

DENDRITES AND SYMMETRIC PRODUCTS

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ABSTRACT. For a given continuum X and a natural number n , we consider the hyperspace $F_n(X)$ of all nonempty subsets of X with at most n points, metrized by the Hausdorff metric. In this paper we show that if X is a dendrite whose set of end points is closed, $n \in \mathbb{N}$ and Y is a continuum such that the hyperspaces $F_n(X)$ and $F_n(Y)$ are homeomorphic, then Y is a dendrite whose set of end points is closed.

1. INTRODUCTION

A *continuum* is a nondegenerate, compact, connected metric space. Let \mathbb{N} represent the set of positive integers. For a given continuum X and $n \in \mathbb{N}$, we consider the following hyperspaces of X :

$$F_n(X) = \{A \subset X : A \text{ is nonempty and has at most } n \text{ points}\}$$

and

$$C_n(X) = \{A \subset X : A \text{ is closed, nonempty and has at most } n \text{ components}\}.$$

We call $C_n(X)$ the *n-fold hyperspace of X* and $F_n(X)$ the *n-th symmetric product of X* . Both $F_n(X)$ and $C_n(X)$ are metrized by the Hausdorff metric H ([17, Definition 2.1]).

If two continua X and Y are homeomorphic, we write $X \approx Y$. Note that if X and Y are continua, then $X \approx Y$ if and only if $F_1(X) \approx F_1(Y)$. Let \mathcal{G} be a class of continua, $n \in \mathbb{N}$ and $X \in \mathcal{G}$. We say that X has *unique hyperspace $F_n(X)$ in \mathcal{G}* if whenever $Y \in \mathcal{G}$ is such that $F_n(X) \approx F_n(Y)$, it follows that $X \approx Y$. Similarly, X has *unique hyperspace $C_n(X)$ in \mathcal{G}* if whenever $Y \in \mathcal{G}$ is such that $C_n(X) \approx C_n(Y)$, we have $X \approx Y$. If \mathcal{G} is the class of all continua, we simply say that X has unique hyperspace $F_n(X)$ or unique hyperspace $C_n(X)$, respectively. Note that each continuum X has unique hyperspace $F_1(X)$.

A *dendrite* is a locally connected continuum that contains no simple closed curves. Throughout this paper we denote by \mathcal{D} the class of dendrites whose set of end points is closed. In [11, Theorem 10] it is shown that if $X \in \mathcal{D}$ is not an arc, then X has unique hyperspace $C_1(X)$. In [16, Theorem 3]

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that every $X \in \mathcal{D}$ has unique hyperspace $C_2(X)$, and in [12, Theorem 5.7] that each $X \in \mathcal{D}$ has also unique hyperspace $C_n(X)$, for $n \geq 3$. In [2, Theorem 5.1] it is proved that if the set of end points of the dendrite Y is not closed, then Y does not have unique hyperspace $C_1(Y)$ in the class of dendrites. This result is not known for $n \geq 2$. By [1, Lemma 11] an arc Y has unique hyperspace $C_1(Y)$ in the class of dendrites, but not in the class of all continua.

In the First Workshop in Hyperspaces and Continuum Theory, celebrated in the city of Puebla, Mexico, July 2-13, 2007, the problem to determine if every element $X \in \mathcal{D}$ has unique hyperspace $F_n(X)$ was asked by A. Illanes. During the workshop, the three authors of this paper showed that if $X \in \mathcal{D}$, $n \in \mathbb{N}$ and Y is a continuum such that $F_n(X) \approx F_n(Y)$, then $Y \in \mathcal{D}$. This is the main result of this paper. In the same workshop, D. Herrera-Carrasco, M. de J. López and F. Macías-Romero proved that every element $X \in \mathcal{D}$ has unique hyperspace $F_n(X)$ in \mathcal{D} ([13, Theorem 3.5]). Combining these results it follows that every element $X \in \mathcal{D}$ has unique hyperspace $F_n(X)$. This is a partial positive answer to the following problem, which remains open.

Question 1.1. *Let X be a dendrite and $n \in \mathbb{N} - \{1\}$. Does X have unique hyperspace $F_n(X)$?*

2. GENERAL NOTIONS AND FACTS

All spaces considered in this paper are assumed to be metric. For a space X , a point $x \in X$ and a positive number ε , we denote by $B_X(x, \varepsilon)$ the open ball in X centered at x and having radius ε . If A is a subset of the space X , we use the symbols $\text{cl}_X(A)$, $\text{int}_X(A)$ and $\text{bd}_X(A)$ to denote the closure, the interior and the boundary of A in X , respectively. We denote the diameter of A by $\text{diam}(A)$, and the cardinality of A by $|A|$. The letter I stands for the unit interval $[0, 1]$ in the real line \mathbb{R} .

A *finite graph* is a continuum that can be written as the union of finitely many arcs, each two of which intersect in a finite set. A *tree* is a finite graph that contains no simple closed curves.

If X is a continuum, $U_1, U_2, \dots, U_m \subset X$ and $n \in \mathbb{N}$ we define:

$$\langle U_1, U_2, \dots, U_m \rangle_n = \left\{ A \in F_n(X) : A \subset \bigcup_{i=1}^m U_i \text{ and } A \cap U_i \neq \emptyset \text{ for each } i \right\}.$$

It is known that the sets of the form $\langle U_1, U_2, \dots, U_m \rangle_n$, where $m \in \mathbb{N}$ and U_1, U_2, \dots, U_m are open in X , form a basis of the topology of $F_n(X)$, i.e., a basis for the topology induced by the Hausdorff metric H on $F_n(X)$ ([17, Theorems 1.2 and 3.1]).

If $n \in \mathbb{N}$, then an *n-cell* is a space homeomorphic to the Cartesian product I^n .

Theorem 2.1. *Let X be a continuum and $n \in \mathbb{N}$. Given $i \in \{1, 2, \dots, n\}$ let J_i be an arc in X with end points a_i and b_i . If the sets J_1, J_2, \dots, J_n are*

pairwise disjoint, then $\langle J_1, J_2, \dots, J_n \rangle_n$ is an n -cell in $F_n(X)$ whose manifold interior is the set $\langle J_1 - \{a_1, b_1\}, \dots, J_n - \{a_n, b_n\} \rangle_n$.

Proof. Given $(x_1, x_2, \dots, x_n) \in J_1 \times J_2 \times \dots \times J_n$, let $g(x_1, x_2, \dots, x_n) = \{x_1, x_2, \dots, x_n\}$. It is easy to see that $g: J_1 \times J_2 \times \dots \times J_n \rightarrow \langle J_1, J_2, \dots, J_n \rangle_n$ is a homeomorphism whose restriction to $\prod_{i=1}^n (J_i - \{a_i, b_i\})$ is a homeomorphism from $\prod_{i=1}^n (J_i - \{a_i, b_i\})$ onto $\langle J_1 - \{a_1, b_1\}, \dots, J_n - \{a_n, b_n\} \rangle_n$. \square

From now on, in this section, the letter X represents a dendrite. For properties of dendrites we refer the reader to [21, Chapter 10]. If $p \in X$ then by the *order of p in X* , denoted by $\text{ord}_p X$, we mean the Menger-Urysohn order (see [21, Definition 9.3] and [23, (1.1), (iv), p. 88]). We say that $p \in X$ is an *end point of X* if $\text{ord}_p X = 1$. The set of all such points is denoted by $E(X)$. Let

$$E_a(X) = \{p \in E(X) : \text{there is a sequence in } E(X) - \{p\} \text{ that converges to } p\}.$$

If $p \in E(X) - E_a(X)$ we call p an *isolated end point of X* . If $\text{ord}_p X = 2$ we say that p is an *ordinary point of X* . The set of all such points is denoted by $O(X)$. By [18, Theorem 8, p. 302], $O(X)$ is dense in X . If $\text{ord}_p X \geq 3$, we say that p is a *ramification point of X* . The set of all such points is denoted by $R(X)$.

