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# On the Dynamics of a Higher Order Nonlinear System of Difference Equations

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**Abstract.** The aim of this paper is to investigate the dynamics of a higher order system of rational difference equations. Our concentration is on boundedness character, the oscillatory, the existence of unbounded solutions and the global behavior of positive solutions for the following system of difference equations

$$x_{n+1} = A + \frac{x_{n-m}}{z_n}, \quad y_{n+1} = A + \frac{y_{n-m}}{z_n}, \quad z_{n+1} = A + \frac{z_{n-m}}{y_n}, \quad n = 0, 1, ...,$$

where $A$ and the initial values $x_{-i}$, $y_{-i}$, $z_{-i}$, for $i = 0, 1, ..., m$, are positive real numbers.

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## 1. Introduction

Difference equations appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations having many applications in economics, population biology, computer science, probability theory, psychology and so forth. Difference equation or discrete dynamical system is a diverse field which impact almost every branch of pure and applied mathematics. Recently, there has been great interest in investigating the behavior of solutions of a system of nonlinear difference equations and discussing the asymptotic stability of their equilibrium points. There are many papers in which systems of difference equations have been studied, see [1-10].

In [2], Camouzis and and Papaschinopoulos studied the system of difference equations

$$x_{n+1} = 1 + \frac{x_n}{y_{n-m}}, \quad y_{n+1} = 1 + \frac{y_n}{x_{n-m}}, \quad n = 0, 1, ..., \tag{1.1}$$

with initial conditions $x_{-i}$, $y_{-i} > 0$, $i = -m, -m+1, ..., 0$, and $m$ is positive integer.

In [7], Yang studied the behavior of positive solutions of the system of difference equations

$$x_{n+1}=A+\frac{y_{n-1}}{x_{n-p}y_{n-q}},\quad y_{n+1}=A+\frac{x_{n-1}}{x_{n-r}y_{n-s}},\quad n=1,2,..., \tag{1.2}$$

where $p\geq 2$, $q\geq 2$, $r\geq 2$, $s\geq 2$, $A$ is a positive constant, and $x_{1-\max\{p,r\}}$, $x_{2-\max\{p,r\}}$, ...,$x_0$, $y_{1-\max\{q,s\}}$, $y_{2-\max\{q,s\}}$, ...,$y_0$ are positive real numbers.

In [5], Papaschinopoulos and Schinas considered the system of difference equations

$$x_{n+1}=A+\frac{y_n}{x_{n-p}},\quad y_{n+1}=A+\frac{x_n}{y_{n-q}},\quad n=0,1,..., \tag{1.3}$$

where $A\in(0,\infty)$, $p$, $q$ are positive integers and $x_{-p},...,x_0$, $y_{-q},...,y_0$ are positive numbers.

In [4], Okumuş and Soykan studied the boundedness, persistence and periodicity of the positive solutions and the global asymptotic stability of the positive equilibrium points of system of the difference equations

$$x_{n+1}=A+\frac{x_{n-1}}{z_n},\ \ y_{n+1}=A+\frac{y_{n-1}}{z_n},\ \ z_{n+1}=A+\frac{z_{n-1}}{y_n},\ \ n=0,1,...,$$

where $A\in(0,\infty)$ and initial conditions $x_i$, $y_i$, $z_i\in(0,\infty)$, $i=-1,0$.

Motivated by all above mentioned studies and in the light of this work in [4], in this paper, we investigate the global asymptotic stability, boundedness character and oscillatory of positive solutions of the system of difference equations

$$x_{n+1}=A+\frac{x_{n-m}}{z_n},\ \ y_{n+1}=A+\frac{y_{n-m}}{z_n},\ \ z_{n+1}=A+\frac{z_{n-m}}{y_n},\ \ n=0,1,..., \tag{1.4}$$

where $A$ and the initial values $x_{-i}$, $y_{-i}$, $z_{-i}$, for $i=0,1,...,m$, are positive real numbers and $m$ is positive integer.

## 2. Preliminaries

We recall some basic definitions that we afterwards need in the paper.

