ON THE BRAVERMAN-KAZHDAN PROPOSAL FOR LOCAL FACTORS: SPHERICAL CASE

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ABSTRACT. In this paper, we study the Braverman-Kazhdan proposal for the local spherical situation. In the p-adic case, we give a definition of the spherical component of conjectural space $\mathcal{S}_\rho(G,K)$ and the ρ-Fourier transform kernel Φ^K_ρ, and verify several conjectures in [BK00] in this situation. In the archimedean case, we study the asymptotic of the basic function $1_{\rho,s}$ and the ρ-Fourier transform kernel $\Phi^K_{\rho,s}$.

CONTENTS

1. INTRODUCTION

The theory of zeta integrals can be traced back to the work of B. Riemann, who first wrote the Riemann zeta function $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$ as the Mellin transform of a theta function. The idea was developed by

J. Tate in his thesis [Tat50] using the theory of zeta integrals. For convenience, in the introduction we restrict to the non-archimedean local fields case. For each character χ of $F^\times$, where F is a non-archimedean local field, one considers a family of distributions given by zeta integrals $Z(s,f,\varphi)$ with parameter $s\in\mathbb{C}$ on the space $\mathcal{S}(F^\times)=C_c^\infty(F)$. Tate shows that the distribution admits meromorphic continuation to $s\in\mathbb{C}$, possibly with a pole at $s=0$. The pole can be described by the L-factor $L(s,\chi)$, in the sense that the distribution $\frac{Z(s,\cdot,\chi)}{L(s,\chi)}$ admits holomorphic continuation to the whole complex plane.

R. Godement and H. Jacquet [GJ72] generalize the work of Tate and study, for any irreducible admissible representation π of $\mathrm{GL}(n)$ over a non-archimedean local field, the family of distributions given by zeta integrals $Z(s,f,\varphi_\pi)$ with parameter $s\in\mathbb{C}$ on the space $\mathcal{S}(\mathrm{GL}(n))=C_c^\infty(\mathrm{M}_n)$, where $\varphi_\pi\in\mathcal{C}(\pi)$ is a matrix coefficient of π. They show that $Z(s,f,\varphi_\pi)$ has meromorphic continuation to $s\in\mathbb{C}$ with a possible pole at $s=0$, and their poles are captured by the standard local L-factor $L(s,\pi)$ attached to π.

According to R. Langlands ([Lan70]), for any reductive algebraic group G defined over F, and for any finite dimensional representation ρ of the Langlands dual group ${}^LG$, one may define the local L-factor $L(s,\pi,\rho)$ associated to an irreducible admissible representation π of $G(F)$. It is natural to ask: Is it possible to find a family of distributions similar to the case of Godement-Jacquet that define the general local L-factor $L(s,\pi,\rho)$? Over the last fifty years, one found various types of global zeta integrals of Rankin-Selberg type, whose local zeta integrals may define local L-factors for a special list of G and ρ. Often, the zeta integrals of Rankin-Selberg type are not the same as that of Godement-Jacquet. In 2000, A. Braverman and D. Kazhdan in [BK00] propose a conjectural construction of families of distributions that may define the general L-factors $L(s,\pi,\rho)$, similar to that in [GJ72]. We will explain their proposal below.

1.1. **Notation and Convention.** Throughout the paper, we fix a local field F of characteristic 0, which can be either a p-adic field or an archimedean field. When F is a p-adic field, we let $\mathcal{O}_F$ be the ring of integers of F with fixed uniformizer ϖ, and we assume that the residue field of F has cardinality q.

We fix a valuation $|\cdot|$ on F. When F is a p-adic field, we normalize $|\cdot|$ so that $|\varpi|=q^{-1}$. When $F\cong\mathbb{R}$, it is the usual valuation on $\mathbb{R}$. When $F\cong\mathbb{C}$, $|z|=z\overline{z}$ for any $z\in\mathbb{C}$, where $\overline{z}$ is the complex conjugate of z.

Let G be a split connected reductive algebraic group over F. Following the notation of [Li17, Section 3.1], we assume that the group G fits into the following short exact sequence

$$1 \longrightarrow G_0 \longrightarrow G \xrightarrow{\sigma} \mathbb{G}_m \longrightarrow 1 \tag{1}$$

Here G_0 is a split connected semisimple algebraic group over F, and σ is a character of G playing the role of determinant as in $\mathrm{GL}(n)$ case.

Let ${}^LG$ be the Langlands dual group of G. We fix an irreducible algebraic representation

$$\rho : {}^LG \to \mathrm{GL}(V_\rho)$$

of dimension $n = \dim V_\rho$. There are similar results for reducible ρ, but for convenience we only work with the case when ρ is irreducible. Following [BK00, Definition 3.13] and [Li17, Section 3.1], we further assume that ρ is faithful, the restriction of ρ to the central torus $\mathbb{G}_m \to {}^LG$ is $z \to z\mathrm{Id}$, and $\ker(\rho)$ is connected.

We require that the representation ρ fits into the following commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{G}_m \arrow[r, "\widehat{\sigma}"] \arrow[d, "\mathrm{Id}"] & {}^LG \arrow[r] \arrow[d, "\rho"] & {}^LG_0 \arrow[r] \arrow[d, "\overline{\rho}"] & 1 \\
1 \arrow[r] & \mathbb{G}_m \arrow[r] & \mathrm{GL}(V, \mathbb{C}) \arrow[r] & \mathrm{PGL}(V, \mathbb{C}) \arrow[r] & 1
\end{tikzcd}$$

The top row is obtained by dualizing the short exact sequence (1), and $\overline{\rho}$ is the projective representation obtained from ρ. By [Li17, Section 3.1], we may assume that the rows are exact and the second square is cartesian.

We fix a Borel pair (B, T) for our group G. Let $X_*(T)$ and $X^*(T)$ be the cocharacter and character group of T respectively. Let $W = W(G, T)$ be the Weyl group. Let ρ_B be the half sum of positive roots. The corresponding modular character is denoted by δ_B. Following the suggestion of [BNS16] and [BNS17], we let $l = 2 < \rho_B, \lambda >$, where λ is the highest weight of the representation ρ.

When F is a p-adic field, we choose a hyperspecial vertex in the Bruhat-Tits building of G which lies in the apartment determined by T. The corresponding hyperspecial subgroup $G(F)$ is denoted by K as usual. When F is an archimedean field, by Cartan-Iwasawa-Malcev theorem [Bor98, Theorem 1.2], we fix a maximal compact subgroup K of G.

When F is a p-adic field, we fix the Cartan decomposition $G(F) = \coprod_{\lambda \in X_*(T)_+} K\lambda(\varpi)K$, where $X_*(T)_+$ is the positive Weyl chamber. When

F is an archimedean field, we also fix the Cartan decomposition $G = K\exp(\mathfrak{a})K$, where $\mathfrak{a}$ is a maximal abelian subaglebra of the Lie algebra $\mathfrak{g}$ of G. Let $T(F)\cap K = T_K$.

We fix a nontrivial additive character ψ of F with conductor $\mathcal{O}_F$. We also fix a Haar measure on F such that the Haar measure is self-dual w.r.t. the additive character ψ.

1.2. **Braverman-Kazhdan Proposal.** In [BK00], the local aspect of the Braverman-Kazhdan proposal is to construct a family of zeta distributions associated to each finite dimensional representation ρ of the Langlands dual group ${}^LG$ that define the general L-factor $L(s,\pi,\rho)$ for every irreducible admissible representation π of $G(F)$ via a generalization of the work of Godement and Jacquet [GJ72]. Roughly speaking, they proposed the existence of a function space $\mathcal{S}_\rho(G)\subset C^\infty(G)$, which should be the space of test functions for the zeta distributions, such that the following conjecture holds

Conjecture 1.2.1. [BK00, Conjecture 5.11] *With the notation above, the following hold.*

(1) *For every $f\in\mathcal{S}_\rho(G)$ and every $\varphi\in\mathcal{C}(\pi)$ the integral*
$$Z(s,f,\varphi)=\int_G f(g)\varphi(g)|\sigma(g)|^{s+\frac{l}{2}}dg$$
is absolutely convergent for $\mathrm{Re}(s)\gg 0$.
(2) *$Z(s,f,\varphi)$ has a meromorphic continuation to $\mathbb{C}$ and defines a rational function of q^s.*
(3) *$I_\pi=\{Z(s,f,\varphi)|\quad f\in\mathcal{S}_\rho(G),\varphi\in\mathcal{C}(\pi)\}$ is a finitely generated non-zero fractional ideal of the ring $\mathbb{C}[q^s,q^{-s}]$, where $\mathcal{C}(\pi)$ is the space of matrix coefficients of π.*

Remark 1.2.2. *In* [BK00]*, Braverman and Kazhdan defined the number l to be the semisimple rank of G. Following the work of* [BNS16] *and* [BNS17]*, it is suggested that the correct normalization should be $l = 2 < \rho_B,\lambda >$, where λ is the highest weight of ρ. In the case where ρ is the standard representation of* $\mathrm{GL}(n)$*, the number $l = n-1$. The definition coincides with the work of Godement and Jacquet* [GJ72].

Assuming that the Conjecture 1.2.1 holds, one may define the local L-factor $L(s,\pi,\rho)$ to be the unique generator of the fractional ideal I_π of the form $P(q^{-s})^{-1}$, where P is a polynomial such that $P(0)=1$. Moreover, they also proposed the existence of a Fourier-type transform $\mathcal{F}_\rho$ [BK00, Section 5.3] that is defined by
$$\mathcal{F}_\rho(f)=|\sigma|^{-l-1}(\Phi_{\psi,\rho}*f^\vee),\quad f\in C_c^\infty(G),$$

and satisfies the following

Conjecture 1.2.3. [BK00, Conjecture 5.9] *The ρ-Fourier transform $\mathcal{F}_\rho$ extends to a unitary operator on $L^2(G, |\sigma|^{l+1}dg)$ and the space $\mathcal{S}_\rho(G)$ is $\mathcal{F}_\rho$-invariant. Here the character σ is defined in (1).*

Here $\Phi_{\psi,\rho}$ is a G-stable σ-compact distribution in the sense of [BK00, Definition 3.8]. After unramified twist, the action of $\Phi_{\psi,\rho,s}$ on the space of $\pi \in$Irr(G) is given by a rational function in s, which is the associated local gamma factor $\gamma(-s-\frac{l}{2}, \pi^\vee, \rho, \psi)$.

Remark 1.2.4. *Here we want to make a remark on the γ-factor. Assuming the local Langlands functoriality for ρ, we can set*

$$\gamma(s, \pi, \rho, \psi) = \gamma(s, \rho(\pi), \psi),$$

where $\rho(\pi)$ is the functorial lifting of π along ρ. The γ-factor is a rational function in s. Hence, for special values of s, for instance $s = -\frac{l}{2}$, there might exist $\pi \in Irr(G)$ such that the constant $\gamma(-\frac{l}{2}, \rho(\pi), \psi)$ does not exist for π. In this case, we can take an unramified twist of $\Phi_{\psi,\rho}$, which we denote as $\Phi_{\psi,\rho,s}$. Then the action of $\Phi_{\psi,\rho,s}$ on the space of π is given by the local gamma factor $\gamma(-s-\frac{l}{2}, \pi^\vee, \rho, \psi)$.

Remark 1.2.5. *In* [BK00, Section 1.2]*, Braverman and Kazhdan define the distribution $\Phi_{\psi,\rho,s}$ with the property that its action on the space of $\pi \in$Irr(G) is given by the local gamma factor $\gamma(s, \pi, \rho, \psi)$ with parameter $s \in \mathbb{C}$. For normalization purpose, we define our G-stable distribution $\Phi_{\psi,\rho,s}$ with action on π via the scalar $\gamma(-s-\frac{l}{2}, \pi^\vee, \rho, \psi)$. In Lemma 2.4.4 below, we show how to derive the relation between γ-factor and $\Phi_{\psi,\rho,s}$ formally from the conjectural functional equation*

$$Z(1-s, \mathcal{F}_\rho(f), \varphi^\vee) = \gamma(s, \pi, \rho, \psi) Z(s, f, \varphi), \quad f \in \mathcal{S}_\rho(G), \varphi \in \mathcal{C}(\pi)$$

In [BK00, Section 7], Braverman and Kazhdan give a conjectural algebro-geometric construction of the distribution $\Phi_{\psi,\rho,s}$. It is not difficult to define the distribution $\Phi_{\psi,\rho\circ i,s}$ on T associated to the representation $\rho \circ i$ of ${}^LT$

$${}^LT \xrightarrow{i} {}^LG \xrightarrow{\rho} \mathrm{GL}(V_\rho) .$$

Since the distribution $\Phi_{\psi,\rho,s}$ is conjectured to be G-stable, using the adjoint quotient map $G^{reg} \to T/W$, one can naturally extend it to a distribution on G once the W-equivariance of the distribution $\Phi_{\psi,\rho\circ i,s}$ is established as conjectured in [BK00, Conjecture 7.11]. Then Braverman and Kazhdan conjectured that the construction gives us the distribution $\Phi_{\psi,\rho,s}$ that we want. There is a parallel conjecture in finite field case, and some recent works ([BK03], [Che16], and [CN17]) confirm the construction.

For the construction of function space $\mathcal{S}_\rho(G)$, Braverman and Kazhdan [BK00, Section 5.5] expect to use the Vinberg's monoids [Vin95]. For each ρ, one can construct a reductive monoid $\overline{G}_\rho$ containing G as an open dense subvariety, whose unit is just the group G, and there is a $G\times G$ equivariant embedding of G into $\overline{G}_\rho$. Here $\overline{G}_\rho$ is expected to play the role of M_n as in [GJ72]. But for almost all ρ, $\overline{G}_\rho$ is a singular variety. Hence one cannot simply use the locally constant compactly supported functions on $\overline{G}_\rho$ as our conjectural function space $\mathcal{S}_\rho(G)$. Recently there are some works in the function field case ([BNS16] and [BNS17]) explaining the relation between the geometry of $\overline{G}_\rho$ and the *basic function* in $\mathcal{S}_\rho(G)$.

Assuming the local Langlands functoriality for ρ, L. Lafforgue [Laf14] proposes the definition of $\mathcal{S}_\rho$ and $\mathcal{F}_\rho$ using Plancherel formula. However, the analytical properties of $\mathcal{S}_\rho$ and $\mathcal{F}_\rho$ may not be easily figured out from such an abstract definition.

By the work of Godement and Jacquet [GJ72], when ρ is the standard representation of $\mathrm{GL}(n)$ the above conjectures hold. We can take $\mathcal{S}_\rho(G)$ to be the restriction to $\mathrm{GL}(n)$ of functions in $C_c^\infty(\mathrm{M}_n)$, and $\mathrm{GL}(n)$ embeds into M_n naturally. Here M_n is the monoid of $n\times n$ matrices which fits into the construction of Vinberg [Vin95]. $\mathcal{F}_\rho$ in this case is the classical Fourier transform on M_n fixing $C_c^\infty(\mathrm{M}_n)$ defined by

$$\begin{aligned}\mathcal{F}(f)(g) &= |\det g|^{-n}(\Phi_{\psi,\mathrm{std}} * f^\vee)(g)\\ &= \int_{\mathrm{M}_n(F)} f(y)\psi(\mathrm{tr}(yg))dy, \quad f\in C_c^\infty(\mathrm{M}_n)\end{aligned}$$

where $\Phi_{\psi,\mathrm{std}}(g) = \psi(\mathrm{tr}(g))|\det(g)|^n$.

Basic Function. Although the structure of the space $\mathcal{S}_\rho(G)$ is still unclear, there is a distinguished element in the space $\mathcal{S}_\rho(G)$, called basic function, which we will introduce below.

In [GJ72], the authors find that the characteristic function $1_{\mathrm{M}_n(\mathcal{O}_F)}$ of $\mathrm{M}_n(\mathcal{O}_F)$ satisfies the following two properties:

(1) For any spherical representation π of $G=\mathrm{GL}(n)$ with Satake parameter $c\in\widehat{T}/W$, let φ_π be the associated zonal spherical function, then

$$\begin{aligned}Z(s,1_{\mathrm{M}_n(\mathcal{O}_F)},\varphi_\pi) &= \int_G 1_{\mathrm{M}_n(\mathcal{O}_F)}(g)\varphi_\pi(g)|\det g|^{s+\frac{n-1}{2}}dg\\ &= \det(1-(c)q^{-s}|V)^{-1} = L(s,\pi,\mathrm{std}).\end{aligned}$$

(2) $\mathcal{F}_{\mathrm{std}}(1_{\mathrm{M}_n(\mathcal{O}_F)}) = 1_{\mathrm{M}_n(\mathcal{O}_F)}$.