The following result is easy to prove.

Theorem 2.2. *Let X be a dendrite and $n \in \mathbb{N}$. Assume that $A \in F_n(X)$ and that \mathcal{U} is a neighborhood of A in $F_n(X)$. Then, for each $k \in \mathbb{N}$ with $|A| \leq k \leq n$, there is $C \subset O(X)$ such that $|C| = k$ and $C \in \mathcal{U}$.*

If $p, q \in X$ and $p \neq q$, then there is only one arc in X joining p and q . We denote such arc by $[p, q]$. We also consider the sets $(p, q) = [p, q] - \{p, q\}$, $[p, q) = [p, q] - \{q\}$ and $(p, q] = [p, q] - \{p\}$. Let $[p, q]$ be an arc in X such that $(p, q) \subset O(X)$. We say that $[p, q]$ is:

- a) *internal* if $p, q \in R(X)$;
- b) *external* if one end point of $[p, q]$ is an end point of X , and the other end point of $[p, q]$ is a ramification point of X .

Note that if $[p, q]$ is an internal arc in X , then $\text{int}_X([p, q]) = (p, q)$. If $[p, q]$ is an external arc in X and $p \in E(X)$, then $\text{int}_X([p, q]) = [p, q)$.

Given $n \in \mathbb{N}$, we consider the following subsets of $F_n(X)$:

$$EA_n(X) = \{A \in F_n(X) : A \cap E_a(X) \neq \emptyset\},$$

$$R_n(X) = \{A \in F_n(X) : A \cap R(X) \neq \emptyset\}$$

and

$$\Lambda_n(X) = F_n(X) - (R_n(X) \cup EA_n(X)).$$

Note that $A \in \Lambda_n(X)$ if and only if $A \in F_n(X)$ and A is contained in $O(X) \cup (E(X) - E_a(X))$.

3. THE CLASS \mathcal{D}

Recall that \mathcal{D} is the class of all dendrites whose set of end points is closed. Let $X \in \mathcal{D}$. By [3, Theorem 3.3] the order of every point of X is finite. Let us assume that $s \in X$ is the limit of a sequence $(s_n)_n$ of distinct ramification points of X and that $s \neq s_1$. By [3, Proposition 3.4] s is both the limit of a sequence of ramification points of X , all in the arc $[s, s_1]$, and the limit of a sequence of end points, all different than s . Now assume that $e \in X$ is the limit of a sequence $(e_n)_n$ of distinct end points of X and that $e \neq e_1$. Then e is also the limit of a sequence of ramification points of X , all in the arc $[e, e_1]$. Hence $e \in E_a(X)$ if and only if e is the limit of a sequence of ramification points of X .

Theorem 3.1. *Let $X \in \mathcal{D}$ and $n \in \mathbb{N}$. If $A \in F_n(X) - EA_n(X)$, then there exists a tree T in X such that:*

$$(*) \quad A \subset \text{int}_X(T) \text{ and } T \cap E_a(X) = \emptyset.$$

Proof. Let $A \in F_n(X) - EA_n(X)$. We proceed by induction over $|A|$. If $|A| = 1$, then $A = \{x\}$. Let $k = \text{ord}_x X$. Since $X \in \mathcal{D}$, k is finite, so $X - \{x\}$ has exactly k components C_1, C_2, \dots, C_k . Since $x \notin E_a(X)$, for each $i = \{1, 2, \dots, k\}$, there is $p_i \in O(X) \cap C_i$ such that if $T = \bigcup_{i=1}^k [p_i, x]$, then $T - \{x\} \subset O(X)$. Hence T is a tree in X that satisfies $(*)$.

Now suppose that if $B \in F_n(X) - EA_n(X)$ contains i points, with $i < n$, then there is a tree G in X that satisfies $(*)$, replacing A by B and T by G , respectively. Assume that $|A| = i + 1$ and let $A = \{x_1, x_2, \dots, x_{i+1}\}$. Let T_1 be a tree in X such that $A - \{x_{i+1}\} \subset \text{int}_X(T_1)$ and $T_1 \cap E_a(X) = \emptyset$. By the first part of this proof, there exists a tree T_2 in X such that $x_{i+1} \in \text{int}_X(T_2)$ and $T_2 \cap E_a(X) = \emptyset$. Thus $T = T_1 \cup T_2 \cup [x_i, x_{i+1}]$ is a tree in X that satisfies $(*)$. \square

Theorem 3.2. *Let $X \in \mathcal{D}$ and $m, n \in \mathbb{N}$ so that $m \leq n$. Let $\mathcal{U} = \langle U_1, U_2, \dots, U_m \rangle_n$ be an open subset of $F_n(X)$ such that:*

- 1) U_i is an open connected subset of X , for each $i \in \{1, 2, \dots, m\}$;
- 2) $U_i \cap U_j = \emptyset$ if $i, j \in \{1, 2, \dots, m\}$ and $i \neq j$.

For each $i \in \{1, 2, \dots, m\}$, let $\{J_\alpha^i : \alpha \in \mathcal{A}_i\}$ be the set of components of $U_i \cap [O(X) \cup (E(X) - E_a(X))]$. Then the components of $\mathcal{U} \cap \Lambda_n(X)$ are the nonempty sets of the form:

$$\langle J_{\alpha_1}^{r_1}, J_{\alpha_2}^{r_2}, \dots, J_{\alpha_k}^{r_k} \rangle_n,$$

where $\{r_1, r_2, \dots, r_k\} = \{1, 2, \dots, m\}$, the sets $J_{\alpha_1}^{r_1}, J_{\alpha_2}^{r_2}, \dots, J_{\alpha_k}^{r_k}$ are pairwise different and $\alpha_t \in \mathcal{A}_{r_t}$, for every $t \in \{1, 2, \dots, k\}$.

Proof. It is easy to see that for each $i \in \{1, 2, \dots, m\}$ and every $\alpha \in \mathcal{A}_i$, J_α^i is an open connected subset of X . Let $J_{\alpha_1}^{r_1}, J_{\alpha_2}^{r_2}, \dots, J_{\alpha_k}^{r_k}$ be a finite collection of pairwise different sets such that $\{r_1, r_2, \dots, r_k\} = \{1, 2, \dots, m\}$ and $\alpha_t \in \mathcal{A}_{r_t}$, for every $t \in \{1, 2, \dots, k\}$. Since the sets $J_{\alpha_1}^{r_1}, J_{\alpha_2}^{r_2}, \dots, J_{\alpha_k}^{r_k}$ are open and

connected, by [19, Lemma 1],

$$\langle J_{\alpha_1}^{r_1}, J_{\alpha_2}^{r_2}, \dots, J_{\alpha_k}^{r_k} \rangle_n$$

is an open connected subset of $F_n(X)$. Let $J_{\epsilon_1}^{s_1}, J_{\epsilon_2}^{s_2}, \dots, J_{\epsilon_l}^{s_l}$ be a finite collection of pairwise different sets such that: $\{s_1, s_2, \dots, s_l\} = \{1, 2, \dots, m\}$, $\epsilon_v \in \mathcal{A}_{s_v}$, for every $v \in \{1, 2, \dots, l\}$, and

$$\{J_{\alpha_1}^{r_1}, J_{\alpha_2}^{r_2}, \dots, J_{\alpha_k}^{r_k}\} \neq \{J_{\epsilon_1}^{s_1}, J_{\epsilon_2}^{s_2}, \dots, J_{\epsilon_l}^{s_l}\}.$$

