

# The Almost Complex Structure on $\mathbb{S}^6$ and Related Schrödinger Flows

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## Abstract

In this paper, by using the  $G_2$ -structure on  $\text{Im}(\mathbb{O}) \cong \mathbb{R}^7$  from the octonions  $\mathbb{O}$ , the  $G_2$ -binormal motion of curves  $\gamma(t, s)$  in  $\mathbb{R}^7$  associated to the almost complex structure on  $\mathbb{S}^6$  is studied. The motion is proved to be equivalent to Schrödinger flows from  $\mathbb{R}^1$  to  $\mathbb{S}^6$ , and also to a nonlinear Schrödinger-type system in three unknown complex functions that generalizes the famous correspondence between the binormal motion of curves in  $\mathbb{R}^3$  and the focusing nonlinear Schrödinger equation. Some related geometric properties of the surface  $\Sigma$  in  $\text{Im}(\mathbb{O})$  swept by  $\gamma(t, s)$  are determined.

Mathematics Subject Classification: 32Q60, 53C44, 53C14, 53A04

Keywords: Almost complex structure, Schrödinger flow,  $G_2$ -structure, octonions

## §1. Introduction

The study of moving curves in a Riemannian or pseudo-Riemannian manifold, especially in the Euclidean or Minkowski spaces, is an attractive topic in differential geometry, as it has applications in physics, such as the deformation of a thin vortex filament in inviscid fluid [1, 20], kinematics of interfaces in crystal growth [4, 17], viscous fingering in a Hele-shaw cell [25], etc. It is well-known that in considering a motion of a vortex filament in  $\mathbb{R}^3$ , one encounters the Da Rios equation as follows (see [9], [1]):

$$\gamma_t = \gamma_s \times \gamma_{ss}, \quad (1)$$

where  $\gamma = \gamma(t, s) \in \mathbb{R}^3$  is the centerline (curve) of the vortex filament represented by a vector-valued function with respect to arclength  $s$  and time  $t$ , the subscript stands for

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partial derivative with respect to the indicated variable and  $\times$  denotes the cross product between vectors in  $\mathbb{R}^3$ . Eq.(1) is also called the binormal motion of space curves in  $\mathbb{R}^3$ , since the right-hand-side of (1) is equal to a multiple of the curvature and the binormal vector at  $\gamma$ . The distinguishing features of Eq.(1) are its complete integrability and the fact that it is equivalent to the focusing nonlinear Schrödinger equation (NLS):

$$\varphi_t + \varphi_{ss} + 2|\varphi|^2\varphi = 0.$$

This produces the so-called Da Rios-NLS correspondence (refer to [15], [18]), that is, if a curve  $\gamma(t, s)$  evolves according to the Da Rios equation (1), then the associated complex function  $\varphi = \kappa(t, s) \exp(i \int^s \tau(t, x) dx)$  by Hasimoto transform evolves according to the NLS equation, where  $\kappa$  and  $\tau$  stand for the curvature and the torsion curvature at  $\gamma(t, s)$  respectively.

On the other hand, Euclidean 3-space  $\mathbb{R}^3$  can be regarded as the imaginary part  $\text{Im}(\mathbb{H})$  of the quaternions  $\mathbb{H}$  and the cross product  $\times$  on  $\mathbb{R}^3 \cong \text{Im}(\mathbb{H})$  is created by the quaternion algebraic structure in  $\text{Im}(\mathbb{H})$ . It is also well-known that the quaternions are contained in the octonions  $\mathbb{O}$  and thus the imaginary part  $\text{Im}(\mathbb{H})$  in the imaginary part  $\text{Im}(\mathbb{O})$  of the octonions  $\mathbb{O}$ . It is not a surprise that there is a cross product on  $\text{Im}(\mathbb{O}) \cong \mathbb{R}^7$  induced by the octonion algebraic structure, which includes the cross product on  $\text{Im}(\mathbb{H}) \cong \mathbb{R}^3$  as a special case. Therefore, as a natural generalization of Eq.(1) in higher dimensions, the following equation in  $\text{Im}(\mathbb{O}) \cong \mathbb{R}^7$ :

$$\gamma_t = \gamma_s \times \gamma_{ss}, \quad \gamma(t, s) \in \mathbb{R}^7 \tag{2}$$

looks very interesting from the purely mathematical point of view. Eq.(2) is in fact the  $G_2$ -binormal motion of curves in  $\mathbb{R}^7$ , as we shall see below. However, to our disappointment, we haven't found physical applications of Eq.(2) yet in the literature.

The aim of this paper is to give a geometric interpretation of Eq.(2) associated to the almost complex structure on  $\mathbb{S}^6$ . To our surprise, Eq.(2) is proved to be equivalent to Schrödinger flows of maps from  $\mathbb{R}^1$  to the 6-sphere  $\mathbb{S}^6 \hookrightarrow \mathbb{R}^7$ , where  $\mathbb{S}^6$  is equipped with the standard almost complex structure called Kirchhoff's almost complex structure that is not integrable. It was proved by Borel and Serre in 1953 in [3] that  $\mathbb{S}^{2n}$  admits an almost complex structure if and only if  $n = 1$  or  $3$ . In 1993, Calabi and Gluck [8] proved that the best almost complex structure on  $\mathbb{S}^6$  is the one constructed by Kirchhoff in the sense that it has the smallest volume in a class of sections of the bundle  $O(8) = U(4)$  over  $\mathbb{S}^6$ . From the viewpoint of Schrödinger flows, one may obtain different equations (2) by choosing different almost complex structures on  $\mathbb{S}^6$  in advance. This indicates that Eq.(2) not only is a higher dimensional generalization of Eq.(1), but also relates to almost complex structures on  $\mathbb{S}^6$ . An old problem in this aspect is whether or not there is a complex structure on  $\mathbb{S}^6$  (refer to [6, 19]). This gives us further motivations for studying Eq.(2). The aim of this paper can also be regarded as a contribution to our understanding of almost complex structures on  $\mathbb{S}^6$  and the  $G_2$ -structure on  $\text{Im}(\mathbb{O}) = \mathbb{R}^7$  via Schrödinger flows. Furthermore, Eq.(2) is also shown to be equivalent to a nonlinear Schrödinger-type system in three unknown complex functions, which sets the famous Da

Rios-NLS correspondence as a special case. Some geometric properties of the surface  $\Sigma$  in  $\text{Im}(\mathbb{O})$  swept by  $\gamma(t, s)$  are characterized by the unknown functions in the nonlinear Schrödinger-type system.

The paper is organized as follows. Section §2 gives preliminaries about Schrödinger flows from a Riemannian manifold to an almost Hermitian manifold, the Cayley-Dickson construction, the exceptional simple Lie group  $G_2$  and  $G_2$ -frame in  $\text{Im}(\mathbb{O}) \cong \mathbb{R}^7$ . In §3, we construct the complexified  $G_2$ -frame and establish a related Frenet formula along curves. In §4, we give a proof of the correspondence between Eq.(2) and a nonlinear Schrödinger-type system. In §5 we exploit the geometric properties of the surface swept by  $G_2$ -binormal moving curves in terms of the unknown-functions in the nonlinear Schrödinger-type system.

## §2. Preliminaries

In this section, we recall briefly the geometric concept of Schrödinger flows from a Riemannian manifold to an almost Hermitian manifold. We also give some facts about the Cayley-Dickson construction, the octonions, the exceptional simple Lie group  $G_2$  and  $G_2$ -frame.

### §2.1 Schrödinger flows to almost Hermitian manifolds

The motivation for introducing Schrödinger flows comes from the fact that the Heisenberg model in condensed matter physics is described as the equation of Schrödinger flows from  $\mathbb{R}^1$  to the 2-sphere  $\mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ , in which the standard complex structure on  $\mathbb{S}^2$  is used. First of all, we recall the definition of Schrödinger flows from the Riemannian manifold  $(M, g)$  to the almost Hermitian manifold  $(N, J, h)$ , where  $J$  is an almost complex structure compatible to the metric  $h$  on  $N$  (refer to [10, 13, 21, 26], for example). Some other related geometric flows, such as KdV geometric flows, are described in [11, 12].

**Definition 1.** *A map  $u = u(t, x) : [0, T) \times M \rightarrow N$ , where  $0 < T \leq \infty$ , is called a Schrödinger flow from  $(M, g)$  to  $(N, J, h)$  if  $u$  satisfies the following equation of the Hamiltonian gradient flow*

$$u_t = J_u \nabla E(u),$$

where  $E(u)$  is the energy functional of  $u : M \rightarrow N$ .

Recall that the energy  $E(u)$  of  $u : M \rightarrow N$  is defined by

$$E(u) = \int_M e(u) dv_g,$$

where, in a local chart  $(x_\alpha)$  of  $M$ ,  $e(u) = \frac{1}{2} g^{\alpha\beta} h_{jk}(u) \frac{\partial u^j}{\partial x_\alpha} \frac{\partial u^k}{\partial x_\beta}$ . It is easy to verify that the gradient  $\nabla E(u)$  is exactly the tension field  $\tau(u)$  of map  $u$ , which is expressed in local

coordinates as

$$\tau(u)^i = \Delta_M u^i + g^{\alpha\beta} \Gamma_{jk}^i(u) \frac{\partial u^j}{\partial x_\alpha} \frac{\partial u^k}{\partial x_\beta},$$

where  $\Delta_M$  is the Laplace-Beltrami operator on  $M$  and  $\Gamma_{jk}^i$  are the Christoffel symbols of  $N$ . Hence, the equation of the Schrödinger flows of  $u : M \rightarrow N$  can be rewritten as

$$u_t = J_u \tau(u). \quad (3)$$

Next, it is easy to see that when the target  $N = \mathbb{C}$ , the complex plane, Eq.(3) is nothing but the linear Schrodinger equation. When  $N = \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$  ( $n = 2, 6$ ), the tension field of map  $u : M \rightarrow \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$  is given by  $\tau(u) = \Delta u + |\nabla u|^2 u$ . Noting that the standard almost complex structure  $J$  at  $u \in \mathbb{S}^n$  ( $n = 2, 6$ ) is given by  $J_u = u \times : T_u \mathbb{S}^n \rightarrow T_u \mathbb{S}^n$ , in which  $J$  is integral only when  $n = 2$ , we obtain the equation of Schrödinger flows from  $M$  to  $\mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$  as follows

$$u_t = u \times \Delta u, \quad u : M \rightarrow \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1} \quad (n = 2, 6).$$

## §2.2 The Cayley-Dickson construction and $G_2$ -frame

Let  $\mathbb{A}$  be an algebra over the field  $\mathbb{R}$  which is not necessarily associative but finite-dimensional. A linear mapping  $a \rightarrow \bar{a}$  of  $\mathbb{A}$  to itself is said to be a conjugation or involutory anti-automorphism if  $\bar{\bar{a}} = a$  and  $\overline{ab} = \bar{b}\bar{a}$  for any elements  $a, b \in \mathbb{A}$  (an case  $\bar{a} = a$  is not excluded).

