

Sugihara Algebras: Admissibility Algebras via the Test Spaces Method

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Abstract

This paper studies finitely generated quasivarieties of Sugihara algebras. These quasivarieties provide complete algebraic semantics for certain propositional logics associated with the relevant logic R-mingle. The motivation for the paper comes from the study of admissible rules. Recent earlier work by the present authors, jointly with Freisberg and Metcalfe, laid the theoretical foundations for a feasible approach to this problem for a range of logics—the Test Spaces Method. The method, based on natural duality theory, provides an algorithm to obtain the algebra of minimum size on which admissibility of sets of rules can be tested. (In the most general case a set of such algebras may be needed rather than just one.) The method enables us to identify this ‘admissibility algebra’ for each quasivariety of Sugihara algebras which is generated by an algebra whose underlying lattice is a finite chain. To achieve our goals, it was first necessary to develop a (strong) duality for each of these quasivarieties. The dualities promise also to also provide a valuable new tool for studying the structure of Sugihara algebras more widely.

Keywords: Sugihara algebra, natural duality, quasivariety, admissibility, free algebra

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1. Introduction

This paper investigates finitely generated quasivarieties of Sugihara algebras. These algebras have attracted interest because they are associated with the logic R-mingle, for which they provide complete algebraic semantics (see [12, 1, 21]). The definition of these algebras is given in Section 2.

In a wider context, algebraic methods have been used for a long time to solve general problems in logic for a variety of propositional logics, most famously classes of modal and intuitionistic logics. However, often an algebraic semantics is too close to the logic it models to provide a powerful tool. An exception is seen with the study of admissible rules when a suitable algebraic semantics is available. Once it is observed that some logical system has admissible non-derivable rules, it is natural to seek a description of such admissible rules. Usually, this is done through an axiomatization. That is, the provision of a set of rules that, once these are added to those of the original system, its admissible rules become derivable; see [16, 25] (intuitionistic logic), [17, 7] (intermediate logics), [18] (transitive modal logics), [19, 20] (Łukasiewicz many-valued logics). In

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particular, Metcalfe [21] presented axiomatizations for the admissible rules of various fragments of the logic R-mingle. However, the problem of finding such axiomatizations for R-mingle remains open. This paper does not directly address this problem but does develop tools whereby a rule’s admissibility may be tested.

Assume that we have a quasivariety of the form $\mathcal{A} = \mathbb{ISP}(\mathbf{M})$, where \mathbf{M} is a finite algebra which has an s -element subset which generates it. Then the free algebra $\mathbf{F}_{\mathcal{A}}(s)$ on s generators can be used to test whether some given rule for the associated logic is admissible (see for example [5, Theorem 2]). However, unless both \mathbf{M} and s are very small, this free algebra is likely to be extremely large: no explicit description may be available and often even the size cannot be computed. Hence this result on validating admissibility algebraically is likely to be principally of theoretical interest. It does prove that the admissibility problem is decidable, but it will not always be feasible to solve it in practice.

A breakthrough came with the demonstration by Metcalfe and Röthlisberger [22] and Röthlisberger [24] that validity of a rule could be tested on a much smaller algebra (or set of algebras). Proceeding syntactically, one searches for the minimal set of algebras, in a suitable multiset order, that generate the same quasivariety as does $\mathbf{F}_{\mathcal{A}}(s)$. This *admissibility set* for \mathcal{A} has been proved to exist and to be unique up to isomorphism [22, Theorem 4]. When this set is a singleton, we refer to its unique element as the *admissibility algebra*. In [22] and [24] an algorithm was developed to calculate the admissibility set. This algorithm is implemented in the *TAF*A package¹. However, the success of *TAF*A in a given case hinges on the free algebra $\mathbf{F}_{\mathcal{A}}(s)$ not being too large and a search involving all subalgebras of that free algebra being computationally feasible.

A further advance in methods to determine the admissibility set came with the exploitation of duality theory. It is very well known that the development of relational semantics revolutionised the investigation of modal and intuitionistic logics. Relational semantics, and topological relational semantics, are especially powerful when the underlying algebraic semantics are based on distributive lattices with additional operations which model, for example, a non-classical implication or negation. In this situation there will exist a dual equivalence between \mathcal{A} and a category \mathcal{Y} of enriched Priestley spaces; an associated relational semantics is obtained from \mathcal{Y} by suppressing the topology. What is significant is the computational advantages this passage to a dually equivalent category brings: the functors setting up the dual adjunction between \mathcal{A} and \mathcal{Y} act like a ‘logarithm’ from algebras to dual structures and an ‘exponential’ in the other direction. However, in general we cannot expect a duality based on enriched Priestley duality to yield a smooth translation into an equivalent dual form of the strategy devised in [22].

This problem can be overcome. Instead of a hand-me-down duality based on Priestley duality, we need a duality more closely tailored to the finitely generated quasivariety $\mathbb{ISP}(\mathbf{M})$. The theory of natural dualities is available, and supplies exactly the right machinery, provided we can set up a duality based on a strongly dualising alter ego $\underline{\mathbf{M}}$. A preliminary exploration of the idea of using natural dualities to study admissible rules was undertaken by Cabrer and Metcalfe [5], drawing on well-known natural dualities (for De Morgan algebras, in particular). Encouraged by the evidence in [5], Cabrer *et al.* [6] undertook a more extensive study. The setting is finitely generated quasivarieties $\mathbb{ISP}(\mathbf{M})$ for which strong dualities are available. Under a strong duality, injective homomorphisms correspond to surjective morphisms on the dual side, and surjective homomorphisms correspond to embeddings. In addition, the dual space of the free algebra on s generators is the s^{th} power of the alter ego $\underline{\mathbf{M}}$. These properties enable `MINGENSET` and `SUBPREHOM`, the

¹Available at: <https://sites.google.com/site/admissibility/downloads>

component algorithms from *TAFAs*, to be recast in equivalent dual forms. The Test Spaces Method (TSM for short) can thereby be formulated and validated. In outline TSM is an algorithm for determining an admissibility algebra (or admissibility set): it tells us how to find a *test space* that is the dual space of the admissibility algebra we seek (or, likewise, a set of test spaces). Only at the final step does one pass back from the dual category to the original category $\mathbb{ISP}(\mathbf{M})$.

The theory in [6] is accompanied by a suite of case studies illustrating the Test Spaces Method in action, with the focus on computational feasibility. The studies include a reprise on De Morgan algebras and progress to more complex examples of similar type. This confirmed that TSM, with or without computer assistance, can successfully find admissibility algebras in examples which have been shown to be beyond *TAFAs* reach. The largest of the admissibility algebras found in [6], that for involutive Stone algebras, has 20 elements. There the free algebra on two generators with which *TAFAs* had to contend has 1 483 648 elements. Small wonder that *TAFAs* was not up to the task.

We shall reveal in this paper that the variety \mathcal{SA} of Sugihara algebras provides a powerful demonstration of the capabilities of the Test Spaces Method. We seek to describe the admissibility algebra for each subquasivariety \mathcal{SA}_k of \mathcal{SA} which is generated by a finite subdirectly irreducible algebra: $\mathcal{SA}_k = \mathbb{ISP}(\mathbf{Z}_k)$, where the lattice reduct of \mathbf{Z}_k is a k -element chain. The family of quasivarieties $\{\mathcal{SA}_k\}$ can be viewed (see Figure 1) as forming two interlocking chains $\{\mathcal{SA}_{2n+1}\}_{n \geq 0}$ (the *odd case*) and $\{\mathcal{SA}_{2n}\}_{n \geq 1}$ (the *even case*). A minimal generating set for \mathbf{Z}_{2n+1} has $(n + 1)$ elements and one for \mathbf{Z}_{2n} has n elements. So both the size of the generating algebra \mathbf{Z}_k for \mathcal{SA}_k and the size of a minimal generating subset for this algebra tend to infinity with k . Nevertheless, we are able to achieve our goal of identifying the admissibility algebra for each \mathcal{SA}_k . To accomplish this we first have to develop the strong dualities we need. The dualities we obtain are pleasingly simple and exhibit a uniform pattern in the odd case and in the even case. This uniformity works to our advantage in executing TSM. Moreover, these dualities are of potential value beyond the application that led us to derive them.

The existing literature includes studies of Sugihara monoids, in which the language contains a constant, as well as of Sugihara algebras. Note in particular [3, 14]. Our techniques apply equally well to the monoid case. Only minor adaptations to the results and their proofs would be needed.

The paper is organised as follows. Section 2 assembles definitions and algebraic facts about Sugihara algebras. Some basic results are well known, but the discussion of partial endomorphism monoids, which underpin our dualities, is new. We treat the theory of natural dualities in black-box fashion: Section 3 summarises the bare essentials and supplies references. This primer is followed by a brief exposition of the Test Spaces Method as presented in [6]. Thereafter, we work exclusively with Sugihara algebras, separating the odd and even cases since the differences between them are great enough to make this advantageous. Sections 4 (duality) and 5 (TSM) cover the odd case and, likewise, Sections 6 and 7 the even case. Propositions 5.5 and 7.5 present our admissibility algebras. Section 8 treats the odd and even cases together. We take stock of what we have achieved by employing the Test Spaces Method as opposed to its algebraic counterpart. Table 2 compares the sizes of our admissibility algebras with the sizes of the test spaces from which we derived them and with lower bounds for the sizes of the associated free algebras. Already for small values of k the data are very striking. We see here the double benefit that the TSM approach has conferred: working with a ‘logarithmic’ duality and the ability to test for admissibility on an algebra of minimum size.

Having, we believe, conclusively demonstrated the virtues of admissibility algebras we analyse their structure more closely. We present a canonical generating set with $s = \lfloor \frac{k+1}{2} \rfloor$ elements for

the admissibility algebra \mathbf{B}_k for the quasivariety $\mathcal{S}\mathcal{A}_k$. This both allows us to view \mathbf{B}_k in a standalone manner and also allows us to capitalise on the way that \mathbf{B}_k arises both as a quotient and as a subalgebra of the free algebra on s generators in $\mathcal{S}\mathcal{A}_k$. We conclude by offering a glimpse of the application of our results to admissible rules.

We would like to thank Prof. George Metcalfe twice over. Firstly, he introduced us to the problem of finding admissibility algebras for Sugihara algebras. Secondly, we have benefitted from his ongoing interest in our work as this has progressed and from comments that have influenced the presentation of our results. In particular he has contributed to our understanding of the consequences of these results for the study of admissible rules of R-mingle logics.

2. Sugihara algebras: preliminaries

Here we introduce the quasivarieties $\mathcal{S}\mathcal{A}_k$ of Sugihara algebras in which we are interested and present algebraic facts about their generating algebras.

We assume familiarity with the basic notions of universal algebra, for which we recommend [4] for reference. Let \mathcal{A} be a non-trivial class of algebras over some common language. Then \mathcal{A} is a *variety* if it is the class of models for a set of equations and a *quasivariety* if it is the class of models for a set of quasi-identities. Our focus in this paper is on quasivarieties (for which [15] provides a background reference). We are interested in the special case of a class obtained from a non-trivial *finite* algebra \mathbf{M} . Specifically, the class $\mathbb{I}\mathbb{S}\mathbb{P}(\mathbf{M})$ is a quasivariety (where the class operators \mathbb{I} , \mathbb{S} and \mathbb{P} have their usual meanings: they denote respectively the formation of isomorphic images, subalgebras and products over a non-empty index set). We note that natural duality, in its simplest form and as we employed it in [6], applies to quasivarieties $\mathbb{I}\mathbb{S}\mathbb{P}(\mathbf{M})$. Free algebras play a central role in our investigations. A well-known result from universal algebra tells us that free algebras exist in any class $\mathbb{I}\mathbb{S}\mathbb{P}(\mathbf{M})$ and serve also as free algebras in the variety generated by \mathbf{M} [4, Chapter II].

We now proceed to definitions. Let $\mathbf{Z} = (Z; \wedge, \vee, \rightarrow, \neg)$ be the algebra whose universe Z is the set of integers, $(Z; \wedge, \vee)$ is the lattice derived from the natural order on Z , and \neg and \rightarrow are defined by $\neg a = -a$ and

$$a \rightarrow b = \begin{cases} (-a) \vee b & \text{if } a \leq b, \\ (-a) \wedge b & \text{otherwise.} \end{cases}$$

Modulus, given by $|a| := a \vee \neg a$, or alternatively by $a \rightarrow a$, defines a term function. This elementary fact is important later. Usually a fusion operator \cdot is also considered. It can be defined by $a \cdot b = \neg(a \rightarrow \neg b)$. We shall not make use of it.

The variety of *Sugihara algebras* may be defined to be the variety generated by \mathbf{Z} (see [2]). We denote it by $\mathcal{S}\mathcal{A}$. We define a family $\{\mathbf{Z}_k\}_{k \geq 1}$ of Sugihara subalgebras of \mathbf{Z} as follows:

subalgebra	universe
\mathbf{Z}_{2n+1}	$\{a \in Z \mid -n \leq a \leq n\}$ $(n = 0, 1, 2, \dots)$
\mathbf{Z}_{2n}	$\mathbf{Z}_{2n+1} \setminus \{0\}$ $(n = 1, 2, \dots)$

We shall henceforth not distinguish in our notation between an algebra \mathbf{Z}_k and its universe. Later we shall often encounter products $(\mathbf{Z}_k)^2$, with $k = 2n$ or $k = 2n+1$. To avoid cumbersome notation we shall write $(\mathbf{Z}_k)^2$ as \mathbf{Z}_k^2 . A similar abuse of notation will be adopted for other powers.

Let $\mathcal{S}\mathcal{A}_k = \mathbb{I}\mathbb{S}\mathbb{P}(\mathbf{Z}_k)$ denote the quasivariety generated \mathbf{Z}_k . Here $\mathcal{S}\mathcal{A}_1$ is the trivial quasivariety and $\mathcal{S}\mathcal{A}_2$ is term-equivalent to Boolean algebras. The quasivariety $\mathcal{S}\mathcal{A}_k$ is the algebraic

counterpart of R-mingle's axiomatic extension RM_k ; see [12]. Because of degeneracies, the quasivarieties $\mathcal{S}\mathcal{A}_k$ for $k < 6$ do not fully exhibit the features seen with larger values of k . For convenience we shall refer to consideration of the classes $\mathcal{S}\mathcal{A}_{2n}$, for $n \geq 1$, as the even case and the study of $\mathcal{S}\mathcal{A}_{2n+1}$, for $n \geq 0$, as the odd case. (We note that there are references in the literature to even (respectively, odd) Sugihara algebras for members of these classes.) We shall now proceed to a detailed discussion of basic algebraic properties, separating the even and odd cases as necessary.

By the definition of \rightarrow , a non-empty subset A of Z is the universe of a subalgebra of \mathbf{Z} if and only if it is closed under \wedge, \vee , and \neg . Hence, any union of sets of the form $\{a, \neg a\}$, for $a \in \mathbf{Z}_{2n}$, is the universe of a subalgebra of \mathbf{Z}_{2n} . This observation leads to the following proposition.

Proposition 2.1 (subalgebras). *Let $n \geq 1$.*

- (i) *There is an isomorphism from the lattice of subalgebras of \mathbf{Z}_{2n} , augmented with the empty set, to the powerset $\mathcal{P}(\{1, 2, \dots, n\})$. Moreover, each proper subalgebra of \mathbf{Z}_{2n} is isomorphic to \mathbf{Z}_{2m} for some $m < n$.*
- (ii) *There is an isomorphism from the lattice of subalgebras of \mathbf{Z}_{2n+1} , augmented with the empty set, to $\mathcal{P}(\{0, 1, 2, \dots, n\})$. Moreover, each proper subalgebra of \mathbf{Z}_{2n+1} is isomorphic to \mathbf{Z}_k for some $k < 2n + 1$.*

Proposition 2.2 (homomorphisms, even case). *Let $m, n \geq 1$ and h be a homomorphism from \mathbf{Z}_{2m} into \mathbf{Z}_{2n} . Then h is injective and $m \leq n$.*

Furthermore, the only endomorphism of \mathbf{Z}_{2n} is the identity map.

Proof. Assume there exist $k < \ell \in A$ with $h(k) = h(\ell) = r \in \mathbf{Z}_{2n}$. Then

$$\neg r \vee r = h(\neg k \vee \ell) = h(k \rightarrow \ell) = r \rightarrow r = h(\ell \rightarrow k) = h(\neg \ell \wedge k) = \neg r \wedge r,$$

that is, $\neg r = r$, which is impossible in \mathbf{Z}_{2n} . So h is injective, and hence an isomorphism from \mathbf{Z}_{2m} onto a subalgebra of \mathbf{Z}_{2n} isomorphic to \mathbf{Z}_{2m} . Necessarily $m \leq n$.

The final statement is immediate from what we have proved already. \square

Proposition 2.3 (homomorphisms, odd case). *Let $m, k \geq 1$ and h be a homomorphism from \mathbf{Z}_{2m+1} into \mathbf{Z}_k . Then $k = 2n + 1$ for some $n \geq 0$ and if $a, b \in \mathbf{Z}_{2m+1}$ are distinct and such that $h(a) = h(b)$, then $h(a) = 0 = h(b)$. Moreover, two endomorphisms $h_1, h_2 \in \text{End}(\mathbf{Z}_{2m+1})$ are equal if and only if their images coincide and there is a bijection between $\text{End}(\mathbf{Z}_{2m+1})$ and subalgebras of \mathbf{Z}_{2m+1} that contain 0.*

Proof. Since $0 \in \mathbf{Z}_{2m+1}$ is the unique element x such that $\neg \neg x = x$ necessarily $h(0) = 0$, whence $k = 2n + 1$ for some $n \geq 0$.