Let $\mathcal{S} : \mathcal{H}(G,K) \to \mathbb{C}[\widehat{T}/W]$ be the Satake transform, which is an isomorphism of algebras. Using the Cartan decomposition, the zeta integral $Z(s, 1_{\mathrm{M}_n(\mathcal{O}_F)}, \varphi_\pi)$ is equal to the Satake transform of the function $1_{\mathrm{M}_n(\mathcal{O}_F)}|\det|^{s+\frac{n-1}{2}}$ evaluated at the Satake parameter $c \in \widehat{T}/W$ of π. For general ρ, one is naturally led to the following definition of the *basic function* $1_{\rho,s}$ with parameter $s \in \mathbb{C}$.

Definition 1.2.6. [Li17, Defintion 3.2.1] *The basic function $1_{\rho,s} = 1_\rho|\sigma|^s$ with parameter $s \in \mathbb{C}$ is the smooth bi-K-invariant function on G such that*

$$\mathcal{S}(1_{\rho,s})(c) = L(s,\pi,\rho)$$

for any spherical representation π of G with Satake parameter c, where σ is the character defined in (1).

Following the work of Godement-Jacquet [GJ72], one hopes that the function $1_{\rho,-\frac{l}{2}} = 1_\rho|\sigma|^{-\frac{l}{2}}$ lies in the function space $\mathcal{S}_\rho(G)$ and has the following property

Conjecture 1.2.7. $\mathcal{F}_\rho(1_{\rho,-\frac{l}{2}}) = 1_{\rho,-\frac{l}{2}}$.

It is shown in [BK00, Lemma 5.8] that Conjecture 1.2.7 holds assuming the compatibility of parabolic descent and ρ-Fourier transform [BK00, Conjecture 3.15].

One of the reasons that we care about the function $1_{\rho,s}$ is its role in Langlands' Beyond endoscopy program [Lan04]. When $\mathrm{Re}(s)$ is sufficiently large, we expect to plug it into the Arthur-Selberg trace formula [FLM11]. On the spectral side, we would get a partial automorphic L-function. On the geometric side, the weighted orbital integrals of the basic function can tell us information about the automorphic L-function. For details the reader is recommended to read [Ngô16] and the last section of [Get15].

1.3. **Our Results.**

We obtain results uniformly for both p-adic and archimedean local field. For convenience, we treat them separately in the following.

p-Adic Case: We give a construction of the spherical component of the function space $\mathcal{S}_\rho(G)$ and the distribution kernel of ρ-Fourier transform $\Phi_{\psi,\rho}$, which we denote by $\mathcal{S}_\rho(G,K)$ and $\Phi^K_{\psi,\rho}$. Here we need to use the extension of *Satake isomorphism* $\mathcal{S} : \mathcal{H}(G,K) \to \mathbb{C}[\widehat{T}/W]$ to *almost compactly supported functions* $\mathcal{H}_{ac}(G,K)$ in the sense of [Li17, Proposition 2.3.2], since the L-functions and γ-factors are rational functions rather than polynomial functions on $\widehat{T}/W$. The functions in $\mathcal{S}_\rho(G,K)$ are not always compactly supported, but always *almost compactly supported*.

Definition 1.3.1. *Define the function space $\mathcal{S}_\rho(G,K)$ to be*

$$\mathcal{S}_\rho(G,K) = 1_{\rho,-\frac{l}{2}} * \mathcal{H}(G,K).$$

Define the distribution kernel of ρ-Fourier transform $\Phi^K_{\psi,\rho,s}$ to be

$$\Phi^K_{\psi,\rho,s} = 1_{\rho,1+s+\frac{l}{2}} * \mathcal{S}^{-1}(\frac{1}{L(-s-\frac{l}{2},\pi,\rho^\vee)}).$$

In Proposition 2.2.2, we show that when ρ is the standard representation of $G = \mathrm{GL}(n)$, we actually have

$$\mathcal{S}_{\mathrm{std}}(G,K) = 1_{\mathrm{std},-\frac{n-1}{2}} * \mathcal{H}(G,K) = 1_{\mathrm{M}_n(\mathcal{O}_F)} * \mathcal{H}(G,K).$$

Here $\mathcal{S}_{\mathrm{std}}(G,K)$ is the restriction of functions in $C_c^\infty(\mathrm{M}_n,K)$, the bi-$K$-invariant functions in $C_c^\infty(\mathrm{M}_n)$, to $\mathrm{GL}(n)$. The structure for the standard case will be our main ingredient for introducing Definition 1.3.1.

Based on Definition 1.3.1 we can verify that the Conjecture 1.2.1 and Conjecture 1.2.3 hold under the assumption that the functions and representations are spherical. We can also verify Conjecture 1.2.7 without referring to [BK00, Conjecture 3.15]. More precisely, the following theorems holds

Theorem 1.3.2. *Let π be a spherical representation of G. For every $f \in \mathcal{S}_\rho(G,K)$, $\varphi \in \mathcal{C}(\pi)$ the integral*

$$Z(s,f,\varphi) = \int_G f(g)\varphi(g)|\sigma(g)|^{s+\frac{l}{2}}dg$$

is a rational function in q^s, and the fractional ideal $I_\pi = \{Z(s,f,\varphi) | f \in \mathcal{S}_\rho(G,K), \varphi \in \mathcal{C}(\pi)\}$ is equal to $L(s,\pi,\rho)\mathbb{C}[q^s,q^{-s}]$.

The idea for the proof of Theorem 1.3.2 is as follows. We notice that the function $f \in \mathcal{S}_\rho(G,K)$ is bi-K-invariant. Following the proof of Proposition 2.2.2, we can actually assume that φ is bi-K-invariant, which means that φ is a scalar multiple of the zonal spherical function associated to π. Then, up to multiplying by a constant, the zeta integral $Z(s,f,\varphi)$ is equal to $\mathcal{S}(f_{s+\frac{l}{2}})(c)$, where $c \in \widehat{T}/W$ is the Satake parameter associated to π. Now Theorem 1.3.2 follows from the definition of $\mathcal{S}_\rho(G,K)$ and Remark 2.2.3.

Theorem 1.3.3. *For any $f \in \mathcal{S}_\rho(G,K)$, define the ρ-Fourier transform $\mathcal{F}_\rho$ as in* [BK00] *by the formula*

$$\mathcal{F}_\rho(f) = |\sigma|^{-l-1}(\Phi_{\psi,\rho} * f^\vee).$$

Then $\mathcal{F}_\rho$ extends to a unitary operator on $L^2(G,K,|\sigma|^{l+1}dg)$ and the space $\mathcal{S}_\rho(G,K)$ is $\mathcal{F}_\rho$-invariant.

The idea for the proof of Theorem 1.3.3 is as follows. To show that $\mathcal{F}_\rho$ extends to a unitary operator on $L^2(G,K,|\sigma|^{l+1}dg)$, equivalently we need to show the following equality

$$< \mathcal{F}_\rho(f), \mathcal{F}_\rho(h) >_{L^2(G,K,|\sigma|^{l+1}dg)} = < f, h >_{L^2(G,K,|\sigma|^{l+1}dg)}$$

for any $f, h \in \mathcal{H}(G,K)$, since the smooth compactly supported functions are dense in $L^2(G,K,|\sigma|^{l+1}dg)$.

We first rewrite the integration as follows

$$< \mathcal{F}_\rho(f), \mathcal{F}_\rho(h) >_{L^2(G,K,|\sigma|^{l+1}dg)} = \mathcal{F}_{\rho,l+1}(f) * \overline{\mathcal{F}_\rho(h)}^\vee(e)$$
$$< f, h >_{L^2(G,K,|\sigma|^{l+1}dg)} = \overline{h}_{l+1} * f^\vee(e).$$

Then as in the proof of Proposition 2.4.8 we can show that after the Satake transform, the functions $\mathcal{F}_{\rho,l+1}(f) * \overline{\mathcal{F}_\rho(h)}^\vee$ and $\overline{h}_{l+1} * f^\vee$ are equal to each other as a rational function on $\widehat{T}/W$. Hence we get the first part of Theorem 1.3.3. To show that the space $\mathcal{S}_\rho(G,K)$ is $\mathcal{F}_\rho$-invariant, we show that $\mathcal{F}_\rho(\mathcal{S}_\rho(G,K))$ and $\mathcal{S}_\rho(G,K)$ have the same image under Satake transform.

Theorem 1.3.4. $\mathcal{F}_\rho(1_{\rho,-\frac{l}{2}}) = 1_{\rho,-\frac{l}{2}}$.

The idea for the proof of Theorem 1.3.4 follows from the direct computation of the Satake transform of $\mathcal{F}_\rho(1_{\rho,-\frac{l}{2}})$ and $1_{\rho,-\frac{l}{2}}$. We show that they coincide with each other after Satake transform as a rational function on $\widehat{T}/W$, from which we deduce that they are equal to each other.

The detailed proof of the theorems are given in Section 2.4.

Archimedean Case: We give a construction of $\Phi^K_{\psi,\rho}$ using the *spherical Plancherel transform.* More precisely,

Definition 1.3.5. *We define* $\Phi^K_{\psi,\rho,s} = 1_{\rho,1+s+\frac{l}{2}} * \mathcal{H}^{-1}(\frac{1}{L(-s-\frac{l}{2},\pi,\rho^\vee)})$. *Here $\mathcal{H}$ is the* spherical Plancherel transform.

Parallel to the p-adic case, we can verify that Conjecture 1.2.7 holds through showing that $\mathcal{F}_\rho(1_{\rho,-\frac{l}{2}})$ and $1_{\rho,-\frac{l}{2}}$ have the same image under spherical Plancherel transform $\mathcal{H}$.

We also study asymptotic properties of $1_{\rho,s}$ and $\Phi^K_{\psi,\rho,s}$. We let $S^p(K\backslash G/K)$ be the L^p-Harish-Chandra Schwartz space, where $0 < p \leq 2$ is any real number. Then we can prove the following theorem.

Theorem 1.3.6. (1) *If $F \cong \mathbb{R}$, and* $\mathrm{Re}(s)$ *satisfies the following inequality*

$$\mathrm{Re}(s) > \max\{\varpi_k(\mu)| \quad 1 \leq k \leq n, \mu \in C^{\varepsilon\rho_B}\},$$

or

(2) *If $F \cong \mathbb{C}$, and* $\mathrm{Re}(s)$ *satisfies the following inequality*

$$\mathrm{Re}(s) > \max\{\frac{\varpi_k(\mu)}{2}| \quad 1 \leq k \leq n, \mu \in C^{\varepsilon\rho_B}\},$$

then the function $1_{\rho,s}$ lies in $S^p(K\backslash G/K)$.

Here $\{\varpi_k\}_{k=1}^n$ are the weights of the reprentation ρ, $\varepsilon = \frac{2}{p} - 1$, and $C^{\varepsilon\rho_B}$ is the convex hull in $\mathfrak{a}^*$ generated by elements $W \cdot \varepsilon\rho_B$.

Theorem 1.3.7. (1) *If $F \cong \mathbb{R}$, and* $\mathrm{Re}(s)$ *satisfies the following inequality*

$$\mathrm{Re}(s) > -1 - \frac{l}{2} + \max\{\varpi_k(\mu)| \quad 1 \leq k \leq n, \mu \in C^{\varepsilon\rho_B}\},$$

or

(2) *If $F \cong \mathbb{C}$, and* $\mathrm{Re}(s)$ *satisfies the following inequality*

$$\mathrm{Re}(s) > -\frac{1}{2} - \frac{l}{4} + \max\{\frac{\varpi_k(\mu)}{2}| \quad 1 \leq k \leq n, \mu \in C^{\varepsilon\rho_B}\},$$

then the function $\Phi^K_{\psi,\rho,s}$ lies in $S^p(K\backslash G/K)$.

The idea for proving the asymptotic theorems is based on several asymptotic estimations for classical Γ-function and its derivatives, which are recalled and proved in the beginning of Section 3.4.

The details are presented in Section 3.4 and Section 3.5.

Jayce Getz [Get15] also has similar descriptions for $\mathcal{F}_\rho(f)$, where f lies in $C_c^\infty(G, K)$. His description of the Fourier transform uses the relation between $\mathcal{F}_\rho$ and the standard one on $\mathrm{GL}(n)$ also via *spherical Plancherel transform*, in which the ρ-Fourier transform is not written as an explicit kernel function. Using the functional equation, one can observe that his definition coincides with our definition of $\mathcal{F}_\rho$. On the other hand, using the explicit estimation for the kernel function $\Phi^K_{\psi,\rho,s}$, we find that our domain for the Fourier transform $\mathcal{F}_\rho$ is bigger than $C_c^\infty(G, K)$. For instance, we can take Fourier transform for the basic function $1_{\rho,s}$.

Organization of Paper. In Section 2.1, we have a quick review of the Satake isomorphism. In Section 2.2, we give a description of the structure of $\mathcal{S}_{\mathrm{std}}(G,K)$, which are the restriction of functions in $C_c^\infty(\mathrm{M}_n,K)$ to $G=\mathrm{GL}(n)$. In Section 2.3, we briefly review the theory of basic functions. In Section 2.4 we prove the unramified part of the conjectures mentioned in the introduction.

In Section 3.1, we review the theory of spherical plancherel transform. In Section 3.2 and 3.3, we review the Langlands classification and Langlands correspondence of spherical representations for $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$. From the Langlands classification and Langlands correspondence, we obtain the explicit formula of local L-factors. In Section 3.4 and 3.5 we prove asymptotic properties of $1_{\rho,s}$ and $\Phi^K_{\psi,\rho,s}$, from which we can deduce the theorems mentioned in the introduction.

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2. p-Adic Case

The theory of spherical functions and spherical representations for p-adic groups are developed by I. Satake in [Sat63]. In particular, Satake proves that under the Satake transform $\mathcal{S}$, the spherical Hecke algebra $\mathcal{H}(G,K)$ is isomorphic to $\mathbb{C}[\widehat{T}/W]$, which nowadays is called the *Satake isomorphism*.

On the other hand, $\mathcal{H}(G,K)$ is contained in the conjectural function space $\mathcal{S}_\rho(G,K)$ as a proper subspace. In order to obtain a similar description for $\mathcal{S}_\rho(G,K)$, we need to extend the Satake isomorphism to $\mathcal{S}_\rho(G,K)$. This is achieved in [Li17, Proposition 2.3.2]. For the basic function $1_{\rho,-\frac{l}{2}}$, although it is not compactly supported on G, it is compactly supported on the sets $\{g\in G| \quad |\sigma(g)|=q^{-n}\}_{n\in\mathbb{Z}}$. For different n, the sets are disjoint. This means that the function $1_{\rho,-\frac{l}{2}}$ is *almost compactly supported* as defined in [Li17, Definition 2.3.1]. In particular we can apply the Satake isomorphism to $1_{\rho,-\frac{l}{2}}$.

Using the Satake isomorphism, we will give a definition of $\mathcal{S}_\rho(G,K)$ and $\Phi^K_{\psi,\rho,s}$, and we can verify several conjectures in this case as mentioned in the introduction.

2.1. **Satake Isomorphism.** In this section, we review the *Satake isomorphism*. The main references are [Car79], [Gro98] and [Sat63].

First we give the definition of Satake transform.

Definition 2.1.1 (Satake Transform)**.** *For $f \in \mathcal{H}(G,K)$, the function $\mathcal{S}(f)$ is defined to be*

$$\mathcal{S}(f)(t) = \delta_B^{\frac{1}{2}}(t)\int_N f(tn)dn.$$

In [Sat63], Satake proves the fact that $\mathcal{S}$ is an algebra isomorphism from $\mathcal{H}(G,K)$ to $\mathcal{H}(T,T_K)^W$, where both algebras are equipped with convolution structure.