**Definition 2.** (Cayley-Dickson construction [2], [14]) Consider the vector space of the direct sum of two copies of  $\mathbb{A}$ :  $\mathbb{A}^2 = \mathbb{A} \oplus \mathbb{A}$ . A multiplication on  $\mathbb{A}^2$  is defined as:

$$(a, b)(c, d) = (ac - d\bar{b}, \bar{a}d + cb).$$

It is easy to check that relative to the multiplication the vector space  $\mathbb{A}^2$  is an algebra of dimension  $2 \cdot \dim(\mathbb{A})$ . This is called the doubling of  $\mathbb{A}$ .

**Remark 1.** The correspondence  $a \rightarrow (a, 0)$  is a monomorphism of  $\mathbb{A}$  into  $\mathbb{A}^2$ . Therefore we will identify elements  $a$  and  $(a, 0)$  and consequently assume  $\mathbb{A}$  is a subalgebra of  $\mathbb{A}^2$ . If  $\mathbb{A}$  has an identity element, then the element  $1 = (1, 0)$  is obviously an identity element in  $\mathbb{A}^2$ .

An important element in  $\mathbb{A}^2$  is  $e = (0, 1)$ . It follows from the definition of multiplication that  $be = (0, b)$  and hence  $(a, b) = a + be$  for all  $a, b \in \mathbb{A}$ . Thus every element of the algebra  $\mathbb{A}^2$  is uniquely written as  $a + be$ . Moreover, as can be easily checked, the following identities are true:

$$a(be) = (ba)e, \quad (ae)b = (a\bar{b})e, \quad (ae)(be) = -\bar{b}a. \quad (4)$$

In particular  $e^2 = -1$ .

To iterate the Cayley-Dickson construction it is necessary to define a conjugation in  $\mathbb{A}^2$ . This will be done by the formula

$$\overline{a + b} = \bar{a} - be.$$

The doubling  $\mathbb{R}^2$  of the field  $\mathbb{R}$  is the algebra  $\mathbb{C}$  of complex numbers and the doubling  $\mathbb{C}^2$  of  $\mathbb{C}$  is the algebra of quaternions  $\mathbb{H}$ . In the latter case  $e$  is denoted by  $j$  and  $ie$  is denoted by  $k$ , so a general quaternion is of the form  $r = r_1 + r_2i + r_3j + r_4k$ , where  $r_i \in \mathbb{R}, i = 1, 2, 3, 4$ . Due to the identity (4)  $ea = \bar{a}e$  for all  $a \in \mathbb{A}$ , one may verify that  $\mathbb{H}$  is not commutative.

The doubling algebra  $\mathbb{O} = \mathbb{H}^2$  of  $\mathbb{H}$  is the Cayley algebra which is not commutative and associative, and its elements are called octonions or Cayley numbers. By definition every octonion is of the form  $\xi = a + be$ , where  $a$  and  $b$  are quaternions. The basis of  $\mathbb{O}$  consists of  $\{1, i, j, k, l, il, jl, kl\}$  in which we replace  $e$  by  $l$ . The square of each of these elements is  $-1$  except the unit element 1. The full multiplication table is summarized in Table 1.

	$i$	$j$	$k$	$l$	$il$	$jl$	$kl$
$i$	$-1$	$k$	$-j$	$il$	$-l$	$-kl$	$jl$
$j$	$-k$	$-1$	$i$	$jl$	$kl$	$-l$	$-il$
$k$	$j$	$-i$	$-1$	$kl$	$-jl$	$il$	$-l$
$l$	$-il$	$-jl$	$-kl$	$-1$	$i$	$j$	$k$
$il$	$l$	$-kl$	$jl$	$-i$	$-1$	$-k$	$j$
$jl$	$kl$	$l$	$-il$	$-j$	$k$	$-1$	$-i$
$kl$	$-jl$	$il$	$l$	$-k$	$-j$	$i$	$-1$

Table 1: The multiplication table of  $\mathbb{O}$

The group  $G_2$  is defined to be the automorphism group of the octonions  $\mathbb{O}$  :

$$G_2 = \{g \in Iso_{\mathbb{R}}\mathbb{O} \mid g(xy) = g(x)g(y), \forall x, y \in \mathbb{O}\},$$

where  $Iso_{\mathbb{R}}\mathbb{O} = O(8)$  denotes the set of all  $\mathbb{R}$ -linear isomorphisms of  $\mathbb{O}$ . The cross product  $x \times y$  and scalar product  $\langle x, y \rangle$  of  $\mathbb{O}$  are determined respectively by the multiplicity on  $\mathbb{O}$  as follows:

$$x \times y = (1/2)(\bar{y}x - \bar{x}y), \quad \langle x, y \rangle = (1/2)(\bar{x}y + \bar{y}x),$$

where  $\bar{x} = 2\langle x, 1 \rangle - x$  is the conjugation of  $x \in \mathbb{O}$ . One may verify directly that  $x \times y \in \text{Im}(\mathbb{O}), \forall x, y \in \text{Im}(\mathbb{O})$ , where  $\text{Im}(\mathbb{O}) = \{x \in \mathbb{O} \mid \langle x, 1 \rangle = 0\}$ . This induces a cross product  $\times$  among vectors in  $\text{Im}(\mathbb{O}) \cong \mathbb{R}^7$ .

The differential version of  $G_2$  realization leads to the so-called  $G_2$ -frame of  $\mathbb{R}^7 = \text{Im}(\mathbb{O})$ . In 1958, Calabi [7] first constructed  $G_2$ -structure equations of a submanifold in  $\mathbb{R}^7$ . In 1982, Bryant [5, 6] gave a more concrete representation of  $G_2$  by taking account of the algebraic properties of the octonions  $\mathbb{O}$ .

From the standard basis of  $\text{Im}(\mathbb{O}) = \text{Span}_{\mathbb{R}}\{i, j, k, l, il, jl, kl\}$ , we define a basis of the complexification of  $\text{Im}(\mathbb{O})$  over  $\mathbb{C}$ :

$$N = \frac{1}{\sqrt{2}}(1 - \sqrt{-1}l), \quad \bar{N} = \frac{1}{\sqrt{2}}(1 + \sqrt{-1}l),$$

$$E_1 = iN, \quad E_2 = jN, \quad E_3 = -kN, \quad \bar{E}_1 = i\bar{N}, \quad \bar{E}_2 = j\bar{N}, \quad \bar{E}_3 = -k\bar{N}.$$

A basis  $(e_4 \ f \ \bar{f})$  of  $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{O}$  is said to be admissible, if there exists  $g \in G_2$  such that  $(e_4 \ f \ \bar{f})^T = g(l \ E \ \bar{E})^T$ , where  $E = (E_1, E_2, E_3)$ . Usually,  $(e_4 \ f \ \bar{f})$  is called a complexified  $G_2$ -frame.

**Theorem 1.** (Bryant [5]) For a complexified  $G_2$ -frame  $(e_4 \ f \ \bar{f})$ , we have

$$d \begin{pmatrix} e_4 \\ f \\ \bar{f} \end{pmatrix} = \left( \begin{array}{c|c|c} 0 & -\sqrt{2}\sqrt{-1}\theta & \sqrt{2}\sqrt{-1}\bar{\theta} \\ \hline -\sqrt{2}\sqrt{-1}\bar{\theta}^T & \kappa & [\theta] \\ \hline \sqrt{2}\sqrt{-1}\theta^T & [\bar{\theta}] & \bar{\kappa} \end{array} \right) \begin{pmatrix} e_4 \\ f \\ \bar{f} \end{pmatrix},$$

where  $\theta = (\theta^1 \ \theta^2 \ \theta^3)$  is an  $M_{1 \times 3}(\mathbb{C})$  valued 1-form,  $\kappa$  is an  $su(3)$  valued 1-form which satisfies

$$\kappa + \bar{\kappa}^T = 0_{3 \times 3}, \quad \text{tr} \kappa = 0$$

and

$$[\theta] = \begin{pmatrix} 0 & -\theta^3 & \theta^2 \\ \theta^3 & 0 & -\theta^1 \\ -\theta^2 & \theta^1 & 0 \end{pmatrix}.$$

The above structure of the  $G_2$ -frame will play a crucial role in the proof of Theorem 4, as we shall see below. We should mention that the standard Kirchhoff's complex structure  $J$  on  $\mathbb{S}^6$  is explicitly given as follows: for  $u \in \mathbb{S}^6$ ,

$$J_u : T_u \mathbb{S}^6 \rightarrow T_u \mathbb{S}^6, \quad X \mapsto J_u(X) = u \times X, \quad X \in T_u \mathbb{S}^6.$$

### §3 $G_2$ -structure equations of curves in $\text{Im}(\mathbb{O})$

In this section, we describe the construction of  $G_2$ -frame along curves in  $\text{Im}(\mathbb{O}) \cong \mathbb{R}^7$  and then take its complexification. One may refer to [16, 22, 23] for details. Based on the complexification, we present a Frenet formula of the complexified  $G_2$ -frame along a curve in  $\text{Im}(\mathbb{O}) \cong \mathbb{R}^7$ .

