AN EFFICIENT ABSTRACT METHOD FOR THE STUDY OF AN INITIAL BOUNDARY VALUE PROBLEM ON SINGULAR DOMAIN

BELKACEM CHAOUCHI AND MARKO KOSTIĆ

ABSTRACT. The present work is devoted to the study of a boundary value problem for second order linear differential equation set on singular cylindrical domain. This problem can be regarded via a natural change of variables as an elliptic abstract differential equation with variable operators coefficients subject to some anti-periodic conditions. The complete study of this abstract version allows us to establish some interesting regularity results for our problem. The study is performed in the framework of Hölder spaces.

1. INTRODUCTION AND PRELIMINARIES

In [2] and [3], the solvability of some BVP's set on cusp domain was discussed. The authors opted for the use of the abstract differential equations theory and some regularity results for these problems were successfully established in the framework of Little Hölder spaces. In the same direction, we will show that this approach can be exploited in order to give a complete study of an initial boundary value problem involving the Laplace operator and which is also posed on nonsmooth domain. More precisely, we consider a particular conical domain Ω given by

$$\Pi = [0, T] \times \Omega$$

where

$$\Omega = \left\{ (x,y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} \le \varphi\left(t\right) \right\}.$$

Here, φ is a positive real-valued function of parametrization defined on [0,1] such that

$$\varphi\left(0\right) = \varphi'\left(0\right) = 0.$$

In Π , we consider the following problem

(1.1)
$$\partial_t^2 u + \Delta u - \lambda u = h, \ \lambda > 0,$$

corresponding to the following initial conditions

(1.2)
$$u|_{t=0} + u|_{D(T,\varphi(T))} = 0, \quad \partial_t u|_{t=0} + \partial_t u|_{D(T,\varphi(T))} = 0,$$

where $D(T, \varphi(T))$ denotes the disc of radius $\varphi(T)$ centred at (T, 0, 0). We accompany (1.1)-(1.2) with following additional conditions

(1.3)
$$u|_{\partial\Pi\setminus D(T,\varphi(T))} = 0, \quad u|_{\partial\Pi\setminus\{0\}} = 0.$$

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We assume that the right hand term of (1.1) is taken in the anistropic Hölder space $C^{2\theta}([0,T]; C(\Omega)), 0 < 2\theta < 1$, defined by

$$\left\{\phi \in C([0,T];C(\Omega)): \lim_{\varepsilon \to 0^+} \sup_{0 < |t-t'| \le \varepsilon} \frac{\|\phi(t) - \phi(t')\|_{C(\Omega)}}{|t-t'|^{2\theta}} < \infty\right\}.$$

Now, we consider the following change of variables

(1.4)
$$T: \quad \Pi \to Q,$$
$$(t, x, y) \mapsto (t, \xi, \eta) = \left(t, \frac{x}{\varphi(t)}, \frac{y}{\varphi(t)}\right),$$

 \sim

where

$$Q = [0, T] \times D,$$

with

$$D := D(0,1) = \left\{ (\xi,\eta) \in \mathbb{R}^2 : \xi^2 + \eta^2 \le 1 \right\}$$

Define the following change of functions

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(1.5)
$$u(t, x, y) = v(t, \xi, \eta) \text{ and } h(t, x, y) = f(t, \xi, \eta).$$

It follows from the change of functions (1.5) that the new version of problem (1.1)is given by

(1.6)
$$\begin{aligned} \partial_t^2 v + L(t) v - \lambda v &= f, & \text{in } Q \\ v|_{\{0\} \times D} + v|_{\{T\} \times D} &= 0, \\ \partial_t v|_{\{0\} \times D} + \partial_t v|_{\{T\} \times D} &= 0, \\ v|_{[0,T] \times \partial D} &= 0, \end{aligned}$$

where L is the linear operator with singular coefficients given by

$$L(t) = \frac{1}{\varphi^2(t)} \Delta + \frac{\varphi'(t)}{\varphi(t)} \left\{ \xi \partial_{\xi} + \eta \partial_{\eta} \right\}, \ 0 \le t \le T.$$

The following lemma is needed in order to clarify the impact of the change of variables (1.4) on the functional framework of Hölder Spaces.

Lemma 1.1. Let $0 < 2\theta < 1$. Then

(1)
$$h \in C^{2\theta}([0,T]; C(\Omega)) \Rightarrow f \in C^{2\theta}([0,T]; C(D)).$$

(2) $f \in C^{2\theta}([0,T]; C(D)) \Rightarrow h \in C^{2\theta}_{w}([0,T]; C(\Omega))$ with
 $C^{2\theta}_{w}([0,T]; C(\Omega))) = \left\{ h \in C^{2\theta}([0,T]; C(\Omega)) : (\varphi(.))^{2\theta} h \in C^{2\theta}([0,T]; C(\Omega)) \right\}.$
Proof. See Proposition 3.1 in [2].

Proof. See Proposition 3.1 in [2].