Using the canonical W-equivariant isomorphisms

$$T/T_K \cong X_*(T) \cong X^*(\widehat{T}),$$

we have

$$\mathcal{H}(T,T_K)^W \cong \mathbb{C}[X_*(T)]^W \cong \mathbb{C}[X^*(\widehat{T})]^W.$$

Since $\mathbb{C}[X^*(\widehat{T})]$ consists of $\mathbb{C}$-linear combinations of algebraic characters of $\widehat{T}$, it can naturally be identified with algebraic functions on $\widehat{T}$. Therefore $\mathbb{C}[X^*(\widehat{T})]^W \cong \mathbb{C}[\widehat{T}/W]$. Sometimes we abuse the notation of Satake transform $\mathcal{S}$ with the image identified with $\mathbb{C}[\widehat{T}/W]$.

2.2. **Structure of $\mathcal{S}_{\mathrm{std}}(G,K)$.** In this section, we review the theory of the zeta integrals for the standard L-function of $\mathrm{GL}(n)$ over a non-archimedean local field following the approach of Godement-Jacquet. The main references are [GJ72] and [Jac79]. In the end we give a description of the structure of $\mathcal{S}_{\mathrm{std}}(G,K)$.

In [GJ72], Godement and Jacquet established the theory of standard L-function for multiplicative group of central simple algebras following the approach of [Tat50]. For our purpose, we only focus on $G = \mathrm{GL}(n)$, though the story for multiplicative group of central simple algebras is almost the same.

Let (π, V) be an admissible representation of G with smooth admissible contragredient dual $(\pi^\vee, \widetilde{V})$. Let

$$\begin{aligned} <,>: \widetilde{V} \times V &\to \mathbb{C} \\ (\widetilde{v}, v) &\to <\widetilde{v}, v> \end{aligned}$$

be the canonical linear pairing between $\widetilde{V}$ and V.

Let $\mathcal{C}(\pi)$ be the $\mathbb{C}$-linear span of the following functions

$$\pi_{\widetilde{v},v} : g \to <\widetilde{v}, \pi(g)v>, v \in V, \widetilde{v} \in \widetilde{V}.$$

Elements in $\mathcal{C}(\pi)$ are called the matrix coefficients of π.

By the admissibility of π, the smooth contragredient of $\pi^\vee$ is canonically isomorphic to π. It follows that for any $\varphi \in \mathcal{C}(\pi)$, the function

$$\varphi^\vee(g) = \varphi(g^{-1})$$

is a matrix coefficient of $\pi^\vee$.

Let $\mathrm{M}_n(F)$ be the space of $n \times n$ matrices over F. Let $C_c^\infty(\mathrm{M}_n)$ be the space of smooth compactly supported functions on $\mathrm{M}_n(F)$.

For $\varphi \in \mathcal{C}(\pi)$, $f \in C_c^\infty(\mathrm{M}_n)$, $s \in \mathbb{C}$, one set

$$Z(s, f, \varphi) = \int_G f(g)\varphi(g)|\det g|^{s+\frac{n-1}{2}} d^\times g. \tag{2}$$

In [GJ72], the following proposition was proved.

Proposition 2.2.1. [Jac79, Proposition (1.2)] *Suppose that π is an irreducible and admissible representation of G, then*

(1) *There exists $s_0 \in \mathbb{C}$ such that the integral (2) converges absolutely for $\mathrm{Re}(s) > \mathrm{Re}(s_0)$.*
(2) *The integral (2) is given by a rational function in q^{-s}, where q is the cardinality of the residue field of F. Moreover, the family of rational functions in q^{-s}*
$$I(\pi) = \{Z(s, f, \varphi)| \quad f \in C_c^\infty(\mathrm{M}_n), \varphi \in \mathcal{C}(\pi)\}$$
admits a common denominator which does not depend on f or φ.
(3) *Let $\psi \neq 1$ be an additive character of F. There exists a rational function $\gamma(s, \pi, \psi)$ such that for any $\varphi \in \mathcal{C}(\pi)$ and $f \in C_c^\infty(\mathrm{M}_n)$, we have the following functional equation*
$$Z(1-s, \mathcal{F}(f), \varphi^\vee) = \gamma(s, \pi, \psi) Z(s, f, \varphi), \tag{3}$$
where $\mathcal{F}(f)$ is the Fourier transform of f w.r.t. ψ
$$\mathcal{F}(f)(x) = \int_{\mathrm{M}_n} f(y)\psi(\mathrm{tr}(yx))dy.$$
Here we choose dy to be the self-dual Haar measure on $\mathrm{M}_n(F)$, in the sense that
$$\mathcal{F}(\mathcal{F}(f))(x) = f(-x).$$

Now we prove the claim in the introduction, that the space $\mathcal{S}_{\mathrm{std}}(G, K)$, which consists of the restriction to $G = \mathrm{GL}(n)$ of bi-K-invariant functions in the space $C_c^\infty(\mathrm{M}_n(F))$, has the following simple expression

$$1_{\mathrm{M}_n(\mathcal{O})} * \mathcal{H}(G, K) = 1_{\mathrm{std}, -\frac{n-1}{2}} * \mathcal{H}(G, K).$$

Proposition 2.2.2. $\mathcal{S}_{\mathrm{std}}(G, K) = 1_{\mathrm{M}_n(\mathcal{O}_F)} * \mathcal{H}(G, K)$.

Proof. Let $\pi = \pi_c$ be a spherical representation of G with Satake parameter $c \in \widehat{T}/W$. By Proposition 2.2.1

$$\{\frac{Z(s,f,\varphi_\pi)}{L(s,\pi)} | \quad f \in C_c^\infty(\mathrm{M}_n(F)), \varphi_\pi \in \mathcal{C}(\pi)\} = \mathbb{C}[q^{-s}, q^s].$$

Now for any matrix coefficient $\varphi_\pi(g) =< \widetilde{v}, \pi(g)v >$ in $\mathcal{C}(\pi)$, there exists finitely many constant numbers c_i in $\mathbb{C}$, h_0^i and g_0^i ($1 \leq i \leq n$) in G, such that $\varphi_\pi(g) = \sum_{i=1}^n c_i \Gamma_\chi(h_0^i g g_0^i)$, where Γ_χ is the zonal spherical function associated to π. Therefore up to translation and scaling, we can assume that our φ_π is just the zonal spherical function Γ_χ. Moreover

$$\begin{aligned} Z(s,f,\Gamma_\chi) &= \int_G f(g)\Gamma_\chi(g)|\det g|^{s+\frac{n-1}{2}} dg \\ &= f|\det|^{s+\frac{n-1}{2}} * \Gamma_\chi^\vee(e), \end{aligned}$$

and by the fact that G is unimodular

$$\begin{aligned} Z(s,f,\Gamma_\chi) &= \int_G f(g^{-1})\Gamma_\chi(g^{-1})|\det g^{-1}|^{s+\frac{n-1}{2}} dg \\ &= \Gamma_\chi^\vee * f|\det|^{s+\frac{n-1}{2}}(e). \end{aligned}$$

Since $\Gamma_\chi^\vee$ is bi-K-invariant, we can assume that f is bi-K-invariant as well. It follows that Proposition 2.2.1 in the spherical case can be restated as

$$\{Z(s,f,\Gamma_\chi)| \quad f \in \mathcal{S}_{\mathrm{std}}(G,K)\} = L(s,\pi)\mathbb{C}[q^{-s}, q^s].$$

Now we notice that $Z(s,f,\Gamma_\chi) = \mathcal{S}(f|\det|^{s+\frac{n-1}{2}})(c)$. If we let $\Gamma_{\chi,s}$ be the zonal spherical function associated to $\pi_s = \pi|\det|^s$, then we have $Z(s,f,\Gamma_\chi) = Z(0,f,\Gamma_{\chi,s})$, and

$$Z(0,f,\Gamma_{\chi,s}) = \mathcal{S}(f|\det|^{\frac{n-1}{2}})(c \cdot q^{-s}) = \mathcal{S}(f)(c \cdot q^{-s-\frac{n-1}{2}}),$$

where $c \cdot q^{-s}$ is the Satake parameter of $\pi_{c,s} = \pi_c|\det|^s$.

Therefore

$$Z(s,\mathcal{H}(G,K),\Gamma_\chi) = \mathcal{S}(\mathcal{H}(G,K))(c \cdot q^{-s-\frac{n-1}{2}}) = \mathbb{C}[\widehat{T}/W](c \cdot q^{-s-\frac{n-1}{2}}).$$

The space $\mathbb{C}[\widehat{T}/W](c \cdot q^{-s-\frac{n-1}{2}})$ is contained in $\mathbb{C}[q^s, q^{-s}]$ naturally.

On the other hand, the space

$$\mathbb{C}[\widehat{T}/W](c \cdot q^{-s-\frac{n-1}{2}}) = \{Z(s,f,\Gamma_\chi)| \quad f \in \mathcal{H}(G,K)\}$$

can be identified with

$$\{Z(s,f,\varphi_\pi)| \quad f \in C_c^\infty(G), \varphi_\pi \in \mathcal{C}(\pi)\}$$

using the same argument as the beginning of the proof. Moreover, the space $\{Z(s,f,\varphi_\pi)| \quad f \in C_c^\infty(G), \varphi_\pi \in \mathcal{C}(\pi)\}$ is a fractional ideal of $\mathbb{C}[q^s,q^{-s}]$ containing the constants, it follows that $\{Z(s,f,\varphi_\pi)| \quad f \in C_c^\infty(G), \varphi_\pi \in \mathcal{C}(\pi)\} = \mathbb{C}[q^s,q^{-s}]$, and we have proved that $\mathbb{C}[\widehat{T}/W](c \cdot q^{-s-\frac{n-1}{2}}) = \mathbb{C}[q^{-s},q^s]$. Therefore we get

$$\frac{Z(s,f,\Gamma_\chi)}{L(s,\pi)} \in \mathbb{C}[\widehat{T}/W](c\cdot q^{-s-\frac{n-1}{2}}), \quad f \in \mathcal{S}_{\mathrm{std}}(G,K).$$

Letting $s = \frac{1-n}{2}$, we get

$$\mathcal{S}(f) \in \mathcal{S}(1_{\mathrm{M}_n(\mathcal{O}_F)})\mathbb{C}[\widehat{T}/W] = \mathcal{S}(1_{\mathrm{M}_n(\mathcal{O}_F)} * \mathcal{H}(G,K)).$$

From this we get $\mathcal{S}_{\mathrm{std}}(G,K) \subset 1_{\mathrm{M}_n(\mathcal{O}_F)} * \mathcal{H}(G,K)$, and therefore we have proved the equality

$$\mathcal{S}_{\mathrm{std}}(G,K) = 1_{\mathrm{M}_n(\mathcal{O}_F)} * \mathcal{H}(G,K).$$

□

Remark 2.2.3. *Actually from the proof of Proposition 2.2.2 we find that if a smooth bi-K-invariant function f satisfies the condition*

$$Z(s,f,\Gamma_\chi) \subset \mathbb{C}[q^s,q^{-s}], \qquad \text{for any unramified character } \chi,$$

then the function f lies in $\mathcal{H}(G,K)$.

Theorem 2.2.2 will be our basic ingredient for introducing the space $\mathcal{S}_\rho(G,K)$.

2.3. **Unramified L-Factors and the Basic Function.** In this section, we review basic results of *basic function*. The main references are [Li17] and [Sak14].

Let π_c be the spherical representation of G with Satake parameter $c \in \widehat{T}/W$.

First we recall the definition of unramified local L-factor.

Definition 2.3.1. *The unramified local L-factor attached to π_c and ρ is defined by*

$$L(\pi_c,\rho,X) = \det(1-\rho(c)X)^{-1},$$

which is a rational function in X.

The usual L-factors are obtained by specializing X, namely

$$L(s,\pi_c,\rho) = L(\pi_c,\rho,q^{-s}), s \in \mathbb{C}.$$

Then we recall the following identity.

Lemma 2.3.2. [Bum13, Proposition 43.5]

$$L(s,\pi_c,\rho)=[\sum_{i=0}^{n}(-1)^i\mathrm{tr}(\bigwedge^{i}\rho(c))q^{-is}]^{-1}=\sum_{k\geq 0}\mathrm{tr}(\mathrm{Sym}^k\rho(c))q^{-ks}.$$

Here we notice that, by assumption $\rho\circ\widehat{\sigma}$ can be identified with the standard embedding of $\mathbb{G}_m$ into $\mathrm{GL}(V_\rho)$ via $z\to z\mathrm{Id}$. Moreover, following the assumption in [Li17, Section 3.2], the restriction of ρ to the central torus is $z\to z\mathrm{Id}$, $z\in\mathbb{C}$. Therefore we find that for all $s\in\mathbb{C}$,

$$\begin{aligned}L(\pi_c\otimes|\sigma|^s,\rho,X)&=\det(1-\rho(c\cdot q^{-s})X)^{-1}\\ =\det(1-\rho(c\cdot q^{-s}\mathrm{Id})X)^{-1}&=\det(1-\rho(c)q^{-s}X)^{-1}\\ &=L(\pi_c,\rho,q^{-s}X).\end{aligned}$$

Now to define the basic function $1_{\rho,s}$, we want to apply the inverse Satake isomorphism to $L(\pi_c,\rho,X)$. But $L(\pi_c,\rho,X)$ is a rational function rather than polynomial on $\widehat{T}/W$, hence we need to analyze the support of the inverse Satake transform of $L(\pi_c,\rho,X)$. Following [Li17, Section 3.2], we give an argument showing that the basic function is a formal sum of compactly supported functions on G with disjoint support.

We recall the Kato-Lusztig formula for inverse Satake transform

Theorem 2.3.3 ([Kat82], [Lus83])**.** *For $\lambda\in X_*(T)_+=X^*(\widehat{T})_+$, let $V(\lambda)$ be the irreducible representation of ${}^LG$ of highest weight λ, then*

$$\mathrm{tr}V(\lambda)=\sum_{\mu\in X_*(T)_+,\mu\leq\lambda}q^{-<\rho_B,\mu>}K_{\lambda,\mu}(q^{-1})\mathcal{S}(1_{K\mu(\varpi)K})$$

as an element in $\mathcal{H}(T,T_K)^W$. Here the function $K_{\lambda,\mu}$ is the Lusztig's q-analogue *of Kostant's partition function as mentioned in* [Li17, Section 2.2].

If we let $\mathrm{mult}(\mathrm{Sym}^k\rho:V(\lambda))$ be the multiplicity of $V(\lambda)$ in $\mathrm{Sym}^k\rho$, then

$$L(\pi_c,\rho,X)=\sum_{k\geq 0}\sum_{\lambda\in X_*(T)_+}\mathrm{mult}(\mathrm{Sym}^k\rho:V(\lambda))\mathrm{tr}V(\lambda)(c)X^k.$$

By the Kato-Lusztig formula, it equals

$$\sum_{k\geq 0}\Bigg\{\sum_{\lambda,\mu\in X_*(T)_+,\mu\leq\lambda}\mathrm{mult}(\mathrm{Sym}^k\rho:V(\lambda))q^{-<\rho_B,\mu>} K_{\lambda,\mu}(q^{-1})\mathcal{S}(1_{K\mu(\varpi)K})(c)\Bigg\}X^k$$

$$=\sum_{\mu\in X_*(T)_+}\Bigg\{\sum_{k\geq 0}\sum_{\lambda\in X_*(T)_+,\lambda\geq\mu}K_{\lambda,\mu}(q^{-1})\mathrm{mult}(\mathrm{Sym}^k\rho:V(\lambda))X^k\Bigg\} q^{-<\rho_B,\mu>}\mathcal{S}(1_{K\mu(\varpi)K})(c).$$

Here we observe that each weight ν of $\mathrm{Sym}^k\rho$ satisfies $\sigma(\nu)=k$, where k is identified with the character of $\mathbb{G}_m: z\to z^k$. Thus for each $\mu\in X_*(T)_+$, the inner sum can be taken over $k=\sigma(\mu)$.

For $\mu\in X_*(T)_+$, we set

$$c_\mu(q)=\sum_{\lambda\in X_*(T)_+,\lambda\geq\mu}K_{\lambda,\mu}(q^{-1})\mathrm{mult}(\mathrm{Sym}^k\rho:V(\lambda)),\tag{4}$$

if $\sigma(\mu)\geq 0$, and 0 otherwise.