Let us introduce the discrete dynamical system:

$$\begin{aligned} x_{n+1} &= f_1\left(x_n,x_{n-1},...,x_{n-k},y_n,y_{n-1},...,y_{n-k},z_n,z_{n-1},...,z_{n-k}\right), \\ y_{n+1} &= f_2\left(x_n,x_{n-1},...,x_{n-k},y_n,y_{n-1},...,y_{n-k},z_n,z_{n-1},...,z_{n-k}\right), \\ z_{n+1} &= f_3\left(x_n,x_{n-1},...,x_{n-k},y_n,y_{n-1},...,y_{n-k},z_n,z_{n-1},...,z_{n-k}\right), \end{aligned} \tag{2.1}$$

$n\in\mathbb{N}$, where $f_1: I_1^{k+1}\times I_2^{k+1}\times I_3^{k+1}\to I_1$, $f_2: I_1^{k+1}\times I_2^{k+1}\times I_3^{k+1}\to I_2$ and $f_3: I_1^{k+1}\times I_2^{k+1}\times I_3^{k+1}\to I_3$ are continuously differentiable functions and $I_1$, $I_2$, $I_3$ are some intervals of real numbers. Also, a solution $\{(x_n,y_n,z_n)\}_{n=-k}^{\infty}$ of system (2.1) is uniquely determined by initial values $(x_{-i},y_{-i},z_{-i})\in I_1\times I_2\times I_3$ for $i\in\{0,1,...,k\}$.

DEFINITION 2.1. *An equilibrium point of system (2.1) is a point* $(\overline{x},\overline{y},\overline{z})$ *that satisfies*

$$\begin{aligned} \overline{x} &= f_1\left(\overline{x},\overline{x},...,\overline{x},\overline{y},\overline{y},...,\overline{y},\overline{z},\overline{z},...,\overline{z}\right), \\ \overline{y} &= f_2\left(\overline{x},\overline{x},...,\overline{x},\overline{y},\overline{y},...,\overline{y},\overline{z},\overline{z},...,\overline{z}\right), \\ \overline{z} &= f_3\left(\overline{x},\overline{x},...,\overline{x},\overline{y},\overline{y},...,\overline{y},\overline{z},\overline{z},...,\overline{z}\right). \end{aligned}$$

Together with system (2.1), if we consider the associated vector map

$$F = \left(f_1, x_n, x_{n-1}, ..., x_{n-k}, f_2, y_n, y_{n-1}, ..., y_{n-k}, f_3, z_{n-1}, ..., z_{n-k}\right),$$

then the point $(\overline{x},\overline{y},\overline{z})$ is also called a fixed point of the vector map $F$.

DEFINITION 2.2. *Let* $(\overline{x},\overline{y},\overline{z})$ *be an equilibrium point of system (2.1).*

**(a):** *An equilibrium point* $(\overline{x},\overline{y},\overline{z})$ *is called stable if, for every* $\varepsilon > 0$*; there exists* $\delta > 0$ *such that for every initial value* $(x_{-i}, y_{-i}, z_{-i}) \in I_1 \times I_2 \times I_3$*, with*

$$\sum\nolimits_{i=-k}^{0} |x_i - \overline{x}| < \delta, \ \sum\nolimits_{i=-k}^{0} |y_i - \overline{y}| < \delta, \ \sum\nolimits_{i=-k}^{0} |z_i - \overline{z}| < \delta$$

*implying* $|x_n - \overline{x}| < \varepsilon$, $|y_n - \overline{y}| < \varepsilon$, $|z_n - \overline{z}| < \varepsilon$*for* $n \in \mathbb{N}$.

**(b):** *If an equilibrium point* $(\overline{x},\overline{y},\overline{z})$ *of system (2.1) is called unstable if it is not stable.*

**(c):** *An equilibrium point* $(\overline{x},\overline{y},\overline{z})$ *of system (2.1) is called locally asymptotically stable if, it is stable, and if in addition there exists* $\gamma > 0$ *such that*

$$\sum\nolimits_{i=-k}^{0} |x_i - \overline{x}| < \gamma, \ \sum\nolimits_{i=-k}^{0} |y_i - \overline{y}| < \gamma, \ \sum\nolimits_{i=-k}^{0} |z_i - \overline{z}| < \gamma$$

*and* $(x_n, y_n, z_n) \to (\overline{x},\overline{y},\overline{z})$ *as* $n \to \infty$.

**(d):** *An equilibrium point* $(\overline{x},\overline{y},\overline{z})$ *of system (2.1) is called a global attractor if,* $(x_n, y_n, z_n) \to (\overline{x},\overline{y},\overline{z})$ *as* $n \to \infty$.