Due to the presence of a singular coefficients, we must approximate the cylinder Π by a sequence of regular subdomains. As in [7], we perform the following regular change of variables given by

$$\Pi_n := [t_n, T] \times \Omega \to Q_n := [t_n, T] \times D(0, 1),$$

$$(t, x, y) \mapsto (t, \xi, \eta) = \left(t, \frac{x}{\varphi(t)}, \frac{y}{\varphi(t)}\right).$$

Here, $(t_n)_{n \in \mathbb{N}}$ is a decreasing sequence such that $0 \le t_n \le 1$ and $\lim_{n \to +\infty} t_n = 0$. Set

$$\begin{cases} v_n = v|_{Q_n}, \\ f_n = f|_{Q_n}. \end{cases}$$

Summing up, we are confronted to the study of following problems

(1.7)
$$\begin{aligned} \partial_t^2 v_n + L(t) v_n - \lambda v_n &= f_n, & \text{in } Q_n, \\ v_n|_{\{t_n\} \times D} + v_n|_{\{T\} \times D} &= 0, \\ \partial_t v_n|_{\{t_n\} \times D} + \partial_t v_n|_{\{T\} \times D} &= 0, \\ v_n|_{[t_n,T] \times \partial D} &= 0. \end{aligned}$$

Here, we just briefly note that

$$\lim_{n \to +\infty} v_n = v|_{\lim_{n \to +\infty} Q_n} = v|_Q = v.$$

In the next section, we will show that our transformed problems (1.7) can be formulated as a second order abstract differential equation of elliptic type with variable operators coefficients.

2. The abstract formulation of the problems (1.7)

Let us introduce the following vector-valued functions:

$$\begin{aligned} v_n &: \quad [t_n,T] \to E; \ t \longrightarrow v_n(t); \quad v_n(t)(\xi,\eta) = v_n\left(t,\xi,\eta\right), \\ f_n &: \quad [t_n,T] \to E; \ t \longrightarrow f_n(t); \quad f_n(t)(\xi,\eta) = f_n\left(t,\xi,\eta\right), \end{aligned}$$

with E = C(D). So, the transformed problem (1.7) can be formulated as follows

(2.1)
$$\begin{aligned} v_n''(t) + A(t) v_n(t) - \lambda v_n(t) &= f_n(t), \quad t_n \le t \le T, \\ v_n(t_n) + v_n(T) &= 0, \\ v_n'(t_n) + v_n'(T) &= 0. \end{aligned}$$

Here, $(A(t))_{t_n \le t \le T}$ is a family of closed linear operators with domains D(A(t)) (which are not dense) defined by (2.2)

$$\begin{cases} D(A(t)) := \left\{ \phi \in W^{2,p}(D) \cap C_0(D), \ p > 2 : L(t) \ \phi \in C_0(D) \right\}, \ t_n \le t \le T, \\ (A(t)) \ \phi(\xi, \eta) := (L(t)) \ \phi(\xi, \eta). \end{cases}$$

Consider the natural change of function

$$w_n(t) = v_n(t+t_n)$$
 and $g_n(t) = f_n(t+t_n);$

then

(2.3)
$$g_n \in C^{2\theta}([0,T]; C(D));$$

and w_n is the eventual solution of

(2.4)
$$\begin{cases} w_n''(t) + A(t+t_n)w_n(t) - \lambda w_n(t) = g_n(t), \ 0 \le t \le T, \\ w_n(0) + w_n(T) = 0, \\ w_n'(0) + w_n'(T) = 0, \end{cases}$$

From [1] p. 60, we know that the family $(A(t+t_n))_{0 \le t \le T}$ enjoys the following three properties:

(1)

(2.5)
$$\exists M > 0, \ \forall z \ge 0, \forall t \in [0,T] \quad \left\| (A_n(t+t_n) - z)^{-1} \right\|_{L(E)} \leqslant \frac{M}{z+1};$$

(2) For all $z \ge 0$, the application $t \mapsto (A_n(t+t_n) - \lambda - z)^{-1}$ defined on [0,T] is in $C^2([0,T]; L(E))$ and there exist C > 0 such that :

(2.6)
$$\forall z \ge 0, \forall t \in [0,T] \quad \left\| \frac{\partial}{\partial t} (A(t+t_n) - \lambda - zI)^{-1} \right\|_{L(E)} \le \frac{C}{z+1},$$

and

(2.7)
$$\forall z \ge 0, \forall t \in [0,T] \quad \left\| \frac{\partial^2}{\partial t^2} (A(t+t_n) - \lambda - zI)^{-1} \right\|_{L(E)} \le \frac{C}{z+1};$$

(3) Moreover, one has : $\forall z \ge 0, \forall t, s \in [0, T]$ (2.8)

$$\begin{cases} \left\| \frac{\partial^2}{\partial t^2} (A(t+t_n) - \lambda - z)^{-1} - \frac{\partial^2}{\partial s^2} (A(s+s_n) - \lambda - z)^{-1} \right\|_{L(E)} \leqslant \frac{C |t-s|^{2\theta}}{z+1}, \\ \left\| \frac{\partial^2}{\partial t^2} (A(t+t_n) - \lambda - z)^{-1} - \frac{\partial^2}{\partial s^2} (A(s+s_n) - \lambda - z)^{-1} \right\|_{L(E)} \leqslant \frac{C |t-s|^{2\theta}}{z+1}. \end{cases}$$

Remark 2.1. In the sequel the symbol C stands for a generic positive constant except when other dependence is stated explicitly. On the other hand, it is important to note that all the constants given above are independent of t and and consequently of n.