Let  $\gamma(s)$  be a unit speed curve in  $\text{Im}(\mathbb{O}) \cong \mathbb{R}^7$ . We set  $k_1(s) = \|\gamma_{ss}(s)\|$  and assume that this function does not vanish anywhere. Now we define a  $G_2$ -frame along the curve as follows

$$I_4(s) = \gamma_s(s), \quad I_1(s) = \frac{1}{k_1} I_{4s}, \quad I_5(s) = I_1 \times I_4,$$

$$I_2(s) = \frac{1}{\kappa_2} (I_{1s} - \langle I_{1s}, I_4 \rangle I_4 - \langle I_{1s}, I_5 \rangle I_5),$$

$$I_3(s) = I_1 \times I_2, \quad I_6(s) = I_2 \times I_4, \quad I_7(s) = I_3 \times I_4,$$

where

$$\kappa_2(s) = \sqrt{\|I_{1s}\|^2 - \langle I_{1s}, I_4 \rangle^2 - \langle I_{1s}, I_5 \rangle^2} > 0$$

is assumed and  $I_5 = I_1 \times I_4$  is usually regarded as the  $G_2$ -binormal vector along the curve.

The multiplication table of  $(I_4 I_1 I_2 I_3 I_5 I_6 I_7)$  coincides with that of  $(l i j k il jl kl)$ . In other words, there exists a  $G_2$ -valued function  $g$  such that

$$(I_4 I_1 I_2 I_3 I_5 I_6 I_7) = (g(l) g(i) g(j) g(k) g(il) g(jl) g(kl)).$$

If  $\kappa_2 = 0$  and we take  $I_2(s) \in (\text{span}_{\mathbb{R}}\{I_4, I_1, I_5\})^\perp$  with  $|I_2(s)| = 1$ ,  $I_3(s) = I_1 \times I_2$ ,  $I_6(s) = I_2 \times I_4$ , and  $I_7(s) = I_3 \times I_4$ , then  $(I_4 I_1 I_2 I_3 I_5 I_6 I_7)$  also consists of a  $G_2$ -frame along the curve in  $\text{Im}(\mathbb{O}) \cong \mathbb{R}^7$ , in which  $\{I_4, I_1, I_5\}$  consists of an autonomy system, that is, it satisfies the formula

$$\begin{pmatrix} I_4 \\ I_1 \\ I_5 \end{pmatrix}_s = \begin{pmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & \rho_1 \\ 0 & -\rho_1 & 0 \end{pmatrix} \begin{pmatrix} I_4 \\ I_1 \\ I_5 \end{pmatrix},$$

where  $\rho_1 = \langle I_{1s}, I_5 \rangle$ .

**Proposition 1.** (*[22, 23]*) *Let  $\gamma : I = (0, 1) \rightarrow \text{Im}(\mathbb{O})$  be a curve with  $k_1 > 0$ . The associated  $G_2$ -frame  $(I_4 I_1 I_2 I_3 I_5 I_6 I_7)$  satisfies the following differential equation*

$$\begin{pmatrix} I_4 \\ I_1 \\ I_2 \\ I_3 \\ I_5 \\ I_6 \\ I_7 \end{pmatrix}_s = \begin{pmatrix} 0 & k_1 & 0 & 0 & 0 & 0 & 0 \\ -k_1 & 0 & \kappa_2 & 0 & \rho_1 & 0 & 0 \\ 0 & -\kappa_2 & 0 & \alpha & 0 & \rho_2 & \beta_1 \\ 0 & 0 & -\alpha & 0 & 0 & \beta_2 & \rho_3 \\ 0 & -\rho_1 & 0 & 0 & 0 & \kappa_2 & 0 \\ 0 & 0 & -\rho_2 & -\beta_2 & -\kappa_2 & 0 & \alpha \\ 0 & 0 & -\beta_1 & -\rho_3 & 0 & -\alpha & 0 \end{pmatrix} \begin{pmatrix} I_4 \\ I_1 \\ I_2 \\ I_3 \\ I_5 \\ I_6 \\ I_7 \end{pmatrix} \quad (5)$$

with  $\rho_1 = \langle I_{1s}, I_5 \rangle$ ,  $\rho_2 = \langle I_{2s}, I_6 \rangle$ ,  $\rho_3 = \langle I_{3s}, I_7 \rangle$ ,  $\alpha = \langle I_{2s}, I_3 \rangle$ ,  $\beta_1 = \langle I_{2s}, I_7 \rangle$  and  $\beta_2 = \langle I_{3s}, I_6 \rangle$ . These functions satisfy

$$\rho_1 + \rho_2 + \rho_3 = 0, \quad (6)$$

$$\beta_1 - \beta_2 + k_1 = 0. \quad (7)$$

**Remark 2.** *One notes from (5) that Eq.(2) can be rewritten as  $\gamma_t = -k_1 I_5$ . Hence, Eq.(2) is the  $G_2$ -binormal motion of curves in  $\mathbb{R}^7$ . We mention that for a given curve  $\gamma(s)$  in  $\text{Im}(\mathbb{O}) \cong \mathbb{R}^7$ , one may also establish the associated  $SO(7)$ -frame  $\{V_1, V_2, \dots, V_7\}$  and corresponding Frenet formula, as was done by Ohashi in [24]. For the relationship between the  $G_2$ -frame and  $SO(7)$ -frame, Ohashi proved in [24] that*

$$I_5 = \cos \sigma V_3 + \sum_{i=1}^4 a_{i1} V_{3+i},$$

where

$$\cos \sigma = \langle V_3, V_3 \times V_1 \rangle, \quad a_{i1} = \langle V_{i+3}, V_2 \times V_1 \rangle, \quad i \in \{1, 2, 3, 4\}.$$

Therefore, from the view point of  $SO(7)$ -frame, Eq.(2) is a motion of curves along the direction in a combination of  $\{V_3, V_4, V_5, V_6, V_7\}$  in  $\mathbb{R}^7$ .

The six functions  $(k_1, \kappa_2, \rho_1, \rho_3, \alpha, \beta_1)$  are the complete  $G_2$ -invariants of  $\gamma$ . Now, we give the complexification of the  $G_2$ -frame  $(I_4, I_1, I_2, I_3, I_5, I_6, I_7)$  along  $\gamma(s)$  according to Bryant in [5] as follows. Let

$$\begin{aligned} e_4 &= I_4, \\ e_1 &= r(I_1 - \sqrt{-1}I_5), & \bar{e}_1 &= \bar{r}(I_1 + \sqrt{-1}I_5), \\ e_2 &= q(I_2 - \sqrt{-1}I_6), & \bar{e}_2 &= \bar{q}(I_2 + \sqrt{-1}I_6), \\ e_3 &= -p(I_3 - \sqrt{-1}I_7), & \bar{e}_3 &= -\bar{p}(I_3 + \sqrt{-1}I_7), \end{aligned} \quad (8)$$

where

$$\begin{aligned} r &= \frac{1}{\sqrt{2}} \exp(-\sqrt{-1} \int_0^s \rho_1 d\tilde{s}), \\ q &= \frac{1}{\sqrt{2}} \exp(-\sqrt{-1} \int_0^s \rho_2 d\tilde{s}), \\ p &= \frac{1}{\sqrt{2}} \exp(-\sqrt{-1} \int_0^s \rho_3 d\tilde{s}) = \sqrt{2} \bar{q} \bar{r} \end{aligned} \quad (9)$$

and bar denotes the complex conjugation of elements in  $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{O}$ , namely,

$$\overline{x_1 + \sqrt{-1}x_2} = x_1 - \sqrt{-1}x_2, \quad x_1, x_2 \in \mathbb{O}.$$

One notes that the complex conjugation here is different from the conjugation over  $\mathbb{O}$ . The complex-conjugate on  $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{O}$  satisfies

$$\overline{xy} = \bar{x} \bar{y}, \quad x, y \in \mathbb{C} \otimes_{\mathbb{R}} \mathbb{O}.$$

In the sequel, the bar is used to denote the complex conjugation on  $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{O}$ , unless otherwise specified.

Eq.(8) can be rewritten as

$$\begin{pmatrix} e_4 \\ f \\ \bar{f} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A & -\sqrt{-1}A \\ 0 & \bar{A} & \sqrt{-1}\bar{A} \end{pmatrix} \begin{pmatrix} I_4 \\ J_1 \\ J_2 \end{pmatrix} := N_1 \begin{pmatrix} I_4 \\ J_1 \\ J_2 \end{pmatrix}, \quad (10)$$

where

$$f = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \quad J_1 = \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix}, \quad J_2 = \begin{pmatrix} I_5 \\ I_6 \\ I_7 \end{pmatrix}, \quad A = \begin{pmatrix} r & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & -p \end{pmatrix}. \quad (11)$$



**Theorem 2.** For the complexified  $G_2$ -frame  $(e_4 \ e_1 \ e_2 \ e_3 \ \bar{e}_1 \ \bar{e}_2 \ \bar{e}_3)$  along  $\gamma$ , we have the following Frenet formula:

$$\begin{pmatrix} e_4 \\ e_1 \\ e_2 \\ e_3 \\ \bar{e}_1 \\ \bar{e}_2 \\ \bar{e}_3 \end{pmatrix}_s = \begin{pmatrix} 0 & \varphi_1 & 0 & 0 & \bar{\varphi}_1 & 0 & 0 \\ -\bar{\varphi}_1 & 0 & \varphi_2 & 0 & 0 & 0 & 0 \\ 0 & -\bar{\varphi}_2 & 0 & \varphi_3 & 0 & 0 & -\frac{\sqrt{-1}}{\sqrt{2}}\varphi_1 \\ 0 & 0 & -\bar{\varphi}_3 & 0 & 0 & \frac{\sqrt{-1}}{\sqrt{2}}\varphi_1 & 0 \\ -\varphi_1 & 0 & 0 & 0 & 0 & \bar{\varphi}_2 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{-1}}{\sqrt{2}}\bar{\varphi}_1 & -\varphi_2 & 0 & \bar{\varphi}_3 \\ 0 & 0 & -\frac{\sqrt{-1}}{\sqrt{2}}\bar{\varphi}_1 & 0 & 0 & -\varphi_3 & 0 \end{pmatrix} \begin{pmatrix} e_4 \\ e_1 \\ e_2 \\ e_3 \\ \bar{e}_1 \\ \bar{e}_2 \\ \bar{e}_3 \end{pmatrix}, \quad (12)$$

where

$$\varphi_1 = k_1\bar{r}, \quad \varphi_2 = 2\kappa_2r\bar{q}, \quad \varphi_3 = -\sqrt{2}q^2r[2\alpha + \sqrt{-1}(\beta_1 + \beta_2)]. \quad (13)$$