**(e):** *An equilibrium point* $(\overline{x},\overline{y},\overline{z})$ *of system (2.1) is called globally asymptotically stable if it is stable, and a global attractor.*

DEFINITION 2.3. *Let* $(\overline{x},\overline{y},\overline{z})$ *be an equilibrium point of the map* $F$ *where* $f_1$*,* $f_2$ *and* $f_3$ *are continuously differentiable functions at* $(\overline{x},\overline{y},\overline{z})$*. The linearized system of system (2.1) about the equilibrium point* $(\overline{x},\overline{y},\overline{z})$ *is*

$$X_{n+1} = F\left(X_n\right) = BX_n,$$

*where*

$$X_n = \begin{pmatrix} x_n \\ \vdots \\ x_{n-k} \\ y_n \\ \vdots \\ y_{n-k} \\ z_n \\ \vdots \\ z_{n-k} \end{pmatrix}$$

*and $B$ is a Jacobian matrix of system (2.1) about the equilibrium point $(\overline{x}, \overline{y}, \overline{z})$.*

Definition 2.4. *Assume that*

$$X_{n+1} = F(X_n), n = 0, 1, ...,$$

*be a system of difference equations such that $\overline{X}$ is a fixed point of $F$. If no eigenvalues of the Jacobian matrix $B$ about $\overline{X}$ have absolute value equal to one, then $\overline{X}$ is called hyperbolic. If there exists an eigenvalue of the Jacobian matrix $B$ about $\overline{X}$ with absolute value equal to one, then $\overline{X}$ is called nonhyperbolic.*

Theorem 2.5 (The Linearized Stability Theorem). *Assume that*

$$X_{n+1} = F(X_n), n = 0, 1, ...,$$

*be a system of difference equations such that $\overline{X}$ is a fixed point of $F$.*

- **(a):** *If all eigenvalues of the Jacobian matrix $B$ about $\overline{X}$ lie inside the open unit disk $|\lambda| < 1$, that is, if all of them have absolute value less than one, then $\overline{X}$ is locally asymptotically stable.*
- **(b):** *If at least one of them has a modulus greater than one, then $\overline{X}$ is unstable.*

## 3. Main Results

In this section, we prove our main results. We deal with the following cases of $0 < A < 1$, $A = 1$, and $A > 1$.

Theorem 3.1. *If $(\overline{x}, \overline{y}, \overline{z})$ is a positive equilibrium point of system (1.4), then*

$$(\overline{x}, \overline{y}, \overline{z}) = \begin{cases} (A+1, A+1, A+1), & \text{if } A \neq 1, \\ \left(\mu, \mu, \frac{\mu}{\mu-1}\right), \ \mu \in (1, \infty) & \text{if } A = 1. \end{cases}$$

**Proof.** It is easily seen from the definition of equilibrium point that the equilibrium points of system (1.4) are the nonnegative solution of the equations

$$\overline{x} = A + \frac{\overline{x}}{\overline{z}}, \ \overline{y} = A + \frac{\overline{y}}{\overline{z}}, \ \overline{z} = A + \frac{\overline{z}}{\overline{y}}.$$

From this, we get

$$\begin{aligned}\overline{xz} &= A\overline{z}+\overline{x},\ \ \overline{yz}=A\overline{z}+\overline{y},\ \ \overline{zy}=A\overline{y}+\overline{z}\\ &\Rightarrow\ \overline{xz}-\overline{x}=\overline{yz}-\overline{y},\ \ A\overline{z}+\overline{y}=A\overline{y}+\overline{z}\\ &\Rightarrow\ \overline{x}\left(\overline{z}-1\right)=\overline{y}\left(\overline{z}-1\right),\ \ \overline{z}\left(A-1\right)=\overline{y}\left(A-1\right).\end{aligned}$$

From which it follows that if $A\neq 1$,

$$\overline{x}=\overline{y}=\overline{z}=A+1\Rightarrow\left(\overline{x},\overline{y},\overline{z}\right)=\left(A+1,A+1,A+1\right).$$

Also, we have

$$\begin{aligned}\frac{\overline{xz}-\overline{x}}{\overline{z}} &= A,\ \ \frac{\overline{yz}-\overline{y}}{\overline{z}}=A,\ \ \frac{\overline{zy}-\overline{z}}{\overline{y}}=A\\ &\Rightarrow\ \frac{\overline{xz}-\overline{x}}{\overline{z}}=\frac{\overline{yz}-\overline{y}}{\overline{z}},\ \ \frac{\overline{yz}-\overline{y}}{\overline{z}}=\frac{\overline{zy}-\overline{z}}{\overline{y}}\\ &\Rightarrow\ \overline{xz}-\overline{x}=\overline{yz}-\overline{y},\ \ \overline{y}^2\overline{z}-\overline{y}^2=\overline{z}^2\overline{y}-\overline{z}^2\\ &\Rightarrow\ \overline{x}\left(\overline{z}-1\right)=\overline{y}\left(\overline{z}-1\right),\ \ \overline{yz}\left(\overline{y}-\overline{z}\right)=\left(\overline{y}-\overline{z}\right)\left(\overline{y}+\overline{z}\right).\end{aligned}$$

From which it follows that if $A=1$,

$$\overline{x}=\overline{y}\text{ and }\overline{yz}=\overline{y}+\overline{z}\Rightarrow\left(\overline{x},\overline{y},\overline{z}\right)=\left(\mu,\mu,\frac{\mu}{\mu-1}\right),\ \mu\in\left(1,\infty\right).$$

In that case, we have a continuous of positive equilibriums which lie on the hyperboloid $\overline{yz}=\overline{y}+\overline{z}$.