3. Some Regularity Results

We are concerned with a study of the problem

(3.1)
$$\begin{cases} w_n''(t) + A_n(t)w_n(t) - \lambda w_n(t) = g_n(t), \ 0 \le t \le T, \\ w_n(0) + w_n(T) = 0, \\ w_n'(0) + w_n'(T) = 0, \end{cases}$$

where :

•
$$g_n \in C^{2\theta}([0,T], E), 2\theta \in]0,1[,$$

•
$$A_n(t) := A(t+t_n).$$

Our purpose is to establish some results about the existence, uniqueness and maximal regularity of a strict solution w_n for Problem (3.1) by building explicitly a representation of the solution $w_n(t)$ and studying its optimal regularity. Recall here that a strict solution is a function w_n such that

$$\begin{cases} w_{n} \in C^{2}\left(\left[0,T\right],E\right), \\ w_{n}\left(t\right) \in D\left(A\left(t\right)\right) \text{ for every } t \in \left[0,T\right], \\ t \mapsto \left(A_{n}(t) - \lambda\right) w_{n}(t) \in C\left(\left[0,T\right],E\right), \end{cases}$$

and satisfying the anti-periodic boundary conditions

$$w_n(0) + w_n(T) = 0$$

 $w'_n(0) + w'_n(T) = 0$

The techniques used here are essentially based on the Dunford functional calculus and the methods applied in [1], [5]-[6] and [9]. We know that if $A_n(t)$ is a constant operator satisfying (2.5), the representation of the solution w_n is given by the formula

(3.2)
$$w_n(t) = -\frac{1}{2i\pi} \int_{\gamma} \int_0^T K_{\sqrt{-z}}(t,s) \left(A_n - \lambda - z\right)^{-1} g_n(s) ds dz$$

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where

(3.3)
$$K_{\sqrt{-z}}(t,s) = \begin{cases} \frac{e^{-\sqrt{-z}(t-s)} - e^{-\sqrt{-z}(T-t+s)}}{2\sqrt{-z}\left(1+e^{-T\sqrt{-z}}\right)}, & 0 \le s \le t, \\ \frac{e^{-\sqrt{-z}(s-t)} - e^{-\sqrt{-z}(T+t-s)}}{2\sqrt{-z}\left(1+e^{-T\sqrt{-z}}\right)}, & t \le s \le T, \end{cases}$$

and the curve γ is the retrograde oriented boundary of the sector Π_{θ_0, r_0} of the form

(3.4)
$$\Pi_{\delta_0,r_0} = \{ z \in \mathbb{C} - \{ 0 \} : |\operatorname{arg}(z)| \leq \delta_0 \} \cup \{ z \in \mathbb{C} : |z| \leq r_0 \},$$

with some small $\delta_0 > 0$, and $r_0 > 0$. (Here, $\rho(A_n)$ is the resolvent set of A_n). First, it is necessary to note here that

Lemma 3.1. There exists $C(\delta_0) > 0$ such that for all $z \in \prod_{\delta_0, r_0}$, one has :

$$\left|1+e^{-T\sqrt{-z}}\right| \ge C\left(\delta_0\right).$$

Proof. See Lemma 3 in [4].

Remark 3.2. By using a classical argument of analytic continuation on the resolvent, the previous assumptions hold true in the sector Π_{δ_0, r_0} , and then, on γ . Furthermore, we can replace z by $z + \lambda$.

Keeping in mind the constant case (see formula (3.2)), we look for a solution of Problem (3.1) in the following form :

(3.5)
$$w_n(t) = -\frac{1}{2i\pi} \int_{\gamma} \int_0^T K_{\sqrt{-z}}(t,s) \left(A_n(t) - \lambda - z\right)^{-1} g_n^*(s) ds dz,$$

where g_n^* is an unknown function to be determined in some adequate space in order to obtain a strict solution w_n of Problem (3.1), when $g_n \in C^{2\theta}([0,T], E)$.