**Proof:** First of all, we can rewrite (5) as

$$\begin{pmatrix} I_4 \\ J_1 \\ J_2 \end{pmatrix}_s = \begin{pmatrix} 0 & u & 0 \\ -u^T & B & C \\ 0 & -C^T & B \end{pmatrix} \begin{pmatrix} I_4 \\ J_1 \\ J_2 \end{pmatrix} := N_2 \begin{pmatrix} I_4 \\ J_1 \\ J_2 \end{pmatrix},$$

where

$$u = \begin{pmatrix} k_1 \\ 0 \\ 0 \end{pmatrix}^T, \quad B = \begin{pmatrix} 0 & \kappa_2 & 0 \\ -\kappa_2 & 0 & \alpha \\ 0 & -\alpha & 0 \end{pmatrix}, \quad C = \begin{pmatrix} \rho_1 & 0 & 0 \\ 0 & \rho_2 & \beta_1 \\ 0 & \beta_2 & \rho_3 \end{pmatrix}.$$

From (10) and (14), we obtain

$$(e_4 \ f \ \bar{f})_s^T = (N_{1s} + N_1N_2)N_1^{-1}(e_4 \ f \ \bar{f})^T,$$

with

$$(N_{1s} + N_1N_2)N_1^{-1} = \begin{pmatrix} 0 & u\bar{A} & uA \\ -Au^T & \eta_1 & \eta_2 \\ -\bar{A}u^T & \bar{\eta}_2 & \bar{\eta}_1 \end{pmatrix},$$

$$\eta_1 = [2A_s + 2AB + \sqrt{-1}A(C^T + C)]\bar{A},$$

$$\eta_2 = -\sqrt{-1}A(C - C^T)A.$$

It follows from (6), (7), (11) and (14) that

$$u\bar{A} = (k_1\bar{r} \ 0 \ 0),$$

$$\eta_1 = \begin{pmatrix} 0 & 2\kappa_2r\bar{q} & 0 \\ -2\kappa_2q\bar{r} & 0 & -\sqrt{2}q^2r[2\alpha + \sqrt{-1}(\beta_1 + \beta_2)] \\ 0 & \sqrt{2}q^2\bar{r}[2\alpha - \sqrt{-1}(\beta_1 + \beta_2)] & 0 \end{pmatrix},$$

$$\eta_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{-1}}{\sqrt{2}}k_1\bar{r} \\ 0 & \frac{\sqrt{-1}}{\sqrt{2}}k_1\bar{r} & 0 \end{pmatrix}.$$

By setting

$$\varphi_1 = k_1 \bar{r}, \quad \varphi_2 = 2\kappa_2 r \bar{q} \quad \text{and} \quad \varphi_3 = -\sqrt{2}q^2 r [2\alpha + \sqrt{-1}(\beta_1 + \beta_2)],$$

we arrive at (12).  $\square$

Based on the complexified  $G_2$ -frame  $(e_4 \ e_1 \ e_2 \ e_3 \ \bar{e}_1 \ \bar{e}_2 \ \bar{e}_3)$  along the curve  $\gamma(s)$ , we shall establish the multiplication and cross product tables with respect to  $(e_4 \ e_1 \ e_2 \ e_3 \ \bar{e}_1 \ \bar{e}_2 \ \bar{e}_3)$ , which are well-suited for the development in the next section. We have the following tables, in which  $e_0 = -1 - \sqrt{-1}e_4$ :

$A \setminus B$	$e_4$	$e_1$	$e_2$	$e_3$	$\bar{e}_1$	$\bar{e}_2$	$\bar{e}_3$
$e_4$	-1	$-\sqrt{-1}e_1$	$-\sqrt{-1}e_2$	$-\sqrt{-1}e_3$	$\sqrt{-1}\bar{e}_1$	$\sqrt{-1}\bar{e}_2$	$\sqrt{-1}\bar{e}_3$
$e_1$	$\sqrt{-1}e_1$	0	$-\sqrt{2}\bar{e}_3$	$\sqrt{2}\bar{e}_2$	$e_0$	0	0
$e_2$	$\sqrt{-1}e_2$	$\sqrt{2}\bar{e}_3$	0	$-\sqrt{2}\bar{e}_1$	0	$e_0$	0
$e_3$	$\sqrt{-1}e_3$	$-\sqrt{2}\bar{e}_2$	$\sqrt{2}\bar{e}_1$	0	0	0	$e_0$
$\bar{e}_1$	$-\sqrt{-1}\bar{e}_1$	$\bar{e}_0$	0	0	0	$-\sqrt{2}e_3$	$\sqrt{2}e_2$
$\bar{e}_2$	$-\sqrt{-1}\bar{e}_2$	0	$\bar{e}_0$	0	$\sqrt{2}e_3$	0	$-\sqrt{2}e_1$
$\bar{e}_3$	$-\sqrt{-1}\bar{e}_3$	0	0	$\bar{e}_0$	$-\sqrt{2}e_2$	$\sqrt{2}e_1$	0

Table 2: The multiplication table  $AB$

$A \setminus B$	$e_4$	$e_1$	$e_2$	$e_3$	$\bar{e}_1$	$\bar{e}_2$	$\bar{e}_3$
$e_4$	0	$-\sqrt{-1}e_1$	$-\sqrt{-1}e_2$	$-\sqrt{-1}e_3$	$\sqrt{-1}\bar{e}_1$	$\sqrt{-1}\bar{e}_2$	$\sqrt{-1}\bar{e}_3$
$e_1$	$\sqrt{-1}e_1$	0	$-\sqrt{2}\bar{e}_3$	$\sqrt{2}\bar{e}_2$	$-\sqrt{-1}e_4$	0	0
$e_2$	$\sqrt{-1}e_2$	$\sqrt{2}\bar{e}_3$	0	$-\sqrt{2}\bar{e}_1$	0	$-\sqrt{-1}e_4$	0
$e_3$	$\sqrt{-1}e_3$	$-\sqrt{2}\bar{e}_2$	$\sqrt{2}\bar{e}_1$	0	0	0	$-\sqrt{-1}e_4$
$\bar{e}_1$	$-\sqrt{-1}\bar{e}_1$	$\sqrt{-1}e_4$	0	0	0	$-\sqrt{2}e_3$	$\sqrt{2}e_2$
$\bar{e}_2$	$-\sqrt{-1}\bar{e}_2$	0	$\sqrt{-1}e_4$	0	$\sqrt{2}e_3$	0	$-\sqrt{2}e_1$
$\bar{e}_3$	$-\sqrt{-1}\bar{e}_3$	0	0	$\sqrt{-1}e_4$	$-\sqrt{2}e_2$	$\sqrt{2}e_1$	0

Table 3: The multiplication table of the cross product  $A \times B$

Furthermore, the complexified  $G_2$ -frame  $(e_4 \ f \ \bar{f})^T$  satisfies

$$\langle e_4, e_i \rangle = 0, \quad \langle e_i, e_j \rangle = \langle \bar{e}_i, \bar{e}_j \rangle = 0, \quad \langle e_i, \bar{e}_j \rangle = \delta_{ij}, \quad (14)$$

$$e_i \times e_4 = \sqrt{-1}e_i, \quad \langle e_1 \times e_2, e_3 \rangle = -\sqrt{2}, \quad (15)$$

for any  $i \in \{1, 2, 3\}$ . By using Table 2, Table 3, (14) and (15), we may directly deduce Theorem 1.

## §4 Schrödinger flows to 6-sphere

We have already seen that the equation of Schrödinger flows from  $\mathbb{R}^1$  to  $N = \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$  is actually the Heisenberg ferromagnet model:

$$T_t = T \times T_{ss},$$

which is equivalent to the Da Rios equation (1). For  $\mathbb{S}^6 = \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7) : \sum_{i=1}^7 x_i^2 = 1\}$  in  $\mathbb{R}^7$ , we know that the equation (3) of Schrödinger flows from  $\mathbb{R}^1$  to  $(\mathbb{S}^6, J)$  reads

$$u_t = u \times u_{ss}, \quad (16)$$

where  $u = (u_1, u_2, u_3, u_4, u_5, u_6, u_7) \in \mathbb{R}^7$  with  $\sum_{i=1}^7 u_i^2 = 1$ .

