null.

Also R_k can be linked to a symmetric hypercubic tensor of order k and dimension $n+k-1$ written $\boldsymbol{R_k}$ whose elements are $r_{(k)\,i_1\,\dots\,i_k}$.

The coefficients c_k, $k \in \llbracket k_{\max} \rrbracket$ are chosen so that the number of edges calculated by the generalized handshake lemma is valid.

We choose: $c_j = \dfrac{k_{\max}}{j}$ as:

$|E| = \dfrac{1}{k_{\max}} \sum\limits_{i_1,\dots,i_{k_{\max}} \in \llbracket n+k_{\max}-1 \rrbracket} r_{i_1 \dots i_{k_{\max}}} = \sum\limits_{j=1}^{k_{\max}} \dfrac{1}{j} \sum\limits_{i_1,\dots,i_j \in \llbracket n \rrbracket} a_{(j)\,i_1 \dots i_j}.$

Hence, combining above with the fact that $a_{(j)\,i_1 \dots i_j} = \dfrac{1}{(j-1)!}$ when $\left\{ v_{i_1}, \dots, v_{i_j} \right\} \in E$ and 0 otherwise: $r_{i_1 \dots i_{k_{\max}}} = \dfrac{1}{(k_{\max}-1)!}$ for nonzero elements of $\boldsymbol{R_{k_{\max}}}$.

Definition 2.2 The hypermatrix $\boldsymbol{R_{k_{\max}}}$ is called the **layered e-adjacency tensor** of the hypergraph $\mathcal{H}$. We write it later $\mathcal{A}_{\mathcal{H}}$.

3 Further comments and results

The HUP adds vertices in the IPs; they give indication on the original cardinality of the hyperedge they are added to as well as the level of k-adjacency possible in this hyperedge. The resulting tensor is symmetric and is bijectively associated to the original hypergraph, containing its overall structure.

We consider in the following propositions a hypergraph $\mathcal{H} = (V, E)$ with no repeated hyperedge with layered e-adjacency tensor $\mathcal{A}_{\mathcal{H}} = \left(a_{i_1 \dots i_{k_{\max}}} \right)$.

Proposition 3.1 *It holds:* $\sum\limits_{\substack{i_2,\dots,i_{k_{\max}}=1 \\ \delta_{ii_2\dots i_{k_{\max}}}=0}}^{n+k_{\max}-1} a_{ii_2 \dots i_{k_{\max}}} = d_i$

where: $\forall i \in \llbracket n \rrbracket : \ d_i = \deg(v_i)$ *and* $\forall i \in \llbracket k_{\max}-1 \rrbracket : \ d_{n+i} = \deg(y_i)$.
Moreover: $\forall j \in \llbracket 2; k_{\max} \rrbracket$*:* $|\{e : |e| = j\}| = d_{n+j} - d_{n+j-1}$
and: $|\{e : |e| = 1\}| = d_{n+1}$

Using the definition of eigenvalue of [2], we state:

Theorem 3.2 *The e-adjacency tensor $\mathcal{A}_{\mathcal{H}}$ has its eigenvalues λ such that:*

$$|\lambda| \leqslant \max(\Delta, \Delta^\star) \tag{1}$$

where $\Delta = \max\limits_{1 \leqslant i \leqslant n} (d_i)$ *and* $\Delta^\star = \max\limits_{1 \leqslant i \leqslant k_{\max}-1} (d_{n+i})$.

Proposition 3.3 *Let $\mathcal{H}$ be a r-regular*[8] *r-uniform hypergraph with no repeated hyperedge. Then this maximum is reached.*

4 Conclusion

Properly defining the concept of adjacency in a hypergraph is important to build a proper e-adjacency tensor that preverves the information on the structure of the hypergraph. The resulting tensor allows to reconstruct with no ambiguity the original hypergraph. First results on spectral analysis show that additional vertices inflate the spectral radius bound. The HUP is a strong basis for further proposals: to allow repetition of vertices, we introduce hb-graphs, family of multisets and, propose two other e-adjacency tensors [8].

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[8] A hypergraph is said r-regular if all vertices have same degree r.