Now consider $\mathbf{Z}_{2m+1} \setminus h^{-1}(0)$. This is a subalgebra \mathbf{A} of \mathbf{Z}_{2m+1} with $0 \notin \mathbf{A}$. By Proposition 2.1, $\mathbf{A} \cong \mathbf{Z}_{2\ell}$ for some $\ell > 0$. The remaining claims follow from this together with Proposition 2.2. \square

We now have enough information to describe the lattices of congruences of \mathbf{Z}_{2n} and \mathbf{Z}_{2n+1} . For $m = 0, \dots, n - 1$, let $\approx_m \subseteq \mathbf{Z}_{2n+1}^2$ be the equivalence relation defined by $a \approx_m b$ if and only if $a = b$ or $a, b \in \mathbf{Z}_{2m+1}$; here \approx_0 is just the diagonal relation. Each \approx_m is algebraic, that is, a subalgebra of \mathbf{Z}_{2n+1}^2 . Indeed, each \approx_m is the kernel of a homomorphism from \mathbf{Z}_{2n+1} into \mathbf{Z}_{2m+1} . To avoid overloading the notation, \approx_m will denote both the relation on \mathbf{Z}_{2n+1} defined above and also its restriction to \mathbf{Z}_{2n} .

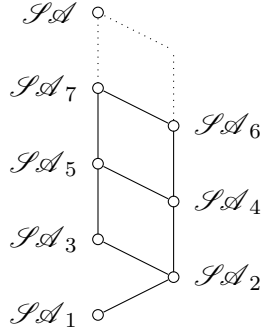


Figure 1: Quasivarieties $\mathcal{S}\mathcal{A}_k$

Proposition 2.4 (congruences). *For each $n \geq 1$ and $k \in \{2n, 2n + 1\}$ the lattice of congruences of \mathbf{Z}_k is the chain*

$$\approx_0 \subseteq \approx_1 \subseteq \cdots \subseteq \approx_{n-1} \subseteq \approx_n = \mathbf{Z}_k^2.$$

Proof. For k odd, the result follows straightforwardly from Proposition 2.3.

Now assume that $k = 2n$ for some $n > 1$ (the case $n = 1$ is trivial). Let θ be a non-trivial congruence of \mathbf{Z}_{2n} . Let $r = \max\{m \mid m\theta \neg m\}$. By construction $\mathbf{Z}_{2r}^2 \subseteq \theta$. If $r = n$, then $\theta = \mathbf{Z}_{2n}^2$. If $r < n$, it follows that $\approx_r \subseteq \theta$. To complete the proof we derive the reverse inclusion. Let $x\theta y$ be such that $x \neq y$. Without loss of generality assume that $x > y \geq \neg x$. Then

$$x = \neg y \vee x = y \rightarrow x \theta x \rightarrow y = \neg x \wedge y = \neg x.$$

Hence $x \leq r$ and $x \approx_r \neg x$. Therefore $y = y \wedge x \approx_r y \wedge \neg x = \neg x$, and so $x \approx_r y$. \square

Proposition 2.4 leads to the next result. It complements [2, Lemma 1.1] which asserts, *inter alia*, that every finite subdirectly irreducible subalgebra of $\mathcal{S}\mathcal{A}$ is of the form \mathbf{Z}_k for some $k \geq 1$.

Proposition 2.5. *For each $k \geq 1$ every subalgebra of \mathbf{Z}_k is subdirectly irreducible.*

We now describe the lattice structure of the family of subquasivarieties $\{\mathcal{S}\mathcal{A}_k\}$. From our results so far we may anticipate some connections between the odd and even cases. Observe that $\mathbf{Z}_{2n-1} \notin \text{ISP}(\mathbf{Z}_{2n}) = \mathcal{S}\mathcal{A}_{2n}$. It follows that the quasivarieties $\mathcal{S}\mathcal{A}_k$ are ordered as indicated in Figure 1. We remark as an aside that $\mathcal{S}\mathcal{A}_{2n+1} = \text{ISP}(\mathbf{Z}_{2n+1})$ coincides with $\text{HSP}(\mathbf{Z}_{2n+1})$ and $\text{HSP}(\mathbf{Z}_{2n})$ coincides with $\text{ISP}(\mathbf{Z}_{2n}, \mathbf{Z}_{2n-1})$. We do not need these facts in the present paper.

We shall now study partial endomorphisms. When we refer to a map e as a *partial endomorphism* of $Z\mathbf{Z}_k$ we mean that $h: \text{dom } h \rightarrow \text{im } h$, where $\text{dom } h$ and $\text{im } h$ are subalgebras of \mathbf{Z}_k and thereby non-empty; total maps (the endomorphisms) are included. We shall denote by $\text{End}_p(\mathbf{Z}_k)$ the set of partial endomorphisms of \mathbf{Z}_k . The composition of two elements of $\text{End}_p(\mathbf{Z}_k)$ may have empty domain. By a slight abuse of notation we shall when expedient refer to $\text{End}_p(\mathbf{Z}_k)$ as a monoid, under the operation \circ of composition. In this situation we tacitly add in the empty map. This allows us also to talk about the submonoid of $\text{End}_p(\mathbf{Z}_k)$ generated by a given subset.

We first record an easy corollary of Proposition 2.1.

Corollary 2.6. *Let $1 \leq m \leq n$. Let $0 < b_1 < b_2 < \cdots < b_m \leq n$ and $0 < c_1 < c_2 < \cdots < c_m \leq n$.*

- (i) *There exists an invertible partial endomorphism e of \mathbf{Z}_{2n} such that $e(b_i) = c_i$ for $1 \leq i \leq m$.*
- (ii) *There exists an invertible partial endomorphism e of \mathbf{Z}_{2n+1} such that $e(b_i) = c_i$ for $1 \leq i \leq m$ and $e(0) = 0$.*

Proof. By Proposition 2.1(i), the subalgebras \mathbf{B} and \mathbf{C} of \mathbf{Z}_{2n} generated by $\{b_1, b_2, \dots, b_m\}$ and $\{c_1, c_2, \dots, c_m\}$, respectively, have $2m$ elements and each is isomorphic to \mathbf{Z}_{2m} . Hence they are isomorphic. Since morphisms are order preserving any isomorphism $e: \mathbf{B} \rightarrow \mathbf{C}$ is such that $0 < e(b_1) < \dots < e(b_m)$. Therefore, $e(b_i) = c_i$ for $1 \leq i \leq m$. Hence (i) holds.

The partial endomorphism e of \mathbf{Z}_{2n} in (i) also belongs to $\text{End}_p(\mathbf{Z}_{2n+1})$. No conflict arises if we extend this map by defining $e(0) = 0$. Hence (ii) holds. \square

We now analyse partial endomorphisms more closely, with the objective of identifying amenable generating sets for the monoids $\text{End}_p(\mathbf{Z}_{2n})$ and $\text{End}_p(\mathbf{Z}_{2n+1})$. For each $1 < i \leq n$, we define $f_i: \mathbf{Z}_{2n} \setminus \{i, -i\} \rightarrow \mathbf{Z}_{2n}$ (see Figure 2) by

$$f_i(a) = \begin{cases} i & \text{if } a = i - 1, \\ -i & \text{if } a = -(i - 1), \\ a & \text{otherwise} \end{cases}$$

and $g: \mathbf{Z}_{2n} \setminus \{1, -1\} \rightarrow \mathbf{Z}_{2n}$ by

$$g(a) = \begin{cases} a - 1 & \text{if } a > 0, \\ a + 1 & \text{otherwise.} \end{cases}$$

We have excluded the case $n = 1$ here; the only element of $\text{End}_p(\mathbf{Z}_2)$ is the identity endomorphism. Henceforth we adopt a commonsense convention in relation to degeneracies of this sort, not explicitly excluding vacuous scenarios for example.

Let F_n denote the submonoid of $\text{End}_p(\mathbf{Z}_{2n})$ which is generated by f_2, \dots, f_n, g .

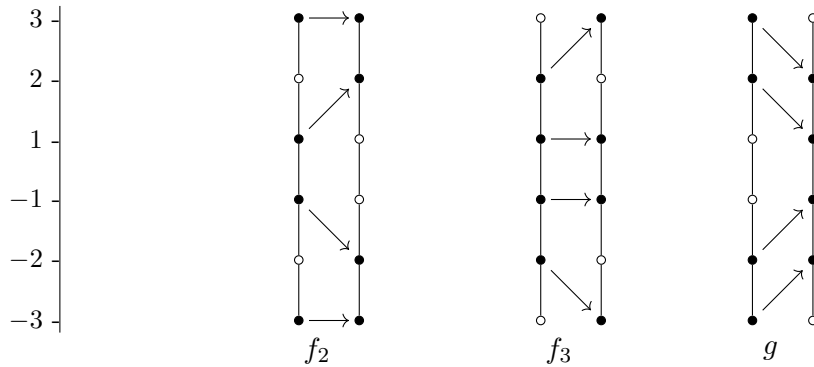


Figure 2: A generating set for $\text{End}_p(\mathbf{Z}_6)$

Proposition 2.7. *Let $n \geq 1$. Then for each $m \leq n$ and for any $2m$ -element subalgebra \mathbf{A} of \mathbf{Z}_{2n} , there exists $\varphi_{\mathbf{A}} \in F_n$ such that $\varphi_{\mathbf{A}}|_{\mathbf{A}}$ is an isomorphism from \mathbf{A} onto \mathbf{Z}_{2m} . Moreover, $\varphi_{\mathbf{A}}^{-1}$ also belongs to F_n .*

Proof. If $n = 1$ the only subalgebra \mathbf{A} of \mathbf{Z}_2 is \mathbf{Z}_2 itself, and then the result is trivial. We henceforth assume that $n \geq 2$.

The existence of $\varphi_{\mathbf{A}} \in \text{End}_p(\mathbf{Z}_{2n})$ that maps \mathbf{A} isomorphically onto \mathbf{Z}_{2m} is ensured by Corollary 2.6. Hence the point at issue is that $\varphi_{\mathbf{A}}$ and its inverse belong to F_n . We shall prove this by induction on n . Assume that the claim is valid when n is replaced by $n - 1$. Our first task is one of reconciliation: in order to apply our inductive hypothesis we shall need to relate elements of $\text{End}_p(\mathbf{Z}_{2n})$ to elements of $\text{End}_p(\mathbf{Z}_{2(n-1)})$. We keep the notation f_2, \dots, f_n, g for the partial endomorphisms of \mathbf{Z}_{2n} defined already and adopt the notation h_2, \dots, h_{n-1}, p for the corresponding maps obtained when n is replaced by $n - 1$. Observe that for $1 < i \leq n - 1$ we have the following compatibilities: $f_i \upharpoonright_{\mathbf{Z}_{2n-2}} = h_i$ and $g \upharpoonright_{\mathbf{Z}_{2n-2}} = p$. Consequently, $\eta_n: F_{n-1} \hookrightarrow F_n$ defined by

$$\begin{aligned}\eta_n(h_i) &= f_i \text{ if } 1 \leq i < n - 1, \\ \eta_n(p) &= g, \\ \eta_n(\text{id}_{\mathbf{Z}_{2n-2}}) &= g \circ f_2 \circ \dots \circ f_{n-2} \circ f_{n-1} \circ f_n\end{aligned}$$

determines an embedding from F_{n-1} into F_n .

Let \mathbf{A} be a $2m$ -element subalgebra of \mathbf{Z}_{2n} , where $m \leq n$. If $m = n$ then $\mathbf{A} = \mathbf{Z}_{2n} = \mathbf{Z}_{2m}$. Then $\varphi_{\mathbf{A}}$ is the identity map on \mathbf{Z}_{2n} and necessarily belongs to F_n . Now assume $m < n$.

Case 1: Assume $n \notin \mathbf{A}$. Then $\mathbf{A} \subseteq \mathbf{Z}_{2n-2}$ and, by the inductive hypothesis, we can find an isomorphism ι from \mathbf{A} onto \mathbf{Z}_{2m} such that both ι and ι^{-1} belong to F_{n-1} . Then $\eta_n(\iota) \in F_n$ is an isomorphism from \mathbf{A} onto \mathbf{Z}_{2m} and also $\eta_n(\iota^{-1}) = (\eta_n(\iota))^{-1} \in F_n$.

Case 2: Assume $n \in \mathbf{A}$. Since $m < n$, there exists a largest $k > 0$ such that $k \notin \mathbf{A}$. Thus all of $k+1, \dots, n$ belong to \mathbf{A} . The composite map $g \circ f_2 \circ \dots \circ f_k$ is an isomorphism from $\mathbf{Z}_{2n} \setminus \{k, -k\}$ onto \mathbf{Z}_{2n-2} ; both this map and its inverse, *viz.* $f_k \circ \dots \circ f_{n-1}$, belong to F_n . This allows us to reduce the problem to that considered in Case 1. \square

Proposition 2.8 (generation of partial endomorphisms, even case). *For $n = 2, 3, \dots$ the monoid $\text{End}_p(\mathbf{Z}_{2n})$ of partial endomorphisms of \mathbf{Z}_{2n} is generated by the maps f_2, \dots, f_n, g .*

Proof. Let $f \in \text{End}_p(\mathbf{Z}_{2n})$. By Proposition 2.1 and Lemma 2.7, f is an isomorphism from its domain $\text{dom } f$ onto its image $\text{im } f$. and $m = |\text{dom } f| = |\text{im } f|$. By Proposition 2.7, the maps $\varphi_{\text{dom } f}$ and $\varphi_{\text{im } f}$ constructed there restrict to isomorphisms onto \mathbf{Z}_{2m} from $\text{dom } f$ and $\text{im } f$ respectively, and they and their inverses lie in F_n . It follows immediately that $f = \varphi_{\text{im } f} \circ (\varphi_{\text{dom } f})^{-1} \in F_n$. \square

We now consider partial endomorphisms in the odd case. This time we are able to take advan-

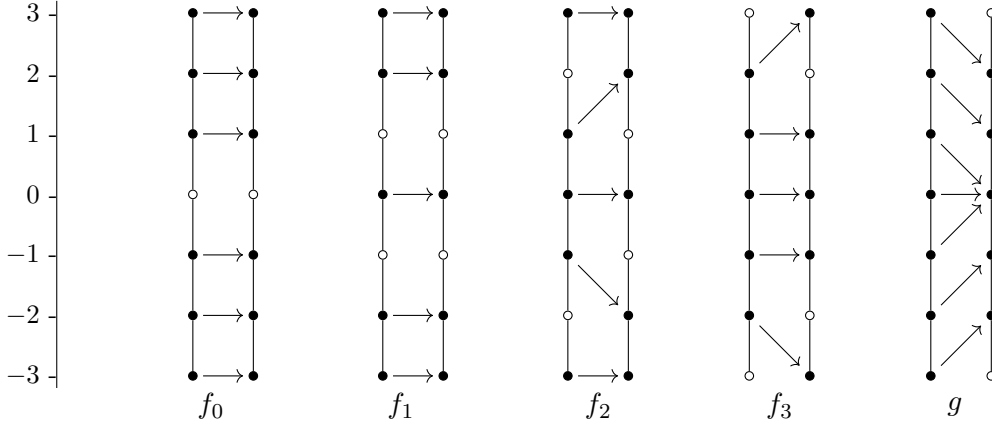


Figure 3: A set of generators for $\text{End}_p(\mathbf{Z}_7)$

tage of the existence of non-identity endomorphisms. We make the following definitions:

partial endomorphisms: $f_0: \mathbf{Z}_{2n+1} \setminus \{0\} \rightarrow \mathbf{Z}_{2n+1}, \quad f_0(a) = a \text{ for } a \neq 0;$

$f_1: \mathbf{Z}_{2n+1} \setminus \{1, -1\} \rightarrow \mathbf{Z}_{2n+1}, \quad f_1(a) = a \text{ for } a \neq \pm 1;$

and for $1 < i \leq n,$

$$f_i: \mathbf{Z}_{2n+1} \setminus \{i, -i\} \rightarrow \mathbf{Z}_{2n+1}, \quad f_i(a) = \begin{cases} i & \text{if } a = i - 1, \\ -i & \text{if } a = -(i - 1), \\ a & \text{otherwise;} \end{cases}$$

endomorphism: $g: \mathbf{Z}_{2n+1} \rightarrow \mathbf{Z}_{2n+1}, \quad g(a) = \begin{cases} a - 1 & \text{if } a > 0, \\ a + 1 & \text{if } a < 0, \\ 0 & \text{if } x = 0. \end{cases}$

Proposition 2.9 (generation of partial endomorphisms, odd case). *For $n = 1, 2, \dots,$ the partial endomorphism monoid $\text{End}_p(\mathbf{Z}_{2n+1})$ is generated by the maps $f_0, \dots, f_n, g.$*

Proof. If $n = 1,$ it is easy to see that $\text{End}_p(\mathbf{Z}_3) = \{\text{id}_{\mathbf{Z}_3}, f_0, f_1, g, g \circ f_0\}.$

Assume now that $n > 1.$ We provide only a sketch of the proof. We denote by F_{2n+1} the subset of $\text{End}_p(\mathbf{Z}_{2n+1})$ generated by $f_0, f_1, f_2, \dots, f_n, g.$ Given $p \in \text{End}_p(\mathbf{Z}_{2n+1})$ the first step is to find a p_0 in F_{2n+1} that has the same domain as $p.$ This is easily achieved using only $f_0, f_1, f_2, \dots, f_n.$ Secondly one applies g sufficiently many times to p_0 to obtain p_1 such that $p_1(a) = 0$ if and only if $p(a) = 0.$ Observe that now the sizes of the images of p and p_1 are the same. Hence these images are isomorphic as subalgebras of $\mathbf{Z}_{2n+1}.$ Finally, an adaptation of Lemma 2.7 (now using g, f_2, \dots, f_n as defined for \mathbf{Z}_{2n+1}) will provide a partial endomorphism in F_{2n+1} that is defined at 0 and that maps the image of p_1 to the image of $p.$ \square

3. Natural dualities and the Test Spaces Method

We first lay the groundwork for the presentation of our Test Spaces Method by recalling very briefly the theory of natural dualities as we shall use it. A textbook treatment can be found in [9]. Alternative sources which jointly cover the material on which we shall draw in black-box fashion are: [10] (an introductory survey of natural duality theory in general, aimed at novices) and [11, 23] (survey articles focusing on the theory as it applies to classes of algebras with distributive lattice reducts). In addition [8] provides a detailed contextualised account of strong dualities.