It is not difficult to see that:

$$\langle J_{\alpha_1}^{r_1}, J_{\alpha_2}^{r_2}, \dots, J_{\alpha_k}^{r_k} \rangle_n \cap \langle J_{\epsilon_1}^{s_1}, J_{\epsilon_2}^{s_2}, \dots, J_{\epsilon_l}^{s_l} \rangle_n = \emptyset.$$

Now assume that \mathcal{C} is a component of $\mathcal{U} \cap \Lambda_n(X)$. Note that, for every $A \in \mathcal{C}$, there is a unique finite collection

$$J_{\sigma_1}^{s_1}, J_{\sigma_2}^{s_2}, \dots, J_{\sigma_w}^{s_w}$$

of pairwise different sets such that: $\{s_1, s_2, \dots, s_w\} = \{1, 2, \dots, m\}$, $\sigma_j \in \mathcal{A}_{s_j}$, for each $j \in \{1, 2, \dots, w\}$, and

$$A \in \mathcal{V}_A = \langle J_{\sigma_1}^{s_1}, J_{\sigma_2}^{s_2}, \dots, J_{\sigma_w}^{s_w} \rangle_n.$$

Hence $\mathcal{C} = \bigcup_{A \in \mathcal{C}} \mathcal{V}_A$, which expresses the open connected set \mathcal{C} as a union of nonempty pairwise disjoint open connected sets. Thus \mathcal{C} is of the form $\langle J_{\alpha_1}^{r_1}, J_{\alpha_2}^{r_2}, \dots, J_{\alpha_k}^{r_k} \rangle_n$ where $\{r_1, r_2, \dots, r_k\} = \{1, 2, \dots, m\}$, the sets $J_{\alpha_1}^{r_1}, J_{\alpha_2}^{r_2}, \dots, J_{\alpha_k}^{r_k}$ are pairwise different and $\alpha_t \in \mathcal{A}_{r_t}$, for every $t \in \{1, 2, \dots, k\}$. \square

Assume that $X \in \mathcal{D}$. It is not difficult to see that $\Lambda_n(X)$ is an open subset of $F_n(X)$. As a particular case of Theorem 3.2 we obtain the following result, which is the equivalent version of [7, Lemma 4.1] for elements of \mathcal{D} .

Theorem 3.3. *Let $X \in \mathcal{D}$ such that X is not an arc and $n \in \mathbb{N}$. Then the components of $\Lambda_n(X)$ are exactly the sets of the form:*

$$\langle \text{int}_X(I_1), \text{int}_X(I_2), \dots, \text{int}_X(I_m) \rangle_n,$$

where $m \leq n$, I_j is either an internal or an external arc in X for every $j \in \{1, 2, \dots, m\}$, and the sets $\text{int}_X(I_1), \text{int}_X(I_2), \dots, \text{int}_X(I_m)$ are pairwise disjoint.

The following result is the equivalent version of [7, Lemma 4.3], for elements of \mathcal{D} .

Theorem 3.4. *Let $X \in \mathcal{D}$ and $n \geq 4$. If $A \in F_{n-1}(X)$, then no neighborhood of A in $F_n(X)$ can be embedded in \mathbb{R}^n .*

Proof. We show first that:

- (*) if $C \in F_{n-1}(X) - EA_n(X)$, then no neighborhood of C in $F_n(X)$ can be embedded in \mathbb{R}^n .

To show (*) let $C \in F_{n-1}(X) - EA_n(X)$ and assume that there is a neighborhood \mathcal{V} of C in $F_n(X)$ that can be embedded in \mathbb{R}^n . By Theorem 3.1, there is a tree T in X such that $C \subset \text{int}_X(T)$ and $T \cap E_a(X) = \emptyset$. Then $\mathcal{V} \cap F_n(T)$ is a neighborhood of C in $F_n(T)$ that can be embedded in \mathbb{R}^n . Since this contradicts [7, Lemma 4.3], claim (*) holds.

To show the theorem let $A \in F_{n-1}(X)$. Assume that there is a neighborhood \mathcal{U} of A in $F_n(X)$ that can be embedded in \mathbb{R}^n . By Theorem 2.2, there is $C \subset O(X)$ such that $|C| = |A|$ and $C \in \text{int}_{F_n(X)}(\mathcal{U})$. Since $A \in F_{n-1}(X)$ it follows that $C \in F_{n-1}(X) - EA_n(X)$. Then, by (*), no neighborhood of C in $F_n(X)$ can be embedded in \mathbb{R}^n . However, since $C \in \text{int}_{F_n(X)}(\mathcal{U})$, the set \mathcal{U} is a neighborhood of C in $F_n(X)$ that can be embedded in \mathbb{R}^n . This contradiction completes the proof of the theorem. \square

4. THE SET $\mathcal{E}_n(X)$

Given a continuum X and a natural number n , we consider the following set:

$$\mathcal{E}_n(X) = \{A \in F_n(X) : A \text{ has a neighborhood in } F_n(X) \text{ which is an } n\text{-cell}\}.$$

In this section we prove some properties of $\mathcal{E}_n(X)$.

Theorem 4.1. *Let X and Y be continua and $n \in \mathbb{N}$. If $h: F_n(X) \rightarrow F_n(Y)$ is a homeomorphism, then $h(\mathcal{E}_n(X)) = \mathcal{E}_n(Y)$.*

A *simple triod* is a continuum G that can be written as the union of three arcs I_1, I_2 and I_3 such that: $I_1 \cap I_2 \cap I_3 = \{p\}$, p is an end point of each arc I_i and $(I_i - \{p\}) \cap (I_j - \{p\}) = \emptyset$, if $i \neq j$. The point p is called the *core* of G .

Given a continuum X let:

$$T(X) = \{p \in X : p \text{ is the core of a simple triod in } X\}.$$

Let X be a locally connected continuum and $A \in \mathcal{E}_n(X)$. In [7, Lemma 3.1] it is shown that $A \cap T(X) = \emptyset$. A straightforward modification can be applied to obtain the following result.