Our first result concerning the vector valued function $w_n(t)$ given by (3.5) is

Proposition 3.3. Suppose that $g_n^* \in C^{2\theta}([0,T], E), 0 < 2\theta < 1$. Then, for all $t \in [0,T]$:

$$w_n \in C^2([0,T], E),$$

and

$$w_n(t) \in D\left(A_n\left(t\right)\right).$$

Proof. First, observe that the vector valued function w_n is well defined. In fact, using a direct computation on the kernel (3.3), one has

$$\begin{aligned} \left\| \int_{0}^{T} K_{\sqrt{-z}}(t,s) (A_{n}(t) - \lambda - z)^{-1} g_{n}^{*}(s) ds \right\| \\ &\leqslant \left(\sup_{t \in [0,T]} \int_{0}^{T} \left| K_{\sqrt{-z}}(t,s) \right| \left\| (A_{n}(t) - \lambda - z)^{-1} \right\|_{L(E)} ds \right) \|g_{n}^{*}\|_{C([0,T];E)} \\ &\leqslant \frac{C}{|z|} \|g_{n}^{*}\|_{C([0,T];E)} . \end{aligned}$$

Now, we write w_n as follows :

$$w_n(t) = -\frac{1}{2i\pi} \int_{\gamma} \int_0^T K_{\sqrt{-z}}(t,s) (A_n(t) - \lambda - z)^{-1} (g_n^*(s) - g_n^*(t)) \, ds dz - \frac{1}{2i\pi} \int_{\gamma} \int_0^T K_{\sqrt{-z}}(t,s) (A_n(t) - \lambda - z)^{-1} g_n^*(t) \, ds dz,$$

which becomes :

$$w_{n}(t) = -\frac{1}{2i\pi} \int_{\gamma} \int_{0}^{T} K_{\sqrt{-z}}(t,s) (A_{n}(t) - \lambda - z)^{-1} (g_{n}^{*}(s) - g_{n}^{*}(t)) ds dz$$

$$-\frac{1}{2i\pi} \int_{\gamma} c_{\sqrt{-z}}(t) \frac{(A_{n}(t) - \lambda - z)^{-1}}{z} g_{n}^{*}(t) dz$$

$$-\frac{1}{2i\pi} \int_{\gamma} \frac{(A_{n}(t) - \lambda - z)^{-1}}{z} g_{n}^{*}(t) dz,$$

where

$$c_{\sqrt{-z}}(t) = \frac{e^{-\sqrt{-z}t} + e^{-\sqrt{-z}(T-t)}}{\left(1 + e^{-T\sqrt{-z}}\right)}.$$

Thanks to (2.5), we have

$$\begin{split} & \left\| \frac{1}{2i\pi} \int_{\gamma} \int_{0}^{T} K_{\sqrt{-z}}(t,s) (A_{n}(t) - \lambda - z)^{-1} \left(g_{n}^{*}(s) - g_{n}^{*}(t) \right) ds dz \right\| \\ & \leq \int_{\gamma} \frac{C}{|z|} \int_{0}^{T} \left| K_{\sqrt{-z}}(t,s) \right| |t - s|^{2\theta} \left\| g_{n}^{*} \right\|_{C^{2\theta}([0,T];E)} ds \left| dz \right| \\ & \leq C \int_{\gamma} \frac{|dz|}{|z|^{2+\theta}} \left\| g_{n}^{*} \right\|_{C^{2\theta}([0,T];E)} \\ & \leq C \left\| g_{n}^{*} \right\|_{C^{2\theta}([0,T];E)} . \end{split}$$

Concerning the second integral, we have

$$\left\|\frac{1}{2i\pi}\int_{\gamma} c_{\sqrt{-z}}(t) \frac{(A_n(t)-\lambda)^{-1}(A_n(t)-\lambda-z)^{-1}}{z} g_n^*(t) dz\right\| \leqslant C \, \|g_n^*\|_{C([0,T];E)} \, .$$

On the other hand, by Cauchy Theorem, we deduce that

$$\frac{1}{2i\pi} \int_{\gamma} \frac{(A_n(t) - \lambda - z)^{-1}}{z} g_n^*(t) dz = -(A_n(t) - \lambda)^{-1} g_n^*(t).$$

Summing up, we deduce that

$$\forall t \in [0,T], \ w_n(t) \in D(A_n(t))$$

and

$$\begin{split} &(A_n(t) - \lambda)^{-1} w_n(t) \\ &= -\frac{1}{2i\pi} \int_{\gamma} \int_0^{\delta} K_{\sqrt{-z}}(t,s) (A_n(t) - \lambda)^{-1} (A_n(t) - \lambda - z)^{-1} \left(g_n^*(s) - g_n^*(t) \right) ds dz \\ &- \frac{1}{2i\pi} \int_{\gamma} c_{\sqrt{-z}}(t) \frac{(A_n(t) - \lambda)^{-1} (A_n(t) - \lambda - z)^{-1}}{z} g_n^*(t) dz \\ &- g_n^*(t). \end{split}$$

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Proposition 3.4. Suppose that $g_n^* \in C^{2\theta}([0,T], E), 0 < 2\theta < 1$. Then, the abstract equation

$$w_n'(t) + A_n(t) w_n(t) - \lambda w_n(t) = g_n^*(t) - R_\lambda(g_n^*)(t)$$

 $is \ satisfied, \ where$

$$R_{\lambda}(g_{n}^{*})(t) = +\frac{1}{i\pi} \int_{\gamma} \int_{0}^{T} \frac{\partial}{\partial t} K_{\sqrt{-z}}(t,s) \frac{\partial}{\partial t} (A_{n}(t) - \lambda - z)^{-1} g_{n}^{*}(s) ds dz$$
$$-\frac{1}{2i\pi} \int_{\gamma} \int_{0}^{T} K_{\sqrt{-z}}(t,s) \frac{\partial^{2}}{\partial t^{2}} (A_{n}(t) - \lambda - z)^{-1} g_{n}^{*}(s) ds dz.$$