We shall tailor our exposition to our intended applications. In particular we shall restrict arities of operations and relations to those we shall require. Let \mathcal{A} be the quasivariety generated by a finite algebra \mathbf{M} , that is, $\mathcal{A} = \mathbb{ISP}(\mathbf{M})$, regarded as a category by taking the morphisms to be all homomorphisms. Our aim is to find a second category \mathcal{X} whose objects are topological structures of common type and which is dually equivalent to \mathcal{A} via functors $\mathbf{D}: \mathcal{A} \rightarrow \mathcal{X}$ and $\mathbf{E}: \mathcal{X} \rightarrow \mathcal{A}$.

Definition 3.1 (alter ego). We consider a topological structure $\underline{\mathbf{M}} = (M; G, H, K, R, \mathcal{T})$, where

- \mathcal{T} is the discrete topology on M ;
- G is a set of endomorphisms of \mathbf{M} ;
- H is a set of (non-total) partial endomorphisms on \mathbf{M} ;
- K is a set of one element subalgebras of \mathbf{M} (considered as constants);
- R is a set of binary relations on M such that each $r \in R$ is the universe of a subalgebra \mathbf{r} of \mathbf{M}^2 .

We refer to such a topological structure $\underline{\mathbf{M}}$ as an *alter ego* for \mathbf{M} and say that $\underline{\mathbf{M}}$ and \mathbf{M} are *compatible*.

Using an alter ego $\underline{\mathbf{M}}$ we build the desired category \mathcal{X} of structured topological spaces. We define the topological quasivariety generated by $\underline{\mathbf{M}}$ to be $\mathcal{X} := \mathbb{IS}_c\mathbb{P}^+(\underline{\mathbf{M}})$, the class of isomorphic copies of closed substructures of non-empty powers of $\underline{\mathbf{M}}$, with $+$ indicating that the empty structure is included. Here a non-empty power $\underline{\mathbf{M}}^S$ of $\underline{\mathbf{M}}$ carries the product topology and is equipped with the pointwise liftings of the members of $G \cup H \cup K \cup R$. Closed substructures and isomorphic copies are defined in the expected way; Thus a member \mathbf{X} of \mathcal{X} is a structure $(X; G^{\mathbf{X}}, H^{\mathbf{X}}, K^{\mathbf{X}}, R^{\mathbf{X}}, \mathcal{T}^{\mathbf{X}})$ of the same type as $\underline{\mathbf{M}}$. Details are given in [9, Section 1.4]. We make \mathcal{X} into a category by taking all continuous structure-preserving maps as the morphisms.

The best-known natural dualities—and in particular those employed hitherto in the study of admissible rules—have alter egos which contain no partial endomorphisms which are not total, that is, $H = \emptyset$. Except in very special cases, partial operations play a crucial role in the dualities we construct for Sugihara algebras; the exceptions are the quasivarieties $\mathcal{S}\mathcal{A}_k$ for $k \leq 3$. A full discussion of the technical niceties that arise when an alter ego contains partial operations is given in [9, Chapter 2]. We draw attention here to the constraints on the domain and range of a morphism $\varphi: \mathbf{Y} \rightarrow \mathbf{Z}$ in \mathcal{X} when partial operations are present, that is, $H \neq \emptyset$. Given any $h \in H$ and $y \in \text{dom } h^{\mathbf{Y}}$, we must restrict $\varphi(y)$ to lie in $\text{dom } h^{\mathbf{Z}}$; the preservation condition becomes $h^{\mathbf{Z}}(\varphi(y)) = \varphi(h^{\mathbf{Y}}(y))$, for all $y \in \text{dom } h^{\mathbf{Y}}$. The morphism φ is said to be an *embedding* if $\varphi(\mathbf{Y})$ is a substructure of \mathbf{Z} and $\varphi: \mathbf{Y} \rightarrow \varphi(\mathbf{Y})$ is an isomorphism. This implies in particular that $y \in \text{dom } h^{\mathbf{Y}}$ whenever $\varphi(y) \in h^{\mathbf{Z}}$. We henceforth use the same symbol for an operation $g \in G$ and for its pointwise lifting to a power of $\underline{\mathbf{M}}$ and more generally for its interpretation on any $\mathbf{X} \in \mathbb{IS}_c\mathbb{P}^+(\underline{\mathbf{M}})$. We do likewise for members of H and R . This will cause no confusion in practice, since the meanings attributed to the various symbols will be dictated by the context.

Assume, as above, that $\underline{\mathbf{M}}$ is an alter ego for \mathbf{M} and that $\mathcal{A} = \mathbb{ISP}(\mathbf{M})$ and $\mathcal{X} = \mathbb{IS}_c\mathbb{P}^+(\underline{\mathbf{M}})$. Given this scenario we can set up a dual adjunction between \mathcal{A} and \mathcal{X} , based on hom-functors into \mathbf{M} and into $\underline{\mathbf{M}}$. Let $\mathbf{A} \in \mathcal{A}$ and $\mathbf{X} \in \mathcal{X}$. Then $\mathcal{A}(\mathbf{A}, \mathbf{M})$ is the universe of a closed substructure of $\underline{\mathbf{M}}^{\mathbf{A}}$ and $\mathcal{X}(\mathbf{X}, \underline{\mathbf{M}})$ is the universe of a subalgebra of $\underline{\mathbf{M}}^{\mathbf{X}}$. As a consequence of compatibility, there exist well-defined contravariant hom-functors $\mathbf{D}: \mathcal{A} \rightarrow \mathcal{X}$ and $\mathbf{E}: \mathcal{X} \rightarrow \mathcal{A}$:

$$\begin{array}{ll} \text{on objects:} & \mathbf{D}: \mathbf{A} \mapsto \mathcal{A}(\mathbf{A}, \mathbf{M}), \\ \text{on morphisms:} & \mathbf{D}: x \mapsto - \circ x \end{array}$$

and

$$\begin{array}{ll} \text{on objects:} & \mathbf{E}: \mathbf{X} \mapsto \mathcal{X}(\mathbf{X}, \underline{\mathbf{M}}), \\ \text{on morphisms:} & \mathbf{E}: \varphi \mapsto - \circ \varphi. \end{array}$$

Given $\mathbf{A} \in \mathcal{A}$ we refer to $\mathbf{D}(\mathbf{A})$ as the (*natural*) *dual space* of \mathbf{A} .

Let $\mathbf{A} \in \mathcal{A}$ and $\mathbf{X} \in \mathcal{X}$. There exist natural evaluation maps $e_{\mathbf{A}}: \mathbf{A} \rightarrow \mathbf{E}\mathbf{D}(\mathbf{A})$ and $\varepsilon_{\mathbf{X}}: \mathbf{X} \rightarrow \mathbf{D}\mathbf{E}(\mathbf{X})$, with $e_{\mathbf{A}}(a): f \mapsto f(a)$ and $\varepsilon_{\mathbf{X}}(x): g \mapsto g(x)$. Moreover, $(\mathbf{D}, \mathbf{E}, e, \varepsilon)$ is a dual adjunction (see [?]Chapter 2]CD98). Each of the maps $e_{\mathbf{A}}$ and $\varepsilon_{\mathbf{X}}$ is an embedding. These claims hinge on the compatibility of $\underline{\mathbf{M}}$ and \mathbf{M} . We say that $\underline{\mathbf{M}}$ *yields a duality on* \mathcal{A} if each $e_{\mathbf{A}}$ is also surjective. If in addition each $\varepsilon_{\mathbf{X}}$ is surjective and so an isomorphism, we say that the duality yielded by $\underline{\mathbf{M}}$ is *full*. In this case \mathcal{A} and \mathcal{X} are dually equivalent.

With the basic categorical framework now in place, we interpose an example to illustrate the calculation of dual spaces when partial operations are present.

Example 3.2. We work with the quasivariety $\mathcal{SA}_6 = \mathbb{ISP}(\mathbf{Z}_6)$ and take our alter ego $\underline{\mathbf{Z}}_6$ to include the generating set $\{f_2, f_3, g\}$ for $\text{End}_p(\mathbf{Z}_6)$ (as shown in Figure 2) and the relations \approx_1 and \approx_2 (as in Proposition 2.4). (This will turn out to be a dualising alter ego, but this fact is not needed here.) Denote by $\mathbf{D}_6: \mathcal{SA}_6 \rightarrow \mathbb{IS}_c\mathbb{P}^+(\underline{\mathbf{Z}}_6)$ the functor determined by $\underline{\mathbf{Z}}_6$.

Note that $\mathbf{Z}_4 \in \mathcal{SA}_4 \subset \mathcal{SA}_6$. We shall calculate $\mathbf{D}_6(\mathbf{Z}_4)$ as a member of $\mathcal{X} = \mathbb{IS}_c\mathbb{P}^+(\underline{\mathbf{Z}}_6)$. Its universe consists of all the homomorphisms from \mathbf{Z}_4 into $\underline{\mathbf{Z}}_6$. By 2.2 every such map is injective. From this we easily see that $\mathcal{SA}_6(\mathbf{Z}_4, \underline{\mathbf{Z}}_6)$ consists of three maps e_1, e_2 , and e_3 . These are given by $e_1 = (-3, -2, 2, 3)$, $e_2 = (-3, -1, 1, 3)$ and $e_3 = (-2, -1, 1, 2)$. Here we use the tuple $(\varphi(-2), \varphi(-1), \varphi(1), \varphi(2))$ to depict $\varphi \in \mathcal{SA}_6(\mathbf{Z}_4, \underline{\mathbf{Z}}_6) = \mathbf{D}_6(\mathbf{Z}_4)$.

Consider the action on $\mathbf{D}_6(\mathbf{Z}_4)$ of f_2, f_3 and g . For each $\varphi \in \{e_1, e_2, e_3\}$ and any $y \in \text{dom } \varphi$, we require $\varphi(y) \in \text{dom } f_2 \cap \text{dom } f_3 \cap \text{dom } g$. We deduce that $\text{dom } f_2 = \{e_1\}$, $\text{dom } f_3 = \{e_2\}$ and $\text{dom } g = \{e_3\}$, and $f_2(e_1) = e_2$, $f_3(e_2) = e_3$ and $g(e_3) = e_1$.

Similarly, the binary relations are lifted pointwise. The equivalence classes of \approx_1 consist only of singletons and those of \approx_2 are $\{e_1, e_2\}$ and $\{e_3\}$.

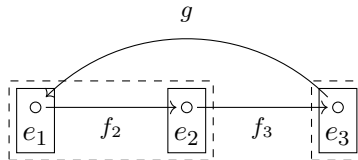


Figure 4: $\mathbf{D}_6(\mathbf{Z}_4)$.

We now return to general theory. Our objective in setting up a natural duality for a given quasivariety \mathcal{A} is thereby to transfer algebraic problems about \mathcal{A} into problems about the dual category \mathcal{X} using the hom-functors \mathbf{D} and \mathbf{E} to toggle backwards and forwards.

Our needs in this paper are very specific, tailored as they are to the Test Spaces Method (which we shall recall shortly). First of all, we need a dual description of finitely generated free algebras. Here we can call on a fundamental fact. A dualising alter ego $\underline{\mathbf{M}}$ plays a special role in the duality it sets up: it is the dual space of $\mathbf{F}_{\mathcal{A}}(1)$. More generally, the free algebra $\mathbf{F}_{\mathcal{A}}(\kappa)$ generated by a non-empty set S of cardinality κ has dual space $\underline{\mathbf{M}}^S$ [9, Section 2.2, Lemma 2.1]. Assuming the duality is full the free algebra $\mathbf{F}_{\mathcal{A}}(\kappa)$ is concretely realised as $\mathbf{E}(\underline{\mathbf{M}}^\kappa)$, with the coordinate projections as the free generators.

We also require a duality for \mathcal{A} for which there are dual characterisations of homomorphisms which are injective and of those which are surjective. This is not a categorical triviality since, for morphisms in \mathcal{X} , epi (mono) may not equate to surjective (injective). We need the notion of a duality which is *strong*, which may most concisely be defined as one in which $\underline{\mathbf{M}}$ is injective in the topological quasivariety \mathcal{X} that it generates. The technical details, and equivalent definitions, need not concern us here (they can be found in [9, Chapter 3] or [8]). We shall exploit without proof two key facts. Firstly, any strong duality is full. Secondly, in a strong duality, each of \mathbf{D} and \mathbf{E} has the property that it converts embeddings to surjections and surjections to embeddings, and vice versa; see [9, Chapter 3, Lemma 2.4] or [8, Section 3].

The Test Spaces Method (TSM) is an algorithm that takes as input a finite algebra \mathbf{M} and returns a set \mathcal{K} of algebras in $\mathbb{ISP}(\mathbf{M})$ such that $\mathbb{ISP}(\mathbf{M}) = \mathbb{ISP}(\mathcal{K})$ and \mathcal{K} is a minimal set of finite algebras each of minimum size. So, according to the definitions in Section 1, \mathcal{K} is the admissibility set and, if $\mathcal{K} = \{\mathbf{B}\}$, as occurs in our applications in this paper, then \mathbf{B} is the admissibility algebra. The method relies on the availability of a strong duality for $\mathcal{A} := \mathbb{ISP}(\mathbf{M})$.

To describe TSM, as it is given in [6, Section 4] and as we shall apply it, we first recall some definitions. Assume that \mathbf{D} and \mathbf{E} set up a strong duality between $\mathcal{A} := \mathbb{ISP}(\mathbf{M})$ and \mathcal{X} and that \mathbf{M} is a finite s -generated algebra in \mathcal{A} . We refer to a triple $(\mathbf{X}, \gamma, \eta)$ as a *Test Space configuration*, or *TS-configuration* for short, if

$$\mathbf{D}(\mathbf{M}) \xrightarrow{\eta} \mathbf{X} \xleftarrow{\gamma} \mathbf{D}(\mathbf{F}_{\mathcal{A}}(s)) = \underline{\mathbf{M}}^s,$$

where the morphism $\eta: \mathbf{D}(\mathbf{M}) \hookrightarrow \mathbf{X}$ is an embedding and the morphism $\gamma: \underline{\mathbf{M}}^s \twoheadrightarrow \mathbf{X}$ is surjective. Since we are assuming the duality is strong, fullness then ensures that any TS-configuration gives rise to an algebra $\mathbf{A} := \mathbf{E}(\mathbf{X})$ in \mathcal{A} which is both a subalgebra of $\mathbf{F}_{\mathcal{A}}(s)$ and such that \mathbf{M} is a homomorphic image of \mathbf{A} . (See [6, Section 2] for a contextual discussion.)

Let \mathbf{X} be a finite structure in $\mathbb{IS}_{\mathbf{c}}\mathbb{P}^+(\underline{\mathbf{M}})$. We denote by $\mathcal{S}_{\mathbf{X}}$ the set of substructures \mathbf{Z} of \mathbf{X} with the property that \mathbf{Z} is generated (as a substructure of \mathbf{X}) by $\text{im } \varphi_1 \cup \dots \cup \text{im } \varphi_m$, where $\varphi_1, \dots, \varphi_m: \mathbf{X} \rightarrow \mathbf{X}$ are morphisms. It was shown in [6, Proposition 5.3] that $\mathcal{S}_{\mathbf{X}}$ is a lattice for the inclusion order.