Theorem 4.2. *Let X be a locally connected continuum and $n \in \mathbb{N}$. If $A \in \mathcal{E}_n(X)$, then $A \cap \text{cl}_X(T(X)) = \emptyset$.*

Theorem 4.3. *Let $X \in \mathcal{D}$ and $n \in \mathbb{N}$. Then $\Lambda_n(X) - F_{n-1}(X) \subset \mathcal{E}_n(X)$.*

Proof. Take $A \in \Lambda_n(X) - F_{n-1}(X)$. Then $|A| = n$, so we can write

$$A = \{x_1, x_2, \dots, x_n\}.$$

Since $A \in \Lambda_n(X)$, we have $A \subset O(X) \cup (E(X) - E_a(X))$. Then there exist n pairwise disjoint arcs J_1, J_2, \dots, J_n in X such that $x_i \in \text{int}_X(J_i)$, for each $i \in \{1, 2, \dots, n\}$, and

$$J_1 \cup J_2 \cup \dots \cup J_n \subset O(X) \cup (E(X) - E_a(X)).$$

Note that $\langle J_1, J_2, \dots, J_n \rangle_n$ is a neighborhood of A in $F_n(X)$ which is an n -cell, by Theorem 2.1. Then $A \in \mathcal{E}_n(X)$. \square

Theorem 4.4. *Let $X \in \mathcal{D}$ and $n \in \mathbb{N}$. Then $\mathcal{E}_n(X)$ is dense in $F_n(X)$.*

Proof. Let \mathcal{U} be a nonempty open subset of $F_n(X)$. By Theorem 2.2 there is $D \subset O(X)$ such that $|D| = n$ and $D \in \mathcal{U}$. Note that $D \in \Lambda_n(X) - F_{n-1}(X)$ so, by Theorem 4.3, $D \in \mathcal{E}_n(X)$. This shows that $\mathcal{E}_n(X)$ is dense in $F_n(X)$. \square

Theorem 4.5. *Let $X \in \mathcal{D}$ and $n \in \mathbb{N}$. Then*

- a) $\mathcal{E}_n(X) \subset \Lambda_n(X)$;
- b) if $n \in \{2, 3\}$, then $\mathcal{E}_n(X) = \Lambda_n(X)$;
- c) if $n \geq 4$, then $\mathcal{E}_n(X) = \Lambda_n(X) - F_{n-1}(X)$.

Proof. To show a) let $A \in \mathcal{E}_n(X)$. By Theorem 4.2, $A \cap \text{cl}_X(T(X)) = \emptyset$. Since $X \in \mathcal{D}$, this implies that $A \cap (R(X) \cup E_a(X)) = \emptyset$. Thus $A \in \Lambda_n(X)$, so a) holds. Assertion b) follows from a) and the proof of [7, Lemma 5.1]. To show c) assume that $n \geq 4$. Take $A \in \mathcal{E}_n(X)$. By a), $A \in \Lambda_n(X)$. Let \mathcal{U} be a neighborhood of A in $F_n(X)$ which is an n -cell. Then \mathcal{U} can be embedded in \mathbb{R}^n so, by Theorem 3.4, $A \notin F_{n-1}(X)$. This shows that $\mathcal{E}_n(X) \subset \Lambda_n(X) - F_{n-1}(X)$. The other inclusion holds by Theorem 4.3. \square

Theorem 4.6. *Let $X \in \mathcal{D}$ and $A \in F_n(X)$. If $A \cap E_a(X) = \emptyset$, then there exists a basis \mathfrak{B} of open neighborhoods of A in $F_n(X)$ such that for each $\mathcal{V} \in \mathfrak{B}$, the set $\mathcal{V} \cap \mathcal{E}_n(X)$ is nonempty and has a finite number of components.*

Proof. Since $A \cap E_a(X) = \emptyset$, we have $A \in F_n(X) - EA_n(X)$. Thus, by Theorem 3.1, there is a tree T in X such that $A \subset \text{int}_X(T)$ and $T \cap E_a(X) = \emptyset$. Let $A = \{x_1, x_2, \dots, x_m\}$ and consider that A has exactly m points. Let $\varepsilon > 0$. Choose a finite collection U_1, U_2, \dots, U_m of pairwise disjoint open connected subsets of X with the following properties:

- 1) $x_i \in U_i \subset \text{int}_X(T) \cap B_X(x_i, \varepsilon)$, for each $i \in \{1, 2, \dots, m\}$;
- 2) $U_i - \{x_i\} \subset O(X)$, for each $i \in \{1, 2, \dots, m\}$.

Let $\mathcal{V}_\varepsilon = \langle U_1, U_2, \dots, U_m \rangle_n$. By 1) we have $\mathcal{V}_\varepsilon \subset B_{F_n(X)}(A, \varepsilon)$ and, by Theorem 4.4, $\mathcal{V}_\varepsilon \cap \mathcal{E}_n(X) \neq \emptyset$. Given $i \in \{1, 2, \dots, m\}$, since $X \in \mathcal{D}$, the order of x_i in X is finite. From this and 2), the set

$$U_i \cap [O(X) \cup (E(X) - E_a(X))]$$

has a finite number of components. Let $\{J_1^i, J_2^i, \dots, J_{l_i}^i\}$ be the set of components of $U_i \cap [O(X) \cup (E(X) - E_a(X))]$. By Theorem 3.2 the components of $\mathcal{V}_\varepsilon \cap \Lambda_n(X)$ are the nonempty sets of the form:

$$(4.1) \quad \langle J_{s_1}^{r_1}, J_{s_2}^{r_2}, \dots, J_{s_k}^{r_k} \rangle_n$$

where $\{r_1, r_2, \dots, r_k\} = \{1, 2, \dots, m\}$, the sets $J_{s_1}^{r_1}, J_{s_2}^{r_2}, \dots, J_{s_k}^{r_k}$ are pairwise different and $s_t \in \{1, 2, \dots, l_{r_t}\}$, for every $t \in \{1, 2, \dots, k\}$. Since we have a finite number of elements of the form $J_{s_t}^{r_t}$, the number of nonempty sets of the form (4.1) is finite.

If $n \in \{2, 3\}$ then, by part b) of Theorem 4.5, the nonempty sets of the form (4.1) are the components of $\mathcal{V}_\varepsilon \cap \mathcal{E}_n(X)$. Assume then that $n \geq 4$. Then,

by part c) of Theorem 4.5, $\mathcal{E}_n(X) = \Lambda_n(X) - F_{n-1}(X)$. Given a component $\mathcal{C} = \langle J_{s_1}^{r_1}, J_{s_2}^{r_2}, \dots, J_{s_k}^{r_k} \rangle_n$ of $\mathcal{V}_\varepsilon \cap \Lambda_n(X)$ and $(q_1, q_2, \dots, q_k) \in \mathbb{N}^k$ such that $q_1 + q_2 + \dots + q_k = n$ let:

$$\mathcal{C}(q_1, q_2, \dots, q_k) = \{C \in \mathcal{C}: |C \cap J_{s_t}^{r_t}| = q_t \text{ for each } t \in \{1, 2, \dots, k\}\}.$$

Note that $\mathcal{C}(q_1, q_2, \dots, q_k) \subset \mathcal{V}_\varepsilon \cap \mathcal{E}_n(X)$. It is not difficult to see that $\mathcal{C}(q_1, q_2, \dots, q_k)$ is homeomorphic to

$$(F_{q_1}(J_{s_1}^{r_1}) - F_{q_1-1}(J_{s_1}^{r_1})) \times \dots \times (F_{q_k}(J_{s_k}^{r_k}) - F_{q_k-1}(J_{s_k}^{r_k})),$$

where we agree that $F_0(R) = \emptyset$ for each continuum R . Since the sets

$$F_{q_1}(J_{s_1}^{r_1}) - F_{q_1-1}(J_{s_1}^{r_1}), \dots, F_{q_k}(J_{s_k}^{r_k}) - F_{q_k-1}(J_{s_k}^{r_k})$$

are connected, $\mathcal{C}(q_1, q_2, \dots, q_k)$ is a connected subset of $\mathcal{V}_\varepsilon \cap \mathcal{E}_n(X)$. Moreover

$$\mathcal{C} \cap \mathcal{E}_n(X) = \bigcup \left\{ \mathcal{C}(q_1, \dots, q_k) : (q_1, \dots, q_k) \in \mathbb{N}^k \text{ and } q_1 + \dots + q_k = n \right\}.$$

This implies that $\mathcal{C} \cap \mathcal{E}_n(X)$ has a finite number of components. Since each component of $\mathcal{C} \cap \mathcal{E}_n(X)$ is a component of $\mathcal{V}_\varepsilon \cap \mathcal{E}_n(X)$ and $\mathcal{V}_\varepsilon \cap \Lambda_n(X)$ has a finite number of components, the set $\mathcal{V}_\varepsilon \cap \mathcal{E}_n(X)$ has a finite number of components as well.