Returning to Eq.(2), we have

**Proposition 2.** *Suppose that  $\gamma(t, s)$  evolves according to Eq.(2). Then the arclength parameter  $s$  is independent of time  $t$  for all  $t > 0$ .*

**Proof:** It suffices to prove that  $\frac{d}{dt}|\gamma_s|^2 = 0$ . In fact, from Eq.(2), we have

$$\frac{d}{dt}|\gamma_s|^2 = \langle \gamma_s, \gamma_s \rangle_t = 2\langle \gamma_{st}, \gamma_s \rangle = 2\langle \gamma_{ts}, \gamma_s \rangle = 2\langle (\gamma_s \times \gamma_{ss})_s, \gamma_s \rangle = 2\langle \gamma_s \times \gamma_{sss}, \gamma_s \rangle = 0. \quad \square$$

Let  $e_4 = \gamma_s$  and we obtain from Eq.(2) that

$$e_{4t} = \gamma_{st} = \gamma_{ts} = (\gamma_s \times \gamma_{ss})_s = (e_4 \times e_{4s})_s = e_4 \times e_{4ss},$$

which is exactly Eq.(16). Thus we have showed the following

**Theorem 3.** *Eq.(2) in  $Im(\mathbb{O}) \cong \mathbb{R}^7$  is equivalent to Eq.(16) of Schrödinger flows from  $\mathbb{R}^1$  into  $\mathbb{S}^6$ .*

We shall transform Eq.(16), and hence Eq.(2), to a nonlinear Schrödinger-type system, like the Da Rios-NLS correspondence. In fact, from Eq.(2), the fact:  $e_{1s} = -\bar{\varphi}_1 e_4 + \varphi_2 e_2$  and Table 3, we have that

$$\begin{aligned} e_{4t} = \gamma_{st} = \gamma_{ts} &= (-\sqrt{-1}\varphi_1 e_1 + \sqrt{-1}\bar{\varphi}_1 \bar{e}_1)_s \\ &= -\sqrt{-1}\varphi_{1s} e_1 + \sqrt{-1}\bar{\varphi}_{1s} \bar{e}_1 - \sqrt{-1}\varphi_1 \varphi_2 e_2 + \sqrt{-1}\bar{\varphi}_1 \bar{\varphi}_2 \bar{e}_2. \end{aligned}$$

Hence, the complexified  $G_2$ -frame by  $(e_4 \ f \ \bar{f})^T$  admits

$$\begin{pmatrix} e_4 \\ f \\ \bar{f} \end{pmatrix}_t = \begin{pmatrix} 0 & \omega & \bar{\omega} \\ -\bar{\omega}^T & \kappa & [\omega] \\ -\omega^T & [\bar{\omega}] & \bar{\kappa} \end{pmatrix} \begin{pmatrix} e_4 \\ f \\ \bar{f} \end{pmatrix},$$

where,  $\omega = (\omega^1 \ \omega^2 \ \omega^3) = (-\sqrt{-1} \varphi_{1s} \ -\sqrt{-1} \varphi_1 \varphi_2 \ 0)$ ,  $R_1 + R_2 + R_3 = 0$ ,

$$\kappa = \begin{pmatrix} \sqrt{-1}R_1 & a_1 & a_2 \\ -\bar{a}_1 & \sqrt{-1}R_2 & a_3 \\ -\bar{a}_2 & -\bar{a}_3 & \sqrt{-1}R_3 \end{pmatrix},$$

$$[\omega] = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & \varphi_1 \varphi_2 \\ 0 & 0 & -\varphi_{1s} \\ -\varphi_1 \varphi_2 & \varphi_{1s} & 0 \end{pmatrix},$$

and  $R_i \in \mathbb{R}, a_i \in \mathbb{C}, i \in \{1, 2, 3\}$  are functions with respect to  $s$  and  $t$ , which will be determined later.

One the other hand, Eq.(12) can be rewritten as

$$\begin{pmatrix} e_4 \\ f \\ f \end{pmatrix}_s = \begin{pmatrix} 0 & g & \bar{g} \\ -\bar{g}^T & M & [G] \\ -g^T & [G] & M \end{pmatrix} \begin{pmatrix} e_4 \\ f \\ f \end{pmatrix},$$

where

$$g = \begin{pmatrix} \varphi_1 \\ 0 \\ 0 \end{pmatrix}^T, \quad [G] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{-1}}{\sqrt{2}}\varphi_1 \\ 0 & \frac{\sqrt{-1}}{\sqrt{2}}\varphi_1 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & \varphi_2 & 0 \\ -\bar{\varphi}_2 & 0 & \varphi_3 \\ 0 & -\bar{\varphi}_3 & 0 \end{pmatrix}.$$

From the integrability condition:  $\begin{pmatrix} e_4 \\ f \\ f \end{pmatrix}_{ts} = \begin{pmatrix} e_4 \\ f \\ f \end{pmatrix}_{st}$ , we have:

$$g_t = \omega_s + \omega M + \bar{\omega} [G] - g\kappa - \bar{g} [\omega], \quad (17a)$$

$$M_t = \kappa_s - \bar{\omega}^T g + \kappa M + [\omega] [G] + \bar{g}^T \omega - M\kappa - [G] [\omega], \quad (17b)$$

$$[G]_t = [\omega]_s - \bar{\omega}^T \bar{g} + \kappa [G] + [\omega] \bar{M} + \bar{g}^T \bar{\omega} - M[\omega] - [G] \bar{\kappa}, \quad (17c)$$

From (17a), (17b) and (17c) we have

$$\begin{cases} \varphi_{1t} = -\sqrt{-1}\varphi_{1ss} + \sqrt{-1}\varphi_1\varphi_2\bar{\varphi}_2 - \sqrt{-1}R_1\varphi_1, \\ a_1 = -\frac{\sqrt{-1}}{\varphi_1}(2\varphi_{1s}\varphi_2 + \varphi_1\varphi_{2s}), \\ a_2 = -\frac{1}{\varphi_1}(\sqrt{-1}\varphi_1\varphi_2\varphi_3 + \sqrt{2}\bar{\varphi}_1\bar{\varphi}_1\bar{\varphi}_2), \end{cases} \quad (18)$$

and

$$\begin{cases} \varphi_{2t} = a_{1s} - \frac{3}{2}\sqrt{-1}\varphi_1\bar{\varphi}_1\varphi_2 + \sqrt{-1}\varphi_2(R_1 - R_2) - a_2\bar{\varphi}_3, \\ \varphi_{3t} = a_{3s} + \sqrt{-1}\varphi_3(R_2 - R_3) + \bar{\varphi}_2 a_2, \\ \sqrt{-1}R_{1s} = \sqrt{-1}(\varphi_1\bar{\varphi}_1)_s - \varphi_2\bar{a}_1 + \bar{\varphi}_2 a_1, \\ \sqrt{-1}R_{2s} = -\frac{\sqrt{-1}}{2}(\varphi_1\bar{\varphi}_1)_s - \bar{\varphi}_2 a_1 + \varphi_2\bar{a}_1 + \bar{\varphi}_3 a_3 - \varphi_3\bar{a}_3, \\ \sqrt{-1}R_{3s} = -\frac{\sqrt{-1}}{2}(\varphi_1\bar{\varphi}_1)_s - \bar{\varphi}_3 a_3 + \varphi_3\bar{a}_3, \\ \varphi_2 a_3 = a_{2s} + a_1\varphi_3. \end{cases} \quad (19)$$

One notes that  $R_1 + R_2 + R_3 = 0$  in (19) which is just compatible with the requirement. Furthermore, from (18) (19), we obtain

$$\begin{cases} \varphi_{1t} &= -\sqrt{-1}\varphi_{1ss} + \sqrt{-1}\varphi_1\varphi_2\bar{\varphi}_2 - \sqrt{-1}R_1\varphi_1, \\ \varphi_{2t} &= a_{1s} - \frac{3}{2}\sqrt{-1}\varphi_1\bar{\varphi}_1\varphi_2 + \sqrt{-1}\varphi_2(R_1 - R_2) - a_2\bar{\varphi}_3, \\ \varphi_{3t} &= a_{3s} + \sqrt{-1}\varphi_3(R_1 + 2R_2) + \bar{\varphi}_2 a_2, \end{cases}$$

where

$$a_1 = -\frac{\sqrt{-1}}{\varphi_1}(2\varphi_{1s}\varphi_2 + \varphi_1\varphi_{2s}), \quad (20a)$$

$$a_2 = -\frac{1}{\varphi_1}(\sqrt{-1}\varphi_1\varphi_2\varphi_3 + \sqrt{2}\bar{\varphi}_1\bar{\varphi}_1\bar{\varphi}_2), \quad (20b)$$

$$a_3 = -2\sqrt{-1}\varphi_3(\ln \varphi_1\varphi_2)_s - \sqrt{-1}\varphi_{3s} - \frac{\sqrt{2}}{\varphi_2}\left(\frac{\bar{\varphi}_1\bar{\varphi}_1\bar{\varphi}_2}{\varphi_1}\right)_s, \quad (20c)$$

$$\sqrt{-1}R_{1s} = \sqrt{-1}(\varphi_1\bar{\varphi}_1)_s - \sqrt{-1}(\varphi_2\bar{\varphi}_2)_s - 2\sqrt{-1}\varphi_2\bar{\varphi}_2(\ln \varphi_1\bar{\varphi}_1)_s, \quad (20d)$$

$$\begin{aligned} \sqrt{-1}R_{2s} &= -\frac{\sqrt{-1}}{2}(\varphi_1\bar{\varphi}_1)_s + \sqrt{-1}(\varphi_2\bar{\varphi}_2)_s - \sqrt{-1}(\varphi_3\bar{\varphi}_3)_s + 2\sqrt{-1}\varphi_2\bar{\varphi}_2(\ln \varphi_1\bar{\varphi}_1)_s \\ &\quad - 2\sqrt{-1}\varphi_3\bar{\varphi}_3(\ln \varphi_1\bar{\varphi}_1\varphi_2\bar{\varphi}_2)_s + \sqrt{2}\left[\frac{\varphi_3}{\bar{\varphi}_2}\left(\frac{\varphi_1\varphi_1\varphi_2}{\bar{\varphi}_1}\right)_s - \frac{\bar{\varphi}_3}{\varphi_2}\left(\frac{\bar{\varphi}_1\bar{\varphi}_1\bar{\varphi}_2}{\varphi_1}\right)_s\right]. \end{aligned} \quad (20e)$$