Suppose we have a candidate TS-configuration $(\mathbf{X}, \gamma, \eta)$ and that \mathbf{X} is join-irreducible in the lattice $\mathcal{S}_{\mathbf{X}}$. Then \mathbf{X} itself is the unique maximal join-irreducible. Observe that this happens if, for some specified element of \mathbf{X} , any morphism $\varphi: \mathbf{X} \rightarrow \mathbf{X}$ whose image contains that element is such that the substructure generated by $\text{im } \varphi$ is \mathbf{X} . In this situation, Step 3 is accomplished without the need to check in Step 2 that \mathbf{X} is of minimum size. Moreover, Step 4 is bypassed, and we proceed straight to Step 5. Under this scenario, the outcome of the Test Spaces Method is a TS-configuration $(\mathbf{X}, \gamma, \eta)$ such that $\mathbf{E}(\mathbf{X})$ is the admissibility algebra we seek, with the property

0. Find \mathbf{M} that yields a strong duality for $\mathcal{A} = \mathbb{ISP}(\mathbf{M})$.
1. Compute $D(\mathbf{M})$.
2. Find a TS-configuration $(\mathbf{X}, \gamma, \eta)$ with X of minimum size.
3. Determine the set \mathcal{M} of maximal join-irreducible elements of $\mathcal{S}_{\mathbf{X}}$.
4. Construct a set \mathcal{V} by repeatedly removing from \mathcal{M} any structure that is a morphic image of some other structure in the set.
5. Compute $\mathcal{K} = \{E(\mathbf{X}) \mid \mathbf{X} \in \mathcal{V}\}$.

Table 1: Test Spaces Method

that \mathbf{X} is both a substructure and a quotient in \mathcal{X} of \mathbf{M}^s . Dually (because the duality is strong), $E(\mathbf{X})$ is both a quotient and a subalgebra of $\mathbf{F}_{\mathcal{A}}(s)$. See Corollary 8.1 below.

Step 0 demands a strong duality for the quasivariety under consideration. The class of Sugihara algebras is lattice-based. In this setting the *existence* of some strongly dualising alter ego for any finitely generated subquasivariety is not at issue: general theory ensures this (see the discussion in [9] that we be able to *exhibit* an alter ego for each of the quasivarieties \mathcal{SA}_{2n} and \mathcal{SA}_{2n+1} . We also wish to do this in an efficient manner, discarding from the set $G \cup H \cup R$ any elements which are not needed for the duality to work. This will make the duality easier to understand and so to apply. The result we call on here is the specialisation to distributive-lattice-based algebras of the Piggyback Duality Theorem, as given in [9, Chapter 7, Theorem 2.1], combined, for the strongness assertion, with the NU Strong Duality Theorem Corollary [9, Chapter 3, Corollary 3.9]. In the form in which we apply piggybacking, the statement in Theorem 3.3 will precisely meet our needs.

Let \mathcal{D} denote the category of distributive lattices with lattice homomorphisms as the morphisms and let $\mathbf{2} \in \mathcal{D}$ have universe $\{0, 1\}$, where $0 < 1$.

Theorem 3.3 (Piggyback Strong Duality Theorem). *Let \mathbf{M} be a finite algebra having a reduct $\mathbf{U}(\mathbf{M})$ in \mathcal{D} . Assume further that \mathbf{M} is such that every non-trivial subalgebra of \mathbf{M} is subdirectly irreducible. Let $\mathbf{M} = (M, G, H, K, R, \mathcal{T})$ where \mathcal{T} is the discrete topology and G, H and R are chosen to satisfy the following conditions, for some subset Ω of $\mathcal{D}(\mathbf{U}(\mathbf{M}), \mathbf{2})$:*

- (i) *given $a \neq b$ in M there exists $\omega \in \Omega$ and an endomorphism h which is a composite of finitely many maps in G such that $\omega(h(a)) \neq \omega(h(b))$;*
- (ii) *R is the set of subalgebras which are maximal, with respect to inclusion, in sublattices of $\mathbf{U}(\mathbf{M})^2$ of the form $(\omega, \omega')^{-1}(\leq) := \{(a, b) \in M^2 \mid \omega(a) \leq \omega'(b)\}$, where ω, ω' range over Ω ;*
- (iii) *$G \cup H$ is the monoid of partial endomorphisms of \mathbf{M} ;*
- (iv) *K is the set of one-element subalgebras of \mathbf{M} (if any).*

Then \mathbf{M} is an alter ego for \mathbf{M} which strongly dualises $\mathbb{ISP}(\mathbf{M})$.

Conditions (i) and (ii) in Theorem 3.3 suffice to yield a duality. Including conditions (iii) and (iv) is a brute force way to ensure that the duality is in fact strong.

In Section 5 we shall apply Theorem 3.3 with \mathbf{M} as \mathbf{Z}_{2n+1} ($n \geq 0$) and in Section 7 we apply the theorem with \mathbf{M} as \mathbf{Z}_{2n} ($n \geq 1$). This enables us to identify a strongly dualising alter ego for any \mathbf{Z}_k , in a uniform manner for the odd case and for the even case. Before we embark on identifying the alter egos for these two families of dualities some comments about the conditions in the piggyback theorem should be made. In the even case, in which $\text{End}(\mathbf{M})$ contains only the

identity map, we need the elements of Ω to separate the points of \mathbf{M} . We shall see that for the odd case a more economical choice is available, courtesy of the endomorphism g .

We shall reveal a close connection between the graphs of the maps in condition (iii) and the relations in condition (ii). This works to our advantage, enabling us to streamline our alter egos. To strip down an alter ego by removing superfluous elements of $G \cup H \cup R$ we rely on the notion of entailment, as discussed in [9, Chapter 2, Section 3], together with an important technical result (see [9, Chapter 3, Subsection 2.3] or [8, Lemma 3.1]). We record a simplified version.

Lemma 3.4 ($\underline{\mathbf{M}}$ -Shift Strong Duality Lemma). *Assume that a finitely generated quasivariety $\mathcal{A} = \mathbb{ISP}(\mathbf{M})$ is strongly dualised by $\underline{\mathbf{M}} = (M; G, H, K, R, \mathcal{T})$. Then $\underline{\mathbf{M}}' = (M; G', H', K, R', \mathcal{T})$ will also yield a strong duality if it is obtained from $\underline{\mathbf{M}}$ by*

- (a) *enlarging R , and/or*
- (b) *deleting from $G \cup H$ any elements expressible as compositions of the elements that remain.*

Moreover, if $\underline{\mathbf{M}}'$ yields a duality on \mathcal{A} and is obtained from $\underline{\mathbf{M}}$ by deleting members of R then $\underline{\mathbf{M}}$ yields a strong duality on \mathcal{A} .

In the statement of Theorem 3.3 the restriction in (ii) to subalgebras which are *maximal* is customary, and this provides a device for simplifying an alter ego from the outset. However this restriction is optional. This is shown by examination of the proof of the theorem (or more circuitously by an appeal to Lemma 3.4). (Any duality has an alternative alter ego in which operations and partial operations are replaced by their graphs. But we warn that this process may destroy strongness, so it is not relevant to our study.)

4. Strong duality: odd case

In this section we establish the strong duality for $\mathcal{S}\mathcal{A}_{2n+1}$ which we shall use in our application of the Test Spaces Method. We wish to apply Theorem 3.3 with \mathbf{M} as \mathbf{Z}_{2n+1} . We still have some work to do to identify the alter ego we want.

Define lattice homomorphisms α^+ and α^- from $\mathbf{U}(\mathbf{Z}_{2n+1})$ to $\mathbf{2}$ by

$$\alpha^+(a) = 1 \Leftrightarrow a \geq 1 \quad \text{and} \quad \alpha^-(a) = 1 \Leftrightarrow a \geq 0.$$

Note that $\alpha^-(a) = 1 - \alpha^+(-a)$, so that the apparent asymmetry in the definition, caused by the presence of 0, is illusory

Lemma 4.1. *Given $a \neq b$ in \mathbf{Z}_{2n+1} there exists an endomorphism h such that either $\alpha^+(h(a)) \neq \alpha^+(h(b))$ or $\alpha^-(h(a)) \neq \alpha^-(h(b))$.*

Proof. If $a < 0 \leq b$, then $\alpha^-(a) \neq \alpha^-(b)$. Similarly, if $a \leq 0 < b$, then $\alpha^+(a) \neq \alpha^+(b)$. Assume now that $0 \leq a < b$. Then $\alpha^+(g^a(a)) \neq \alpha^+(g^a(b))$. Likewise, if $a < b \leq 0$, then $\alpha^-(g^{-b}(a)) \neq \alpha^-(g^{-b}(b))$. \square

We next investigate the relations in Theorem 3.3(ii) associated with the choice $\Omega = \{\alpha^+, \alpha^-\}$. Let $\omega, \omega' \in \Omega$. For each choice of ω, ω' we wish to describe the subalgebras S of \mathbf{Z}_{2n+1}^2 for which

$$S \subseteq (\omega, \omega')^{-1}(\leq) = \{ (a, b) \in \mathbf{Z}_{2n+1}^2 \mid \omega(a) \leq \omega'(b) \}.$$

Necessarily, S is disjoint from the rectangle $\omega^{-1}(1) \times \omega'^{-1}(0)$. Further constraints arise because S has to be closed under \neg and also under the implication \rightarrow . Proposition 4.2 identifies the permitted subalgebras for the four possible cases. We illustrate the proofs in Figures 5 and 6. In the diagrams the shaded regions indicate exclusion zones. For $Q \subseteq \mathbf{Z}_{2n+1}^2$ we write $\{\neg q \mid q \in Q\}$ as $\neg Q$. Our starting point is Figure 5(a), in which we include labels for points of reference on the axes. These labels are suppressed on subsequent diagrams. We denote the converse of a binary relation r by r^\smile .

Proposition 4.2. *Let S be a subalgebra of \mathbf{Z}_{2n+1}^2 .*

- (i) *If $S \subseteq (\alpha^-, \alpha^-)^{-1}(\leq)$ then there exists $e \in \text{End}_p(\mathbf{Z}_{2n+1})$ such that $S = \text{graph } e$.*
- (ii) *If $S \subseteq (\alpha^+, \alpha^+)^{-1}(\leq)$ then there exists $e \in \text{End}_p(\mathbf{Z}_{2n+1})$ such that $S^\smile = \text{graph } e$.*
- (iii) *If $S \subseteq (\alpha^-, \alpha^+)^{-1}(\leq)$ then there exists $e \in \text{End}_p(\mathbf{Z}_{2n+1})$ such that $0 \notin \text{im } e \cup \text{dom } e$ and $S = \text{graph } e$.*
- (iv) *If $S \subseteq (\alpha^+, \alpha^-)^{-1}(\leq)$ then S or S^\smile is the graph of a partial endomorphism.*

Proof. We adopt the notation $[i, j]$ for the set of elements k for which $i \leq k \leq j$ ($i, j, k \in \mathbf{Z}_{2n+1}$).

Consider (i). Since S is closed under \neg and \neg is an involution,

$$S \subseteq (\alpha^-, \alpha^-)^{-1}(\leq) \cap \neg(\alpha^-, \alpha^-)^{-1}(\leq) = [-n, -1] \times [-n, 0] \cup \{(0, 0)\} \cup [1, n] \times [0, n]$$

(see Figure 5, (a) and (b)).

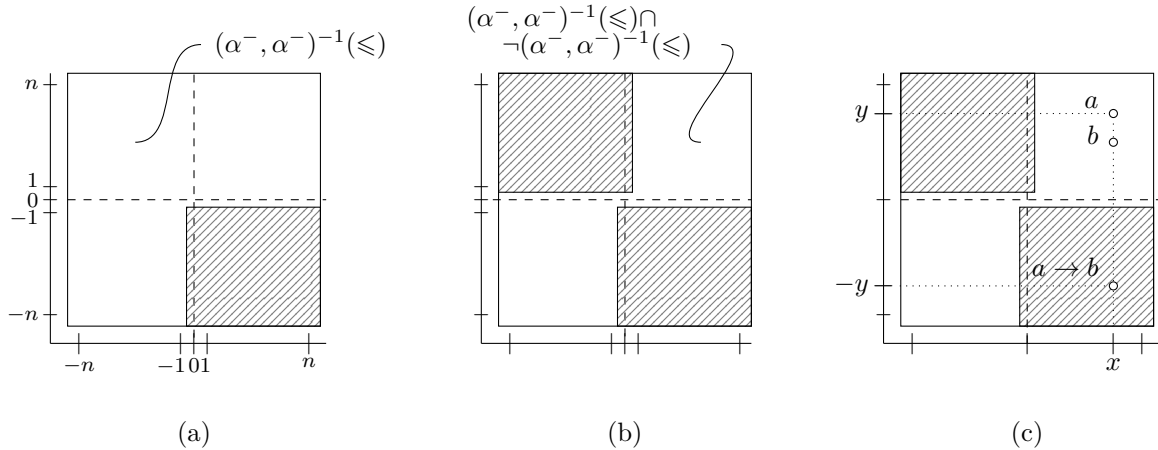


Figure 5: Illustration of proof of Proposition 4.2(i)

Let $a = (x, y) \in S$ where x is fixed. We wish to show that y is unique. If $x = 0$ both $(0, y)$ and $(0, -y)$ belong to $S \subseteq (\alpha^-, \alpha^-)^{-1}(\leq)$. Then $\alpha^-(y) = \alpha^-(-y) = 0$ and hence $y = 0$. We may now assume without loss of generality that $x > 0$ (otherwise we can consider instead $\neg a = (-x, -y)$, which also belongs to S). Suppose for contradiction that there exists $y' \neq y$ such that $b = (x, y') \in S$. Since $\alpha^-(x) = 1$, we know that $y, y' \in \alpha^{-1}(1) = [0, n]$. If $0 \leq y' < y$ then

$$a \rightarrow b = (x \rightarrow x, y \rightarrow y') = (x, -y \wedge y') = (x, -y).$$

Since $-y < 0$ we must have $\alpha^-(-y) = 0 \not\leq \alpha^-(x)$. But this is incompatible with closure of S under implication and S being contained in $(\alpha^-, \alpha^-)^{-1}(\leq)$ (see Figure 5(c)). If $y < y'$, a similar

argument considering $b \rightarrow a$ leads to a contradiction. Therefore S is the graph of a (possibly partial) map e from \mathbf{Z}_{2n+1} to \mathbf{Z}_{2n+1} . Since S is a subalgebra of \mathbf{Z}_{2n+1}^2 , it follows that e is a partial endomorphism.

We now prove (ii). First observe that

$$((\alpha^+, \alpha^+)^{-1}(\leq) \cap \neg(\alpha^+, \alpha^+)^{-1}(\leq))^\sim = (\alpha^-, \alpha^-)^{-1}(\leq) \cap \neg(\alpha^-, \alpha^-)^{-1}(\leq)$$

(see Figures 5(b) and 6(a)). Hence, $S \subseteq (\alpha^+, \alpha^+)^{-1}(\leq)$ implies $S^\sim \subseteq (\alpha^-, \alpha^-)^{-1}(\leq)$. Now (ii) follows from (i).

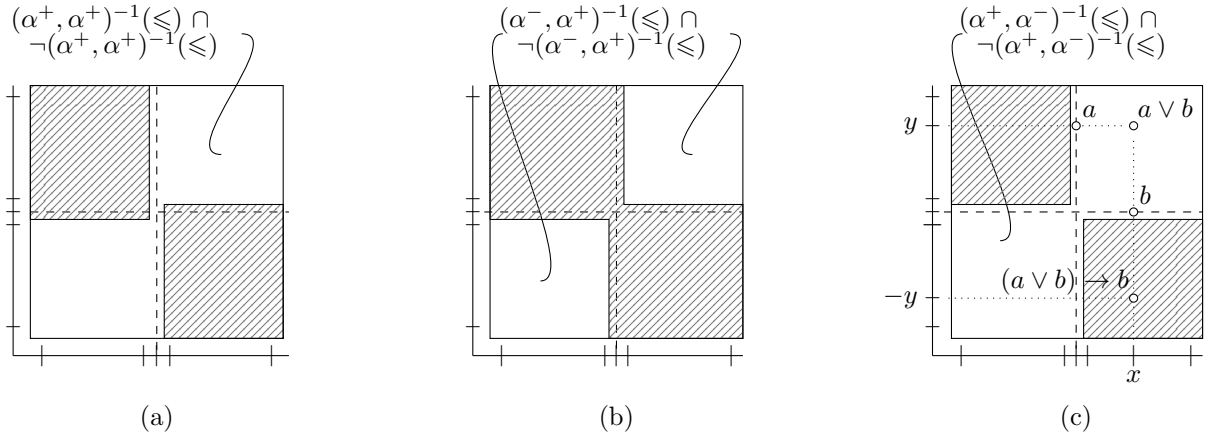


Figure 6: Proofs of Proposition 4.2(ii)-(iv)

For (iii), observe that

$$(\alpha^-, \alpha^+)^{-1}(\leq) \cap \neg(\alpha^-, \alpha^+)^{-1}(\leq) = (\alpha^-, \alpha^-)^{-1}(\leq) \cap \neg(\alpha^-, \alpha^-)^{-1}(\leq) \setminus (\mathbf{Z}_{2n} \times \{0\})$$

(see Figure 6(b)). Hence, by (i), S is the graph of some partial endomorphism e . Since

$$(\alpha^-, \alpha^+)^{-1}(\leq) \cap \neg(\alpha^-, \alpha^+)^{-1}(\leq) \cap \left((\mathbf{Z}_{2n} \times \{0\}) \cup (\{0\} \times \mathbf{Z}_{2n}) \right) = \emptyset,$$

it follows that $0 \notin \text{im } e \cup \text{dom } e$, which concludes the proof of (iii).