 $\mathit{Proof.}$ Step 1. First, regarding the derivative $w_n'(t),$ we have :

$$\begin{split} w_n'(t) &= \frac{1}{2i\pi} \int_{\gamma} \int_0^t \frac{e^{-\sqrt{-z}(t-s)} + e^{-\sqrt{-z}(T-t+s)}}{2\left(1+e^{-T\sqrt{-z}}\right)} (A_n(t) - \lambda - z)^{-1} g_n^*(s) ds dz \\ &- \frac{1}{2i\pi} \int_{\gamma} \int_t^T \frac{e^{-\sqrt{-z}(s-t)} + e^{-\sqrt{-z}(T+t-s)}}{2\left(1+e^{-T\sqrt{-z}}\right)} (A_n(t) - \lambda - z)^{-1} g_n^*(s) ds dz \\ &- \frac{1}{2i\pi} \int_{\gamma} \int_0^t \frac{e^{-\sqrt{-z}(t-s)} - e^{-\sqrt{-z}(T-t+s)}}{2\sqrt{-z}\left(1+e^{-T\sqrt{-z}}\right)} \frac{\partial}{\partial t} (A_n(t) - \lambda - z)^{-1} g_n^*(s) ds dz \\ &- \frac{1}{2i\pi} \int_{\gamma} \int_t^T \frac{e^{-\sqrt{-z}(s-t)} - e^{-\sqrt{-z}(T+t-s)}}{2\sqrt{-z}\left(1+e^{-T\sqrt{-z}}\right)} \frac{\partial}{\partial t} (A_n(t) - \lambda - z)^{-1} g_n^*(s) ds dz \end{split}$$

Step 2. Let us study the second derivative $w''_n(t)$. We follow the approach used in [8]. Let ε be a very small positive number and t be such that

$$0 < \varepsilon \leqslant t \leqslant T - \varepsilon < T,$$

and $w'_{n,\varepsilon}$ be the function defined by

$$\begin{split} w_{n,\varepsilon}'(t) &= \frac{1}{2i\pi} \int_{\gamma} \int_{0}^{t-\varepsilon} \frac{e^{-\sqrt{-z}(t-s)} + e^{-\sqrt{-z}(T-t+s)}}{2\left(1+e^{-T\sqrt{-z}}\right)} (A_{n}(t) - \lambda - z)^{-1} g_{n}^{*}(s) ds dz \\ &- \frac{1}{2i\pi} \int_{\gamma} \int_{t+\varepsilon}^{T} \frac{e^{-\sqrt{-z}(s-t)} + e^{-\sqrt{-z}(T+t-s)}}{2\left(1+e^{-T\sqrt{-z}}\right)} (A_{n}(t) - \lambda - z)^{-1} g_{n}^{*}(s) ds dz \\ &- \frac{1}{2i\pi} \int_{\gamma} \int_{0}^{t-\varepsilon} \frac{e^{-\sqrt{-z}(t-s)} - e^{-\sqrt{-z}(T-t+s)}}{2\sqrt{-z}\left(1+e^{-T\sqrt{-z}}\right)} \frac{\partial}{\partial t} (A_{n}(t) - \lambda - z)^{-1} g_{n}^{*}(s) ds dz \\ &- \frac{1}{2i\pi} \int_{\gamma} \int_{t+\varepsilon}^{T} \frac{e^{-\sqrt{-z}(s-t)} - e^{-\sqrt{-z}(T+t-s)}}{2\sqrt{-z}\left(1+e^{-T\sqrt{-z}}\right)} \frac{\partial}{\partial t} (A_{n}(t) - \lambda - z)^{-1} g_{n}^{*}(s) ds dz. \end{split}$$

Observe here that all these integrals are absolutely convergent and

$$w'_{n,\varepsilon}(t) \to w'_n(t),$$

strongly as $\varepsilon \to 0$. In addition, we have

$$w_{n,\varepsilon}^{\prime\prime}(t) := \Pi_{n,\varepsilon}^{1}\left(t\right) + \Pi_{n,\varepsilon}^{2}\left(t\right) + \Pi_{n,\varepsilon}^{3}\left(t\right) + \Pi_{n,\varepsilon}^{4}\left(t\right),$$

where

$$\Pi_{n,\varepsilon}^{1}(t) = -\frac{1}{2i\pi} \int_{\gamma} \int_{0}^{t-\varepsilon} \sqrt{-z} \frac{e^{-\sqrt{-z}(t-s)} - e^{-\sqrt{-z}(T-t+s)}}{2\left(1+e^{-T\sqrt{-z}}\right)} (A_{n}(t) - \lambda - z)^{-1} g_{n}^{*}(s) ds dz$$
$$-\frac{1}{2i\pi} \int_{\gamma} \int_{t+\varepsilon}^{T} \sqrt{-z} \frac{e^{-\sqrt{-z}(s-t)} - e^{-\sqrt{-z}(T+t-s)}}{2\left(1+e^{-T\sqrt{-z}}\right)} (A_{n}(t) - \lambda - z)^{-1} g_{n}^{*}(s) ds dz,$$