The equations (20d) and (20e) imply that  $R_1$  and  $R_2$  are of the forms as follows

$$\begin{aligned} R_1 &= |\varphi_1|^2 - |\varphi_2|^2 - 2\int_0^s |\varphi_2|^2(\ln |\varphi_1|^2)_s d\tilde{s} + R_{10}(t), \\ R_2 &= \int_0^s \{2|\varphi_2|^2(\ln |\varphi_1|^2)_s - 2|\varphi_3|^2(\ln(|\varphi_1|^2|\varphi_2|^2))_s - \sqrt{2}\sqrt{-1}\left[\frac{\varphi_3}{\bar{\varphi}_2}\left(\frac{\varphi_1^2\varphi_2}{\bar{\varphi}_1}\right)_s - \frac{\bar{\varphi}_3}{\varphi_2}\left(\frac{\bar{\varphi}_1^2\bar{\varphi}_2}{\varphi_1}\right)_s\right]\} d\tilde{s} \\ &\quad - \frac{1}{2}|\varphi_1|^2 + |\varphi_2|^2 - |\varphi_3|^2 + R_{20}(t), \end{aligned}$$

where  $R_{10}$  and  $R_{20}$  depend only on  $t$ . Now, by the following transformations:

$$\begin{aligned} \varphi_1 &\mapsto \sqrt{2}\varphi_1 \exp(-\sqrt{-1}\int_0^t R_{10}d\tilde{t}), \\ \varphi_2 &\mapsto \varphi_2 \exp(\sqrt{-1}\int_0^t (R_{10} - R_{20})d\tilde{t}), \\ \varphi_3 &\mapsto \varphi_3 \exp(\sqrt{-1}\int_0^t (R_{10} + 2R_{20})d\tilde{t}), \end{aligned}$$

we arrive at

**Theorem 4.** *If  $\kappa_2 \neq 0$ , then the equation of Schrödinger flows to  $\mathbb{S}^6 \subset \mathbb{R}^7$  is equivalent to*

$$\begin{cases} \sqrt{-1}\varphi_{1t} &= \varphi_{1ss} + 2\varphi_1|\varphi_1|^2 - 2\varphi_1|\varphi_2|^2 - 2\varphi_1\int_0^s |\varphi_2|^2(\ln |\varphi_1|^2)_s d\tilde{s}, \\ \sqrt{-1}\varphi_{2t} &= \varphi_{2ss} + 2\varphi_2|\varphi_2|^2 + 2(\varphi_2(\ln \varphi_1)_s)_s - 2\varphi_2|\varphi_3|^2 + 2\sqrt{-1}\frac{\bar{\varphi}_1^2\bar{\varphi}_2\bar{\varphi}_3}{\varphi_1} \\ &\quad + 2\varphi_2\int_0^s \{2|\varphi_2|^2(\ln |\varphi_1|^2)_s - |\varphi_3|^2(\ln(|\varphi_1|^2|\varphi_2|^2))_s + 2\text{Im}\left[\frac{\varphi_3}{\bar{\varphi}_2}\left(\frac{\varphi_1^2\varphi_2}{\bar{\varphi}_1}\right)_s\right]\} d\tilde{s}, \\ \sqrt{-1}\varphi_{3t} &= \varphi_{3ss} + 2\varphi_3|\varphi_3|^2 + 2\left[\varphi_3(\ln(\varphi_1\varphi_2))_s - \frac{\sqrt{-1}}{\varphi_2}\left(\frac{\bar{\varphi}_1^2\bar{\varphi}_2}{\varphi_1}\right)_s\right]_s - 2\sqrt{-1}\frac{\bar{\varphi}_1^2\bar{\varphi}_2^2}{\varphi_1} \\ &\quad + 2\varphi_3\int_0^s \{2|\varphi_3|^2(\ln |\varphi_1|^2|\varphi_2|^2)_s - |\varphi_2|^2(\ln |\varphi_1|^2)_s - 4\text{Im}\left[\frac{\varphi_3}{\bar{\varphi}_2}\left(\frac{\varphi_1^2\varphi_2}{\bar{\varphi}_1}\right)_s\right]\} d\tilde{s}, \end{cases} \quad (21)$$

which is a nonlinear Schrödinger-type system (NLSS) in three unknown complex functions.

**Remark 3.** From the proof of Theorem 4, one sees that it is by an application of the  $G_2$  structure displayed in Theorem 1 that we obtain the nonlinear Schrödinger-type system (NLSS) (21). Moreover, one also notes that the same sign in the terms  $e_1$  and  $e_2$  in the equation  $e_{4t} = e_4 \times e_{4ss}$  produces an integration term in  $a_1$  (see (20a)) and hence in the first equation in (21). The additional integration terms in  $a_2$  and  $a_3$  (see (20b) and (20c)) (i.e. the second and third equations in (21)) are in fact produced by the  $G_2$ -structure itself. Moreover, if  $\kappa_2 = 0$ , i.e.  $\varphi_2 = 0$ , Eq.(18) and (19) are clearly reduced to

$$\begin{cases} \sqrt{-1}\varphi_{1t} &= \varphi_{1ss} + 2\varphi_1|\varphi_1|^2, \\ \sqrt{-1}\varphi_{3t} &= \varphi_{3ss} + 2\varphi_3|\varphi_3|^2, \end{cases}$$

by choosing  $a_3 = -\sqrt{-1}\varphi_{3s}$ , and hence Eq.(21) is reduced to the focusing nonlinear Schrödinger equation (NLS).

Conversely, if  $\varphi_1(t, s)$ ,  $\varphi_2(t, s)$  and  $\varphi_3(t, s)$  are a solution to Eq.(21), one may verify that there is a corresponding family of curves  $\gamma(t, s)$  in  $\mathbb{R}^7$  satisfying Eq.(2). The details are omitted here. This produces Eq.(2)-NLSS correspondence, that is, if a curves  $\gamma(t, s)$  evolves according to Eq.(2), then the associated complex functions  $\varphi_1, \varphi_2$  and  $\varphi_3$  given by (13) evolve according to Eq.(21). This generalizes the Da Rios-NLS correspondence mentioned in the introduction, since when  $\varphi_2 = 0$  the correspondence is reduced to the Da Rios-NLS correspondence. One realizes that though Kirchhoff's almost complex structure on  $\mathbb{S}^6$  is not integrable, the Eq.(2)-NLSS correspondence is still valid.

It seems that it is the non-integrability of Kirchhoff's almost complex structure on  $\mathbb{S}^6$  that creates the integration terms in NLSS (21), as there is no an integration term involved in the NLS equation. If there is an almost complex structure  $J$  on  $\mathbb{S}^6$  for which the corresponding nonlinear Schrödinger-type system is without integration terms, we believe such an almost complex structure is integrable. Does there exist such an almost complex structure on  $\mathbb{S}^6$ ?

To complete this section, we come to find an almost complex structure structure on  $\mathbb{S}^6$ , which is different from Kirchhoff's almost complex structure, such that the integration terms in the corresponding nonlinear Schrödinger-like system are reduced greatly.

For  $A \in O(7)/G_2$ , Calabi constructed in [7] an almost complex structure  $J^A$  on  $\mathbb{S}^6$ , which is still compatible to the standard round metric on  $\mathbb{S}^6$ . The almost complex structure  $J^A$  reads

$$J^A : T_u\mathbb{S}^6 \rightarrow T_u\mathbb{S}^6, \quad X \mapsto J^A(X) = A^{-1}((Au) \times (AX)), \quad X \in T_u\mathbb{S}^6.$$

One sees that only when  $A \in G_2$ ,  $J^A$  coincides with Kirchhoff's almost complex structure  $J$  on  $\mathbb{S}^6$ . In fact, in this case for  $u \in \mathbb{S}^6$ ,

$$J^A(X) = A^{-1}((Au) \times (AX)) = A^{-1}(Au) \times A^{-1}(AX) = u \times X = J_u(X), \quad \forall X \in T_u\mathbb{S}^6.$$

With the almost complex structure  $J^A$ , one may verify straightforwardly that the equation of Schrödinger flows from  $\mathbb{R}$  to  $\mathbb{S}^6 \hookrightarrow \mathbb{R}^7$  is

$$u_t = A^{-1}(Au \times (A(u_{ss}))), \quad u : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{S}^6 \hookrightarrow \mathbb{R}^7.$$

Correspondingly, Eq.(2) is modified to be

$$\gamma_{4t} = A^{-1}(A\gamma_s \times (A\gamma_{ss})), \quad (22)$$

where  $\gamma = \gamma(t, s) \in \mathbb{R}^7$ .

Now by choosing

$$A = \left( \begin{array}{ccc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \in O(7)/G_2$$

and with a direct but long computation similar to that in the proof of Theorem 4, we have

**Proposition 3.** *Eq.(22) is equivalent to the following nonlinear Schrödinger-like system in three unknown complex-functions:*

$$\left\{ \begin{array}{l} \sqrt{-1}\varphi_{1t} = \varphi_{1ss} + 2\varphi_1|\varphi_1|^2 + 2\varphi_1|\varphi_2|^2. \\ \sqrt{-1}\varphi_{2t} = -\varphi_{2ss} - 2\varphi_2|\varphi_2|^2 - 6\varphi_2|\varphi_1|^2 + 2\varphi_2|\varphi_3|^2 - 2\sqrt{-1}\frac{\bar{\varphi}_1\bar{\varphi}_2\bar{\varphi}_3}{\varphi_1} \\ \quad + 2\varphi_2 \int_0^s \{ |\varphi_3|^2 (\ln(|\varphi_2|^2))_s - 2\text{Im} \left[ \frac{\varphi_3}{\varphi_2} \left( \frac{\varphi_1^2 \varphi_2}{\varphi_1} \right)_s \right] \} d\tilde{s}, \\ \sqrt{-1}\varphi_{3t} = -\varphi_{3ss} - 2\varphi_3|\varphi_3|^2 - 2[\varphi_3(\ln \varphi_2)_s - \frac{\sqrt{-1}}{\varphi_2} \left( \frac{\bar{\varphi}_1^2 \bar{\varphi}_2}{\varphi_1} \right)_s]_s + 2\sqrt{-1}\frac{\bar{\varphi}_1\bar{\varphi}_2^2}{\varphi_1} \\ \quad - 4\varphi_3 \int_0^s \{ 2|\varphi_3|^2 (\ln |\varphi_2|^2)_s - 2\text{Im} \left[ \frac{\varphi_3}{\varphi_2} \left( \frac{\varphi_1^2 \varphi_2}{\varphi_1} \right)_s \right] \} d\tilde{s}. \end{array} \right. \quad (23)$$

We would point out that it is due to the choice of  $A$  indicated above that the last term in (20d) is cancelled and hence the integration terms in the corresponding nonlinear Schrödinger-like system (23) are reduced.