Finally, to prove (iv) we will prove that $S \subseteq (\alpha^+, \alpha^-)^{-1}(\leq)$ implies $S \subseteq (\alpha^-, \alpha^-)^{-1}(\leq)$ or $S \subseteq (\alpha^+, \alpha^+)^{-1}(\leq)$ and the result will then follow from (i) and (ii). Note that

$$\begin{aligned} (\alpha^+, \alpha^-)^{-1}(\leq) \cap \neg(\alpha^+, \alpha^-)^{-1}(\leq) &= ((\alpha^+, \alpha^+)^{-1}(\leq) \cap \neg(\alpha^+, \alpha^+)^{-1}(\leq)) \cup (\{0\} \times \mathbf{Z}_{2n}) \\ &= ((\alpha^-, \alpha^-)^{-1}(\leq) \cap \neg(\alpha^-, \alpha^-)^{-1}(\leq)) \cup (\mathbf{Z}_{2n} \times \{0\}) \end{aligned}$$

(see Figure 6(c)). Suppose there exist $a = (x, 0), b = (0, y) \in S$ such that $x \neq 0 \neq y$. Assume, without loss of generality, that $0 < x, y$. Then $(a \vee b) \rightarrow a = (-x \vee x, -y \wedge 0) = (x, -y)$ (again see Figure 6(c)). However $\alpha^+(x) = 1 \not\leq 0 = \alpha^-(y)$. Hence, $S \cap ((\{0\} \times \mathbf{Z}_{2n}) \cup (\mathbf{Z}_{2n} \times \{0\}))$ is contained in either $\{0\} \times \mathbf{Z}_{2n}$ or in $\{0\} \times \mathbf{Z}_{2n}$. Therefore $S \subseteq (\alpha^+, \alpha^+)^{-1}(\leq)$ or $S \subseteq (\alpha^-, \alpha^-)^{-1}(\leq)$. \square

We are ready to present our duality theorem for the odd case.

Theorem 4.3. For $n = 1, 2, \dots$, the topological structure

$$\underline{\mathbf{Z}}_{2n+1} = (\mathbf{Z}_{2n+1}; f_0, f_1, f_2, \dots, f_n, g, \mathbf{0}, \mathcal{T})$$

is an alter ego for \mathbf{Z}_{2n+1} which yields a strong duality on $\mathcal{S}\mathcal{A}_{2n+1}$.

Proof. Fix n and let $\mathbf{M} = \mathbf{Z}_{2n+1}$. Proposition 2.5 tells us that every non-trivial subalgebra of \mathbf{Z}_{2n+1} is subdirectly irreducible. So Theorem 3.3 is applicable provided we satisfy its conditions (i)–(iv).

By Lemma 4.1 we satisfy (i) by taking $\Omega = \{\alpha^+, \alpha^-\} \subseteq \mathcal{D}(\mathbf{U}(\mathbf{Z}_{2n+1}), \mathbf{2})$. By Proposition 4.2, every relation as in Theorem 3.3(ii) is the graph of a member of $G \cup H = \text{End}_{\mathbf{p}}(\mathbf{Z}_{2n+1})$ or is the converse of such a relation. (For the present application of the theorem the restriction in item (ii) to subalgebras which are maximal could be dispensed with.) The only one-element subalgebra of \mathbf{Z}_{2n+1} is $\{0\}$. It follows from [9, Chapter 2, Section 3] that after discarding all the relations in (ii) from the alter ego we have a new dualising alter ego. Lemma 3.4 tells us that this still yields a strong duality.

Finally, we apply Lemma 3.4 again, now with $\underline{\mathbf{M}}$ having $R = \emptyset$: we may replace $G \cup H = \text{End}_{\mathbf{p}}(\mathbf{Z}_{2n+1})$ by the generating set $\{f_0, f_1, f_2, \dots, f_n, g\}$ given in Proposition 2.9. \square

We remark for subsequent use that any morphism in the dual category $\mathbb{I}\mathbb{S}_{\mathbf{c}}\mathbb{P}^+(\underline{\mathbf{Z}}_{2n+1})$ preserves all elements of $\text{End}_{\mathbf{p}}(\mathbf{Z}_{2n+1})$ and not just those present in the alter ego.

5. Test Spaces Method applied to Sugihara algebras: odd case

In this section we work with $\mathcal{S}\mathcal{A}_{2n+1}$, treating n as fixed. Step 0 of the Test Spaces Method for $\mathcal{S}\mathcal{A}_{2n+1}$ is covered by Theorem 4.3. To accomplish Step 1 of the Test Spaces Method we need to calculate the dual space $\mathbf{D}(\mathbf{Z}_{2n+1})$ for the choice $\underline{\mathbf{Z}}_{2n+1}$ of alter ego given in the theorem.

Proposition 5.1 (Step 1). *Up to an isomorphism of structures, the dual space $\mathbf{D}(\mathbf{Z}_{2n+1})$ has universe $\{U \in \mathcal{P}(\{0, \dots, n\}) \mid 0 \in U\}$. On this set, an element $e \in G \cup H$ acts by restriction on those sets U for which $U \subseteq \text{dom } e$ and is undefined otherwise; also the interpretation of $\mathbf{0} \in K$ is the set $\{0\}$.*

Proof. Together, Propositions 2.3 and 2.1 supply a bijection from $\mathbf{D}(\mathbf{Z}_{2n+1})$ onto the family of subsets of $\{0, 1, \dots, n\}$ which contain 0: this assigns to an endomorphism h the non-negative elements of its image $\text{im } h$. The lifting of $e \in \text{End}_{\mathbf{p}}(\mathbf{Z}_{2n+1})$ to the dual space $\mathbf{D}(\mathbf{Z}_{2n+1})$ is the map which is defined on those endomorphisms h for which $\text{im } h \subseteq \text{dom } e$ and which sends any such h to $e \circ h$. Since $\text{im}(e \circ h) = e(\text{im } h)$ the bijection obtained above is an isomorphism of structures. \square

We now put forward a candidate (\mathbf{Y}, μ, ν) for the TS-configuration that we require for Step 2. The underlying set of any TS-configuration for $\mathcal{S}\mathcal{A}_{2n+1}$ is a set of $(n+1)$ -tuples of elements of $\underline{\mathbf{Z}}_{2n+1}$, equipped with the structure it inherits from $\underline{\mathbf{Z}}_{2n+1}^{n+1}$. Let $U = \{0, b_1, \dots, b_k\}$, where $\{b_1, \dots, b_k\}$ is a (possibly empty) subset of $\{1, \dots, n\}$ whose elements are listed without repetitions and in increasing order. With this convention, U is uniquely determined by the $(n+1)$ -tuple $(0, \dots, 0, b_1, \dots, b_k)$, where there are $n+1-k$ zeros.

Define

$$\mathbf{Y} = \{ \mathbf{a} = (a_1, \dots, a_{n+1}) \mid \exists j \geq 1 [\forall i \leq j (0 \leq a_i = a_j) \text{ and } \forall k \geq j (a_k < a_{k+1})] \}.$$

The value of j depends on \mathbf{a} . When convenient we write j as $j_{\mathbf{a}}$. In what follows, we shall make use several times of the fact (the *uniqueness property*) that, for each choice of j , there is one and only one element (a_1, \dots, a_{n+1}) of \mathbf{Y} whose set of coordinates is $\{a_j, \dots, a_{n+1}\}$.

Define $\nu: D(\mathbf{Z}_{2n+1}) \rightarrow \mathbf{Y}$ by

$$\nu(\{0, b_1, \dots, b_k\}) = (0, \dots, 0, b_1, \dots, b_k), \text{ where the first } (n+1-k) \text{ coordinates are zero} \\ \text{and } 0 < b_p < b_q \text{ when } 1 \leq p < q \leq k.$$

Given $\mathbf{a} = (a_1, \dots, a_{n+1}) \in \mathbf{Z}_{2n+1}^{n+1}$, we define μ to be the map that assigns to \mathbf{a} the unique element of \mathbf{Y} whose set of coordinates is $\{|a_1|, \dots, |a_{n+1}|\}$. Thus if $\{|a_1|, \dots, |a_{n+1}|\} = \{c_1, \dots, c_r\}$ with $0 \leq c_1 < c_2 < \dots < c_r$, then $\mu(a_1, \dots, a_{n+1}) = (c_1, \dots, c_1, c_2, \dots, c_r)$.

Proposition 5.2 (Step 2). *Define \mathbf{Y} , μ and ν as above. Then (\mathbf{Y}, μ, ν) is a TS-configuration.*

Proof. In our alter ego for \mathbf{Z}_{2n+1} , we have $G = \{g\}$, $H = \{f_0, f_1, f_2, \dots, f_n\}$, $K = \{\mathbf{0}\}$ and $R = \emptyset$. Since all the maps in $G \cup H$ are order preserving, it is straightforward to check that \mathbf{Y} is closed under the action of $G \cup H$ and that ν is an embedding. Moreover, $\mathbf{0} = (0, \dots, 0) \in \mathbf{Y}$. Since modulus is a term function on \mathbf{Z}_{2n+1} it is preserved by any element of $\text{End}_p(\mathbf{Z}_{2n+1})$, and so μ is a morphism. Also μ is clearly surjective. \square

In preparation for Steps 3 and 5 we introduce some notation. For each k with $1 \leq k \leq n$ there is a unique element in \mathbf{Y} , *viz.* $\mathbf{k} := (1, \dots, 1, 2, \dots, k)$, whose set of coordinates is $\{1, 2, \dots, k\}$; it has $n - k + 2$ coordinates equal to 1. Also define $\mathbf{n+1} := (0, 1, \dots, n)$.

Let $\sigma: \mathbf{Y} \rightarrow \{1, \dots, n+1\}$ be the map sending $(u_1, \dots, u_{n+1}) \in \mathbf{Y}$ to the cardinality of the set $\{u_1, \dots, u_{n+1}\}$. Observe that $\sigma(\mathbf{k}) = k$ and that $j_{\mathbf{k}} = n - k + 2$, for $1 \leq k \leq n + 1$.

Lemma 5.3. *Let \mathbf{Y} be the structure defined above. Let $\mathbf{u} = (u_1, \dots, u_{n+1})$ belong to \mathbf{Y} . Then*

- (i) *if $u_1 > 0$ then \mathbf{u} lies in the substructure generated by $\{\mathbf{1}, \dots, \mathbf{n}\}$;*
- (ii) *if $u_1 = 0$ then \mathbf{u} lies in the substructure generated by $\mathbf{n+1}$.*

Consequently, $\{\mathbf{1}, \dots, \mathbf{n+1}\}$ generates \mathbf{Y} .

Proof. Consider (i). Here $\sigma(\mathbf{u}) = k$, where $k \leq n$ and $j_{\mathbf{u}} = j_{\mathbf{k}}$. Corollary 2.6 supplies $e \in \text{End}_p(\mathbf{Z}_{2n+1})$ such that $u_i = e(\mathbf{k}_i)$ for each i . Since \mathbf{u} preserves e (coordinatewise) (i) follows.

Now consider (ii). If $\sigma(\mathbf{u}) = n + 1$ then $\mathbf{u} = \mathbf{n+1}$ and there is nothing to prove. Assume $\sigma(\mathbf{u}) = k \leq n$. Then $j_{\mathbf{u}} = n - k + 2$. Also $\mathbf{v} := g^{n-k+1}(\mathbf{n+1}) = (0, \dots, 0, 1, \dots, k-1)$ and $j_{\mathbf{v}} = j_{\mathbf{u}}$. By Corollary 2.6(ii) there exists $e \in \text{End}_p(\mathbf{Z}_{2n+1})$ such that $e(\mathbf{v}) = \mathbf{u}$. We conclude that \mathbf{u} belongs to the substructure of \mathbf{Y} generated by $\mathbf{n+1}$. \square

Proposition 5.4 (Step 3). *\mathbf{Y} is join-irreducible in $\mathcal{S}_{\mathbf{Y}}$.*

Proof. Let $\varphi: \mathbf{Y} \rightarrow \mathbf{Y}$ be a morphism such that $\mathbf{n+1} \in \text{im } \varphi$. We claim that φ is the identity map. Let \mathbf{x} be such that $\varphi(\mathbf{x}) = \mathbf{n+1}$. Since $(0, 1, \dots, n) \notin \text{dom } e$ for any $e \in H$ and it is the only element of \mathbf{Y} with this property, \mathbf{x} is not in the domain of any element of H , so $\mathbf{x} = \mathbf{n+1}$. That is, $\mathbf{n+1}$ is fixed by φ . By Lemma 5.3, the morphism φ fixes any element in \mathbf{Y} having first coordinate zero.

Now let $\mathbf{y} = (b_1, \dots, b_{n+1})$ be any element of \mathbf{Y} . By the uniqueness property, \mathbf{y} is the only element of \mathbf{Y} with set of coordinates $\{b_j, \dots, b_{n+1}\}$, where $j = j_{\mathbf{y}}$. Therefore \mathbf{y} is the only element

of \mathbf{Y} that is in $\text{dom } f_i$ if and only if $i \notin \{b_j, \dots, b_{n+1}\}$. Hence $\{c_1, \dots, c_{n+1}\} \subseteq \{b_j, \dots, b_{n+1}\}$, where $\varphi(\mathbf{y}) = (c_1, \dots, c_{n+1})$. Observe that $g^{b_j}(\mathbf{y}) = (0, \dots, 0, b_{j+1} - b_j, \dots, b_{n+1} - b_j)$, where $j = j_{\mathbf{y}}$, and that this element is fixed by φ . Since $g^{b_j}(b) = b - b_j$ for any $b \in \{b_j, \dots, b_{n+1}\} = \{b_1, \dots, b_{n+1}\}$, and in particular when $b = c_i$ ($1 \leq i \leq n+1$), it follows that

$$(c_1 - b_j, \dots, c_{n+1} - b_j) = g^{b_j}(\varphi(\mathbf{y})) = \varphi(g^{b_j}(\mathbf{y})) = g^{b_j}(\mathbf{y}) = (b_1 - b_j, \dots, b_{n+1} - b_j).$$

Therefore, $c_i = b_i$ for each $i \in \{0, 1, \dots, n\}$. \square

Proposition 5.5 (Step 5: admissibility algebra). *Let $\mathbf{B} \subseteq \mathbf{Z}_{2n+1}^{+1}$ be the subalgebra whose elements $(a_1, \dots, a_n, a_{n+1})$ satisfy the following conditions:*

- (i) $a_1 \in \{1, -1\}$;
- (ii) $a_k \neq 0$ for each $k \in \{1, \dots, n\}$;
- (iii) there exists a value of j , necessarily unique, with $j \leq n$ such that
 - (a) $|a_k| = 1$ if $k \leq j$ and (b) $g(a_{k+1}) = a_k$ if $n > k \geq j$;
- (iv) $g(a_{n+1}) = g(a_n)$.

Then \mathbf{B} is a subalgebra of $\mathbf{Z}_2 \times \mathbf{Z}_4 \times \dots \times \mathbf{Z}_{2n} \times \mathbf{Z}_{2n+1}$ and it is isomorphic to $\mathbf{E}(\mathbf{Y})$.

Proof. The case $n = 1$ follows from a straightforward calculation.

Now fix $n \geq 2$. We define a map t which we shall show is an isomorphism from $\mathbf{E}(\mathbf{Y})$ to \mathbf{B} . For $x \in \mathbf{E}(\mathbf{Y})$, let

$$t(x) := (x(\mathbf{1}), x(\mathbf{2}), \dots, x(\mathbf{n}), x(\mathbf{n}+1)).$$

Claim 1: $t(x) \in \mathbf{B}$.

Proof. If $|x(\mathbf{k})| \leq 1$ for all $k \leq n$, then the claim is true.

Assume now that there exists $k \leq n$ with $|x(\mathbf{k})| \neq 1$. The tuple $\mathbf{1} = (1, 1, \dots, 1)$ is in the domain of every f_i except f_1 . Since x preserves all the partial operations, this constrains $x(\mathbf{1})$ to lie in $\{-1, 1\}$ and so (i) holds. There exists a maximum $j \leq n$ such that $|x(\mathbf{i})| = 1$ for $i \leq j$ and $|x(\mathbf{j}+1)| \neq 1$. Then $g^2(x(\mathbf{j}+1)) = g(x(\mathbf{j})) \in \{g(1), g(-1)\} = \{0\}$. Since $|x(\mathbf{j}+1)| \neq 1$, it follows that $x(\mathbf{j}+1) \in \{-2, 2\}$. If k is such that $j < k < n$ then $g^2(x(\mathbf{k}+1)) = g(x(g(\mathbf{k}+1))) = g(x(\mathbf{k}))$. Therefore, by induction, we can see that $|x(\mathbf{k}+1)| > |x(\mathbf{k})| \geq |x(\mathbf{j}+1)| = 2$. Since g is injective when restricted to $\{-n, \dots, -2, 2, \dots, n\}$ and it sends positive (resp. negative) elements to positive (resp. negative) elements it also follows that $g(x(\mathbf{k}+1)) = x(\mathbf{k})$. We have shown that (ii) and (iii) hold, with j as above. Finally, since $g(\mathbf{n}+1) = g(\mathbf{n})$, it follows that $g(x(\mathbf{n})) = x(g(\mathbf{n})) = x(g(\mathbf{n}+1)) = g(x(\mathbf{n}+1))$. Thus, $t(x)$ satisfies (iv). \square

Claim 2: $t: \mathbf{E}(\mathbf{X}) \rightarrow \mathbf{B}$ is an injective homomorphism.