We have to justify the rearrangement of sums. Given μ with $\sigma(\mu)=k\geq 0$, the expression (4) is a finite sum over those λ with $\sigma(\lambda)=k$ as explained above, and hence is well-defined. On the other hand, given $k\geq 0$, there are only finitely many $V(\lambda)$ that appear in $\mathrm{Sym}^k\rho$. Thus only finitely many $\mu\in X_*(T)_+$ with $\sigma(\mu)=k$ and $c_\mu(q)\neq 0$. To sum up, we arrive at the following equation in $\mathbb{C}[[X]]$

$$L(\pi_c,\rho,X)=\sum_{\mu\in X_*(T)_+}c_\mu(q)q^{-<\rho_B,\mu>}\mathcal{S}(1_{K\mu(\varpi)K})(c)X^{\sigma(\mu)}.$$

Now we define the function $\varphi_{\rho,X}: T(F)/T_K\to\mathbb{C}[X]$ by

$$\varphi_{\rho,X}=\sum_{\mu\in X_*(T)_+}c_\mu(q)q^{-<\rho_B,\mu>}\mathcal{S}(1_{K\mu(\varpi)K})X^{\sigma(\mu)}.$$

By previous argument we find that for fixed k,

$$\sum_{\lambda,\mu\in X_*(T)_+,\mu\leq\lambda}\mathrm{mult}(\mathrm{Sym}^k\rho:V(\lambda))q^{-<\rho_B,\mu>}K_{\lambda,\mu}(q^{-1})\mathcal{S}(1_{K\mu(\varpi)K})(c)X^k$$

lies in $\mathcal{H}(T,T_K)^W$. Hence $\varphi_{\rho,X}$ is a formal sum of functions in $\mathcal{H}(T,T_K)^W$.

Definition 2.3.4. *Define the* basic function $1_{\rho,X}$ *as a formal sum of functions, each is supported on* $\{\mu\in X_*(T)_+|\sigma(\mu)=k\}$ *for some* $k\geq 0$

lying in $\mathcal{H}(G,K)$ *as*

$$1_{\rho,X} = \sum_{\mu\in X_*(T)_+} c_\mu(q) q^{-<\rho_B,\mu>} 1_{K\mu(\varpi)K} X^{\sigma(\mu)}.$$

One may specialize the variable X. Define $1_{\rho,s}$ as the specialization at $X = q^{-s}$. Then

$$1_{\rho,s} = 1_\rho |\sigma|^s.$$

In [Li17], several analytical properties of $1_{\rho,s}$ has been proved. By definition, we have $\mathcal{S}(1_{\rho,X}) = \varphi_{\rho,X}$. Let $c \in \widehat{T}/W$ and π_c be the K-unramified irreducible representation with Satake parameter c. Let V_c denote the underlying $\mathbb{C}$-vector space of π_c. Then

$$\varphi_{\rho,X}(c) = L(\pi_c, \rho, X)$$

is a rational function in $c \in \widehat{T}/W$. For $\mathrm{Re}(s)$ sufficiently large with respect to c, the operator $\pi_c(1_{\rho,s}) : V_c \to V_c$ and its trace are well-defined and

$$\mathrm{tr}(1_{\rho,s}|V_c) = L(s, \pi_c, \rho).$$

Moreover, it is shown in [Li17] that the coefficient $c_\mu(q)$ is of polynomial growth w.r.t μ, and the integrability of $1_{\rho,s}$ when $\mathrm{Re}(s)$ is sufficiently large has also been demonstrated. We refer the reader to the paper [Li17] for further details.

2.4. **Construction of $\mathcal{S}_\rho(G,K)$ and $\mathcal{F}_\rho$.** In this section, we give a definition of the space $\mathcal{S}_\rho(G,K)$ and construct the spherical component of the operator $\mathcal{F}_\rho$ using the inverse Satake transform.

The definition is motivated from the structure of $\mathcal{S}_{\mathrm{std}}(G,K)$ as shown in Proposition 2.2.2.

Definition 2.4.1. *We define the function space* $\mathcal{S}_\rho(G,K)$ *to be* $1_{\rho,-\frac{l}{2}} * \mathcal{H}(G,K)$.

By our definition of $\mathcal{S}_\rho(G,K)$, the spherical part of Conjecture 1.2.1 holds automatically. Moreover, following the proof of Proposition 2.2.2, we find that

$$\{Z(s,f,\Gamma_\chi)| \quad f \in \mathcal{H}(G,K)\} = \mathbb{C}[\widehat{T}/W](c\cdot q^{-s-\frac{l}{2}}) = \mathbb{C}[q^s, q^{-s}]$$

for any spherical representation π_c with Satake parameter $c \in \widehat{T}/W$. From the proof of Proposition 2.2.2, we realize that $\mathcal{S}_\rho(G,K)$ is the largest subspace of $C^\infty(G,K)$ satisfying the spherical part of Conjecture 1.2.1. In other words, the following theorem holds.

Theorem 2.4.2. *Let π be a spherical representation of G. For every $f \in \mathcal{S}_\rho(G,K)$, $\varphi \in \mathcal{C}(\pi)$ the integral*

$$Z(s,f,\varphi) = \int_G f(g)\varphi(g)|\sigma(g)|^{s+\frac{l}{2}}dg$$

is a rational function in q^s, and $I_\pi = \{Z(s,f,\varphi)| \quad f \in \mathcal{S}_\rho(G,K), \varphi \in \mathcal{C}(\pi)\} = L(s,\pi,\rho)\mathbb{C}[q^s,q^{-s}]$.

Using our definition, we can also show the following

Lemma 2.4.3. *$\mathcal{S}_\rho(G,K)$ contains $\mathcal{H}(G,K)$.*

Proof. By Satake isomorphism, the space $\mathcal{S}(1_{\rho,-\frac{l}{2}} * \mathcal{H}(G,K))$ as rational functions on $c \in \widehat{T}/W$ is equal to $L(-\frac{l}{2},\pi_c,\rho)\mathbb{C}[\widehat{T}/W]$, which contains $\mathbb{C}[\widehat{T}/W]$. Applying inverse Satake transform and we get the lemma. $\square$

Then we give our definition of the spherical component of the kernel $\Phi^K_{\psi,\rho}$. Before that we show how to derive the relation between $\gamma(s,\pi,\rho,\psi)$ and $\Phi_{\psi,\rho}$ from the conjectural functional equation

$$Z(1-s,\mathcal{F}_\rho(f),\varphi^\vee) = \gamma(s,\pi,\rho,\psi)Z(s,f,\varphi),$$

where

$$Z(s,f,\varphi) = \int_G f(g)\varphi(g)|\sigma(g)|^{s+\frac{l}{2}}dg$$

and $\varphi(g) = <\widetilde{v},\pi(g)v>$ lies in $\mathcal{C}(\pi)$.

Since the analytical property of $\Phi_{\psi,\rho}$ is still conjectural, the proof of the following lemma is purely formal. But later when restricting to the spherical component, we can make it to be rigorous.

Lemma 2.4.4. *For any irreducible admissible representation π of G*

$$\pi(\Phi_{\psi,\rho,s}) = \gamma(-s-\frac{l}{2},\pi^\vee,\rho,\psi)\mathrm{Id}.$$

Proof. As conjectured in [BK00], the function $\mathcal{F}_\rho(f)$ is defined to be

$$|\sigma|^{-l-1}(\Phi_{\psi,\rho} * f^\vee).$$

We plug the formula into the functional equation, and get

$$<\widetilde{v}, Z(1-s,|\sigma|^{-l-1}(\Phi_{\psi,\rho} * f^\vee),\pi^\vee)v> \tag{5}$$
$$= \gamma(s,\pi,\rho,\psi) <\widetilde{v}, Z(s,f,\pi)v> . \tag{6}$$

Here $Z(s,f,\pi)$ is defined to be the operator $\int_G f(g)\pi(g)|\sigma(g)|^{\frac{l}{2}}dg$ whenever $\mathrm{Re}(s)$ is sufficiently large. For the left hand side of (5), we can further simplify it to be

$$\begin{aligned} Z(1-s, |\sigma|^{-l-1}(\Phi_{\psi,\rho} * f^\vee), \pi^\vee) &= Z(-s-l, \Phi_{\psi,\rho} * f^\vee, \pi^\vee) \\ &= (\pi^\vee)_{-s-\frac{l}{2}}(\Phi_{\psi,\rho})(\pi^\vee)_{-s-\frac{l}{2}}(f^\vee). \end{aligned}$$

Then the conjectural identity can be simplified to be

$$< \widetilde{v}, (\pi^\vee)_{-s-\frac{l}{2}}(\Phi_{\psi,\rho})(\pi^\vee)_{-s-\frac{l}{2}}(f^\vee)v >= \gamma(s,\pi,\rho,\psi) < \widetilde{v}, \pi_{s+\frac{l}{2}}(f)v > .$$

Now by assumption, $\Phi_{\psi,\rho}$ is a G-stable distribution, therefore it should be conjugation-invariant. Then by Schur's lemma the operator $(\pi^\vee)_{-s-\frac{l}{2}}(\Phi_{\psi,\rho})$ should act as a scalar $c(s)$. Hence the identity can be further simplified as

$$c(s) < \widetilde{v}, (\pi^\vee)_{-s-\frac{l}{2}}(f^\vee)v >= \gamma(s,\pi,\rho,\psi) < \widetilde{v}, \pi_{s+\frac{l}{2}}(f)v > .$$

Now we arrive at the equality

$$c(s)Z(-s-l, f^\vee, \varphi^\vee) = \gamma(s,\pi,\rho,\psi)Z(s,f,\varphi).$$

Using the identity $Z(-s-l, f^\vee, \varphi^\vee) = Z(s,f,\varphi)$, we get

$$c(s) = \gamma(s,\pi,\rho,\psi).$$

In other words, we obtain

$$(\pi^\vee)_{-s-\frac{l}{2}}(\Phi_{\psi,\rho}) = \gamma(s,\pi,\rho,\psi)\mathrm{Id},$$

which is equivalent to the desired relation

$$\pi(\Phi_{\psi,\rho,s}) = \gamma(-s-\frac{l}{2}, \pi^\vee, \rho, \psi)\mathrm{Id}.$$

□

Now we restrict our representation π to be a spherical representation.

By the definition of γ-factor in spherical case, we know that

$$\gamma(s,\pi,\rho,\psi) = \varepsilon(s,\pi,\rho,\psi)\frac{L(1-s,\pi^\vee,\rho)}{L(s,\pi,\rho)}.$$

Since we assume that ψ is self-dual, which means that ψ has level 0. By the computations in [GJ72] we know that $\varepsilon(s,\pi,\rho,\psi) = 1$ when ρ is the standard representation of $\mathrm{GL}(n)$. In order to be consistent with the functoriality for general ρ, which means that $\varepsilon(s,\pi,\rho,\psi) = \varepsilon(s,\rho(\pi),\psi)$, where $\rho(\pi)$ is the functorial lifting of π along ρ, we can just let $\varepsilon(s,\pi,\rho,\psi) = 1$ for general ρ whenever ψ is of level 0.

Therefore $\gamma(s,\pi,\rho,\psi)$ can be simplified as

$$\gamma(s,\pi,\rho,\psi) = \frac{L(1-s,\pi^\vee,\rho)}{L(s,\pi,\rho)}.$$

If we assume that the spherical representation π has Satake parameter $c \in \widehat{T}/W$, then $\pi^\vee$ has Satake parameter $c^{-1} \in \widehat{T}/W$. For convenience, we write π_c to mean that the spherical representation has Satake parameter $c \in \widehat{T}/W$.

Using the definition of unramified L-factor, we find that

$$L(s, \pi, \rho) = \det(1 - \rho(c)q^{-s})^{-1},$$
$$L(1-s, \pi^\vee, \rho) = \det(1 - \rho(c^{-1})q^{1-s})^{-1}.$$

On the other hand, we know that

$$\det(1 - \rho(c^{-1})q^{1-s}) = \det(1 - \rho^\vee(c)q^{1-s}),$$

where $\rho^\vee$ is the contragredient of ρ.

It follows that $\gamma(s, \pi, \rho, \psi)$ can be further simplified to be

$$\gamma(s, \pi, \rho, \psi) = \frac{L(1-s, \pi, \rho^\vee)}{L(s, \pi, \rho)}.$$

Now by previous discussion, we know that

$$\pi(\Phi_{\psi,\rho,s}) = \gamma(-s - \frac{l}{2}, \pi^\vee, \rho, \psi)\mathrm{Id}.$$

Using the inverse Satake isomorphism, we get the spherical component of the distribution $\Phi_{\psi,\rho,s}$, which we denote by $\Phi^K_{\psi,\rho,s}$,

$$\begin{aligned}\Phi^K_{\psi,\rho,s} &= \mathcal{S}^{-1}(\gamma(-s - \frac{l}{2}, \pi^\vee, \rho, \psi)) \\ &= \mathcal{S}^{-1}(L(1+s+\frac{l}{2}, \pi^\vee, \rho^\vee)) * \mathcal{S}^{-1}(\frac{1}{L(-s-\frac{l}{2}, \pi^\vee, \rho)}).\end{aligned}$$

Since $L(1+s+\frac{l}{2}, \pi^\vee, \rho^\vee) = L(1+s+\frac{l}{2}, \pi, (\rho^\vee)^\vee) = L(1+s+\frac{l}{2}, \pi, \rho)$, and $L(-s-\frac{l}{2}, \pi^\vee, \rho) = L(-s-\frac{l}{2}, \pi, \rho^\vee)$, we get

$$\begin{aligned}\Phi^K_{\psi,\rho,s} &= \mathcal{S}^{-1}(L(1+s+\frac{l}{2}, \pi, \rho)) * \mathcal{S}^{-1}(\frac{1}{L(-s-\frac{l}{2}, \pi, \rho^\vee)}) \\ &= 1_{\rho,1+s+\frac{l}{2}} * \mathcal{S}^{-1}(\frac{1}{L(-s-\frac{l}{2}, \pi, \rho^\vee)}).\end{aligned}$$

Remark 2.4.5. *We notice that for a fixed $s \in \mathbb{C}$, as a function in Satake parameter $c \in \widehat{T}/W$, $\frac{1}{L(-s-\frac{l}{2},\pi,\rho^\vee)} = \frac{1}{L(-s-\frac{l}{2},\pi_c,\rho^\vee)}$ lies in $\mathbb{C}[\widehat{T}/W]$, therefore $\mathcal{S}^{-1}(\frac{1}{L(-s-\frac{l}{2},\pi,\rho^\vee)})$ lies in $\mathcal{H}(G,K)$. We also notice that the spectral property of $\Phi^K_{\psi,\rho,s}$ is really determined by the basic function $1_{\rho,s}$. On the other hand, we find that when writing the function $\Phi^K_{\psi,\rho}$ as expansion via basis $\{1_{K\lambda K}\}_{\lambda \in X_*(T)_+}$, all its coefficients are real numbers, from which we deduce that the complex conjugate of $\Phi^K_{\psi,\rho}$, which* *we denote as* $\overline{\Phi^K_{\psi,\rho}}$ *is equal to* $\Phi^K_{\psi,\rho}$. *This will be useful for proving Proposition 2.4.8.*

By construction, our definition of $\Phi^K_{\psi,\rho}$ does give us the functional equation

$$Z(1-s,\mathcal{F}_\rho(f),\varphi)=\gamma(s,\pi,\rho,\psi)Z(s,f,\varphi),\quad f\in\mathcal{S}_\rho(G,K),$$

where $\mathcal{F}_\rho(f)=|\sigma|^{-l-1}(\Phi^K_{\psi,\rho}*f^\vee)$.

Proposition 2.4.6. *Conjecture 1.2.7 holds, i.e.* $\mathcal{F}_\rho$ *sends basic function* $1_{\rho,-\frac{l}{2}}$ *to* $1_{\rho,-\frac{l}{2}}$.