$$\Pi_{n,\varepsilon}^{2}(t) = +\frac{1}{2i\pi} \int_{\gamma} \frac{e^{-\sqrt{-z\varepsilon}} + e^{-\sqrt{-z(T-\varepsilon)}}}{2\left(1 + e^{-T\sqrt{-z}}\right)} (A_{n}(t) - \lambda - z)^{-1} g_{n}^{*}(t-\varepsilon) dz$$
$$+\frac{1}{2i\pi} \int_{\gamma} \frac{e^{-\sqrt{-z\varepsilon}} + e^{-\sqrt{-z(T-\varepsilon)}}}{2\left(1 + e^{-T\sqrt{-z}}\right)} (A_{n}(t) - \lambda - z)^{-1} g_{n}^{*}(t+\varepsilon) dz,$$

$$\Pi_{n,\varepsilon}^{3}(t) = -\frac{1}{2i\pi} \int_{\gamma} \frac{e^{-\sqrt{-z\varepsilon}} - e^{-\sqrt{-z}(T-\varepsilon)}}{2\sqrt{-z}\left(1 + e^{-T\sqrt{-z}}\right)} \frac{\partial}{\partial t} (A_{n}(t) - \lambda - z)^{-1} g_{n}^{*}(t-\varepsilon) ds dz$$
$$+ \frac{1}{2i\pi} \int_{\gamma} \frac{e^{-\sqrt{-z\varepsilon}} - e^{-\sqrt{-z}(T-\varepsilon)}}{2\sqrt{-z}\left(1 + e^{-T\sqrt{-z}}\right)} \frac{\partial}{\partial t} (A_{n}(t) - \lambda - z)^{-1} g_{n}^{*}(t+\varepsilon) dz,$$

$$\begin{split} \Pi_{n,\varepsilon}^{4}(t) &= \frac{1}{2i\pi} \int_{\gamma} \int_{0}^{t-\varepsilon} \frac{e^{-\sqrt{-z(t-s)}} + e^{-\sqrt{-z(T-t+s)}}}{2\left(1+e^{-T\sqrt{-z}}\right)} \frac{\partial}{\partial t} (A_{n}(t) - \lambda - z)^{-1} g_{n}^{*}(s) ds dz \\ &- \frac{1}{2i\pi} \int_{\gamma} \int_{t+\varepsilon}^{T} \frac{e^{-\sqrt{-z(s-t)}} + e^{-\sqrt{-z(T+t-s)}}}{2\left(1+e^{-T\sqrt{-z}}\right)} \frac{\partial}{\partial t} (A_{n}(t) - \lambda - z)^{-1} g_{n}^{*}(s) ds dz \\ &+ \frac{1}{2i\pi} \int_{\gamma} \int_{0}^{t-\varepsilon} \frac{e^{-\sqrt{-z(t-s)}} + e^{-\sqrt{-z(T-t+s)}}}{2\left(1+e^{-T\sqrt{-z}}\right)} \frac{\partial}{\partial t} (A_{n}(t) - \lambda - z)^{-1} g_{n}^{*}(s) ds dz \\ &- \frac{1}{2i\pi} \int_{\gamma} \int_{t+\varepsilon}^{T} \frac{e^{-\sqrt{-z(s-t)}} + e^{-\sqrt{-z(T+t-s)}}}{2\left(1+e^{-T\sqrt{-z}}\right)} \frac{\partial}{\partial t} (A_{n}(t) - \lambda - z)^{-1} g_{n}^{*}(s) ds dz, \end{split}$$

$$\begin{aligned} \Pi_{n,\varepsilon}^{5}(t) \\ &= -\frac{1}{2i\pi} \int_{\gamma} \int_{0}^{t-\varepsilon} \frac{e^{-\sqrt{-z}(t-s)} - e^{-\sqrt{-z}(T-t+s)}}{2\sqrt{-z}\left(1+e^{-T\sqrt{-z}}\right)} \frac{\partial^{2}}{\partial t^{2}} (A_{n}(t) - \lambda - z)^{-1} g_{n}^{*}(s) ds dz \\ &- \frac{1}{2i\pi} \int_{\gamma} \int_{t+\varepsilon}^{T} \frac{e^{-\sqrt{-z}(s-t)} - e^{-\sqrt{-z}(T+t-s)}}{2\sqrt{-z}\left(1+e^{-T\sqrt{-z}}\right)} \frac{\partial^{2}}{\partial t^{2}} (A_{n}(t) - \lambda - z)^{-1} g_{n}^{*}(s) ds dz. \end{aligned}$$