Proof. By Lemma 5.3, $\{\mathbf{1}, \dots, \mathbf{n}+1\}$ generates \mathbf{Y} , hence any $x \in \mathbf{E}(\mathbf{Y})$ is uniquely determined by $x(\mathbf{1}), \dots, x(\mathbf{n}+1)$. Therefore t is injective. Moreover, $t: \mathbf{E}(\mathbf{Y}) \rightarrow \mathbf{B}$ is coordinatewise an evaluation map, hence a homomorphism. \square

Claim 3: The homomorphism t maps $\mathbf{E}(\mathbf{Y})$ onto \mathbf{B} .

Proof. The following function enables us manipulate signs to create elements of \mathbf{Y} from suitable $(n+1)$ -tuples: For a real number r , we let

$$\operatorname{sgn} r = \begin{cases} 0 & \text{if } r = 0, \\ \frac{r}{|r|} & \text{otherwise.} \end{cases}$$

On \mathbf{Z}_{2n+1} , partial endomorphisms commute with sgn .

Let $\mathbf{a} = (a_1, \dots, a_{n+1}) \in \mathbf{B}$. Here $\mathbf{a} = (a_1, \dots, a_{n+1})$ can take two possible forms. The first form is $(\pm 1, \dots, \pm 1, 2, \dots, n-j, a_{n+1})$. Here j can take any value for which $1 \leq j \leq n$, and the choices of sign in the first j coordinates are arbitrary. The last coordinate, a_{n+1} , lies between 0 and $n-1$, its value linked to that of $n-j+1$ by condition (iv). The second form is the same, except that the signs of the last $n+1-j$ coordinates are reversed. We need to define a morphism $x_{\mathbf{a}}: \mathbf{Y} \rightarrow \mathbf{Z}_{2n+1}$ such that $x_{\mathbf{a}}(\mathbf{i}) = a_i$ for $i \in \{1, \dots, n+1\}$. Assume without loss of generality that \mathbf{a} takes the first form; the other case is handled likewise. We define

$$x_{\mathbf{a}}(\mathbf{u}) := \begin{cases} \operatorname{sgn} \mathbf{a}_{\sigma(\mathbf{u})} \cdot \mathbf{u}_{n-j+2} & \text{if } \mathbf{u} \neq \mathbf{n}+1, \\ a_{n+1} + 1 & \text{if } \mathbf{u} = \mathbf{n}+1. \end{cases}$$

We first confirm that $x_{\mathbf{a}}(\mathbf{k}) = a_k$ for each k . If $k \leq j$ then

$$x_{\mathbf{a}}(\mathbf{k}) = \operatorname{sgn} a_k \cdot \mathbf{k}_{n+2-j} = \operatorname{sgn} a_k \cdot 1 = \operatorname{sgn} a_k \cdot |a_k| = a_k,$$

as required. If $k > j$ then $\mathbf{k}_{n+2-j} = k-j+1$. Also, $a_k - (k-j-1) = g^{k-j-1}(a_k) = a_{j+1} = 2$, by (iii)(b). Hence, $a_k = k-j+1$. Therefore $x_{\mathbf{a}}(\mathbf{k}) = a_k$. Moreover, $x_{\mathbf{a}}(\mathbf{n}+1) = a_{n+1}$.

It remains to prove that $x_{\mathbf{a}} \in \mathbf{E}(\mathbf{Y})$. Note that $x_{\mathbf{a}}(\mathbf{0}) = \operatorname{sgn} \mathbf{a}_{\sigma(\mathbf{0})} \cdot \mathbf{0}_{n-j+2} = 0$. Therefore $x_{\mathbf{a}}$ preserves $\mathbf{0}$. Now, let $i \in \{0, \dots, n\}$ and $\mathbf{u} = (u_1, \dots, u_{n+1}) \in \operatorname{dom} f_i$. Since $\mathbf{n}+1$ is not in the domain of f_i , necessarily $\mathbf{u} \neq \mathbf{n}+1$. Since f_i is injective, $\sigma(\mathbf{u}) = \sigma(f_i(\mathbf{u}))$ and $f_i(\mathbf{u}) \neq \mathbf{n}+1$. Hence

$$x_{\mathbf{a}}(f_i(\mathbf{u})) = \operatorname{sgn} a_{\sigma(\mathbf{u})} \cdot f_i(u_{n+j-2}) = f_i(\operatorname{sgn} a_{k_{\mathbf{u}}} \cdot u_{n+j-2}) = f_i(x_{\mathbf{a}}(\mathbf{u})).$$

We now prove that $x_{\mathbf{a}}$ preserves g . Assume first that $\mathbf{u} = \mathbf{n}+1$. Note that $g(\mathbf{n}+1) = g(\mathbf{n})$ and that $g(a_n) = g(a_{n+1})$ by (iv). Therefore

$$\begin{aligned} x_{\mathbf{a}}(g(\mathbf{n}+1)) &= x_{\mathbf{a}}(g(\mathbf{n})) = \operatorname{sgn} a_n \cdot g(\mathbf{n})_{n+2-j} \\ &= x_{\mathbf{a}}(g(\mathbf{n})) = g(x_{\mathbf{a}}(\mathbf{n})) = g(a_n) = g(a_{n+1}) = g(x_{\mathbf{a}}(\mathbf{n}+1)). \end{aligned}$$

Assume now that $\mathbf{u} \in \mathbf{Y} \setminus \{\mathbf{n}+1\}$. If $\sigma(\mathbf{u}) = \sigma(g(\mathbf{u}))$ then $x_{\mathbf{a}}(g(\mathbf{u})) = g(x_{\mathbf{a}}(\mathbf{u}))$. Suppose $\sigma(\mathbf{u}) \neq \sigma(g(\mathbf{u}))$. We have three cases to consider.

Case 1: $\sigma(\mathbf{u}) \leq j$. Since $\sigma(g(\mathbf{u})) \leq \sigma(\mathbf{u}) \leq j$, we have $\operatorname{sgn} \mathbf{a}_{\sigma(\mathbf{u})} = 0 = \operatorname{sgn} \mathbf{a}_{\sigma(g(\mathbf{u}))}$. Then

$$x_{\mathbf{a}}(g(\mathbf{u})) = \operatorname{sgn} \mathbf{a}_{\sigma(g(\mathbf{u}))} \cdot g(u)_{n+2-j} = 0 = g(0 \cdot \mathbf{u}_{n+2-j}) = g(\sigma(\mathbf{u}) \cdot \mathbf{u}_{n+2-j}) = g(x_{\mathbf{a}}(\mathbf{u})).$$

Case 2: $\sigma(\mathbf{u}) > j+1$. Then $\operatorname{sgn} \mathbf{a}_{\sigma(\mathbf{u})} = \operatorname{sgn} \mathbf{a}_{\sigma(g(\mathbf{u}))}$. Again $x_{\mathbf{a}}(g(\mathbf{u})) = g(x_{\mathbf{a}}(\mathbf{u}))$.

Case 3: $\sigma(\mathbf{u}) = j+1$. Then $\sigma(g(\mathbf{u})) = j$. Observe that $\sigma(\mathbf{u}) = 1 + \sigma(g(\mathbf{u}))$, which implies that $0, 1 \in \{u_1, \dots, u_j\}$. Note that $\mathbf{u}_{n+2-j} = \mathbf{u}_{n+1-(j-1)}$. Then $0 = \mathbf{u}_i$ for $i \leq n+1 - \sigma(\mathbf{u}) + 1$ and

$1 = \mathbf{u}_{n+1-\sigma(\mathbf{u})+2} = \mathbf{u}_{n+1-(j+1)+2} = \mathbf{u}_{n+2-j}$. It follows that $g(\mathbf{u})_{n+2-j} = 0$. Hence

$$x_{\mathbf{a}}(g(\mathbf{u})) = \text{sgn } \mathbf{a}_{\sigma(g(\mathbf{u}))} \cdot g(\mathbf{u})_{n+2-j} = 0 = g(\text{sgn } \mathbf{a}_{\sigma(\mathbf{u})} \cdot 1) = g(\text{sgn } \mathbf{a}_{\sigma(\mathbf{u})} \cdot \mathbf{u}_{n+2-j}) = g(x_{\mathbf{a}}(\mathbf{u})). \quad \square$$

Claims 1–3. establish that $\mathbf{E}(\mathbf{Y}) \cong \mathbf{B}$. \square

We refer to Section 8 for more information about \mathbf{B} and its relationship to $\mathbf{F}_{\mathcal{S}\mathcal{A}_{2n+1}}(n+1)$.

6. Strong duality: even case

For the odd Sugihara quasivarieties $\mathcal{S}\mathcal{A}_{2n+1}$ we were able to base our piggyback dualities on just two ‘carrier maps’, α^+ and α^- , from \mathbf{Z}_{2n+1} into $\{0, 1\}$, thanks to the presence of non-trivial endomorphisms; recall Lemma 4.1. In the even case we have the opposite extreme: $\text{End } \mathbf{Z}_{2n} = \{\text{id}_{\mathbf{Z}_{2n}}\}$. To satisfy the separation condition (i) in Theorem 3.3 we take $\Omega = \mathcal{D}(\mathbf{U}(\mathbf{Z}_{2n}), \mathbf{2}) \setminus \{0, 1\}$, where the excluded maps are those taking constant value 0 or 1.

We label the elements of Ω as $\beta_{n-1}^-, \dots, \beta_1^-; \beta; \beta_1^+, \dots, \beta_{n-1}^+$, where for $a \in \mathbf{Z}_{2n}$,

$$\beta_i^-(a) = 1 \Leftrightarrow a \geq -i; \quad \beta(a) = 1 \Leftrightarrow a > 0; \quad \beta_i^+(a) = 1 \Leftrightarrow a > i.$$

As compared with the odd case we have a proliferation of piggyback relations to describe. Moreover, the analogues of our exclusion diagrams in Figures 5 and 6 involve additional possible scenarios; see Figure 7. Lemma 6.1 sets out elementary facts which simplify our analysis.

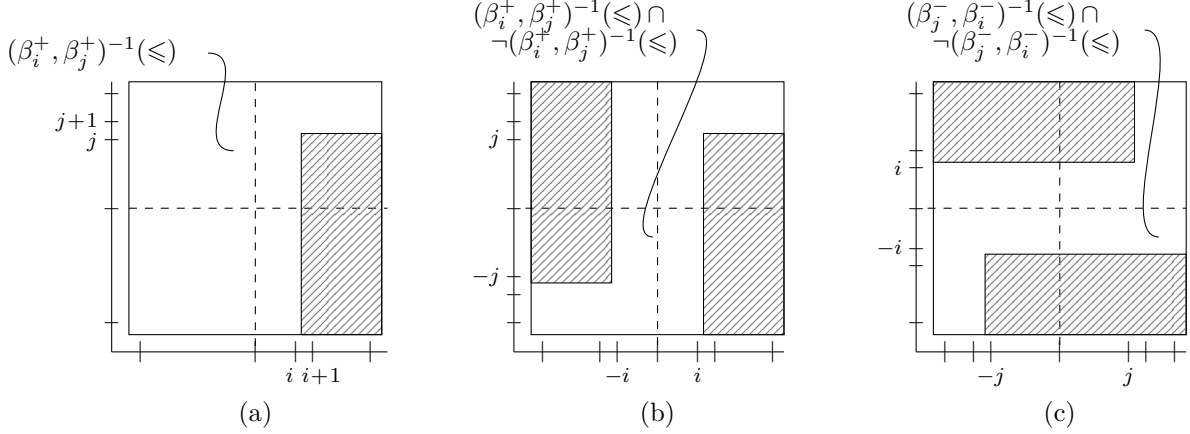


Figure 7: Illustration of proof of Proposition 6.3

Lemma 6.1. *Let S be a subalgebra of \mathbf{Z}_{2n}^2 . Let $i, j \in \{1, \dots, n-1\}$. Then the following hold:*

- (i) $S \subseteq (\beta_i^+, \beta_j^+)^{-1}(\leq)$ if and only if $S^\smile \subseteq (\beta_j^-, \beta_i^-)^{-1}(\leq)$.
- (ii) $S \subseteq (\beta_i^\pm, \beta)^{-1}(\leq)$ if and only if $S^\smile \subseteq (\beta, \beta_i^\mp)^{-1}(\leq)$.
- (iii) $S \subseteq (\beta_i^+, \beta_j^-)^{-1}(\leq)$ if and only if $S \subseteq (\beta_i^+, \beta_j^+)^{-1}(\leq)$ or $S \subseteq (\beta_i^-, \beta_j^-)^{-1}(\leq)$.

Proof. Consider (i). Let $(a, b) \in \mathbf{Z}_{2n}^2$. Then

$$\begin{aligned}
(a, b) &\in (\beta_i^+, \beta_j^+)^{-1}(\leq) \cap \neg(\beta_i^+, \beta_j^+)^{-1}(\leq) \\
&\iff \beta_i^+(a) \leq \beta_j^+(b) \text{ and } \beta_i^+(\neg a) \leq \beta_j^+(\neg b) \\
&\iff 1 - \beta_i^-(\neg a) \leq 1 - \beta_j^-(\neg b) \text{ and } 1 - \beta_i^-(a) \leq 1 - \beta_j^-(b) \\
&\iff \beta_j^-(\neg b) \leq \beta_i^-(\neg a) \text{ and } \beta_j^-(b) \leq \beta_i^-(a) \\
&\iff (b, a) \in (\beta_j^-, \beta_i^-)^{-1}(\leq) \cap \neg((\beta_j^-, \beta_i^-)^{-1}(\leq)).
\end{aligned}$$

Item (ii) follows by the same argument but replacing β_i^+ and β_i^- (β_j^+ and β_j^-) with β .

To prove (iii) assume that there exist (a, b) and (c, d) in S such that $(a, b) \notin (\beta_i^+, \beta_j^+)^{-1}(\leq)$ and $(c, d) \notin (\beta_i^-, \beta_j^-)^{-1}(\leq)$. Since $S \subseteq (\beta_i^+, \beta_j^-)^{-1}(\leq)$, we have $i < a$, $-j < b \leq j$, $-i < c \leq i$, and $d < -j$. Hence

$$(\neg(a, b) \wedge (c, d)) \rightarrow (c, d) = (-a \wedge c, -b \wedge d) \rightarrow (c, d) = (-a, d) \rightarrow (c, d) = (a \vee c, d) = (a, d).$$

However, $(a, d) \notin (\beta_i^+, \beta_j^-)^{-1}(\leq)$ and this contradicts the fact that S is a subalgebra of \mathbf{Z}_{2n}^2 . \square

Compositions of partial endomorphisms h and relations \approx_m are central to our characterisation of piggyback relations. We define $h \circ \approx_m$ by composition in the expected way. Lemma 6.2(i) relies on the last statement in Proposition 2.7. In (ii) and (iii) the relations are those we would get if we replaced h by $\text{graph } h$ and formed the relational product. However, as we indicated in Section 3, we cannot blithely replace a partial operation by its graph.

Lemma 6.2. *Let $h \in \text{End}_p(\mathbf{Z}_{2n})$ and let \approx_m ($m \geq 1$) be as defined earlier.*

- (i) h is invertible and $(\text{graph } h)^\smile = \text{graph } h^{-1}$.
- (ii) $h \circ \approx_m := \{ (x, y) \in \mathbf{Z}_{2n}^2 \mid x \in \text{dom } h \ \& \ h(x) \approx_m y \}$ is a subalgebra of \mathbf{Z}_{2n}^2 .
- (iii) $\approx_m \circ h := \{ (x, y) \in \mathbf{Z}_{2n}^2 \mid \exists z (x \approx_m z \ \& \ (z \in \text{dom } h \ \& \ y = h(z))) \}$ and $\approx_m \circ h = h^{-1} \circ \approx_m$.

Proposition 6.3. *Let S be a given subalgebra of \mathbf{Z}_{2n} which is maximal in $(\omega, \omega')^{-1}(\leq)$, where $\omega, \omega' \in \Omega$. The possible forms for S are indicated below.*

	β_k^-	β	β_ℓ^+
β_i^-	$\approx_m \circ h$	$\text{graph } h$	$\text{graph } h$
β	$\approx_m \circ h$	$\text{graph } h$	$\text{graph } h$
β_j^+	$\approx_m \circ h$ or $h \circ \approx_m$	$h \circ \approx_m$	$h \circ \approx_m$

Here $i, j, k, \ell \in \{1, \dots, n-1\}$. The row label specifies the choice of ω , the column label specifies ω' ; h denotes some element of $\text{End}_p(\mathbf{Z}_{2n})$ and $m \in \{1, \dots, n-1\}$, where both h and m will depend on S and on ω and ω' .

Proof. *Case 1:* Consider the choices $\omega = \beta_l^-$ and $\omega' = \beta_k^-$. Let

$$s = \max\{ r \mid (r, b) \in S \implies -r \leq b \leq r \}.$$

Observe that necessarily $s \geq i$.