To finish the proof note that $\mathfrak{B} = \{\mathcal{V}_\varepsilon : \varepsilon > 0\}$ is a basis of open neighborhoods of A in $F_n(X)$. \square

In [4] and [20] it is proved that locally connected continua admit a convex metric d . This means that every two points $x, y \in X$ can be joined by an arc J in X , in such a way that J is isometric to the closed interval $[0, d(x, y)]$.

Theorem 4.7. *Let $X \in \mathcal{D}$ and $A \in F_n(X)$. Assume that $A \cap E(X) \neq \emptyset$. Then there exists a basis \mathfrak{B} of open neighborhoods of A in $F_n(X)$ such that, for each $\mathcal{V} \in \mathfrak{B}$, the set $\mathcal{V} - \{A\}$ is contractible. Moreover if $A \cap E_a(X) \neq \emptyset$ we can choose \mathfrak{B} with the additional property that, for each $\mathcal{V} \in \mathfrak{B}$, the set $\mathcal{V} \cap \mathcal{E}_n(X)$ has infinitely many components.*

Proof. Let d be a convex metric on X . Assume that $|A| = m$. Let $A = \{x_1, x_2, \dots, x_m\}$ and assume that $x_1 \in E(X)$. Let $\varepsilon > 0$. Choose a finite collection U_1, U_2, \dots, U_m of pairwise disjoint open connected subsets of X such that $x_i \in U_i \subset B_X(x_i, \varepsilon)$, for each $i \in \{1, 2, \dots, m\}$. Let $\mathcal{V}_\varepsilon = \langle U_1, U_2, \dots, U_m \rangle_n$. Clearly $A \in \mathcal{V}_\varepsilon \subset B_{F_n(X)}(A, \varepsilon)$. Assume that $\text{diam}(U_i) < 1$, for each $i \in \{1, 2, \dots, m\}$. Fix $B = \{b_1, b_2, \dots, b_m\}$ so that $b_1 \in U_1 - \{x_1\}$ and $b_i \in U_i$ for each $i \in \{2, 3, \dots, m\}$. Note that $B \in \mathcal{V}_\varepsilon - \{A\}$. Given $i \in \{1, 2, \dots, m\}$ and $(x, t) \in U_i \times I$, by [21, Theorem 8.26], $[x, b_i] \subset U_i$. We also have that $[x, b_i]$ is isometric to the closed interval $[0, d(x, b_i)]$. Hence if $d(x, b_i) \geq t$ there is a unique point $y_x \in [x, b_i]$ such that $d(x, y_x) = t$. We can then define a function $g_i: U_i \times I \rightarrow U_i$ by:

$$g_i(x, t) = \begin{cases} b_i, & \text{if } d(x, b_i) \leq t; \\ y_x, & \text{if } d(x, b_i) \geq t. \end{cases}$$

It is not difficult to prove that g_i is a continuous function. Note that $g_i(x, 0) = x$ and $g_i(x, 1) = b_i$, for all $x \in U_i$. If $x \in U_1 - \{x_1\}$ then $[x, b_1] \subset U_1 - \{x_1\}$ so, by the definition of g_1 , we have $g_1(x, t) \in U_1 - \{x_1\}$ for every $t \in I$.

Define $G: (\mathcal{V}_\varepsilon - \{A\}) \times I \rightarrow \mathcal{V}_\varepsilon - \{A\}$ so that if $(D, t) \in (\mathcal{V}_\varepsilon - \{A\}) \times I$, then:

$$G(D, t) = \bigcup_{i=1}^m g_i((D \cap U_i) \times \{t\}).$$

It is not difficult to see that G is well defined and continuous. Since $G(D, 0) = \bigcup_{i=1}^m (D \cap U_i) = D$ and $G(D, 1) = B$, for each $D \in \mathcal{V}_\varepsilon - \{A\}$, the set $\mathcal{V}_\varepsilon - \{A\}$ is contractible.

Let us assume now that $x_1 \in E_a(X)$. Since $X \in \mathcal{D}$, each element of $E_a(X)$ is the limit of a sequence of distinct ramification points of X , all in the same arc. We also have, since $X \in \mathcal{D}$, that $R(X)$ is discrete ([3, Corollary 3.6]). Then we can find a sequence $(r_k)_k$ in $R(X) \cap U_1$ such that:

- 1) $(r_k)_k$ converges to x_1 ;
- 2) (r_{k+1}, r_k) is an internal arc in X , for every $k \in \mathbb{N}$;
- 3) $r_{k+1} \in (r_{k+2}, r_k) \subset U_1$, for each $k \in \mathbb{N}$.

Given $i \in \{2, 3, \dots, m\}$ fix an arc I_i in $\text{cl}_X(U_i)$ which is either external or internal in $\text{cl}_X(U_i)$. Let $J_i = \text{int}_{U_i}(I_i \cap U_i)$. By Theorems 3.2 and 4.5, for every $k \in \mathbb{N}$, the set:

$$\mathcal{W}_k = \langle J_2, J_3, \dots, J_m, (r_{k+1}, r_k), (r_{k+2}, r_{k+1}), \dots, (r_{k+n-m+1}, r_{k+n-m}) \rangle_n$$

is a component of $\mathcal{V}_\varepsilon \cap \mathcal{E}_n(X)$. Since $\mathcal{W}_k \cap \mathcal{W}_l = \emptyset$, if $k \neq l$, the set $\mathcal{V}_\varepsilon \cap \mathcal{E}_n(X)$ has infinitely many components.

To finish the proof, note that $\mathfrak{B} = \{\mathcal{V}_\varepsilon : \varepsilon > 0\}$ is a basis of open neighborhoods of A in $F_n(X)$ as required. \square

Theorem 4.8. *Let X be a locally connected continuum and Z be a non-degenerate subcontinuum of X such that $\text{cl}_X(T(X) \cap Z) = Z$. Assume that there is a point $p \in Z$ such that $p \in \text{int}_X(Z)$. Then there exists a basis \mathfrak{B} of open neighborhoods of $\{p\}$ in $F_n(X)$ such that, for each $\mathcal{V} \in \mathfrak{B}$, the set $\mathcal{V} \cap \mathcal{E}_n(X)$ is empty.*

Proof. Take $\varepsilon > 0$ such that $B_X(p, \varepsilon) \subset \text{int}_X(Z)$. Let

$$\mathfrak{B} = \{B_{F_n(X)}(\{p\}, \delta) : \delta < \varepsilon\},$$

$\mathcal{V} \in \mathfrak{B}$ and $A \in \mathcal{V} \cap F_n(Z)$. Since $\text{cl}_X(T(X) \cap Z) = Z$, we have $A \cap \text{cl}_X(T(X)) \neq \emptyset$. Thus, by Theorem 4.2, $A \notin \mathcal{E}_n(X)$. This implies that $\mathcal{V} \cap \mathcal{E}_n(X) = \emptyset$. \square

Let X be a continuum and A be an arc in X with end points p and q . We say that A is a *free arc* of X if $A - \{p, q\}$ is an open subset of X .