Taking into account all properties (2.5) to (2.8), it is easy to see all these integrals are absolutely convergent and can be treated similarly using the Lebesgue dominated convergence theorem. Then, we obtain the strong convergence

$$w'_{n,\varepsilon}(t) \to w'_n(t)$$
 and $w''_{n,\varepsilon}(t) \to -(A_n(t) - \lambda - z)^{-1}w_n(t) + R_\lambda(g_n^*)(t) + g_n^*(t)$,
as $\varepsilon \to 0$. Hence

$$w_n''(t) = -(A_n(t) - \lambda - z)^{-1}w_n(t) + R_\lambda(g_n^*)(t) + g_n^*(t),$$

or

$$w_n''(t) + (A_n(t) - \lambda - z)^{-1}w_n(t) = g_n^*(t) + R_\lambda(g_n^*)(t).$$

The relationship between the vectorial functions g_n and g_n^\ast is given by the following

Proposition 3.5. Suppose that $g_n^* \in L^{\infty}([0,T]; E)$. Then, there exists $\lambda^* > 0$ such that for all $\lambda \geq \lambda^*$, the equation

$$g_n(t) = g_n^*(t) - R_\lambda(g_n^*)(t),$$

admits a unique solution

$$g_n^* \in L^\infty([0,T];E).$$

Proof. Recall that

$$R_{\lambda}(g_{n}^{*})(t) = +\frac{1}{i\pi} \int_{\gamma} \int_{0}^{T} \frac{\partial}{\partial t} K_{\sqrt{-z}}(t,s) \frac{\partial}{\partial t} (A_{n}(t) - \lambda - z)^{-1} g_{n}^{*}(s) ds dz$$
$$-\frac{1}{2i\pi} \int_{\gamma} \int_{0}^{T} K_{\sqrt{-z}}(t,s) \frac{\partial^{2}}{\partial t^{2}} (A_{n}(t) - \lambda - z)^{-1} g_{n}^{*}(s) ds dz.$$

Then, thanks to (2.6)-(2.8), we see that

$$\left\| \int_{\gamma} \int_{0}^{T} K_{\sqrt{-z}}(t,s) \frac{\partial^{2}}{\partial t^{2}} (A_{n}(t) - \lambda - z)^{-1} g_{n}^{*}(s) ds dz \right\|$$

$$\leq C \int_{\gamma} \frac{1}{|z|^{3/2} |z+\lambda|^{1/2}} |dz| \left\| g_{n}^{*} \right\|_{C(E)} \leq (C/\lambda^{\alpha}) \left\| g_{n}^{*} \right\|_{L^{\infty}([0,T;E))}$$

and

$$\left\| \frac{1}{2i\pi} \int_{\gamma} \int_{0}^{T} \frac{\partial}{\partial t} K_{\sqrt{-z}}(t,s) \frac{\partial}{\partial t} (A_{n}(t) - \lambda - z)^{-1} g_{n}^{*}(s) ds dz \right\|$$

$$\leq C \int_{\gamma} \frac{1}{\left|z\right|^{1/2} \left|z + \lambda\right|^{1/2 + 1/2}} \left| dz \right| \left\|g_{n}^{*}\right\|_{C(E)} \leq \left(C/\lambda^{1/2 + 1/2}\right) \left\|g_{n}^{*}\right\|_{L^{\infty}([0,T;E))}.$$

This implies

(3.6)
$$\|R_{\lambda}\|_{L(L^{\infty}([0,T];E))} = C/\lambda$$

Now, to establish the result it suffices to choose $\lambda^* > 0$ such that for $\lambda \ge \lambda^*$

$$||R_{\lambda}||_{L(L^{\infty}([0,T];E))} < 1.$$

The following proposition is concerned with the regularity of the operator R_{λ} needed in order to study the optimal regularity of the solution we are looking for:

Proposition 3.6. Let $g_n \in C^{2\theta}([0,T]; E)$. Then, there exists $\lambda^* > 0$ such that for all $\lambda \ge \lambda^*$,

$$R_{\lambda}(g_n^*)(t) \in C^{2\theta}([0,T];E).$$

Proof. It suffices to adapt the same techniques delivered in the proof of Proposition 4.3 in [6]. \Box

This justifies the following result:

Theorem 3.7. Let $g_n \in C^{2\theta}([0,T]; E)$. Then, there exist $\lambda^* > 0$ such that for all $\lambda \ge \lambda^*$, the function w_n given in the representation (3.5) is the unique strict solution of Problem (3.1) satisfying

$$w_n(.), (A_n(.) - \lambda)^{-1} w_n(.) \in C^{2\theta}([0,T]; E).$$

As a consequence, we have

Corollary 3.8. Let $g_n^* \in L^{\infty}([0,T]; E)$. Then there exists $\lambda^* > 0$ and C > 0 such that for all $\lambda \ge \lambda^*$, the strict solution w_n given by (3.5) fulfills the estimate