## §5 Discussions and remarks

In order to exploit further geometric properties of  $\varphi_1, \varphi_2$  and  $\varphi_3$ , we introduce a surface  $\Sigma \hookrightarrow \mathbb{R}^7 = \text{Im } \mathbb{O}$  swept by  $\gamma(t, s)$ . Let  $\{E_1 = \gamma_s = I_4, E_2 = \frac{\gamma_t}{k_1} = -I_5, E_3 = I_1, E_4 = I_2, E_5 = I_6, E_6 = I_3, E_7 = I_7\}$  be an orthonormal  $G_2$ -frame associated to  $\Sigma$  in  $\text{Im}(\mathbb{O}) \cong \mathbb{R}^7$  such that  $\{E_1, E_2\}$  spans  $T_{(t,s)}\Sigma$ ,  $\{E_3, E_4, E_5, E_6, E_7\}$  spans  $(T_{(t,s)}\Sigma)^\perp$ , and  $\{\omega^1, \dots, \omega^7\}$  is its co-frame. The structure equations of  $\Sigma$  are given by

$$\left\{ \begin{array}{l} d\gamma = \omega^i E_i, \\ dE_i = \omega_i^k E_k + \omega_i^\alpha E_\alpha, \\ dE_\alpha = \omega_\alpha^k E_k + \omega_\alpha^\beta E_\beta, \\ d\omega^i = -\omega_j^i \wedge \omega^j = 0, \omega_j^i + \omega_i^j, \\ d\omega^\alpha = -\omega_i^\alpha \wedge \omega^i, \end{array} \right. \quad (24)$$

where  $i, j, k = 1, 2$  and  $\alpha, \beta, \eta = 3, 4, 5, 6, 7$ . Restricting to  $\Sigma$ , we have  $\omega^\alpha = 0$ . Hence, from

$$0 = d\omega^\alpha = -\omega_i^\alpha \wedge \omega^i,$$

we have

$$\omega_i^\alpha = h_{ij}^\alpha \omega^j, \quad h_{ij}^\alpha = h_{ji}^\alpha. \quad (25)$$

One knows that  $\{h_{ij}^\alpha\}$  ( $1 \leq i, j \leq 2, 3 \leq \alpha \leq 7$ ) are the coefficients of the second fundamental form of  $\Sigma \hookrightarrow \mathbb{R}^7 \cong \text{Im}(\mathbb{O})$ .

**Lemma 1.** *The non-zero coefficients of the second fundamental form ( $h_{ij}^\alpha$ ) are*

$$\begin{aligned} h_{11}^3 &= \sqrt{2}|\varphi_1|, \quad h_{12}^3 = h_{21}^3 = \frac{\sqrt{-1}}{\sqrt{2}}(\ln \frac{\bar{\varphi}_1}{\varphi_1})_s, \\ h_{22}^3 &= -\frac{\varphi_1 \bar{\varphi}_{1ss} + \bar{\varphi}_1 \varphi_{1ss}}{2\sqrt{2}|\varphi_1|^3} + \frac{|\varphi_2|^2}{\sqrt{2}|\varphi_1|}, \\ h_{22}^4 &= -\frac{2(\ln |\varphi_1|)_s |\varphi_2| + (|\varphi_2|)_s}{\sqrt{2}|\varphi_1|}, \\ h_{22}^5 &= -\frac{\sqrt{-1}[2(\ln \frac{\bar{\varphi}_1}{\varphi_1})_s |\varphi_2|^2 + \varphi_2 \bar{\varphi}_{2s} - \bar{\varphi}_2 \varphi_{2s}]}{2\sqrt{2}|\varphi_1||\varphi_2|}, \\ h_{22}^6 &= \frac{\varphi_1^3 \varphi_2^2 \varphi_3 + \bar{\varphi}_1^3 \bar{\varphi}_2^2 \bar{\varphi}_3}{2\sqrt{2}|\varphi_1|^4 |\varphi_2|}, \quad h_{12}^5 = h_{21}^5 = -|\varphi_2|, \\ h_{22}^7 &= \frac{\sqrt{-1}(\bar{\varphi}_1^3 \bar{\varphi}_2^2 \bar{\varphi}_3 - \varphi_1^3 \varphi_2^2 \varphi_3)}{2\sqrt{2}|\varphi_1|^4 |\varphi_2|} - \frac{3|\varphi_2|}{2}. \end{aligned} \quad (26)$$

**Proof:** The first fundamental form of  $\gamma(t, s)$  is

$$\begin{aligned} ds_M^2 &= \langle \frac{\partial}{\partial s}, \frac{\partial}{\partial s} \rangle ds^2 + \langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \rangle dt^2 = \langle \gamma_s, \gamma_s \rangle ds^2 + \langle \gamma_t, \gamma_t \rangle dt^2 \\ &= ds^2 + 2|\varphi_1|^2 dt^2 = (\omega^1)^2 + (\omega^2)^2, \end{aligned}$$

where  $\omega^1 = ds$  and  $\omega^2 = \sqrt{2}|\varphi_1|dt$ . By Eq.(8) and Theorem 2, a straightforward computation shows that

$$dE_1 = \frac{(|\varphi_1|)_s}{|\varphi_1|} \omega^2 E_2 + [\sqrt{2}|\varphi_1| \omega^1 + \frac{\sqrt{-1}}{\sqrt{2}}(\ln \frac{\bar{\varphi}_1}{\varphi_1})_s \omega^2] E_3 - |\varphi_2| \omega^2 E_5, \quad (27)$$



and

$$\begin{aligned}
dE_2 &= -dI_5 = d(-\sqrt{-1}\bar{r}e_1 + \sqrt{-1}r\bar{e}_1) \\
&= -\frac{(|\varphi_1|)_s}{|\varphi_1|}\omega^2 E_1 + \left[ \frac{\sqrt{-1}}{\sqrt{2}} \left( \ln \frac{\bar{\varphi}_1}{\varphi_1} \right)_s \omega^1 + \left( -\frac{\varphi_1 \bar{\varphi}_{1ss} + \bar{\varphi}_1 \varphi_{1ss}}{2\sqrt{2}|\varphi_1|^3} + \frac{|\varphi_2|^2}{\sqrt{2}|\varphi_1|} \right) \omega^2 \right] E_3 \\
&\quad - \frac{2(\ln |\varphi_1|)_s |\varphi_2| + (|\varphi_2|)_s}{\sqrt{2}|\varphi_1|} \omega^2 E_4 + \frac{\varphi_1^3 \varphi_2^2 \varphi_3 + \bar{\varphi}_1^3 \bar{\varphi}_2^2 \bar{\varphi}_3}{2\sqrt{2}|\varphi_1|^4 |\varphi_2|} \omega^2 E_6 \\
&\quad - \left[ |\varphi_2| \omega^1 + \frac{\sqrt{-1} (2(\ln \frac{\bar{\varphi}_1}{\varphi_1})_s |\varphi_2|^2 + \varphi_2 \bar{\varphi}_{2s} - \bar{\varphi}_2 \varphi_{2s})}{2\sqrt{2}|\varphi_1| |\varphi_2|} \omega^2 \right] E_5 \\
&\quad + \left[ \frac{\sqrt{-1} (\bar{\varphi}_1^3 \bar{\varphi}_2^2 \bar{\varphi}_3 - \varphi_1^3 \varphi_2^2 \varphi_3)}{2\sqrt{2}|\varphi_1|^4 |\varphi_2|} - \frac{3|\varphi_2|}{2} \omega^2 \right] E_7.
\end{aligned} \tag{28}$$

Since

$$dE_i = \sum_{k=1}^2 \omega_i^k E_k + \sum_{\alpha=3}^7 (h_{i1}^\alpha \omega^1 + h_{i2}^\alpha \omega^2) E_\alpha, \tag{29}$$

based on Eqs.(24) and (25), we have (26) from Eqs.(27), (28) and (29).  $\square$

When  $\varphi_2 = 0$ ,  $h_{ij}^\alpha = 0$  for  $\alpha = 4, 5, 6, 7$  indicates that  $\Sigma$  is located completely in an associative 3-dimensional space  $\mathbb{V}^3 = \text{span}_{\mathbb{R}}\{I_4, I_1, I_5\}$ . Therefore, Eq.(21) is reduced to the nonlinear Schrödinger equation (NLS)

$$\sqrt{-1}\varphi_{1t} = \varphi_{1ss} + 2\varphi_1|\varphi_1|^2.$$

In this case, it can be inferred that the almost complex structure  $J$  on  $\mathbb{S}^6$  returns to the complex (integrable) structure  $J$  on  $\mathbb{S}^2$ . We note that a 3-dimensional vector space  $\mathbb{V}$  in  $\text{Im}(\mathbb{O}) = \mathbb{R}^7$  is called associative if  $\mathbb{V} = \text{span}_{\mathbb{R}}\{u, v, u \times v\}$ . Associative 3-planes or co-associative 4-planes in  $\text{Im}(\mathbb{O})$  are well-studied in  $G_2$ -geometry. This gives the geometric characterization of the surface  $\Sigma$  corresponding to the case that  $\varphi_2 = 0$ .

In order to characterize the geometric properties corresponding to  $\varphi_3 = 0$  and  $\varphi_2 \neq 0$ , we set  $\tilde{E}_i = E_i$  except that  $\tilde{E}_4 = \cos \theta E_4 - \sin \theta E_7$  and  $\tilde{E}_7 = \sin \theta E_4 + \cos \theta E_7$ , where  $\theta$  will be determined later. Then  $\{\tilde{E}_1, \tilde{E}_2, \tilde{E}_3, \tilde{E}_4, \tilde{E}_5, \tilde{E}_6, \tilde{E}_7\}$  also consists of an orthonormal  $SO(7)$ -frame associated to  $\Sigma$  in  $\text{Im}(\mathbb{O}) \cong \mathbb{R}^7$ . Let  $\{\tilde{\omega}^1, \dots, \tilde{\omega}^7\}$  be its dual-frame. Since  $\tilde{E}_1 = E_1$ ,  $\tilde{E}_2 = E_2$ , we have  $\tilde{\omega}^1 = \omega^1$ ,  $\tilde{\omega}^2 = \omega^2$ .