Suppose first that $s = n$. Then $S \subseteq \mathbf{Z}_{2n} \times \mathbf{Z}_{2k} \subseteq (\beta_i^-, \beta_k^-)^{-1}(\leq)$. In this case let $m = n$ and $h = \text{id}_{\mathbf{Z}_{2k}}$, and then

$$\approx_m \circ h = (\mathbf{Z}_{2n} \times \mathbf{Z}_{2n}) \circ \text{id}_{\mathbf{Z}_{2k}} = \mathbf{Z}_{2n} \times \mathbf{Z}_{2k}.$$

Since $\mathbf{Z}_{2n} \times \mathbf{Z}_{2k} \subseteq (\beta_i^-, \beta_k^-)^{-1}(\leq)$ and S is maximal we deduce that $S = \mathbf{Z}_{2n} \times \mathbf{Z}_{2k}$.

Now suppose $s < n$. Let $(a, b) \in S$ be such that $b \notin \mathbf{Z}_{2k}$. Assume without loss of generality that $b < -k$. Then $\beta_i^-(a) \leq \beta_k^-(b) = 0$, that is, $a < -i$.

If there exists $a' \in \mathbf{Z}_{2n}$ such that $(a', b) \in S$ and $a < a'$, then

$$(a, b) \rightarrow (a', b) = (a \rightarrow a', b \rightarrow b) = (-a \vee a', b).$$

However $\beta_i^-(-a \vee a') = 1$ and $\beta_k^-(b) = 0$, that is, $(a, b) \rightarrow (a', b) \notin S$. We arrive at a similar contradiction if we assume $a' < a$. Hence a is the unique element such that $(a, b) \in S$. Therefore, for each $b \notin \mathbf{Z}_{2k}$, if there exists $a \in \mathbf{Z}_{2n}$ for which $(a, b) \in S$, then such an a is unique and $a \notin \mathbf{Z}_{2k}$. Now let h be the partial map defined as follows:

$$h(c) = \begin{cases} c & \text{if } c \in \mathbf{Z}_{2k}, \\ a & \text{if } c \notin \mathbf{Z}_{2k} \text{ and } (a, c) \in S, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Since S is a subalgebra, h is indeed a partial endomorphism. Moreover

$$\begin{aligned} \approx_s \circ h &= \{ (a, c) \in \mathbf{Z}_{2n}^2 \mid \exists b \in \mathbf{Z}_{2k} ((a \approx_s b \ \& \ c = b) \text{ or } \exists b \notin \mathbf{Z}_{2k} (a \approx_s b \ \& \ c = h(b))) \} \\ &= (\mathbf{Z}_{2s} \times \mathbf{Z}_{2k}) \cup \{ (a, c) \in \mathbf{Z}_{2n}^2 \mid \exists b \notin \mathbf{Z}_{2k} (a = b \ \& \ (b, c) \in S) \} \\ &= (\mathbf{Z}_{2s} \times \mathbf{Z}_{2k}) \cup \{ (a, c) \in S \mid a \notin \mathbf{Z}_{2s} \}. \end{aligned}$$

Then $S \subseteq \approx_s \circ h \subseteq (\beta_i^-, \beta_k^-)^{-1}(\leq)$. Maximality of S now implies that $S = \approx_s \circ h$.

Case 2: Here we treat the choices $\omega = \beta_j^+$ and $\omega' = \beta_\ell^+$. By Lemma 6.1, $S^\sim \subseteq (\beta_{\ell^-}, \beta_j^-)^{-1}(\leq)$. We now apply Case 1 to S^\sim and make use of Lemma 6.2.

Case 3. The case in which $\omega = \beta_j^+$ and $\omega' = \beta_k^-$ can now be handled by appealing to Lemma 6.1(iii).

Case 4. The proof for the case $\omega = \beta_i^-$ and $\omega' = \beta_\ell^+$ follows the same lines as that for Case 2, but is simpler: consider the converse relation S^\sim and note that $(\text{graph } h)^\sim = \text{graph } h^{-1}$ for any $h \in \text{End}_p(\mathbf{Z}_{2n})$.

Residual cases: It remains to consider the cases in which one or both of ω and ω' is β .

- (a) $\omega = \beta$ and $\omega' = \beta_i^-$: proceed, *mutatis mutandis*, as in Case 1.
- (b) $\omega = \beta_j^+$ and $\omega' = \beta$: make use of Case (a) and Lemma 6.1(ii).
- (c) $\omega = \omega' = \beta$: proceed, *mutatis mutandis*, as in Case 3.
- (d) $\omega = \beta_i^-$ and $\omega' = \beta$: argue as for Case 3.
- (e) $\omega = \beta$ and $\omega' = \beta_\ell^+$: appeal to the result from Case (c) and Lemma 6.1(ii).

This completes our characterisation of the piggyback relations. \square

We now present our strong duality theorem for the even case. Our strategy, as in the proof of Theorem 4.3, is to start from the alter ego that the piggyback theorem supplies and then to adjust

that alter ego to arrive at the one we want. This time we need to take advantage of the restriction in Theorem 3.3(ii) to *maximal* subalgebras of $(\omega, \omega')^{-1}(\leq)$.

Theorem 6.4. *For $n = 1, 2, \dots$, the topological structure*

$$\underline{\mathbf{Z}}_{2n} = (\mathbf{Z}_{2n}; f_2, \dots, f_n, g, \approx_1, \dots, \approx_{n-1}, \mathcal{T})$$

is an alter ego for the algebra \mathbf{Z}_{2n} which yields a strong duality on $\mathcal{SA}_{2n} = \mathbb{ISP}(\mathbf{Z}_{2n})$.

Proof. In applying the Piggyback Strong Duality Theorem we take $\Omega = \{\beta, \beta_1^\pm, \dots, \beta_{n-1}^\pm\}$, take $G \cup H$ to be the entire monoid of partial endomorphisms and let R be the set of piggyback relations specified in condition (ii) of the theorem. Observe that since \mathbf{Z}_{2n} does not have one element subalgebras. $K = \emptyset$. We now add to R the relations $\approx_1, \dots, \approx_{n-1}$ and the graphs of all partial endomorphisms, to form a set R' . The $\underline{\mathbf{M}}$ -Shift Strong Duality Lemma tells us we still have a strong duality. Now we delete redundant relations, again calling on the $\underline{\mathbf{M}}$ -Shift Strong Duality Lemma. By Proposition 6.3, the set $\text{End}_p(\mathbf{Z}_{2n}) \cup \{\approx_1, \dots, \approx_{n-1}\}$ entails all the relations in R' : converse, graph, partial endomorphism action are all admissible constructs. Hence this set supplies a strongly dualising alter ego. Finally, we may delete all partial endomorphisms except those in the generating set $\{f_2, \dots, f_{n-1}, g\}$ for $\text{End}_p(\mathbf{Z}_{2n})$. \square

7. Test Spaces Method for Sugihara algebras: even case

We now apply the Test Spaces algorithm to \mathcal{SA}_{2n} . We work with n fixed and $n \geq 1$. For Step 0, we employ the strong duality set up in Theorem 6.4.

Proposition 7.1 (Step 1). *The dual space $\mathbf{D}(\mathbf{Z}_{2n})$ has universe $\{\text{id}_{\mathbf{Z}_{2n}}\}$. In this structure, any partial endomorphism acts trivially and any relation \approx_m acts as equality.*

Proof. The universe of $\mathbf{D}(\mathbf{M})$ is simply $\text{End } \mathbf{M}$, by definition of the functor \mathbf{D} . Proposition 2.2 shows that \mathbf{Z}_{2n} has no endomorphisms other than the identity. The lifting of any partial endomorphism acts on $\text{id}_{\mathbf{Z}_{2n}}$ by composition, and is empty unless the composition is defined. Since $\text{id}_{\mathbf{Z}_{2n}}$ is surjective, we deduce that the lifting to $\mathbf{D}(\mathbf{Z}_{2n})$ of any non-total partial endomorphism is the empty map. Arguing similarly, the lifting of each \approx_m is the diagonal relation. \square

We now put forward a candidate (\mathbf{Y}, μ, ν) for the TS-configuration required in Step 2. The definition is very similar to the one used for the odd case in Section 5, the key difference being that elements with zero coordinates cannot now appear.

A TS-configuration for \mathcal{SA}_{2n} will be a set of n -tuples of elements of \mathbf{Z}_{2n} to be regarded as a substructure of $\underline{\mathbf{Z}}_{2n}^n$ and into which we need to embed a copy of $\mathbf{D}(\mathbf{Z}_{2n})$. Define

$$\mathbf{Y} = \{(a_1, \dots, a_n) \mid \exists j \geq 1 [\forall i \leq j (0 < a_i = a_j) \text{ and } \forall k \geq j (a_k < a_{k+1})]\}.$$

Define $\nu: \mathbf{D}(\mathbf{Z}_{2n}) \rightarrow \mathbf{Y}$ by $\nu(\text{id}_{\mathbf{Z}_{2n}}) = (1, 2, \dots, n)$.

Define μ to be the map assigning to each $(a_1, \dots, a_n) \in \underline{\mathbf{Z}}_{2n}^n$ the unique element of \mathbf{Y} whose set of coordinates is $\{|a_1|, \dots, |a_n|\}$. Thus, if $\{|a_1|, \dots, |a_n|\} = \{c_1, \dots, c_i\}$ with $0 < c_1 < c_2 < \dots < c_r$, then $\mu(a_1, \dots, a_{n+1}) = (c_1, \dots, c_1, c_2, \dots, c_r)$.

Proposition 7.2 (Step 2). *The triple (\mathbf{Y}, μ, ν) , defined as above, is a TS-configuration.*

Proof. The argument is the same as that given for Proposition 5.2 (Step 2, odd case) except that we also need to confirm that μ preserves $\approx_1, \dots, \approx_{n-1}$. But this is clear because each \approx_m is a congruence and modulus is a term function. \square

We carry over notation from the odd case with minor adaptation. For each $k \leq n$, there is a unique element of \mathbf{Y} with $\{1, 2, \dots, k\}$ as its set of coordinates, *viz.* $\mathbf{k} := (1, \dots, 1, 2, \dots, k)$, in which 1 appears $n - k$ times. Given a general n -tuple $\mathbf{a} = (a_1, \dots, a_n)$ in \mathbf{Y} , we let $\sigma(\mathbf{a})$ denote the number of different coordinates in \mathbf{a} . In particular $\sigma(\mathbf{k}) = k$.

Lemma 7.3. *Let \mathbf{Y} be the structure defined above. Then $\{\mathbf{1}, \dots, \mathbf{n}\}$ generates \mathbf{Y} .*

Proof. Let $\mathbf{a} = (a_1, \dots, a_n) \in \mathbf{Y}$ and denote $\sigma(\mathbf{a})$ by k . Corollary 2.6(i) supplies $e \in \text{End}_{\mathbf{p}}(\mathbf{Z}_{2n})$ such that $e(\mathbf{k}_i) = a_i$ for each i . \square

Proposition 7.4 (Step 3). *\mathbf{Y} is join-irreducible in $\mathcal{S}_{\mathbf{Y}}$.*

Proof. Let $\varphi: \mathbf{Y} \rightarrow \mathbf{Y}$ be a morphism such that $(1, 2, \dots, n) \in \text{im } \varphi$. The required result will follow if we can show that φ acts as the identity on \mathbf{Y} .

Let $x \in \mathbf{Y}$ be such that $\varphi(x) = (1, 2, \dots, n)$. Since $(1, \dots, n) \notin \text{dom } e$ for any $e \in H$ and it is the only element of \mathbf{Y} with this property, $x = (1, 2, \dots, n)$. Consider now $\mathbf{a} = (a_1, \dots, a_n) \in \mathbf{Y} \setminus \{(1, 2, \dots, n)\}$ and write $\varphi(\mathbf{a}) = (b_1, \dots, b_n) = \mathbf{b}$. We claim that $\{a_1, \dots, a_n\} = \{b_1, \dots, b_n\}$. For this it will be enough to show that $\sigma(\mathbf{a}) = \sigma(\mathbf{b})$.

We know that $\mathbf{a} \in \text{dom } f_i$ if and only if $a_j \neq i$ for each j . Similarly, $\mathbf{a} \in \text{dom } g$ if and only if $a_j \neq 1$ for each j . Since φ preserves each $e \in H$ we deduce that $\{b_1, \dots, b_n\} \subseteq \{a_1, \dots, a_n\}$. Suppose $\sigma(\mathbf{a}) = k < n$. We now need to prove that $\sigma(\mathbf{b}) \geq k$.

Let $m := n - k + 1$. By Corollary 2.6 there exists $e \in \text{End}_{\mathbf{p}}(\mathbf{Z}_{2n})$ such that $(a_1, \dots, a_n) = e(m, \dots, m, m+1, \dots, n)$. Then $e(\mathbf{a}) \approx_m (1, 2, \dots, n)$. Since φ preserves e and \approx_m it follows that

$$e(\varphi(\mathbf{a})) = \varphi(e(\mathbf{a})) \approx_m \varphi(1, 2, \dots, n) = (1, 2, \dots, n).$$

Hence $e(\varphi(\mathbf{a})) = (a_1, \dots, a_m, m+1, m+2, \dots, n)$, because $\approx_m \cap (\mathbf{Z}_{2n}^2 \setminus \mathbf{Z}_{2m}^2)$ is the diagonal relation. Therefore $\sigma(e(\varphi(\mathbf{a}))) \geq k$. Since e is invertible, $\sigma((b_1, \dots, b_n)) = \sigma(\varphi(\mathbf{a})) \geq k$. It follows that $|\{b_1, \dots, b_n\}| = k$ since $\{b_1, \dots, b_n\} \subseteq \{a_1, \dots, a_n\}$. \square

Proposition 7.5 (Step 5). *Let $\mathbf{B} \subseteq \mathbf{Z}_{2n}^n$ be the subalgebra whose elements (a_1, \dots, a_n) satisfy the following conditions:*

- (i) $a_1 \in \{1, -1\}$;
- (ii) *there exists a value of j , necessarily unique, and with $j \leq n$ such that*
 - (a) $|a_k| = 1$ if $k \leq j$ and (b) $g(a_{k+1}) = a_k$ if $n > k \geq j$.

Then \mathbf{B} is a subalgebra of $\mathbf{Z}_2 \times \mathbf{Z}_4 \times \dots \times \mathbf{Z}_{2n}$ and it is isomorphic to $\mathbf{E}(\mathbf{Y})$.

Proof. The cases $n = 1, 2$ follow from straightforward calculations.

Now fix $n > 2$.

We define a map $t: \mathbf{E}(\mathbf{Y}) \rightarrow \mathbf{B}$ which we shall show is an isomorphism. Given $x \in \mathbf{E}(\mathbf{Y})$, let

$$t(x) = (x(\mathbf{1}), x(\mathbf{2}), \dots, x(\mathbf{n})).$$

Claim 1: $t(x) \in \mathbf{B}$.

Proof. If $|x(\mathbf{k})| = 1$ for all k , the claim is true. Assume now that there exists k with $|x(\mathbf{k})| \neq 1$. Since $\mathbf{1} = (1, 1, \dots, 1) \in \bigcap_{i=2}^{n-1} \text{dom } f_i$, it follows that $x(\mathbf{1}) \in \bigcap_{i=2}^{n-1} \text{dom } f_i = \{-1, 1\}$. Therefore $K_x := \{k \in [n] \mid |x(\mathbf{k})| = 1 \text{ and } |x(\mathbf{k}+\mathbf{1})| \neq 1\} \neq \emptyset$. Let $j = \min K_x$. Then $|x(\mathbf{i})| = 1$ for $i \leq j$. Observe that $\mathbf{j}+\mathbf{1} \approx_2 h(\mathbf{j})$, where $h = g^{-1} = f_2 \circ \dots \circ f_n$. Hence

$$x(\mathbf{j}+\mathbf{1}) \approx_2 h(x(\mathbf{j})) \in \{h(1), h(-1)\} = \{-2, 2\}.$$

So $x(\mathbf{j}+\mathbf{1}) \in \{-2, 2\}$. Assume that $x(\mathbf{j}+\mathbf{1}) = 2$. For k such that $j \leq k \leq n$,

$$x(\mathbf{k}+\mathbf{1}) \approx_2 x(h(\mathbf{k})) = h(x(\mathbf{k})).$$

This implies $x(\mathbf{k}+\mathbf{1}) \notin \{-1, 1\}$ and $g(x(\mathbf{k}+\mathbf{1})) = g(h(x(\mathbf{k}))) = x(\mathbf{k})$. The case $x(\mathbf{j}+\mathbf{1}) = -2$ is handled similarly. \square

Claim 2: $t: \mathbf{E}(\mathbf{Y}) \rightarrow \mathbf{B}$ is an injective homomorphism.

Proof. By Lemma 7.3, the map x is uniquely determined by the set $\{x(\mathbf{1}), \dots, x(\mathbf{n})\}$. Therefore, t is injective. Moreover, t is a homomorphism because it is given coordinatewise. \square

Claim 3: The homomorphism t maps $\mathbf{E}(\mathbf{Y})$ onto \mathbf{B} .