Proof. By definition

$$\mathcal{F}_\rho(1_{\rho,-\frac{l}{2}})(g)=|\sigma(g)|^{-l-1}(\Phi^K_{\psi,\rho}*1^\vee_{\rho,-\frac{l}{2}})(g)=|\sigma(g)|^{-l-1}(\Phi^K_{\psi,\rho}*(1^\vee_\rho)_{\frac{l}{2}})(g).$$

Applying the Satake isomorphism to the function $\Phi^K_{\psi,\rho}*(1^\vee_\rho)_{\frac{l}{2}}$, one gets that as rational function on $\widehat{T}/W$

$$\mathcal{S}(\Phi^K_{\psi,\rho}*(1^\vee_\rho)_{\frac{l}{2}})(c)=\mathcal{S}(\Phi^K_{\psi,\rho})(c)\mathcal{S}((1^\vee_\rho)_{\frac{l}{2}})(c)=\frac{L(1+\frac{l}{2},\pi,\rho)}{L(-\frac{l}{2},\pi,\rho^\vee)}\mathcal{S}((1^\vee_\rho)_{\frac{l}{2}})(c).$$

Here we notice that if φ_π is the zonal spherical function of π, then $\varphi(g^{-1})$ is exactly the zonal spherical function of $\pi^\vee$, so we get $\mathcal{S}(1^\vee_\rho)(c)=L(0,\pi^\vee_c,\rho)$. Hence

$$\mathcal{S}((1^\vee_\rho)_{\frac{l}{2}})(c)=L(-\frac{l}{2},\pi^\vee,\rho)=L(-\frac{l}{2},\pi,\rho^\vee).$$

Therefore

$$\begin{aligned}\mathcal{S}(|\sigma|^{l+1}\mathcal{F}_\rho(1_{\rho,-\frac{l}{2}}))=\mathcal{S}(\Phi^K_{\psi,\rho}*(1^\vee_\rho)_{\frac{l}{2}})&=\frac{L(1+\frac{l}{2},\pi,\rho)}{L(-\frac{l}{2},\pi,\rho^\vee)}L(-\frac{l}{2},\pi,\rho^\vee)\\&=L(1+\frac{l}{2},\pi,\rho)=\mathcal{S}(1_{\rho,1+\frac{l}{2}})=\mathcal{S}(1_{\rho,-\frac{l}{2}}|\sigma|^{l+1}).\end{aligned}$$

Using the inverse Satake isomorphism, it follows that $\mathcal{F}_\rho(1_{\rho,-\frac{l}{2}})=1_{\rho,-\frac{l}{2}}$. □

Finally we are going to verify the spherical part of Conjecture 1.2.3.

Proposition 2.4.7. $\mathcal{F}_\rho$ *preserves the space* $\mathcal{S}_\rho(G,K)$.

Proof. To show that $\mathcal{F}_\rho$ preserves the space $\mathcal{S}_\rho(G,K)$, we only need to show that for any $f\in\mathcal{H}(G,K)$, as a rational function on $\widehat{T}/W$

$$\frac{\mathcal{S}(\mathcal{F}_\rho(1_{\rho,-\frac{l}{2}}*f))}{L(-\frac{l}{2},\pi_c,\rho)}$$

lies in $\mathbb{C}[\widehat{T}/W]$.

By definition,

$$\mathcal{F}_\rho(1_{\rho,-\frac{l}{2}} * f) = |\sigma|^{-l-1}(\Phi^K_{\psi,\rho} * ((1_{\rho,-\frac{l}{2}} * f)^\vee)) = |\sigma|^{-l-1}\Phi^K_{\psi,\rho} * f^\vee * 1^\vee_{\rho,-\frac{l}{2}}.$$

Since $\mathcal{H}(G,K)$ is commutative, and functions in $\mathcal{H}(G,K)$ also commute with $1_{\rho,s}$, we get

$$\Phi^K_{\psi,\rho} * f^\vee * 1^\vee_{\rho,-\frac{l}{2}} = \Phi^K_{\psi,\rho} * 1^\vee_{\rho,-\frac{l}{2}} * f^\vee.$$

As shown in the proof of Proposition 2.4.6, we know that $\Phi^K_{\psi,\rho} * 1^\vee_{\rho,-\frac{l}{2}} = 1_{\rho,1+\frac{l}{2}}$. Therefore we only need to show

$$|\sigma|^{-l-1}(1_{\rho,1+\frac{l}{2}} * f^\vee) \in \mathcal{S}_\rho(G,K),$$

which, after applying the Satake isomorphism, is equivalent to showing that

$$\mathcal{S}(1_{\rho,1+\frac{l}{2}} * f^\vee) \subset L(1+\frac{l}{2}, \pi_c, \rho)\mathbb{C}[\widehat{T}/W].$$

But this follows from the definition. □

Proposition 2.4.8. *$\mathcal{F}_\rho$ extends to a unitary operator on the space $L^2(G,K,|\sigma|^{l+1}dg)$.*

Proof. To show that that $\mathcal{F}_\rho$ extends to a unitary operator on the space $L^2(G,K,|\sigma|^{l+1}dg)$, we only need to show the equality

$$<\mathcal{F}_\rho(f), \mathcal{F}_\rho(h)>_{L^2(G,K,|\sigma|^{l+1}dg)} = <f,h>_{L^2(G,K,|\sigma|^{l+1}dg)}$$

for all f and h in $\mathcal{H}(G,K)$.

Now

$$\begin{aligned} <\mathcal{F}_\rho(f), \mathcal{F}_\rho(h)>_{L^2(G,K,|\sigma|^{l+1}dg)} &= \int_G \mathcal{F}_\rho(f)(g)\overline{\mathcal{F}_\rho(h)}(g)|\sigma(g)|^{l+1}dg \\ &= \mathcal{F}_{\rho,l+1}(f) * \overline{\mathcal{F}_\rho(h)}^\vee(e) \\ <f,h>_{L^2(G,K,|\sigma|^{l+1}dg)} &= \int_G f(g)\overline{h}(g)|\sigma(g)|^{l+1}dg \\ &= \overline{h}_{l+1} * f^\vee(e). \end{aligned}$$

To show that they are equal to each other, using the Satake isomorphism, it is enough to show that as a rational function in $c \in \widehat{T}/W$, we have

$$\mathcal{S}(\mathcal{F}_{\rho,l+1}(f) * \overline{\mathcal{F}_\rho(h)}^\vee)(c) = \mathcal{S}(\overline{h}_{l+1} * f^\vee)(c).$$

Using the fact that $\mathcal{S}$ is an algebra homomorphism, we get

$$\mathcal{S}(\mathcal{F}_{\rho,l+1}(f) * \overline{\mathcal{F}_\rho(h)}^\vee) = \mathcal{S}(\mathcal{F}_{\rho,l+1}(f))\mathcal{S}(\overline{\mathcal{F}_\rho(h)}^\vee). \tag{7}$$

Now

$$\begin{aligned}\mathcal{F}_{\rho,l+1}(f)(g) &= \mathcal{F}_\rho(f)(g)|\sigma(g)|^{l+1}\\ &= \Phi^K_{\psi,\rho} * f^\vee(g),\\ \overline{\mathcal{F}_\rho(h)}^\vee(g) &= \overline{\mathcal{F}_\rho(h)}(g^{-1}) = |\sigma(g)|^{l+1}\overline{\Phi^K_{\psi,\rho}} * \overline{h}^\vee(g^{-1})\\ &= |\sigma(g)|^{l+1}(\overline{\Phi^K_{\psi,\rho}} * \overline{h}^\vee)^\vee(g)\\ &= |\sigma(g)|^{l+1}(\overline{h} * \overline{\Phi^K_{\psi,\rho}}^\vee)(g).\end{aligned}$$

Plug the calculations into the equation (7), we get that as a rational function in $c \in \widehat{T}/W$, the left hand side of the equation (7) can be written as

$$\mathcal{S}(\Phi^K_{\psi,\rho})(c)\mathcal{S}(f^\vee)(c)\mathcal{S}(\overline{h})(c\cdot q^{-(l+1)})\mathcal{S}(\overline{\Phi^K_{\psi,\rho}}^\vee)(c\cdot q^{-(l+1)}). \tag{8}$$

Similarly, the right hand side of the equation (7) can be written as

$$\mathcal{S}(\overline{h}_{l+1})(c)\mathcal{S}(f^\vee)(c) = \mathcal{S}(\overline{h})(c\cdot q^{-(l+1)})\mathcal{S}(f^\vee)(c). \tag{9}$$

Comparing equations (8) and (9), we only need to show the following equality

$$\mathcal{S}(\Phi^K_{\psi,\rho})(c)\mathcal{S}(\overline{\Phi^K_{\psi,\rho}}^\vee)(c\cdot q^{-(l+1)}) = 1.$$

First we simplify the term

$$\mathcal{S}(\overline{\Phi^K_{\psi,\rho}}^\vee)(c\cdot q^{-(l+1)}) = \mathcal{S}((\overline{\Phi^K_{\psi,\rho}}^\vee)_{l+1})(c).$$

Then using the definition of $\Phi^K_{\psi,\rho,s}$, we have

$$\mathcal{S}(\Phi^K_{\psi,\rho,s})(c) = \gamma(-s-\frac{l}{2},(\pi_c)^\vee,\rho,\psi) = \frac{L(1+s+\frac{l}{2},\pi_c,\rho)}{L(-s-\frac{l}{2},\pi_c,\rho^\vee)}.$$

Letting $s=0$, we get that as a rational function in $c \in \widehat{T}/W$,

$$\mathcal{S}(\Phi^K_{\psi,\rho})(c) = \frac{L(1+\frac{l}{2},\pi_c,\rho)}{L(-\frac{l}{2},\pi_c,\rho^\vee)}$$

By Remark 2.4.5, the function $\Phi^K_{\psi,\rho}$ is real-valued, which means that $\overline{\Phi^K_{\psi,\rho}} = \Phi^K_{\psi,\rho}$, therefore

$$\begin{aligned}\mathcal{S}((\overline{\Phi^K_{\psi,\rho}}^\vee)_{l+1})(c) &= \mathcal{S}(\overline{\Phi^K_{\psi,\rho,-(l+1)}})(c^{-1}) = \mathcal{S}(\Phi^K_{\psi,\rho,-(l+1)})(c^{-1})\\ &= \frac{L(-\frac{l}{2},\pi_c^\vee,\rho)}{L(\frac{l}{2}+1,\pi_c^\vee,\rho^\vee)} = \frac{L(-\frac{l}{2},\pi_c,\rho^\vee)}{L(\frac{l}{2}+1,\pi_c,\rho)} = \mathcal{S}(\Phi^K_{\psi,\rho,s})(c)^{-1}.\end{aligned}$$

It follows that

$$\mathcal{S}(\mathcal{F}_{\rho,l+1}(f) * \overline{\mathcal{F}_\rho(h)}^\vee)(c) = \mathcal{S}(\overline{h}_{l+1} * f^\vee)(c)$$

as a rational function in $c \in \widehat{T}/W$. Using the inverse Satake isomorphism we get the desired equality

$$\mathcal{F}_{\rho,l+1}(f) * \overline{\mathcal{F}_\rho(h)}^\vee = \overline{h}_{l+1} * f^\vee.$$

□

3. Archimedean Case

In this section we study asymptotic properties for $1_{\rho,s}$ and $\Phi^K_{\psi,\rho,s}$ when F is an archimedean field.

First we give the definition of $1_{\rho,s}$ and $\Phi^K_{\psi,\rho,s}$.

Definition 3.0.1. *The basic function $1_{\rho,s}$ is defined to be the smooth bi-K-invariant function on G such that*

$$\int_G 1_{\rho,s}(g)\varphi_\pi(g)dg = L(s,\pi,\rho),$$

where φ_π is the zonal spherical function associated to the spherical representation π of G, and $1_{\rho,s} = 1_\rho|\sigma|^s$.

Definition 3.0.2. *The spherical component of the distribution kernel of ρ-Fourier transform kernel $\Phi_{\psi,\rho,s}$, which we denote by $\Phi^K_{\psi,\rho,s}$, is defined to be the smooth bi-K-invariant function on G such that*

$$\int_G \Phi^K_{\psi,\rho,s}(g)\varphi_\pi(g)dg = \gamma(-s-\frac{l}{2},\pi^\vee,\rho,\psi),$$

where φ_π is the zonal spherical function associated to the spherical representation π of G, and $\Phi^K_{\psi,\rho,s} = \Phi^K_{\psi,\rho}|\sigma|^s$.

By the *spherical Plancherel transform*, we know that the analytical properties of $1_{\rho,s}$ and $\Phi^K_{\psi,\rho,s}$ are completely determined by the corresponding analytical properties of $L(s,\pi,\rho)$ and $\gamma(s,\pi,\rho,\psi)$.

3.1. **Spherical Plancherel Transform.** In this section, we review the theory of spherical plancherel transform for any real reductive Lie group belonging to the *Harish-Chandra class* as defined in [GV88, Definition 2.1.1]. In particular, it applies to our situation. The main references are [Ank91] and [GV88].

Let $\mathfrak{g}$ be the Lie algebra of G. We fix the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ is the Lie algebra of K. For any $\lambda \in \mathfrak{a}^*$, where $\mathfrak{a}$ is the maximal abelian subalgebra of $\mathfrak{p}$, we let π_λ be the spherical representation induced from the character

$$m\exp(H)n \to e^{i\lambda(H)}, H \in \mathfrak{a}$$

of the minimal parabolic subgroup $P = MAN$. Here $A = \exp\mathfrak{a}$, M is the centralizer of A in K, and N is the corresponding unipotent radical.

We denote the zonal spherical function of π_λ by φ_λ.

We fix the norm $|\cdot|$ induced by the Killing form on G as in [Ank91, 1 Preliminaries].

The elements of $U(\mathfrak{g})$ acts on $C^\infty(G)$ as differential operators. Following [Ank91, 1 Preliminaries], for any $(D,E)\in U(\mathfrak{g})\times U(\mathfrak{g})$ and $f\in C^\infty(G)$, $x\in G$, we can define the left D right E derivative $f(D;x;E)$ of f, which again lies in $C^\infty(G)$.

We introduce the function spaces $S^p(K\backslash G/K)$ and $S(\mathfrak{a}^*_\varepsilon)$. Here $0<p\leq 2$ is any real number, and $\varepsilon=\frac{2}{p}-1$.

Definition 3.1.1. *For $0<p\leq 2$, let $S^p(K\backslash G/K)$ be the space of bi-K-invariant functions f in $C^\infty(K\backslash G/K)$ such that the following norm*

$$\sigma^{(p)}_{D,E,s}(f)=\sup_{x\in G}(|x|+1)^s\varphi_0(x)^{-\frac{2}{p}}|f(D;x;E)|$$

is finite for any $D,E\in U(\mathfrak{g})$, $s\in\mathbb{Z}^+$.

Using the natural convolution structure of two bi-K-invariant functions, we can prove that $S^p(K\backslash G/K)$ is a Frechét algebra, where the topology is induced by the semi-norms given by $\{\sigma^{(p)}_{D,E,s}|\quad D,E\in U(\mathfrak{g}), s\in\mathbb{Z}^+\}$. Moreover, as mentioned in [Ank91, Lemma 6], the space $C_c^\infty(K\backslash G/K)$ is a dense subspace of $S^p(K\backslash G/K)$.

Now we introduce the space $S(\mathfrak{a}^*_\varepsilon)$.

Definition 3.1.2. *Let $C^{\varepsilon\rho}$ be the convex hull generated by $W\cdot\varepsilon\rho_B$ in $\mathfrak{a}^*$. Let $\mathfrak{a}^*_\varepsilon=\mathfrak{a}^*+iC^{\varepsilon\rho_B}$. Then $S(\mathfrak{a}^*_\varepsilon)$ consists of complex valued functions h on $\mathfrak{a}^*_\varepsilon$ such that the following holds.*

(1) *h is holomorphic in the interior of $\mathfrak{a}^*_\varepsilon$.*
(2) *h and all its derivatives extend continuously to $\mathfrak{a}^*_\varepsilon$.*
(3) *For any polynomial function P on $\mathfrak{a}^*_\varepsilon$, $t\in\mathbb{Z}^+$,*

$$\tau^{(\varepsilon)}_{P,t}(h)=\sup_{\lambda\in\mathfrak{a}^*_\epsilon}(|\lambda|+1)^t|P(\frac{\partial}{\partial\lambda})h(\lambda)|$$

is finite.

Let $S(\mathfrak{a}^*_\varepsilon)^W$ be the W-invariant elements in $S(\mathfrak{a}^*_\varepsilon)$. We can show that $S(\mathfrak{a}^*_\varepsilon)^W$ is a Frechét algebra, where the algebra structure is given by pointwise multiplication, and the W-invariant Paley-Wiener functions on $\mathfrak{a}^*_\mathbb{C}$, denoted by $\mathcal{P}(\mathfrak{a}^*_\mathbb{C})^W$, is a dense subspace of $S(\mathfrak{a}^*_\varepsilon)^W$ after restricted to $\mathfrak{a}^*_\varepsilon$.