Theorem 4.9. *Let X be a locally connected continuum and $n \in \mathbb{N}$ such that $\mathcal{E}_n(X)$ is dense in $F_n(X)$. Then, for each nonempty open subset U of X , there is a free arc of X contained in U .*

Proof. Assume, to the contrary, that U contains no free arcs. Let V be a nonempty open connected subset of X such that $\text{cl}_X(V) \subset U$. Define $Z = \text{cl}_X(V)$. We prove that $Z = \text{cl}_X(T(X) \cap Z)$. Let $y \in Z$ and W be an open subset of X such that $y \in W$. Let $p \in W \cap V$ and A be an arc such that $p \in A \subset V \cap W$. Since U has no free arcs and open subsets of X are locally arcwise connected (see [21, Definition 8.24 and Theorem 8.25]) it can be shown that there is $a \in A \cap T(X) \cap W$. Thus $W \cap T(X) \cap Z \neq \emptyset$. This shows that $Z \subset \text{cl}_X(T(X) \cap Z)$ and, since the other inclusion also holds, we have $\text{cl}_X(T(X) \cap Z) = Z$. Since the interior of Z is nonempty, by Theorem 4.8, there is an open set \mathcal{V} in $F_n(X)$ such that $\mathcal{V} \cap \mathcal{E}_n(X) = \emptyset$. This contradicts the fact that $\mathcal{E}_n(X)$ is dense in $F_n(X)$. Therefore U contains a free arc. \square

5. THE MAIN THEOREM

We start this section by showing the following result, which is a positive answer to [15, Question 2]. In its proof we will use the fact that a continuum Z is locally connected if and only if $F_n(Z)$ is locally connected ([9, Theorem 6.3]), and also that if Z is a one-dimensional continuum, then $\dim(F_n(Z)) = n$. This follows from [6, Theorem 3] and [10, Proof of Lemma 3.1].

Theorem 5.1. *Let X be a dendrite and $n \in \mathbb{N}$. If Y is a continuum such that $F_n(X) \approx F_n(Y)$, then Y is a dendrite.*

Proof. Since X is locally connected, Y is also locally connected. By [8, Theorem 1.1(19)], $\dim(X) = 1$. Thus $\dim(F_n(Y)) = \dim(F_n(X)) = n$. Assume that $\dim(Y) > 1$. Then there exist $q \in Y$ and a compact neighborhood B of q such that $\dim(B) \geq 2$. Such B can be chosen so that there is a finite collection A_1, A_2, \dots, A_{n-1} of pairwise disjoint arcs in Y such that $B \cap A_i = \emptyset$, for all $i \in \{1, 2, \dots, n-1\}$. Then $\mathcal{B} = \langle B, A_1, A_2, \dots, A_{n-1} \rangle_n$ is a subset of $F_n(Y)$ which is homeomorphic to $B \times A_1 \times A_2 \times \dots \times A_{n-1}$. Since B is compact and $\dim(A_i) = 1$ for every $i \in \{1, 2, \dots, n-1\}$, by [14, Remark, p. 34], $\dim(\mathcal{B}) = \dim(B \times A_1 \times A_2 \times \dots \times A_{n-1}) = \dim(B) + n - 1 \geq n + 1$. Hence $\dim(F_n(Y)) \geq n + 1$. Since this is a contradiction, $\dim(Y) = 1$.

Assume that Y contains a simple closed curve S^1 . Since $\dim(Y) = 1$, by [22, 18.8, p. 104], there is a retraction $r: Y \rightarrow S^1$. Consider the function $R: F_n(Y) \rightarrow F_n(S^1)$ defined, for $A \in F_n(Y)$, by $R(A) = r(A)$. It is not difficult to see that R is a well defined retraction. Since X is contractible ([8, Theorem 1.2(21)]), $F_n(X)$ is contractible. Thus $F_n(S^1)$ is a retract of the contractible space $F_n(Y)$, so $F_n(S^1)$ is contractible as well ([5, Theorem 13.2]). However, in [24] it is shown that there is no $n \in \mathbb{N}$ so that $F_n(S^1)$ is contractible. This contradiction shows that Y does not contain a simple closed curve. We conclude that Y is a dendrite. \square

Let X be a dendrite and K be a subcontinuum of X . Define $r: X \rightarrow K$ as follows: $r(x) = x$ if $x \in K$ and, otherwise, $r(x)$ is the unique point in K such that $r(x)$ is a point of every arc in X from x to any point of K (see [21, Lemma 10.24]). In [21, Lemma 10.25] it is shown that r is a retraction.

Such function is called the *first point map for K* . We use this function in the proof of the following result.

Theorem 5.2. *Let $X \in \mathcal{D}$ and $n \in \mathbb{N}$. If Y is a continuum such that $F_n(X) \approx F_n(Y)$, then $Y \in \mathcal{D}$.*

Proof. Since $X \approx F_1(X)$, the result is true for $n = 1$, so we consider that $n \geq 2$. By Theorem 5.1, Y is a dendrite. Let us assume that the metric d for Y is convex. If $p, q \in \mathbb{R}^2$, we denote by $[p, q]$ the straight line segment in \mathbb{R}^2 joining p and q . We consider that $(p, q) = [p, q] - \{p, q\}$.

Assume, to the contrary, that $Y \notin \mathcal{D}$. Then, by [3, Theorem 3.3], Y contains either a copy of

$$F_\omega = [(-1, 0), (1, 0)] \cup \left(\bigcup_{m=1}^{\infty} \left[(0, 0), \left(\frac{1}{m}, \frac{1}{m^2} \right) \right] \right)$$

or of

$$W = [(-1, 0), (1, 0)] \cup \left(\bigcup_{m=1}^{\infty} \left[\left(-\frac{1}{m}, 0 \right), \left(-\frac{1}{m}, \frac{1}{m} \right) \right] \right).$$

To simplify notation let us assume that either $F_\omega \subset Y$ or $W \subset Y$. Note that $(0, 0) \in \text{cl}_Y(E(Y)) - E(Y)$. Let $x_1 = (0, 0)$. Since $O(Y)$ is dense in Y , we can take $n - 1$ points x_2, x_3, \dots, x_n in $O(Y) \cap ((0, 0), (1, 0))$. Let

$$B = \{x_1, x_2, \dots, x_n\}.$$

Let $h: F_n(X) \rightarrow F_n(Y)$ be a homeomorphism. We will proceed as follows: after proving Claim 1, we consider the cases $h^{-1}(B) \cap E_a(X) = \emptyset$ and $h^{-1}(B) \cap E_a(X) \neq \emptyset$. In both situations we will find a contradiction. Thus the assumption $Y \notin \mathcal{D}$ is not correct and, in this way, the proof of the theorem will be complete.

By Theorems 4.1 and 4.4, $h(\mathcal{E}_n(X)) = \mathcal{E}_n(Y)$ and $\mathcal{E}_n(Y)$ is dense in $F_n(Y)$.

Take $\delta > 0$ such that $B_Y(x_i, \delta) \cap B_Y(x_j, \delta) = \emptyset$ for each $i, j \in \{1, 2, \dots, n\}$ with $i \neq j$.

Claim 1. For each open neighborhood \mathcal{V} of B in $F_n(Y)$ with $\mathcal{V} \subset B_{F_n(Y)}(B, \delta)$, the set $\mathcal{V} \cap \mathcal{E}_n(Y)$ has infinitely many components.