Proof. Here the even case is somewhat simpler than the odd one since we do not have to contend with the complications of the extra, $(n+1)^{\text{st}}$, coordinate that arose in Proposition 5.5; the n coordinates here follow the same pattern as the first n coordinates in the odd case.

Let $\mathbf{b} = (b_1, \dots, b_n) \in B$. Define $x_{\mathbf{b}}: \mathbf{Y} \rightarrow \underline{\mathbf{Z}}_{2n}$ by $x_{\mathbf{b}}(\mathbf{u}) = \text{sgn } b_{\sigma(\mathbf{u})} \cdot u_{n-j+1}$, where $j \in \{1, \dots, n\}$ is as in (ii).

If $k \leq j$ then $\sigma(\mathbf{k}) = k$ and $x_{\mathbf{b}}(\mathbf{k}) = \text{sgn } a_k \cdot \mathbf{k}_{n+1-j} = \text{sgn } b_k \cdot 1 = b_k$. If $n \geq k > j$ then $\mathbf{k}_{n-j+1} = k - j + 1$. Also, by (ii)(b), $|b_k| - (k - j - 1) = |g^{k-j-1}(b_k)| = |b_{j+1}| = 2$. Hence, $|b_k| = k - j + 1$. Then $x_{\mathbf{b}}(\mathbf{k}) = \text{sgn } b_k \cdot \mathbf{k}_{n+1-j} = \text{sgn } b_k \cdot (k - j + 1) = b_k$. Therefore $x_{\mathbf{b}}(\mathbf{i}) = b_i$ for $i \in \{1, \dots, n\}$.

It only remains to prove that $x_{\mathbf{b}} \in \mathbf{E}(\mathbf{Y})$. Let $h \in \text{End}_{\mathbf{p}}(\underline{\mathbf{Z}}_{2n})$ and $\mathbf{u} = (u_1, \dots, u_{n+1}) \in \text{dom } h$. Since h is injective, $\sigma(\mathbf{u}) = \sigma(h(\mathbf{u}))$. Then h preserves sgn coordinatewise and hence

$$x_{\mathbf{b}}(h(\mathbf{u})) = \text{sgn } b_{\sigma(h(\mathbf{u}))} \cdot (h(\mathbf{u}))_{n-j+1} = \text{sgn } b_{\sigma(\mathbf{u})} \cdot h(u_{n-j+1}) = h(\text{sgn } b_{\sigma(\mathbf{u})} \cdot u_{n-j+1}) = h(x_{\mathbf{b}}(\mathbf{u})).$$

Since \approx_m is a congruence and modulus is a term function, $x_{\mathbf{b}}$ preserves \approx_m . \square

We have proved that $\mathbf{E}(\mathbf{Y})$ is isomorphic to \mathbf{B} . \square

As a corollary to this we can identify a set of generators for \mathbf{B} ; see Proposition 8.1.

8. Admissibility algebras for Sugihara algebras: overview and applications

We have achieved the goal we set in this paper: the determination of the admissibility algebra for each quasivariety \mathcal{SA}_k . We now go back to our discussion in Section 1 and review our results, from a computational and from a theoretical standpoint. We already argued that using admissibility algebras instead of free algebras reduces the search space and makes testing logical rules for admissibility a more tractable problem. But just how big is the improvement? To assess this we

need to compare, for a general value of k , the relative sizes of $\mathcal{S}\mathcal{A}_k$'s admissibility algebra and of the free algebra $\mathbf{F}_{\mathcal{S}\mathcal{A}_k}(\lfloor \frac{k+1}{2} \rfloor)$, where $\lfloor \cdot \rfloor$ denotes integer part.

In Propositions 5.5 and 7.5 we described the admissibility algebras for $\mathcal{S}\mathcal{A}_{2n+1}$ and $\mathcal{S}\mathcal{A}_{2n}$, respectively, and with n fixed. Now we wish to consider the quasivarieties $\mathcal{S}\mathcal{A}_k$, for a general variable, k . Accordingly, when working with $\mathcal{S}\mathcal{A}_k$ we now write the admissibility algebra as \mathbf{B}_k rather than \mathbf{B} and write \mathbf{Y}_k for the structure on which the TS-configuration is based. We write $s := \lfloor \frac{k+1}{2} \rfloor$. Introduction of s highlights the similarities between Propositions 5.2 and Proposition 7.2. In both cases \mathbf{Y}_k consists of s -tuples drawn from the non-negative elements of \mathbf{Z}_k and the map μ in the TS-configuration based on \mathbf{Y}_k is defined in the same way whether k is even or odd.

Free algebras can be notoriously large, and this is the case for our quasivarieties. Using *TAF*A we could calculate $|\mathbf{F}_{\mathcal{S}\mathcal{A}_3}(2)|$. We were able not only to confirm the size of $\mathbf{F}_{\mathcal{S}\mathcal{A}_3}(2)$, but also to determine the sizes of $\mathbf{F}_{\mathcal{S}\mathcal{A}_4}(2)$ and $\mathbf{F}_{\mathcal{S}\mathcal{A}_5}(3)$ exactly. This was done by using the natural dualities developed above, knowing that $\mathbf{E}(\mathbf{Z}_k^s)$ is an s -generated free algebra in $\mathcal{S}\mathcal{A}_k$: we calculated the number of morphisms from \mathbf{Z}_k^s into \mathbf{Z}_k , for $k = 3, 4, 5$, and 6. We have not found a general formula for $|\mathbf{F}(\mathcal{S}\mathcal{A}_k(s))|$ but we do give lower bounds for general k , based on numbers of particular morphisms from \mathbf{Z}_k^s into \mathbf{Z}_k .

Proposition 7.5 leads to a recursive specification of (the universes of) the algebras \mathbf{B}_{2n} :

$$B_2 = \{-1, 1\} \quad \text{and} \quad B_{2n} = \{-1, 1\} \times (B_{2n-2} \cup \{(2, 3, \dots, n), (-2, -3, \dots, n)\}) \quad (n > 1).$$

Hence, using the recurrence relation $|B_{2n}| = 2|B_{2n-2}| + 4$ we can prove that $|B_{2n}| = 3 \cdot 2^n - 4$.

Comparing the definition of \mathbf{B}_{2n+1} with that of \mathbf{B}_{2n} , we see that

$$B_{2n+1} = \{(b_1, \dots, b_n, b_n) \mid (b_1, \dots, b_n) \in B_{2n} \setminus \{-1, 1\}^n\} \cup (\{-1, 1\}^n \times \{-1, 0, 1\}).$$

Hence $|B_{2n+1}| = |B_{2n}| - 2^n + 3 \cdot 2^n = 3 \cdot 2^n - 4 + 2 \cdot 2^n = 5 \cdot 2^n - 4$.

Our data are shown in Table 2.

k	$s = \lfloor \frac{k+1}{2} \rfloor$	$ \mathbf{F}_{\mathcal{S}\mathcal{A}_k}(s) $	$ \mathbf{Y}_k $	$ \mathbf{E}(\mathbf{Y}_k) $
3	2	1 296	3	6
4	2	20 736	3	8
5	3	$2^{44} \cdot 3^{36} \cdot (1 + 2^{-3} \cdot 3^{-4})^6$	7	16
6	3	$2^{68} \cdot 3^{36} \cdot (1 + 2^{-7} \cdot 3^{-4})^6$	7	20
$2n$	n	$\geq 2^{2n+1} \cdot 2^{(n-1)^2/2} \cdot 3^{2n}$	$2^n - 1$	$3 \cdot 2^n - 4$
$2n + 1$	$n + 1$	$\geq 2^{2n+1} \cdot 2^{n^2/2} \cdot 3^{2n+1}$	$2^{n+1} - 1$	$5 \cdot 2^n - 4$

Table 2: Cardinalities of free algebras, admissibility algebras and test spaces

Proposition 8.1 is a corollary of Propositions 5.5 and 7.5, for k odd and k even, respectively. We may regard \mathbf{Y}_k as a substructure of \mathbf{Z}_k^s . We denote the natural inclusion map by ι .

Proposition 8.1 (generators for the admissibility algebra for $\mathcal{S}\mathcal{A}_k$). *Fix k , either even or odd.*

- (i) $t \circ \mathbf{E}(\iota): \mathbf{F}_{\mathcal{S}\mathcal{A}_k}(s) \rightarrow \mathbf{B}_k$ is surjective and $\mathbf{E}(\iota)$ acts by restriction on each coordinate projection π_j .

(ii) $E(\mu) \circ t^{-1}: \mathbf{B}_k \rightarrow \mathbf{F}_{\mathcal{S}\mathcal{A}_k}(s)$ is an embedding and $E(\mu)$ acts by composition on the restriction to \mathbf{Y}_k of each coordinate projection.

For $j = 1, \dots, s$, define

$$\begin{aligned} \text{for } k \text{ even:} \quad b_j^k &= \begin{cases} (1, \dots, 1) & \text{if } j = 1, \\ (1, \dots, 1, 2) & \text{if } j = 2, \\ (1, \dots, 1, 2, \dots, j/2) & \text{if } j > 2; \end{cases} \\ \text{for } k \text{ odd:} \quad b_j^k &= \begin{cases} (1, \dots, 1, 0) & \text{if } j = 1, \\ (1, \dots, 1, 1) & \text{if } j = 2, \\ (1, \dots, 1, 2, \dots, (j-1)/2, (j-1)/2) & \text{if } j > 2. \end{cases} \end{aligned}$$

Then $b_j^k = t(\pi_j \upharpoonright_{\mathbf{Y}_k})$ and $\{b_1^k, \dots, b_s^k\}$ generates \mathbf{B}_k .

Proof. Because the duality is strong, the functor E takes surjections (embeddings) to embeddings (surjections). Moreover $E(\varphi) = \varphi \circ -$ for any \mathcal{X} -morphism φ and it is clear that $\pi_j \upharpoonright_{\mathbf{Y}_k} \circ \mu$ equals $\pi_j \circ \mu$ because $\text{im } \mu = \mathbf{Y}_k$. The claims in (i) and (ii) follow directly.

We define $b_j^k := (\pi_j \upharpoonright_{\mathbf{Y}})$ for $j = 1, \dots, s$. These elements are the images under a surjection of the free generators for $\mathbf{F}_{\mathcal{S}\mathcal{A}_k}(s)$, and so generate \mathbf{B}_k . It only remains to confirm, from the description of the elements of \mathbf{B}_k that these elements are as given in the statement of the proposition. This is a straightforward verification. \square

We know from its construction that $\mathbf{B}_k \in \mathbb{S}(\mathbf{F}_{\mathcal{S}\mathcal{A}_k}(s))$. We can say more about how \mathbf{B}_k sits inside $\mathbf{F}_{\mathcal{S}\mathcal{A}_k}(s)$, *alias* $E(\mathbf{Z}_k^s)$. We present the terms in the free variables X_1, \dots, X_s that correspond to the map $\pi_j \circ \mu$, for $j = 1, \dots, s$. To this end we need to find terms G_1, \dots, G_s such that $\mu(a_1, \dots, a_s) = (G_1(a_1, \dots, a_s), \dots, G_s(a_1, \dots, a_s))$ for each element $(a_1, \dots, a_s) \in \mathbf{Z}_k^s$.

Consider the following Sugihara terms:

- $|x| := x \rightarrow x$
- $x \leftrightarrow y := (x \rightarrow y) \wedge (y \rightarrow x)$

and, for $1 \leq i \leq s$,

- $S_i(X_1, \dots, X_s) = \bigwedge \{ \bigvee \{ |X_i| \mid X_i \in S \} \mid S \subseteq \{X_1, \dots, X_s\} \text{ and } |S| = i \}$
- $T_i(X_1, \dots, X_s)$, defined by

$$T_i(X_1, \dots, X_s) = \begin{cases} S_1 & \text{if } i = 1, \\ \neg(S_i \leftrightarrow S_{i-1}) \vee S_1 & \text{if } i > 1 \end{cases}$$

- $G_j(X_1, \dots, X_s) = S_j(T_1(X_1, \dots, X_s), \dots, T_s(X_1, \dots, X_s))$

It is easy to observe that the action of $|x|$ on elements of \mathbf{Z}_k is precisely to send each a to a if $a \geq 0$ and to $-a$ if $a < 0$. To mimic the way μ operates we need to perform coordinate manipulations in order to arrange the components of s -tuples in increasing order. This is exactly what the terms S_j enable us to do. Finally we must replace a tuple with non-negative components which increase non-strictly by one in which only the smallest coordinate is repeated. To understand how the behaviour

of the term T_i allows us to do this, first observe that if $0 \leq a \leq b$ then $\neg(a \leftrightarrow b)$ is equal to $\neg b$ if $a = b$ and to b if $a < b$. Hence the tuple $(T_1(a_1, \dots, a_s), \dots, T_s(a_1, \dots, a_s))$ is obtained from the tuple $(S_1(a_1, \dots, a_s), \dots, S_s(a_1, \dots, a_s))$ by working iteratively to replace any $S_i(a_1, \dots, a_s)$ that coincides with $S_{i-1}(a_1, \dots, a_s)$ by $S_1(a_1, \dots, a_s)$, *viz.* $\bigwedge\{|a_1|, \dots, |a_s|\}$. We finally arrive at $\mu(a_1, \dots, a_s) = (G_1(a_1, \dots, a_s), \dots, G_s(a_1, \dots, a_s))$. Finally, the assignment $b_j^k \mapsto G_j$ extends to an embedding from \mathbf{B}_k to $\mathbf{F}_{\mathcal{S}\mathcal{A}_k}(s)$.

We indicated at the outset that the introduction of the notion of admissibility algebra stemmed from the desire to find a ‘small’ generating algebra of $\mathbb{ISP}(\mathbf{F}_{\mathcal{S}\mathcal{A}_k}(s))$ which can be used to check the admissibility of quasi-equations. Our application of TSM delivers \mathbf{B}_k , the smallest possible such algebra. Observe that we have exhibited an explicit description of each \mathbf{B}_k , in terms of a generating set, of the subalgebra of \mathbf{Z}_k^s in which it sits. This can be seen as a standalone presentation of this algebra, not involving the duality methodology we used to arrive at it. Denote by p_s the restriction to \mathbf{B}_k of the projection map of \mathbf{Z}_k^s onto the last coordinate. It is easy to see that p_s is surjective, so that $\mathbf{Z}_k \in \mathbb{H}(\mathbf{B}_k)$. From this we can prove algebraically that $\mathbb{ISP}(\mathbf{B}_k) = \mathbb{ISP}(\mathbf{F}_{\mathcal{S}\mathcal{A}_k}(s))$ (see [22, Corollary 22]). This direct, duality-free, argument confirms that \mathbf{B}_k can be employed for admissibility testing, but does not yield the stronger result that is the minimal such algebra.

It is fitting that we should end our paper with a brief discussion of the problem that inspired it: admissibility of rules in the context of the logic R-mingle. We shall use the following interpretations of terms into quasi-equations. To any formula α we assign the equation $\alpha \approx \alpha \rightarrow \alpha$. (With this interpretation every logical rule becomes a quasi-equation. Moreover this is the interpretation that proves that R-mingle is algebraizable and that its equivalent algebraic semantics are Sugihara algebras; see Dunn [12, Section 2], Blok and Dziobak [2], and also Font [13, p. 141].) We assume below that $k > 4$ since $\mathcal{S}\mathcal{A}_2$ is structurally complete (that is, every admissible quasi-equation in $\mathcal{S}\mathcal{A}_2$ is valid on every algebra) and for $k = 3, 4$ every admissible quasi-equation that is not derivable is so because its antecedents are not unifiable. A quasivariety with the latter property is called *almost structurally complete*. In [24, Section 4.5] $\mathcal{S}\mathcal{A}_3$ was proved to be almost structurally complete. The proof that $\mathcal{S}\mathcal{A}_4$ is almost structurally complete follows from [22, Theorem 18] combined with the fact that its admissibility algebra is $\mathbf{B}_4 = \mathbf{Z}_4 \times \mathbf{Z}_2$.

In Table 3 we present a selection of examples of quasi-equations and decide their admissibility using the admissibility algebras we have identified.

quasi-equation	admissible?	
	even case	odd case
$p \leftrightarrow \neg p \vdash q \leftrightarrow r$	✓	×
$p, \neg p \vee q \vdash q$	✓	×
$p, (p \rightarrow q) \rightarrow (p \rightarrow q) \vdash p \rightarrow q$	✓	✓
$q, p \rightarrow (q \rightarrow r) \vdash p \rightarrow r$	✓	✓
$\neg p \vee q \vdash q$	✓	×

Table 3: Sugihara algebras: admissible rules

Observe that B_{2n} is (isomorphic to) a subalgebra of B_{2n+1} . Hence a quasi-equation that is admissible in every quasivariety $\mathcal{S}\mathcal{A}_{2n+1}$ must also be admissible on every \mathbf{B}_{2n} . The converse does not hold, as the results in the table show.

We have already noted that the problem of finding axiomatizations for the admissible rules of R-mingle remains open. Similarly, although Metcalfe [21] succeeded in axiomatizing the admissible rules of various fragments of R-mingle, these did not include the extensions \mathbf{RM}_k for $k > 4$. This

unsolved problem could be approached by first axiomatizing the quasivariety generated by the admissibility algebra \mathbf{B}_k for \mathcal{SA}_k . We hope that the recursive specification of this family of algebras might be of assistance here.

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