In particular, when $\varepsilon=0$, $S(\mathfrak{a}^*)$ is the classical Schwartz space on $\mathfrak{a}^*$.

Definition 3.1.3. *For any $f \in S^p(K\backslash G/K)$, $\lambda \in \mathfrak{a}^*$, let $\mathcal{H}$ be the spherical transform defined by*

$$\mathcal{H}(f)(\lambda) = \int_G f(x)\varphi_\lambda(x)dx.$$

Theorem 3.1.4 ([Ank91],[GV88])**.**
(1) *$\mathcal{H}$ is a topological isomorphism of Frechét algebra between $S^p(K\backslash G/K)$ and $S(\mathfrak{a}_\varepsilon^*)^W$, where $0 < p \leq 2$ and $\varepsilon = \frac{2}{p} - 1$.*
(2) *The inverse transform is given by*

$$\mathcal{H}^{-1}(h)(x) = const \int_{\mathfrak{a}^*} d\lambda |c(\lambda)|^{-2} h(-\lambda)\varphi_\lambda(x).$$

3.2. **Langlands Classification for $\mathrm{GL}_n(\mathbb{R})$: Spherical Case.** Before coming to study the analytical properties of L-functions and γ-factors, we need to obtain an explicit formula for L-functions and γ-factors. Therefore we review the Langlands classifications and Langlands correspondence of spherical representations for $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$. The main reference for this and next sections is [Kna94]. For more advanced reference, the reader can consult [Lan89].

The Langlands classification for $\mathrm{GL}_n(\mathbb{R})$ describes all irreducible admissible representations of $\mathrm{GL}_n(\mathbb{R})$ up to infinitesimal equivalence. Since we only care about the spherical representations, we only present the classification and correspondence for spherical representations of $\mathrm{GL}_n(\mathbb{R})$.

The building blocks for spherical representations of $\mathrm{GL}_n(\mathbb{R})$ are the quasi-character $a \to |a|_{\mathbb{R}}^t$ of $\mathrm{GL}_1(\mathbb{R})$. Here $|\cdot|_{\mathbb{R}}$ denotes the ordinary valuation on $\mathbb{R}$, and $t \in \mathbb{C}$.

We have the diagonal torus subgroup

$$T = \mathrm{GL}_1(\mathbb{R}) \times ... \times \mathrm{GL}_1(\mathbb{R}) \cong (\mathrm{GL}_1(\mathbb{R}))^n.$$

For each j with $1 \leq j \leq n$, let σ_j be a quasi-character of $\mathrm{GL}_1(\mathbb{R})$ of the form $a \to |a|_{\mathbb{R}}^{t_j}$. Then by tensor product, $(\sigma_1, ..., \sigma_n)$ defines a representation of the diagonal torus T, and we extend the representation to the corresponding Borel subgroup $B = TN$, where N is the unipotent radical. We set

$$I(\sigma_1, ..., \sigma_n) = \mathrm{Ind}_B^G(\sigma_1, ..., \sigma_n)$$

using unitary induction.

Theorem 3.2.1. [Kna94, Theorem 1] *For $G = \mathrm{GL}_n(\mathbb{R})$,*
(1) *if the parameters t_j of $(\sigma_1, ..., \sigma_n)$ satisfy*

$$\mathrm{Re}\, t_1 \geq \mathrm{Re}\, t_2 \geq ... \geq \mathrm{Re}\, t_n,$$

then $I(\sigma_1,...,\sigma_n)$ has a unique irreducible quotient $J(\sigma_1,...,\sigma_n)$.

(2) *the representations $J(\sigma_1,...,\sigma_n)$ exhaust the spherical representation of G up to infinitesimal equivalence.*

(3) *Two such representations $J(\sigma_1,...,\sigma_n)$ and $J(\sigma'_1,...,\sigma'_{n'})$ are infinitesimally equivalent if and only if $n' = n$ and there exists a permutation $j(i)$ of $\{1,...,n\}$ such that $\sigma'_i = \sigma_{j(i)}$ for $1 \le i \le n$.*

Next we determine the corresponding Langlands parameters of spherical representations, which are given by homomorphisms of the abelianlization of the Weil group, which we denoted by $W_{\mathbb{R}}^{ab} \cong \mathbb{C}^\times$ into $\mathrm{GL}_n(\mathbb{R})$. Following [Kna94, Section 3], the Langlands parameters corresponding to spherical representations of $\mathrm{GL}_n(\mathbb{R})$ are given by the direct sum of n one-dimensional representations of $\mathbb{C}^\times$ of the following form:

$$(+,t): \quad \varphi(z) = |z|_{\mathbb{R}}^t, \quad \varphi(j) = +1.$$

Now let φ be an n-dimensional semisimple complex representation of $W_{\mathbb{R}}$, which is n direct sum of quasi-characters of the form $(+,t)$. For any $1 \le j \le n$, let φ_j be the corresponding irreducible constituent of φ. To φ_j we associate a quasi-character. In this way, we associate a tuple $(\sigma_1,...,\sigma_n)$ of representations to φ. By permutations if necessary, the complex numbers $t_1,...,t_n$ satisfy the assumption of Theorem 3.2.1. Then by Theorem 3.2.1, we can then make the association

$$\varphi \to \rho_{\mathbb{R}}(\varphi) = J(\sigma_1,...,\sigma_n) \tag{10}$$

and come to the following conclusion.

Theorem 3.2.2. [Kna94, Theorem 2] *The association (10) is a well-defined bijection between the set of all equivalence classes of n-dimensional semisimple complex representations of $W_{\mathbb{R}}$ which are n direct sum of one-dimension representations of the form $(+,t)$, and the set of all equivalence classes of spherical representations of* $\mathrm{GL}_n(\mathbb{R})$.

If φ is one-dimensional given by $(+,t)$, the associated L-function and ε-factor are given as follows.

$$\begin{aligned} L(s,\varphi) &= \pi^{-\frac{(s+t)}{2}}\Gamma\Big(\frac{s+t}{2}\Big), \\ \varepsilon(s,\pi,\psi) &= 1. \end{aligned}$$

For φ reducible, $L(s,\varphi)$ and $\varepsilon(s,\varphi,\psi)$ are the product of the L-functions and $\varepsilon(s,\varphi,\psi)$ of the one-dimensional factors of φ.

3.3. **Langlands Classification for $\mathrm{GL}_n(\mathbb{C})$: Spherical Case.** The Langlands classification for $\mathrm{GL}_n(\mathbb{C})$ describes all irreducible admissible representations of $\mathrm{GL}_n(\mathbb{C})$ up to infinitesimal equivalence. Since we only care about the spherical representations, we only present the classification and correspondence for spherical representations.

The building blocks for spherical representations of the group $\mathrm{GL}_n(\mathbb{C})$ are the quasi-character $a \to |a|_{\mathbb{C}}^t$ of $\mathrm{GL}_1(\mathbb{C})$. Here $|\cdot|_{\mathbb{C}}$ denotes the ordinary valuation on $\mathbb{C}$ given by

$$|z|_{\mathbb{C}} = |z\overline{z}| = |z|^2, \quad z \in \mathbb{C},$$

and $t \in \mathbb{C}$.

We have the diagonal torus subgroup

$$T = \mathrm{GL}_1(\mathbb{C}) \times ... \times \mathrm{GL}_1(\mathbb{C}) \cong (\mathrm{GL}_1(\mathbb{C}))^n.$$

For each j with $1 \leq j \leq n$, let σ_j be a quasi-character of $\mathrm{GL}_1(\mathbb{C})$ of the form $a \to |a|_{\mathbb{C}}^{t_j}$. Then by tensor product, $(\sigma_1, ..., \sigma_n)$ defines a representation of the diagonal torus T, and we extend the representation to the corresponding Borel subgroup $B = TN$, where N is the unipotent radical. We then set

$$I(\sigma_1, ..., \sigma_n) = \mathrm{Ind}_B^G(\sigma_1, ..., \sigma_n)$$

using unitary induction.

Theorem 3.3.1. [Kna94, Theorem 4] *For $G = \mathrm{GL}_n(\mathbb{C})$,*

(1) *if the parameters t_j of $(\sigma_1, ..., \sigma_n)$ satisfy*

$$\mathrm{Re}\, t_1 \geq \mathrm{Re}\, t_2 \geq ... \geq \mathrm{Re}\, t_n,$$

then $I(\sigma_1, ..., \sigma_n)$ has a unique irreducible quotient $J(\sigma_1, ..., \sigma_n)$.

(2) *the representations $J(\sigma_1, ..., \sigma_n)$ exhaust the spherical representations of G up to infinitesimal equivalence.*

(3) *Two such representations $J(\sigma_1, ..., \sigma_n)$ and $J(\sigma'_1, ..., \sigma'_{n'})$ are infinitesimally equivalent if and only if $n' = n$ and there exists a permutation $j(i)$ of $\{1, ..., n\}$ such that $\sigma'_i = \sigma_{j(i)}$ for $1 \leq i \leq n$.*

Next we determine the corresponding Langlands parameters of spherical representations, which are given by homomorphisms of the Weil group $W_{\mathbb{C}} \cong \mathbb{C}^\times$ into $\mathrm{GL}_n(\mathbb{C})$. Following [Kna94, Section 4], the Langlands parameters corresponding to spherical representations of $\mathrm{GL}_n(\mathbb{C})$ are given by the direct sum of n one-dimensional representations of $\mathbb{C}^\times$ of the following form:

$$(0, t) : z \in \mathbb{C}^\times \to |z|_{\mathbb{C}}^t, \quad l \in \mathbb{Z}, t \in \mathbb{C}.$$

Now let φ be an n-dimensional semisimple complex representation of $W_{\mathbb{C}}$, which is n direct sum of quasi-characters of the form $(0, t)$. To

φ_j we associate a quasi-character $\sigma_j = |\cdot|_{\mathbb{C}}^{t_j}$ of $\mathrm{GL}_1(\mathbb{C})$. In this way, we associate a tuple $(\sigma_1, ..., \sigma_n)$ of representations to φ. By permutations if necessary, the complex numbers $t_1, ..., t_n$ satisfy the assumption of Theorem 3.3.1. Then by Theorem 3.3.1, we can then make the association

$$\varphi \to \rho_{\mathbb{C}}(\varphi) = J(\sigma_1, ..., \sigma_n) \tag{11}$$

and come to the following conclusion.

Theorem 3.3.2. [Kna94, Theorem 5] *The association (11) is a well-defined bijection between the set of all equivalence classes of n-dimensional semisimple complex representations of $W_{\mathbb{C}}$ which are n direct sum of 1-dimensions of form $(0, t)$, and the set of all equivalence classes of spherical representations of $\mathrm{GL}_n(\mathbb{C})$.*

If φ is given by $(0, t)$, the associated L-function and ε-factor are given as follows

$$L(s, \varphi) = 2(2\pi)^{-(s+t)}\Gamma(s + t),$$
$$\varepsilon(s, \pi, \psi) = 1.$$

For φ reducible, $L(s, \varphi)$ and $\varepsilon(s, \varphi, \psi)$ are the product of the L-functions and $\varepsilon(s, \varphi, \psi)$ of the irreducible constituents of φ.

3.4. **Asymptotic of $1_{\rho,s}$ and $\Phi^K_{\psi,\rho,s}$: Real Case.** Based on the local Langlands correspondence, we know that in order to study the asymptotic of L-functions, we need to study the asymptotic of Γ function, where

$$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx.$$

Here we recall the following estimation from [Bat53, 1.18(6)], which can easily be derived from the classical Stirling formula.

Theorem 3.4.1. *For fixed $x \in \mathbb{R}$,*

$$\Gamma(x + iy) = \sqrt{2\pi}|y|^{x-\frac{1}{2}}e^{-x-\frac{|y|\pi}{2}}[1 + O(\frac{1}{|y|})], \quad |y| \to \infty.$$

Then we give a proof for the following estimation for the derivatives of Γ-function.

Theorem 3.4.2. *We have*

$$\lim_{|z|\to\infty, |\arg z|<\pi} \frac{\Gamma^{(n)}(z)}{\Gamma(z)(\log z)^n} = 1,$$

where $\Gamma^{(n)}(z)$ is the n-th derivative of $\Gamma(z)$.

Proof. We prove the theorem via induction.

Let $D_n(z) = \frac{\Gamma^{(n)}(z)}{\Gamma(z)}$. When $n = 1$, using the classical Stirling formula, we have

$$\log\Gamma(z) = \frac{1}{2}(\log(2\pi) - \log(z)) + z(\log z - 1) + O(\frac{1}{z})$$

for any $|z| \to \infty, |\arg z| < \pi$. Diving both sides by $z\log z$, we get

$$\lim_{|z|\to\infty, |\arg z|<\pi} \frac{\log\Gamma(z)}{z\log z} = 1.$$

By the L'Hôsptial's rule, we get

$$\lim_{|z|\to\infty, |\arg z|<\pi} \frac{D_1(z)}{1+\log z} = 1.$$

Hence we obtain

$$\lim_{|z|\to\infty, |\arg z|<\pi} \frac{D_1(z)}{\log z} = 1.$$

Therefore we complete the proof for $n = 1$.

By definition, $\Gamma(z)D_n(z) = \Gamma^{(n)}(z)$. Taking derivative on both sides, we get

$$\Gamma^{(1)}(z)D_n(z) + D_n'(z)\Gamma(z) = \Gamma^{(n+1)}(z).$$

From this we can deduce the equality

$$D_{n+1}(z) = D_n'(z) + D_n(z)D_1(z).$$

Hence

$$\frac{D_{n+1}(z)}{(\log z)^{n+1}} = \frac{D_n'(z)}{(\log z)^{n+1}} + \frac{D_n(z)D_1(z)}{(\log z)^n(\log z)}.$$

We assume that the limit

$$\lim_{|z|\to\infty, |\arg z|<\pi} \frac{D_k(z)}{(\log z)^k} = 1, \quad 1 \le k \le n$$

holds. To show that the limit formula holds for $k = n+1$, we only need to show that

$$\lim_{|z|\to\infty, |\arg z|<\pi} \frac{D_n'(z)}{(\log z)^{n+1}} = 0.$$

Now we have the formula

$$\lim_{|z|\to\infty, |\arg z|<\pi} \frac{D_n(z)}{(\log z)^n} = 1.$$

By the L'Hôspital's rule, we get

$$\lim_{|z|\to\infty, |\arg z|<\pi} \frac{zD_n'(z)}{n(\log z)^{n-1}} = 1.$$

Hence

$$\lim_{|z|\to\infty, |\arg z|<\pi} \frac{z(\log z)^2 D_n'(z)}{n(\log z)^{n+1}} = 1,$$

and we get

$$\lim_{|z|\to\infty, |\arg z|<\pi} \frac{D_n'(z)}{(\log z)^{n+1}} = 0.$$

Combining the above results we prove the theorem. □

Then we come to describe an explicit formula for $L(s, \pi_\lambda, \rho)$.

By definition, π_λ is induced from the character

$$m \exp(H) n \to e^{i\lambda(H)}.$$

If we assume that $\lambda = (\lambda_1, ..., \lambda_m) \in \mathfrak{a}^*$, where m is 1 plus the semisimple rank of G, then its associated Langlands parameter is of the form

$$t \in W_{\mathbb{R}}^{ab} \cong \mathbb{R}^\times \to \begin{pmatrix} |t|^{i\lambda_1} & & & \\ & |t|^{i\lambda_2} & & \\ & & \ddots & \\ & & & |t|^{i\lambda_m} \end{pmatrix}.$$

Assume that ρ has weights $\varpi_1, \varpi_2, ..., \varpi_n$, where $n = \dim(V_\rho)$. Then the associated parameter for $\rho(\pi_\lambda)$, which is the functorial lifting image of π_λ along ρ, is

$$t \to \begin{pmatrix} |t|^{i\varpi_1(\lambda)} & & & \\ & |t|^{i\varpi_2(\lambda)} & & \\ & & \ddots & \\ & & & |t|^{i\varpi_n(\lambda)} \end{pmatrix},$$

where $\varpi_j(\lambda) = \sum_{k=1}^m n_k^j \lambda_k$, $n_k \in \mathbb{Z}_{\geq 0}$.

In the following, we need to use the following lemma on the representation ρ:

Lemma 3.4.3. *We have the following inequality*

$$\sum_{k=1}^n |\varpi_k(x)| \geq C_\rho \sum_{t=1}^m |x_t|, \quad \textit{for all } x = (x_1, ..., x_m) \in \mathfrak{a}^* \tag{12}$$

for some constant $C_\rho > 0$.