(3.7)
$$\max_{t} \left\| w_n(t) \right\|_E \le C$$

Sketch of the proof. The calculus are very cumbersome, we just give the main line of the demonstration. We have

$$w_n''(s) + (A_n(s) - \lambda)^{-1} w_n(s) = g_n(s),$$

then, we may deduce that

$$\begin{split} w_n(t) &= -\frac{1}{2i\pi} \int_{\gamma} \int_0^T K_{\sqrt{-z}}(t,s) \left(A_n(t) - \lambda - z\right)^{-1} g_n(s) ds dz \\ &= -\frac{1}{2i\pi} \int_{\gamma} \int_0^T K_{\sqrt{-z}}(t,s) \left(A_n(t) - \lambda - z\right)^{-1} w_n'(s) ds dz \\ &- \frac{1}{2i\pi} \int_{\gamma} \int_0^T K_{\sqrt{-z}}(t,s) \left(A_n(t) - \lambda - z\right)^{-1} (A_n(s) - \lambda)^{-1} w_n(s) ds dz \end{split}$$

After using integration by parts, we deduce that

$$-\frac{1}{2i\pi} \int_{\gamma} \int_{0}^{T} K_{\sqrt{-z}}(t,s) \left(A_{n}\left(t\right) - \lambda - z\right)^{-1} g_{n}(s) ds dz$$

$$= w_{n}(t) + \frac{1}{i\pi} \int_{\gamma} \int_{0}^{T} \frac{\partial}{\partial t} K_{\sqrt{-z}}(t,s) \frac{\partial}{\partial s} (A_{n}\left(s\right) - \lambda - z)^{-1} w_{n}(s) ds dz$$

$$-\frac{1}{2i\pi} \int_{\gamma} \int_{0}^{T} K_{\sqrt{-z}}(t,s) \frac{\partial^{2}}{\partial s^{2}} (A_{n}\left(s\right) - \lambda - z)^{-1} w_{n}(s) ds dz,$$

so that

$$-\frac{1}{2i\pi} \int_{\gamma} \int_{0}^{T} K_{\sqrt{-z}}(t,s) \left(A_{n}(t) - \lambda - z\right)^{-1} g_{n}(s) ds dz = (1 + R_{\lambda}(w_{n}))(t).$$

At this level, it is easy to see that the result is a direct consequence of the estimate (3.6).

4. Coming back to the singular cylindrical domain

Coming back to the problem (2.1) one obtains, for $t \ge t_n$,

$$v_n(t) = w_n(t - t_n)$$

Thanks to Proposition 3.7, a classical argument allows us to extract a convergent subsequence

$$v_{nj} := v\left(t_{nj}\right),$$

where

$$\lim_{n \to +\infty} t_{nj} = 0$$

Then, after a passage to the limit, we deduce the following important result

Theorem 4.1. Let $g \in C^{2\theta}([0,T]; E)$. Then, there exists $\lambda^* > 0$ such that for $\lambda \ge \lambda^*$, the problem

$$\begin{split} &v''(t) + A\left(t\right)v(t) - \lambda v(t) = f(t), \quad t \geq 0, \\ &v(0) + v(T) = 0, \\ &v'(0) + v'(T) = 0 \end{split}$$

admits a unique strict solution satisfying

$$v(.), (A(.) - \lambda)^{-1}v(.) \in C^{2\theta}([0, T]; E).$$

Applying all the preceeding abstract results and Lemma 1.1, our main results concerning the transformed problem (1.6) are formulated as follows:

Theorem 4.2. Let $f \in C^{2\theta}(\overline{Q})$, $0 < 2\theta < 1$. Then, there exists $\lambda^* > 0$ such that for $\lambda \ge \lambda^*$, Problem (1.6) has a unique strict solution $v \in C^2(\overline{Q})$. Moreover, vsatisfies the maximal regularity

$$\begin{cases} \partial_t^2 v \in C^{2\theta}(\overline{Q}),\\ and\\ \frac{1}{\varphi^2(t)} \Delta v + \frac{\varphi'(t)}{\varphi^2(t)} \left\{ \xi \partial_{\xi} + \eta \partial_{\eta} \right\} v - \lambda v \in C^{2\theta}(\overline{Q}). \end{cases}$$

Theorem 4.3. Let $h \in C^{2\theta}([0,T]; C(\Omega)), 0 < 2\theta < 1$. Then, there exists $\lambda^* > 0$ such that for $\lambda \ge \lambda^*$, Problem (1.6) has a unique strict solution $v \in C^2(\Pi)$. Moreover, v satisfies the maximal regularity

$$\begin{cases} \begin{array}{l} \partial_t^2 v \in C_w^{2\theta}([0,T];C(\Omega))),\\ and\\ \Delta v - \lambda v \in C_w^{2\theta}([0,T];C(\Omega))) \end{cases} \end{cases}$$

where

$$C_{w}^{2\theta}([0,T];C(\Omega)) = \left\{ h \in C^{2\theta}([0,T];C(\Omega)) : (\varphi(.))^{2\theta} h \in C^{2\theta}([0,T];C(\Omega)) \right\}.$$

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Lab. de l'Energie et des Systèmes Intelligents, Khemis Miliana University, 44225 Khemis Miliana, Algeria

 $E\text{-}mail\ address:\ \texttt{chaouchicukm@gmail.com}$

Faculty of Technical Sciences, University of Novi Sad, Trg D. Obradovića 6, 21125 Novi Sad, Serbia

E-mail address: marco.s@verat.net