To show Claim 1, let \mathcal{V} be an open neighborhood of B in $F_n(Y)$ such that $\mathcal{V} \subset B_{F_n(Y)}(B, \delta)$. Let $0 < \varepsilon < \delta$ be such that the sets $B_Y(x_1, \varepsilon)$, $B_Y(x_2, \varepsilon)$, \dots , $B_Y(x_n, \varepsilon)$ are pairwise disjoint and

$$\langle B_Y(x_1, \varepsilon), B_Y(x_2, \varepsilon), \dots, B_Y(x_n, \varepsilon) \rangle_n \subset \mathcal{V}.$$

Since $x_1 \in B_Y(x_1, \varepsilon)$ and either $F_\omega \subset Y$ or $W \subset Y$, there exists $N \in \mathbb{N}$ such that either

$$(5.1) \quad \bigcup_{m=N}^{\infty} \left[(0, 0), \left(\frac{1}{m}, \frac{1}{m^2} \right) \right] \subset B_Y(x_1, \varepsilon)$$

or

$$(5.2) \quad \bigcup_{m=N}^{\infty} \left[\left(-\frac{1}{m}, 0 \right), \left(-\frac{1}{m}, \frac{1}{m} \right) \right] \subset B_Y(x_1, \varepsilon).$$

Also, since Y is a dendrite, $x_1 \in \text{cly}(E(Y)) - E(Y)$ and, according the case, the sequences $\left(\left(\frac{1}{m}, \frac{1}{m^2}\right)\right)_m$ or $\left(\left(-\frac{1}{m}, 0\right)\right)_m$ and $\left(\left(-\frac{1}{m}, \frac{1}{m}\right)\right)_m$ converge to x_1 , we can take N so that, for every $m \geq N$, if (5.1) holds then the component of $B_Y(x_1, \varepsilon) - \{x_1\}$ that contains $\left(\frac{1}{m}, \frac{1}{m^2}\right)$ coincides with the component of $Y - \{x_1\}$ that contains $\left(\frac{1}{m}, \frac{1}{m^2}\right)$; and if (5.2) holds, then the component of $B_Y(x_1, \varepsilon) - \left\{-\frac{1}{m}, 0\right\}$ that contains $\left(-\frac{1}{m}, \frac{1}{m}\right)$ coincides with the component of $Y - \left\{-\frac{1}{m}, 0\right\}$ that contains $\left(-\frac{1}{m}, \frac{1}{m}\right)$.

Given $m \geq N$ we define Z_m as follows: if (5.1) holds, then Z_m is the component of $Y - \{x_1\}$ that contains $\left(\frac{1}{m}, \frac{1}{m^2}\right)$ and, if (5.2) holds, then Z_m is the component of $Y - \left\{-\frac{1}{m}, 0\right\}$ that contains $\left(-\frac{1}{m}, \frac{1}{m}\right)$. Since each Z_m is open in Y and $\mathcal{E}_n(Y)$ is dense in $F_n(Y)$, by Theorem 4.9, there is a free arc A_m of Y contained in Z_m . Note that $\{\text{int}_Y(A_m) : m \geq N\}$ is a sequence of pairwise disjoint open connected subsets of $B_Y(x_1, \varepsilon)$.

Given $i \in \{2, 3, \dots, n\}$, since $B_Y(x_i, \varepsilon)$ is open in Y and $\mathcal{E}_n(Y)$ is dense in $F_n(Y)$, by Theorem 4.9, there is a free arc J_i of Y contained in $B_Y(x_i, \varepsilon)$. Note that $\text{int}_Y(J_2), \dots, \text{int}_Y(J_n)$ is a finite sequence of pairwise disjoint open connected subsets of Y .

For $m \geq N$ define

$$\mathcal{A}_m = \langle \text{int}_Y(A_m), \text{int}_Y(J_2), \dots, \text{int}_Y(J_n) \rangle_n.$$

Since $\text{int}_Y(A_m), \text{int}_Y(J_2), \dots, \text{int}_Y(J_n)$ are open connected subsets of Y , by Theorem 2.1, \mathcal{A}_m is an open connected subset of $F_n(Y)$. Since $\text{int}_Y(A_m) \cap \text{int}_Y(A_k) = \emptyset$ if $m \neq k$, we have $\mathcal{A}_m \cap \mathcal{A}_k = \emptyset$. Given $C \in \mathcal{A}_m$, by Theorem 2.1, the set $\langle A_m, J_2, \dots, J_n \rangle_n$ is an n -cell in $F_n(Y)$ that contains C in its interior. Thus $C \in \mathcal{E}_n(Y)$, so $\mathcal{A}_m \subset \mathcal{E}_n(Y)$. Moreover, we have

$$\mathcal{A}_m \subset \langle B_Y(x_1, \varepsilon), B_Y(x_2, \varepsilon), \dots, B_Y(x_n, \varepsilon) \rangle_n \subset \mathcal{V},$$

so $\mathcal{A}_m \subset \mathcal{V} \cap \mathcal{E}_n(Y)$.

Let \mathcal{B}_m be the component of $\mathcal{V} \cap \mathcal{E}_n(Y)$ that contains \mathcal{A}_m . We claim that $\mathcal{B}_m \cap \mathcal{B}_k = \emptyset$ for different $m, k \geq N$. Assume, to the contrary, that $\mathcal{B}_m = \mathcal{B}_k$. Let $D_1 \in \mathcal{A}_m$ and $D_2 \in \mathcal{A}_k$. Since $F_n(Y)$ is locally connected and \mathcal{B}_m is a component of the open subset $\mathcal{V} \cap \mathcal{E}_n(Y)$ of $F_n(Y)$, the set \mathcal{B}_m is arcwise connected. Then there is an arc $\alpha: [0, 1] \rightarrow \mathcal{B}_m$ such that $\alpha(0) = D_1$ and $\alpha(1) = D_2$. Let

$$K = \bigcup \{\alpha(t) : t \in [0, 1]\}.$$

Given $j \in \{1, 2, \dots, n\}$, let $K_j = K \cap B_Y(x_j, \delta)$. Since $\alpha([0, 1])$ is connected in $F_n(Y)$, the subset K of Y has at most n components ([9, Lemma 6.1]). Since $D_1, D_2 \subset K$, the sets $B_Y(x_1, \delta), B_Y(x_2, \delta), \dots, B_Y(x_n, \delta)$ are pairwise disjoint and $D_i \cap B_Y(x_j, \delta) \neq \emptyset$, for each $i \in \{1, 2\}$ and every $j \in \{1, 2, \dots, n\}$, it follows that K_1, K_2, \dots, K_n are the components of K .

Note that $K_1 \cap A_m \neq \emptyset$ and $K_1 \cap A_k \neq \emptyset$ so, if (5.1) holds, then $x_1 \in K_1$ and, if (5.2) holds, then $(-\frac{1}{m}, 0) \in K_1$. This implies that $K \cap R(Y) \neq \emptyset$, so one element of \mathcal{B}_m contains a ramification point of Y . This contradicts Theorem 4.2. Hence $\mathcal{B}_m \cap \mathcal{B}_k = \emptyset$.

Therefore $\mathcal{V} \cap \mathcal{E}_n(Y)$ has infinitely many components. This completes the proof of Claim 1.