Proof. The basic ingredient that we use is the fact that the representation ρ is faithful.

We restrict ρ to the split torus $(\mathbb{C}^\times)^m$ of ${}^LG$. Up to conjugation, we can view ρ as an injective homorphism from $(\mathbb{C}^\times)^m$ to $(\mathbb{C}^\times)^n$, where $n = \dim(V_\rho)$. Passing to Lie algebra, we get an injective homomorphism from $(\mathbb{C})^m$ to $(\mathbb{C})^n$, which is given by the direct sum of ϖ_k, $1 \le k \le n$. Here each ϖ_k can be viewed as a character of $(\mathbb{C})^m$.

We notice that the inequality (12) is invariant by scaling, and holds identically when $x = (x_1, ..., x_m) = 0$. Therefore in order to obtain the bound C_ρ, we can assume that $\sum_{t=1}^m |x_t| = 1$. In this case, the following function $f(x)$

$$f(x) = \sum_{k=1}^n |\varpi_k(x)|, \quad \sum_{t=1}^m |x_t| = 1$$

is continuous. Using the fact that the equality $\sum_{t=1}^m |x_t| = 1$ defines a compact set in $(\mathbb{C})^m$, we notice that there exists $x \in (\mathbb{C})^m$ with the property $\sum_{t=1}^m |x_t| = 1$, such that $f(x)$ is maximal. We let C_ρ to be the maximum.

Now if C_ρ is equal to 0, this means that $\varpi_k(x) = 0$ for all $1 \le k \le n$. In particular, it means that the morphism ρ is not injective when restricted to the Lie algebra $(\mathbb{C})^m$, which is a contraction.

It follows that $C_\rho > 0$. This completes the proof. □

Now we are going to state our result on an asymptotic of $1_{\rho,s}$.

Theorem 3.4.4. *If* $\mathrm{Re}(s)$ *satisfies the following inequality*

$$\mathrm{Re}(s) > max\{\varpi_k(\mu)| \quad 1 \le k \le n, \mu \in C^{\varepsilon\rho_B}\},$$

then $1_{\rho,s}$ *belongs to* $S^p(K\backslash G/K)$. *Here* $\varepsilon = \frac{2}{p} - 1$, $0 < p \le 2$, *and* $\{\varpi_k\}_{k=1}^n$ *are the weights of the representation* $\rho : {}^LG \to \mathrm{GL}(V_\rho)$.

Proof. By definition

$$L(s, \pi_\lambda, \rho) = \prod_{k=1}^n \pi^{-(\frac{s+i\varpi_k(\lambda)}{2})}\Gamma\Big(\frac{s + i\varpi_k(\lambda)}{2}\Big).$$

When $\mathrm{Re}(s)$ is sufficiently large, we want to show that the function $L(s, \pi_\lambda, \rho)$, as a function of λ, lies in the space $S(\mathfrak{a}_\varepsilon^*)^W$. The W-invariance of the function follows from the fact that $\pi_{w\lambda} \cong \pi_\lambda$ for any $w \in W$. Therefore we only need to show the following semi-norm for $L(s, \pi_\lambda, \rho)$

$$\tau_{P,t}^{(\varepsilon)}(L(s, \pi_\lambda, t)) = \sup_{\lambda \in \mathfrak{a}_\varepsilon^*} (|\lambda| + 1)^t P(\frac{\partial}{\partial \lambda}) L(s, \pi_\lambda, \rho)$$

is finite if $\mathrm{Re}(s)$ is bigger than $\max\{\varpi_k(\mu)| \quad 1 \leq k \leq n, \mu \in C^{\varepsilon\rho_B}\}$. The reason that we need this bound is to prevent from touching the possible poles of $L(s, \pi_\lambda, \rho)$.

Now we are going to estimate

$$\sup_{\lambda \in \mathfrak{a}_\varepsilon^*} (|\lambda|+1)^t P(\frac{\partial}{\partial \lambda})[\prod_{k=1}^{n} \pi^{-(\frac{s+i\varpi_k(\lambda)}{2})} \Gamma\Big(\frac{s+i\varpi_k(\lambda)}{2}\Big)].$$

The term

$$P(\frac{\partial}{\partial \lambda}) \pi^{-(\frac{s+i\varpi_k(\lambda)}{2})}$$

is dominated by

$$C_1(|\lambda|+1)^a \pi^{-(\frac{s+i\varpi_k(\lambda)}{2})}$$

for some $a > 0$ and constant $C_1 > 0$.

For the term

$$P(\frac{\partial}{\partial \lambda}) \Gamma\Big(\frac{s+i\varpi_k(\lambda)}{2}\Big),$$

using Theorem 3.4.2 for the estimation on the derivative of $\Gamma(z)$, it is dominated by

$$C_2(|\lambda|+1)^b \Gamma\Big(\frac{s+i\varpi_k(\lambda)}{2}\Big)$$

for some $b > 0$ and some constant $C_2 > 0$. Here we use the fact that $\log(z)$ is dominated by $C(|z|+1)$ for some constant C if $\mathrm{Re}(z)$ is bigger than $\max\{\varpi_k(\mu)| \quad 1 \leq k \leq n, \mu \in C^{\varepsilon\rho_B}\}$.

Hence we only need to show that the following term is bounded

$$\sup_{\lambda \in \mathfrak{a}_\varepsilon^*} (|\lambda|+1)^t \prod_{k=1}^{n} \pi^{-(\frac{s+i\varpi_k(\lambda)}{2})} \Gamma\Big(\frac{s+i\varpi_k(\lambda)}{2}\Big).$$

When $\lambda \in \mathfrak{a}_\varepsilon^* = \mathfrak{a}^* + iC^{\varepsilon\rho}$, the real part of $\frac{s+i\varpi_k(\lambda)}{2}$ is bounded and lies in a compact set, so the function $\pi^{-(\frac{s+i\varpi_k(\lambda)}{2})}$ is always bounded. Using Theorem 3.4.1 for the estimation for $\Gamma(x+iy)$ for $x \in \mathbb{R}$ fixed, we have

$$\sup_{\lambda \in \mathfrak{a}_\varepsilon^*} (|\lambda|+1)^t \prod_{k=1}^{n} \pi^{-(\frac{s+i\varpi_k(\lambda)}{2})} \Gamma\Big(\frac{s+i\varpi_k(\lambda)}{2}\Big) \leq$$
$$\sup_{\lambda \in \mathfrak{a}_\varepsilon^*} C(|\lambda|+1)^t (\sqrt{2\pi})^n \prod_{k=1}^{n} [|\frac{\mathrm{Im}(s)+\varpi_k(x)}{2}|^{\frac{\mathrm{Re}(s)-\varpi_k(y)-1}{2}}$$
$$\cdot e^{\frac{\varpi_k(y)}{2} - \frac{\mathrm{Re}(s)}{2} - \frac{|\mathrm{Im}(s)+\varpi_k(x)|\pi}{4}}]$$

for some constant $C > 0$. Here we write $\lambda = x + iy$ with $x \in \mathfrak{a}^*$, $y \in C^{\varepsilon\rho}$.

Now we know that $s \in \mathbb{C}$ is fixed, and y lies in $C^{\varepsilon\rho}$, which is a compact set. The term $\varpi_k(\lambda)$ is also dominated by a polynomial function in $|\lambda| + 1$. Therefore up to a constant and a polynomial in $(|\lambda| + 1)$, we only need to evaluate the following term

$$\sup_{x \in \mathfrak{a}^*} (|x| + 1)^t \prod_{k=1}^{n} e^{-\frac{|\varpi_k(x)|\pi}{4}}.$$

By Lemma 3.4.3, it is bounded by

$$\sup_{x \in \mathfrak{a}^*} (|x| + 1)^t \prod_{k=1}^{m} e^{-\frac{C_\rho |x_k| \pi}{4}}.$$

which is bounded by a constant. This proves the theorem. $\square$

Remark 3.4.5. *As mentioned in* [Get15], *by the recent work on Arthur-Selberg trace formula* [FL11] [FL16] [FLM11], *the Arthur-Selberg trace formula is valid for functions in $\mathcal{S}^p(K\backslash G/K)$ whenever $0 < p \leq 1$. Therefore our result gives an explicit bound of the parameter s when the basic function $1_{\rho,s}$ can be plugged into the Arthur-Selberg trace formula.*

We can also prove an asymptotic for $\Phi^K_{\psi,\rho,s}$. By definition, the spherical component of $\Phi^K_{\psi,\rho,s}$ is determined via the following identity

$$\mathcal{H}(\Phi^K_{\psi,\rho,s}) = \frac{L(1 + s + \frac{l}{2}, \pi, \rho)}{L(-s - \frac{l}{2}, \pi^\vee, \rho)}.$$

Here we notice that if π has Langlands parameter

$$t \to \begin{pmatrix} |t|^{i\lambda_1} & & & \\ & |t|^{i\lambda_2} & & \\ & & \ddots & \\ & & & |t|^{i\lambda_m} \end{pmatrix},$$

then $\pi^\vee$ has Langlands parameter

$$t \to \begin{pmatrix} |t|^{-i\lambda_1} & & & \\ & |t|^{-i\lambda_2} & & \\ & & \ddots & \\ & & & |t|^{-i\lambda_m} \end{pmatrix}.$$

We first simplify the expression for γ-factor by the functional equation of $\Gamma(z)$.

Lemma 3.4.6. *The formula* $\mathcal{H}(\Phi^K_{\psi,\rho,s}) = \frac{L(1+s+\frac{l}{2},\pi,\rho)}{L(-s-\frac{l}{2},\pi^\vee,\rho)}$ *can be simplified to be*

$$\prod_{k=1}^{n}[\pi^{-(\frac{1}{2}+s+\frac{l}{2}+i\varpi_k(\lambda))}\Gamma\Big(\frac{1+s+\frac{l}{2}+i\varpi_k(\lambda)}{2}\Big)$$
$$\frac{1}{\pi}\sin\Big(\pi(\frac{2+s+\frac{l}{2}+i\varpi_k(\lambda)}{2})\Big)\Gamma\Big(\frac{2+s+\frac{l}{2}+i\varpi_k(\lambda)}{2}\Big)].$$

Proof. Using the definition of L-function, we have

$$\frac{L(1+s+\frac{l}{2},\pi,\rho)}{L(-s-\frac{l}{2},\pi^\vee,\rho)} = \frac{\prod_{k=1}^{n}\pi^{-\frac{1+s+\frac{l}{2}+i\varpi_k(\lambda)}{2}}\Gamma\big(\frac{1+s+\frac{l}{2}+i\varpi_k(\lambda)}{2}\big)}{\prod_{k=1}^{n}\pi^{\frac{s+\frac{l}{2}+i\varpi_k(\lambda)}{2}}\Gamma\big(-\frac{s+\frac{l}{2}+i\varpi_k(\lambda)}{2}\big)}$$
$$= \prod_{k=1}^{n}\pi^{-(\frac{1}{2}+s+\frac{l}{2}+i\varpi_k(\lambda))}\frac{\Gamma\big(\frac{1+s+\frac{l}{2}+i\varpi_k(\lambda)}{2}\big)}{\Gamma\big(-\frac{s+\frac{l}{2}+i\varpi_k(\lambda)}{2}\big)}.$$

Using the functional equation for $\Gamma(z)$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)},$$

we obtain

$$\frac{1}{\Gamma\big(-\frac{s+\frac{l}{2}+i\varpi_k(\lambda)}{2}\big)} = \frac{1}{\pi}\sin\Big(\pi(\frac{2+s+\frac{l}{2}+i\varpi_k(\lambda)}{2})\Big)\Gamma\Big(\frac{2+s+\frac{l}{2}+i\varpi_k(\lambda)}{2}\Big).$$

It follows that

$$\prod_{k=1}^{n}\pi^{-(\frac{1}{2}+s+\frac{l}{2}+i\varpi_k(\lambda))}\frac{\Gamma\big(\frac{1+s+\frac{l}{2}+i\varpi_k(\lambda)}{2}\big)}{\Gamma\big(-\frac{s+\frac{l}{2}+i\varpi_k(\lambda)}{2}\big)}$$
$$= \prod_{k=1}^{n}[\pi^{-(\frac{1}{2}+s+\frac{l}{2}+i\varpi_k(\lambda))}\Gamma\Big(\frac{1+s+\frac{l}{2}+i\varpi_k(\lambda)}{2}\Big)$$
$$\cdot\frac{1}{\pi}\sin\Big(\pi(\frac{2+s+\frac{l}{2}+i\varpi_k(\lambda)}{2})\Big)\Gamma\Big(\frac{2+s+\frac{l}{2}+i\varpi_k(\lambda)}{2}\Big)].$$

□

We write $\lambda = x+iy$ with $x \in \mathfrak{a}^*$ and $y \in C^{\varepsilon\rho_B}$, and we notice that the function $\frac{1}{\pi}\sin\Big(\pi(\frac{2+s+\frac{l}{2}+i\varpi_k(\lambda)}{2})\Big)$ is a Paley-Wiener function in λ, hence lies in $\mathcal{S}(\mathfrak{a}^*_\varepsilon)$ as the space $\mathcal{S}(\mathfrak{a}^*_\varepsilon)$ contains all the Paley-Wiener functions. The function $\pi^{-(\frac{1}{2}+s+\frac{l}{2}+i\varpi_k(\lambda))}$ is bounded. Then combining with Theorem 3.4.4 and the fact that $\mathcal{S}(\mathfrak{a}_\varepsilon^*)$ is a Fréchet algebra, we know that if $\mathrm{Re}(s+1+\frac{l}{2})$ is bigger than $\max\{\varpi_k(\mu)| \quad 1 \leq k \leq n, \mu \in C^{\varepsilon\rho_B}\}$ and $\mathrm{Re}(s+2+\frac{l}{2})$ is bigger than $\max\{\varpi_k(\mu)|1 \leq k \leq n, \mu \in C^{\varepsilon\rho_B}\}$, the function $\mathcal{H}(\Phi^K_{\psi,\rho,s})$ lies in $\mathcal{S}(\mathfrak{a}_\varepsilon^*)$. Using the fact that $\pi_\lambda \cong \pi_{w\lambda}$ for $w \in W$, we know that $\mathcal{H}(\Phi^K_{\psi,\rho,s})$ lies in $\mathcal{S}(\mathfrak{a}_\varepsilon^*)^W$.

In other words, we have proved the following asymptotic for $\Phi^K_{\psi,\rho,s}$

Theorem 3.4.7. *If* $\mathrm{Re}(s)$ *satisfies the following inequality*

$$\mathrm{Re}(s) > -1 - \frac{l}{2} + max\{\varpi_k(\mu)| \quad 1 \leq k \leq n, \mu \in C^{\varepsilon\rho_B}\},$$

then the function $\Phi^K_{\psi,\rho,s}$ *lies in* $\mathcal{S}^p(K\backslash G/K)$.

We can also show that the Fourier transform $\mathcal{F}_\rho$ preserves $1_{\rho,-\frac{l}{2}}$. The proof is just the same as the p-adic case by verifying that they have the same image under spherical Plancherel transform.