**Lemma 2.** *By choosing  $\theta = \arccos \frac{h_{22}^4}{\sqrt{(h_{22}^4)^2 + \frac{9}{4}|\varphi_2|^2}}$ , the coefficients of the second funda-*

mental form  $\tilde{h}_{ij}^\alpha$  corresponding to the orthonormal frame  $\{\tilde{E}_1, \dots, \tilde{E}_7\}$  are

$$\begin{aligned}\tilde{h}_{ij}^3 &= h_{ij}^3, \quad \tilde{h}_{ij}^5 = h_{ij}^5, \quad \tilde{h}_{ij}^6 = h_{ij}^6, \\ \tilde{h}_{11}^4 &= \tilde{h}_{12}^4 = \tilde{h}_{21}^4 = 0, \\ \tilde{h}_{22}^4 &= \sqrt{(h_{22}^4)^2 + \frac{9}{4}|\varphi_2|^2} - \frac{3|\varphi_2|(h_{22}^7 + \frac{3}{2}|\varphi_2|)}{2\sqrt{(h_{22}^4)^2 + \frac{9}{4}|\varphi_2|^2}}, \\ \tilde{h}_{11}^7 &= \tilde{h}_{12}^7 = \tilde{h}_{21}^7 = 0, \quad \tilde{h}_{22}^7 = \frac{h_{22}^4(h_{22}^7 + \frac{3}{2}|\varphi_2|)}{\sqrt{(h_{22}^4)^2 + \frac{9}{4}|\varphi_2|^2}}.\end{aligned}$$

**Proof:** By a direct computation. □

**Proposition 4.** *Let  $\varphi_3 = 0$  and  $\varphi_2 \neq 0$ . Then the coefficients of the second fundamental form are:  $\tilde{h}_{ij}^\alpha = 0$  for  $\alpha \in \{6, 7\}$ , namely, the normal bundle of the surface  $\Sigma$  is flat in directions  $E_i$  ( $i = 6, 7$ ).*

**Proof:** From Lemma 1, Lemma 2 and  $\varphi_3 = 0$ , the corresponding nonzero terms for  $\alpha \in \{6, 7\}$  now become

$$\begin{aligned}\tilde{h}_{11}^6 &= \tilde{h}_{12}^6 = \tilde{h}_{21}^6 = \tilde{h}_{11}^7 = \tilde{h}_{12}^7 = \tilde{h}_{21}^7 = 0, \quad \tilde{h}_{22}^6 = \frac{\varphi_1^3 \varphi_2^2 \varphi_3 + \bar{\varphi}_1^3 \bar{\varphi}_2^2 \bar{\varphi}_3}{2\sqrt{2}|\varphi_1|^4|\varphi_2|} = 0, \\ \tilde{h}_{22}^7 &= \frac{\sqrt{-1}(\varphi_1^3 \bar{\varphi}_2^2 \bar{\varphi}_3 - \bar{\varphi}_1^3 \varphi_2^2 \varphi_3)}{2\sqrt{2}|\varphi_1|^4|\varphi_2|} \frac{h_{22}^4}{\sqrt{(h_{22}^4)^2 + \frac{9}{4}|\varphi_2|^2}} = 0.\end{aligned}$$

This completes the proof of Lemma 4. □

The geometric properties of  $\varphi_i$  ( $i = 1, 2, 3$ ) are summarized as follows. When  $\varphi_2 = 0$ , the surface  $\Sigma$  (swept by  $\gamma(t, s)$ ) is located completely in a 3-manifold spanned by  $\{E_1, E_2, E_3\}$  with the algebraic property of associativity; when  $\varphi_3 = 0$  and  $\varphi_2 \neq 0$ , the normal bundle of the surface  $\Sigma$  is flat in directions  $E_i$  ( $i = 6, 7$ ) and roughly speaking, we can regard  $\Sigma$  as located in a 5-manifold spanned by  $\{\tilde{E}_1, \tilde{E}_2, \tilde{E}_3, \tilde{E}_4, \tilde{E}_5\}$ .

We have furthered our understanding of almost complex structures on  $\mathbb{S}^6$  and the  $G_2$ -structure on  $\text{Im}(\mathbb{O}) = \mathbb{R}^7$  via Schrödinger flows. Many related problems remain to be clarified. For example, the 5-manifold spanned by  $\{\tilde{E}_1, \tilde{E}_2, \tilde{E}_3, \tilde{E}_4, \tilde{E}_5\}$  corresponding to  $\varphi_3 = 0$  and  $\varphi_2 \neq 0$  is a new object in relation to the surface  $\Sigma$  swept by  $\gamma(t, s)$  in  $\mathbb{R}^7$ . How might one further characterize its geometric or algebraic properties? It is very interesting to note that like the usual binormal motion equation (1) in  $\mathbb{R}^3$ , whether the  $G_2$ -binormal motion equation (2) in  $\mathbb{R}^7$  is integrable, and furthermore, whether it sits in a hierarchy with other new integrable systems and has relations to various known integrable systems.

## Acknowledgements

The authors would like to express their deep thanks to the referee for his/her helpful comments. This work is supported by the National Natural Science Foundation of China (Grant No. 11271073).

## References

- [1] R.J. Arms and F.R. Hama, Localized-induction concept on a curved vortex and motion of elliptic vortex ring, *Phys. Fluids* **8** (1965), 553-559.
- [2] J.C. Baez, The octonions, *Bull. Amer. Math. Soc.* **39** (2002), 145-206.
- [3] A. Borel and J.P. Serre, Groupes de Lie et puissances réduites de Steenrod, *Amer. J. Math.* **75** (1953), 409-448.
- [4] B.C. Brower, D.A. Kessler, J. Koplik and H. Levine, Geometrical models of interface evolution, *Phys. Rev. A* **29** (1984), 1335-1342.
- [5] R.L Bryant, Submanifolds and special structures on the octonions, *J. Diff. Geom.* **17** (1982), 185-232.
- [6] R.L. Bryant, S.S. Chern's study of almost-complex structure on the six-sphere, arXiv: 14053405.
- [7] E. Calabi, Construction and properties of some 6-dimensional almost complex manifolds, *Trans. A.M.S.* **87** (1958), 407-438.
- [8] E. Calabi and H. Gluck, What are the best almost complex structures on the 6-sphere, *Proceedings of Symposia in Pure Mathematics* **54** (1993), 99-106.
- [9] L.S. Da Rios, On the motion of an unbounded fluid with a vortex filament of any shape, *Rend. Circ. Mat. Palermo.* **22** (1906), 117-135.
- [10] Q. Ding, A note on the NLS and the Schrödinger flow of maps, *Phys. Lett. A* **248** (1998), 49-56.
- [11] Q. Ding and Z.Z. He, The noncommutative KdV equation and its para-Kähler structure, *Sci. in China Math.* **57** (2014) 1505-1516.
- [12] Q. Ding and Y.D. Wang, Geometric KdV flows, motions of curves and the third order system of the AKNS hierarchy, *Inter. J. of Math.* **22** (2011) 1013-1029.
- [13] W.Y. Ding and Y.D. Wang, Schrödinger flows of maps into symplectic manifolds, *Sci. China A* **41** (1998), 746-755.
- [14] R. Harvey and H.B. Lawson, Calibrated geometries, *Acta Math.* **148** (1982), 47-157.

- [15] H. Hasimoto, A soliton on a vortex filament, *J. Fluid. Mech.* **51** (1972), 477-485.
- [16] H. Hashimoto and M. Ohashi, Orthogonal almost complex structures of hypersurfaces of purely imaginary octonions, *Hokkaido Math. J.* **39** (2010), 351-387.
- [17] J.S. Langer, Instabilities and pattern formation in crystal growth, *Rev. Mod. Phys.* **52** (1980), 1-28.
- [18] J. Langer and R. Perline, Geometric realizations of Fordy-Kulish nonlinear Schrödinger systems, *Pacific J. Math.* **195** (2000), 157-178.
- [19] C. LeBrun, Orthogonal complex structures on  $S^6$ , *Proc. A.M.S.* **101** (1987), 136-138.
- [20] V.V. Meleshko, A.A. Gourjii and T.S. Krasnopolskaya, Vortex ring: history and state of the art, *J. of Math. Sciences* **187** (2012), 772-806.
- [21] A. Nahmod, A. Stefanov and K. Uhlenbeck, On Schrödinger maps, *Comm. Pure Appl. Math.* **56** (2003), 114-151; Erratum, On Schrödinger maps, *Comm. Pure Appl. Math.* **57** (2004), 833-839.
- [22] M. Ohashi, On  $G_2$ -invariants of curves of purely imaginary octonions, *Recent Progress in Differential Geometry and Its Related Fields*, 2012, 25-40.
- [23] M. Ohashi,  $G_2$ -congruence theorem for curves in purely imaginary octonions and its application, *Geometriae Dedicata* **163** (2013), 1-17.
- [24] M. Ohashi, A method of determining the  $SO(7)$ -invariants for curves in  $\text{Im}(\mathbb{O})$  by their  $G_2$ -invariants, *Current Developments in Differential Geometry and its Related Fields*, 2016, 201-213.
- [25] P.G. Saffman and G. Taylor, The penetration of a fluid into a porous medium or Hele-Shaw cell containing a more viscous, *Proc. R. Soc. London A* **245** (1958), 312-329.
- [26] C.L. Terng and K. Uhlenbeck, Schrödinger flows on Grassmannians, *AMS/IP Studies in Advanced Mathematics*, Vol. 36 (2006) 235–256.