Let us assume that $h^{-1}(B) \cap E_a(X) = \emptyset$. Then, by Theorem 4.6, there exists a basis \mathfrak{B}_X of open neighborhoods of $h^{-1}(B)$ in $F_n(X)$ such that, for each $\mathcal{U} \in \mathfrak{B}_X$, the set $\mathcal{U} \cap \mathcal{E}_n(X)$ is nonempty and has a finite number of components. Let $\mathfrak{B}_Y = \{h(\mathcal{U}) : \mathcal{U} \in \mathfrak{B}_X\}$. Then \mathfrak{B}_Y is a basis of open neighborhoods of B in $F_n(Y)$ such that, for each $\mathcal{V} \in \mathfrak{B}_Y$, the set $\mathcal{V} \cap \mathcal{E}_n(Y)$ is nonempty and has a finite number of components. Let $\mathcal{V} \in \mathfrak{B}_Y$ such that $\mathcal{V} \subset B_{F_n(Y)}(B, \delta)$. By Claim 1) the set $\mathcal{V} \cap \mathcal{E}_n(Y)$ has infinitely many components. This is a contradiction.

Let us assume now that $h^{-1}(B) \cap E_a(X) \neq \emptyset$. Then, by Theorem 4.7, there is a basis \mathfrak{B} of open neighborhoods of $h^{-1}(B)$ in $F_n(X)$ such that, for each $\mathcal{U} \in \mathfrak{B}$, the set $\mathcal{U} - \{h^{-1}(B)\}$ is contractible. Let $\mathfrak{C} = \{h(\mathcal{U}) : \mathcal{U} \in \mathfrak{B}\}$. Then \mathfrak{C} is a basis of open neighborhoods of B in $F_n(Y)$ such that, for each $\mathcal{V} \in \mathfrak{C}$, the set $\mathcal{V} - \{B\}$ is contractible.

Let $A = [(-1, 0), (1, 0)]$. Note that A is an arc in Y such that $x_1 = (0, 0) \in ((-1, 0), (1, 0))$.

Claim 2. There is a retraction $r: Y \rightarrow A$ such that $r^{-1}(x_i) = \{x_i\}$, for each $i \in \{1, 2, \dots, n\}$.

To show Claim 2, let $r_1: Y \rightarrow A$ be the first point map for A . By [21, Lemma 10.25], r_1 is a retraction. Given $i \in \{2, 3, \dots, n\}$, since $x_i \in O(Y)$, we have $r_1^{-1}(x_i) = \{x_i\}$. If $x_1 \in O(Y)$, then $r_1^{-1}(x_1) = \{x_1\}$ and r_1 has the required properties. If $x_1 \notin O(Y)$, then $r_1^{-1}(x_1) = \{y \in Y : [y, x_1] \cap A = \{x_1\}\}$. Let $A_0 = [(-1, 0), (0, 0)]$. Given $y \in r_1^{-1}(x_1)$, if $d(x_1, y) \leq d(x_1, (-1, 0))$, there is a unique $z_y \in A_0$ such that $d(x_1, z_y) = d(x_1, y)$. Then we can define a function $r_2: r_1^{-1}(x_1) \rightarrow A_0$ so that:

$$r_2(y) = \begin{cases} z_y, & \text{if } d(x_1, y) \leq d(x_1, (-1, 0)); \\ (-1, 0), & \text{if } d(x_1, y) \geq d(x_1, (-1, 0)). \end{cases}$$

It is not difficult to see that r_2 is a well defined continuous function such that $r_2^{-1}(x_1) = x_1$. Now define $r: Y \rightarrow A$ so that, if $y \in Y$, then:

$$r(y) = \begin{cases} r_1(y), & \text{if } y \notin r_1^{-1}(x_1); \\ r_2(y), & \text{if } y \in r_1^{-1}(x_1). \end{cases}$$

Then r is a retraction such that $r^{-1}(y) = \{x_i\}$, for each $i \in \{1, 2, \dots, n\}$. This proves Claim 2.

Let $r: Y \rightarrow A$ as in Claim 2. Define $R: F_n(Y) \rightarrow F_n(A)$, at $D \in F_n(Y)$, by $R(D) = r(D)$. Then R is a retraction such that $R^{-1}(B) = \{B\}$. For

each $\varepsilon > 0$ with $\varepsilon < \delta$, let $U_i^\varepsilon = B_Y(x_i, \varepsilon)$ for $i \in \{1, 2, \dots, n\}$ and $\mathcal{U}^\varepsilon = \langle U_1^\varepsilon, U_2^\varepsilon, \dots, U_n^\varepsilon \rangle_n$.

Claim 3. For each $\varepsilon > 0$ with $\varepsilon < \delta$, the set $R(\mathcal{U}^\varepsilon)$ is a connected open subset of $F_n(A)$ homeomorphic to the Euclidean space \mathbb{R}^n .

Considering the sets $U_1^\varepsilon, U_2^\varepsilon, \dots, U_n^\varepsilon$ are pairwise disjoint, it is not difficult to prove that $R(\mathcal{U}^\varepsilon) = \langle r(U_1^\varepsilon), r(U_2^\varepsilon), \dots, r(U_n^\varepsilon) \rangle_n$. Since the metric for Y is convex, by the definition of r , $r(U_1^\varepsilon), r(U_2^\varepsilon), \dots, r(U_n^\varepsilon)$ are open connected subsets of A . Thus $R(\mathcal{U}^\varepsilon)$ is an open connected subset of $F_n(A)$. Moreover, since A is an arc, by Theorem 2.1, $R(\mathcal{U}^\varepsilon)$ is homeomorphic to \mathbb{R}^n . This proves Claim 3.

Now we are ready to show the final argument. Fix $\gamma > 0$ with $\gamma < \delta$ and take $\mathcal{V} \in \mathfrak{C}$ such that $B \in \mathcal{V} \subset \mathcal{U}^\gamma$. Now let $\sigma > 0$ such that $B \in \mathcal{U}^\sigma \subset \mathcal{V} \subset \mathcal{U}^\gamma$. Since R is a retraction, $\mathcal{V} - \{B\}$ is contractible and $R(\mathcal{V} - \{B\}) = R(\mathcal{V}) - \{B\}$, then the set $R(\mathcal{V}) - \{B\}$ is contractible.

Since $B \in \mathcal{U}^\sigma \subset \mathcal{V} \subset \mathcal{U}^\gamma$, by the definition of R , $B \in R(\mathcal{U}^\sigma) \subset R(\mathcal{V}) \subset R(\mathcal{U}^\gamma)$. Thus, by Claim 3, $R(\mathcal{U}^\sigma)$ is an open neighborhood of B homeomorphic to \mathbb{R}^n and contained in the set $R(\mathcal{V})$. Then there exists an n -cell G such that $B \in G \subset R(\mathcal{V})$ and $B \notin \partial G$, where ∂G is the manifold boundary of G . By Claim 3, $R(\mathcal{U}^\gamma)$ is also homeomorphic to \mathbb{R}^n so there is a retraction $S: R(\mathcal{U}^\gamma) - \{B\} \rightarrow \partial G$. Then $S|_{R(\mathcal{V}) - \{B\}}: R(\mathcal{V}) - \{B\} \rightarrow \partial G$ is also a retraction. Since $R(\mathcal{V}) - \{B\}$ is contractible, the set ∂G is contractible. This is a contradiction to [14, p. 37] that came from the assumption that $h^{-1}(B) \cap E_a(X) \neq \emptyset$.

Since both cases $h^{-1}(B) \cap E_a(X) = \emptyset$ and $h^{-1}(B) \cap E_a(X) \neq \emptyset$ produced a contradiction, the assumption that $Y \notin \mathcal{D}$ is not correct. Therefore $Y \in \mathcal{D}$. \square

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