Remark 3.4.8. *We make a remark on the function space* $\mathcal{S}_\rho(G,K)$. *In* [GJ72], *the authors defined the space* $\mathcal{S}_{\mathrm{std}}(G)$ *to be the derivatives of the basic function* 1_{std}, *which is not the restriction of the classical Schwartz functions on* M_n *to* G. *Using the classical theory of Fourier transform, one can show that* $\mathcal{S}_{\mathrm{std}}(G)$ *is fixed by* $\mathcal{F}_{\mathrm{std}}$. *Moreover, using Casselman's subrepresentation theorem* [CMc82], *one can show that the function space* $\mathcal{S}_{\mathrm{std}}(G)$ *is enough for us to obtain the standard L-factors.*

Let $\mathbb{C}[\mathfrak{g}]$ *be the polynomial ring on* $\mathfrak{g}$ *and let* $U(\mathfrak{g})$ *be the universal enveloping algebra of* $\mathfrak{g}$. *Since* $\mathcal{S}_{\mathrm{std}}(G)$ *is invariant under multiplication by* $\mathbb{C}[\mathfrak{g}]$ *and* $U(\mathfrak{g})$, *the function space* $\mathcal{S}_{\mathrm{std}}(G)$ *is a Weyl algebra module, which means that the space* $\mathcal{S}_{\mathrm{std}}(G)$ *has a nice algebraic structure. It seems that* $\mathcal{S}_{\mathrm{std}}(G)$ *defined in* [GJ72] *does not carry any natural topological structure. In general, we might hope that our function space* $\mathcal{S}_\rho(G)$ *carries natural topological structure like the Fréchet topology on classical Schwartz space.*

On the other hand, one may ask why we do not set up our space $\mathcal{S}_\rho(G,K)$ *to be just* $1_{\rho,-\frac{l}{2}} * C_c^\infty(G,K)$ *as in p-adic case. Here we notice that the L-factor cannot be written as the fraction of two functions in the Paley-Wiener space* $\mathcal{P}(\mathfrak{a}_\mathbb{C}^*)$, *since the function* $\Gamma(z)$ *satisfies the limit*

$$\lim_{|z|\to\infty, |\arg z|<\pi} \frac{\Gamma(z)}{e^{z\log z}} = 1.$$

In other words, the function space $L(-\frac{l}{2}, \pi_\lambda, \rho)\mathcal{P}(\mathfrak{a}_\mathbb{C}^*)$ *does not contain* $\mathcal{P}(\mathfrak{a}_\mathbb{C}^*)$ *as a proper subspace. We can define* $\mathcal{S}_\rho(G,K)$ *to be the space of functions generated additively by* $1_{\rho,-\frac{l}{2}}$, $C_c^\infty(G,K)$ *and* $\mathcal{F}_\rho(C_c^\infty(G,K))$.

Then $\mathcal{S}_\rho(G,K)$ naturally contains $1_{\rho,-\frac{l}{2}}$ and is fixed by $\mathcal{F}_\rho$, but the algebraic and topological structure is not clear as the p-adic case.

3.5. **Asymptotic of $1_{\rho,s}$ and $\Phi^K_{\psi,\rho,s}$: Complex Case.** Following the proof in the real case, we describe an explicit formula for $L(s,\pi_\lambda,\rho)$.

By definition, π_λ is induced from the character

$$m\exp(H)n \to e^{i\lambda(H)}.$$

If we assume that $\lambda = (\lambda_1, ..., \lambda_m) \in \mathfrak{a}^*$, where m is 1 plus the semisimple rank of G, then its associated Langlands parameter is of the form

$$t \in W_{\mathbb{C}} \cong \mathbb{C}^\times \to \begin{pmatrix} |t|^{i\lambda_1} & & & \\ & |t|^{i\lambda_2} & & \\ & & \ddots & \\ & & & |t|^{i\lambda_m} \end{pmatrix}$$

$$= \begin{pmatrix} |t|_{\mathbb{C}}^{\frac{i\lambda_1}{2}} & & & \\ & |t|_{\mathbb{C}}^{\frac{i\lambda_2}{2}} & & \\ & & \ddots & \\ & & & |t|_{\mathbb{C}}^{\frac{i\lambda_m}{2}} \end{pmatrix}.$$

Assume that ρ has weights $\varpi_1, \varpi_2, ..., \varpi_n$, where $n = \dim(V_\rho)$. Then the associated parameter for $\rho(\pi_\lambda)$, which is the functorial lifting image of π_λ along ρ, is

$$t \to \begin{pmatrix} |t|_{\mathbb{C}}^{\frac{i\varpi_1(\lambda)}{2}} & & & \\ & |t|_{\mathbb{C}}^{\frac{i\varpi_2(\lambda)}{2}} & & \\ & & \ddots & \\ & & & |t|_{\mathbb{C}}^{\frac{i\varpi_n(\lambda)}{2}} \end{pmatrix},$$

where $\varpi_j(\lambda) = \sum_{k=1}^m n_k^j \lambda_k$, $n_k \in \mathbb{Z}_{\geq 0}$.

Now we are going to state our result on an asymptotic of $1_{\rho,s}$.

Theorem 3.5.1. *If $\mathrm{Re}(s)$ satisfies the following inequality*

$$\mathrm{Re}(s) > max\{\frac{\varpi_k(\mu)}{2}| \quad 1 \leq k \leq n, \mu \in C^{\varepsilon\rho_B}\},$$

then $1_{\rho,s}$ belongs to $S^p(K\backslash G/K)$. Here $\varepsilon = \frac{2}{p} - 1$, $0 < p \leq 2$, and $\{\varpi_k\}_{k=1}^n$ are the weights of the representation $\rho : {}^LG \to \mathrm{GL}(V_\rho)$.

Proof. By definition

$$L(s,\pi_\lambda,\rho)=\prod_{k=1}^{n}2(2\pi)^{-(\frac{2s+i\varpi_k(\lambda)}{2})}\Gamma\Big(\frac{2s+i\varpi_k(\lambda)}{2}\Big).$$

When $\mathrm{Re}(s)$ is sufficiently large, we want to show that the function $L(s,\pi_\lambda,\rho)$, as a function of λ, lies in the space $S(\mathfrak{a}_\varepsilon^*)^W$. The W-invariance of the function follows from the fact that $\pi_{w\lambda}\cong\pi_\lambda$ for any $w\in W$. Therefore we only need to show the following semi-norm for $L(s,\pi_\lambda,\rho)$

$$\tau_{P,t}^{(\varepsilon)}(L(s,\pi_\lambda,t))=\sup_{\lambda\in\mathfrak{a}_\varepsilon^*}(|\lambda|+1)^tP(\frac{\partial}{\partial\lambda})L(s,\pi_\lambda,\rho)$$

is finite if $\mathrm{Re}(s)$ is bigger than $\max\{\frac{\varpi_k(\mu)}{2}|\quad 1\le k\le n,\mu\in C^{\varepsilon\rho_B}\}$.

Now we are going to estimate

$$\sup_{\lambda\in\mathfrak{a}_\varepsilon^*}(|\lambda|+1)^tP(\frac{\partial}{\partial\lambda})[\prod_{k=1}^{n}(2\pi)^{-(\frac{2s+i\varpi_k(\lambda)}{2})}\Gamma\Big(\frac{2s+i\varpi_k(\lambda)}{2}\Big)].$$

The estimation is almost the same as the real case.

The term

$$P(\frac{\partial}{\partial\lambda})(2\pi)^{-(\frac{2s+i\varpi_k(\lambda)}{2})}$$

is dominated by

$$C_1(|\lambda|+1)^a(2\pi)^{-(\frac{2s+i\varpi_k(\lambda)}{2})}$$

for some $a>0$ and constant $C_1>0$.

For the term

$$P(\frac{\partial}{\partial\lambda})\Gamma\Big(\frac{2s+i\varpi_k(\lambda)}{2}\Big),$$

using Theorem 3.4.2 for the estimation on the derivative of $\Gamma(z)$, it is dominated by

$$C_2(|\lambda|+1)^b\Gamma\Big(\frac{2s+i\varpi_k(\lambda)}{2}\Big)$$

for some $b>0$ and some constant $C_2>0$. Here we use the fact that $\log(z)$ is dominated by $C(|z|+1)$ for some constant C if $\mathrm{Re}(z)$ is bigger than $\max\{\frac{\varpi_k(\mu)}{2}|\quad 1\le k\le n,\mu\in C^{\varepsilon\rho_B}\}$.

Hence we only need to show that the following term is bounded

$$\sup_{\lambda\in\mathfrak{a}_\varepsilon^*}(|\lambda|+1)^t\prod_{k=1}^{n}(2\pi)^{-(\frac{2s+i\varpi_k(\lambda)}{2})}\Gamma\Big(\frac{2s+i\varpi_k(\lambda)}{2}\Big).$$

When $\lambda \in \mathfrak{a}_\varepsilon^* = \mathfrak{a}^* + iC^{\varepsilon\rho}$, the real part of $\frac{2s+i\varpi_k(\lambda)}{2}$ is bounded and lies in a compact set, so the function $(2\pi)^{-(\frac{2s+i\varpi_k(\lambda)}{2})}$ is always bounded. Using Theorem 3.4.1 for the estimation for $\Gamma(x+iy)$ for $x \in \mathbb{R}$ fixed, we have

$$\sup_{\lambda \in \mathfrak{a}_\varepsilon^*}(|\lambda|+1)^t \prod_{k=1}^n (2\pi)^{-(\frac{2s+i\varpi_k(\lambda)}{2})}\Gamma\Big(\frac{2s+i\varpi_k(\lambda)}{2}\Big) \leq$$
$$\sup_{\lambda \in \mathfrak{a}_\varepsilon^*} C(|\lambda|+1)^t(\sqrt{2\pi})^n \prod_{k=1}^n [|\frac{2\mathrm{Im}(s)+\varpi_k(x)}{2}|^{\frac{2\mathrm{Re}(s)-\varpi_k(y)-1}{2}}$$
$$\cdot e^{\frac{\varpi_k(y)}{2}-\frac{2\mathrm{Re}(s)}{2}-\frac{|2\mathrm{Im}(s)+\varpi_k(x)|\pi}{4}}]$$

for some constant $C > 0$. Here we write $\lambda = x + iy$ with $x \in \mathfrak{a}^*$, $y \in C^{\varepsilon\rho}$.

Now we know that $s \in \mathbb{C}$ is fixed, and y lies in $C^{\varepsilon\rho}$, which is a compact set. The term $\varpi_k(\lambda)$ is also dominated by a polynomial function in $|\lambda|+1$. Therefore up to a constant and a polynomial in $(|\lambda|+1)$, we only need to evaluate the following term

$$\sup_{x \in \mathfrak{a}^*}(|x|+1)^t \prod_{k=1}^n e^{-\frac{|\varpi_k(x)|\pi}{4}}.$$

By Lemma 3.4.3, it is bounded by

$$\sup_{x \in \mathfrak{a}^*}(|x|+1)^t \prod_{k=1}^m e^{-\frac{C_\rho|x_k|\pi}{4}}.$$

which is bounded by a constant. This proves the theorem. □

We can also prove an asymptotic for $\Phi^K_{\psi,\rho,s}$. By definition, the spherical component of $\Phi^K_{\psi,\rho,s}$ is determined via the following identity

$$\mathcal{H}(\Phi^K_{\psi,\rho,s}) = \frac{L(1+s+\frac{l}{2},\pi,\rho)}{L(-s-\frac{l}{2},\pi^\vee,\rho)}.$$

Here we notice that if π has Langlands parameter

$$t \to \begin{pmatrix} |t|^{i\lambda_1} & & & \\ & |t|^{i\lambda_2} & & \\ & & \ddots & \\ & & & |t|^{i\lambda_m} \end{pmatrix},$$

then $\pi^\vee$ has Langlands parameter

$$t \to \begin{pmatrix} |t|^{-i\lambda_1} & & & \\ & |t|^{-i\lambda_2} & & \\ & & \ddots & \\ & & & |t|^{-i\lambda_m} \end{pmatrix}.$$

We first simplify the expression for γ-factor

Lemma 3.5.2. *The formula* $\mathcal{H}(\Phi^K_{\psi,\rho,s}) = \frac{L(1+s+\frac{l}{2},\pi,\rho)}{L(-s-\frac{l}{2},\pi^\vee,\rho)}$ *can be simplified to be*

$$\prod_{k=1}^{n}[\pi^{-(\frac{1}{2}+2s+\frac{l}{2}+i\varpi_k(\lambda))}\Gamma\Big(\frac{1+2s+\frac{l}{2}+i\varpi_k(\lambda)}{2}\Big)$$
$$\frac{1}{\pi}\sin\Big(\pi(\frac{2+2s+\frac{l}{2}+i\varpi_k(\lambda)}{2})\Big)\Gamma\Big(\frac{2+2s+\frac{l}{2}+i\varpi_k(\lambda)}{2}\Big)].$$

Proof. Using the definition of L-function, we have

$$\begin{aligned}\frac{L(1+s+\frac{l}{2},\pi,\rho)}{L(-s-\frac{l}{2},\pi^\vee,\rho)} &= \frac{\prod_{k=1}^n \pi^{-\frac{1+2s+\frac{l}{2}+i\varpi_k(\lambda)}{2}}\Gamma(\frac{1+2s+\frac{l}{2}+i\varpi_k(\lambda)}{2})}{\prod_{k=1}^n \pi^{\frac{2s+\frac{l}{2}+i\varpi_k(\lambda)}{2}}\Gamma(-\frac{2s+\frac{l}{2}+i\varpi_k(\lambda)}{2})} \\ &= \prod_{k=1}^n \pi^{-(\frac{1}{2}+2s+\frac{l}{2}+i\varpi_k(\lambda))}\frac{\Gamma(\frac{1+2s+\frac{l}{2}+i\varpi_k(\lambda)}{2})}{\Gamma(-\frac{2s+\frac{l}{2}+i\varpi_k(\lambda)}{2})}.\end{aligned}$$

Using the functional equation for $\Gamma(z)$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

we get

$$\frac{1}{\Gamma\big(-\frac{2s+\frac{l}{2}+i\varpi_k(\lambda)}{2}\big)} = \frac{1}{\pi}\sin\Big(\pi(\frac{2+2s+\frac{l}{2}+i\varpi_k(\lambda)}{2})\Big)\Gamma\Big(\frac{2+2s+\frac{l}{2}+i\varpi_k(\lambda)}{2}\Big).$$

It follows that

$$\prod_{k=1}^n \pi^{-(\frac{1}{2}+2s+\frac{l}{2}+i\varpi_k(\lambda))}\frac{\Gamma\big(\frac{1+2s+\frac{l}{2}+i\varpi_k(\lambda)}{2}\big)}{\Gamma\big(-\frac{2s+\frac{l}{2}+i\varpi_k(\lambda)}{2}\big)}$$
$$= \prod_{k=1}^{n}[\pi^{-(\frac{1}{2}+2s+\frac{l}{2}+i\varpi_k(\lambda))}\Gamma\Big(\frac{1+2s+\frac{l}{2}+i\varpi_k(\lambda)}{2}\Big)$$
$$\frac{1}{\pi}\sin\Big(\pi(\frac{2+2s+\frac{l}{2}+i\varpi_k(\lambda)}{2})\Big)\Gamma\Big(\frac{2+2s+\frac{l}{2}+i\varpi_k(\lambda)}{2}\Big)].$$

□

We write $\lambda = x + iy$ with $x \in \mathfrak{a}^*$ and $y \in C^{\varepsilon\rho_B}$, and we notice that the function $\frac{1}{\pi}\sin\left(\pi(\frac{2+2s+\frac{l}{2}+i\varpi_k(\lambda)}{2})\right)$ is a Paley-Wiener function in λ, hence lies in $\mathcal{S}(\mathfrak{a}_\varepsilon^*)$. The function $\pi^{-(\frac{1}{2}+2s+\frac{l}{2}+i\varpi_k(\lambda))}$ is bounded. Then combining with Theorem 3.5.1 and the fact that $\mathcal{S}(\mathfrak{a}_\varepsilon^*)$ is a Fréchet algebra, we know that if $\mathrm{Re}(2s+1+\frac{l}{2})$ is bigger than $\max\{\varpi_k(\mu)| \quad 1 \leq k \leq n, \mu \in C^{\varepsilon\rho_B}\}$ and $\mathrm{Re}(2s+2+\frac{l}{2})$ is bigger than $\max\{\varpi_k(\mu)|1 \leq k \leq n, \mu \in C^{\varepsilon\rho_B}\}$, the function $\mathcal{H}(\Phi^K_{\psi,\rho,s})$ lies in $\mathcal{S}(\mathfrak{a}_\varepsilon^*)$. Using the fact that $\pi_\lambda \cong \pi_{w\lambda}$ for $w \in W$, we know that $\mathcal{H}(\Phi^K_{\psi,\rho,s})$ lies in $\mathcal{S}(\mathfrak{a}_\varepsilon^*)^W$.

In other words, we have proved the following asymptotic for $\Phi^K_{\psi,\rho,s}$

Theorem 3.5.3. *If* $\mathrm{Re}(s)$ *satisfies the following inequality*

$$\mathrm{Re}(s) > -\frac{1}{2} - \frac{l}{4} + max\{\frac{\varpi_k(\mu)}{2}| \quad 1 \leq k \leq n, \mu \in C^{\varepsilon\rho_B}\},$$

then the function $\Phi^K_{\psi,\rho,s}$ *lies in* $\mathcal{S}^p(K\backslash G/K)$.

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