THEOREM 3.2. *Assume that* $0<A<1$. *Let* $\{(x_n,y_n,z_n)\}$ *be an arbitrary positive solution of the system (1.4). Then, the following statements are true.*

**(i):** *If* $m$ *is odd and* $0<x_{2k-1}<1$, $0<y_{2k-1}<1$, $0<z_{2k-1}<1$, $x_{2k}>\frac{1}{1-A}$, $y_{2k}>\frac{1}{1-A}$, $z_{2k}>\frac{1}{1-A}$ *for* $k=\frac{1-m}{2},\frac{3-m}{2},...,0$, *then*

$$\begin{aligned}\lim_{n\to\infty}x_{2n} &= \infty,\ \ \lim_{n\to\infty}y_{2n}=\infty,\ \ \lim_{n\to\infty}z_{2n}=\infty,\\ \lim_{n\to\infty}x_{2n+1} &= A,\ \ \lim_{n\to\infty}y_{2n+1}=A,\ \ \lim_{n\to\infty}z_{2n+1}=A.\end{aligned}$$

**(ii):** *If* $m$ *is odd and* $0<x_{2k}<1$, $0<y_{2k}<1$, $0<z_{2k}<1$, $x_{2k-1}>\frac{1}{1-A}$, $y_{2k-1}>\frac{1}{1-A}$, $z_{2k-1}>\frac{1}{1-A}$ *for* $k=\frac{1-m}{2},\frac{3-m}{2},...,0$, *then*

$$\begin{aligned}\lim_{n\to\infty}x_{2n} &= A,\ \ \lim_{n\to\infty}y_{2n}=A,\ \ \lim_{n\to\infty}z_{2n}=A,\\ \lim_{n\to\infty}x_{2n+1} &= \infty,\ \ \lim_{n\to\infty}y_{2n+1}=\infty,\ \ \lim_{n\to\infty}z_{2n+1}=\infty.\end{aligned}$$

**(iii):** *If* $m$ *is even, we can not get some useful results.*

**Proof.**

**(i):** Clearly, we get

$$\begin{aligned} 0 &< x_1 = A + \frac{x_{-m}}{z_0} < A + \frac{1}{z_0} < A + (1-A) = 1, \\ 0 &< y_1 = A + \frac{y_{-m}}{z_0} < A + \frac{1}{z_0} < A + (1-A) = 1, \\ 0 &< z_1 = A + \frac{z_{-m}}{y_0} < A + \frac{1}{y_0} < A + (1-A) = 1, \\ x_2 &= A + \frac{x_{1-m}}{z_1} > x_{1-m} > \frac{1}{1-A}, \\ y_2 &= A + \frac{y_{1-m}}{z_1} > y_{1-m} > \frac{1}{1-A}, \\ z_2 &= A + \frac{z_{1-m}}{y_1} > z_{1-m} > \frac{1}{1-A}. \end{aligned}$$

By induction for $n = 1, 2, ...,$ we obtain

$$\begin{aligned} 0 &< x_{2n-1} < 1,\ 0 < y_{2n-1} < 1,\ 0 < z_{2n-1} < 1, \\ x_{2n} &> \frac{1}{1-A},\ y_{2n} > \frac{1}{1-A},\ z_{2n} > \frac{1}{1-A}. \end{aligned} \tag{3.1}$$

Thus, for $n \geq (m+2)/2$,

$$\begin{aligned} x_{2n} &= A + \frac{x_{2n-(m+1)}}{z_{2n-1}} > A + x_{2n-(m+1)} = 2A + \frac{x_{2n-(2m+2)}}{z_{2n-(m+2)}} \\ &> 2A + x_{2n-(2m+2)}, \\ y_{2n} &= A + \frac{y_{2n-(m+1)}}{z_{2n-1}} > A + y_{2n-(m+1)} = 2A + \frac{y_{2n-(2m+2)}}{z_{2n-(m+2)}} \\ &> 2A + y_{2n-(2m+2)}, \\ z_{2n} &= A + \frac{z_{2n-(m+1)}}{y_{2n-1}} > A + z_{2n-(m+1)} = 2A + \frac{z_{2n-(2m+2)}}{y_{2n-(m+2)}} \\ &> 2A + z_{2n-(2m+2)}, \end{aligned}$$

from which we get

$$\lim_{n\to\infty} x_{2n} = \infty, \quad \lim_{n\to\infty} y_{2n} = \infty, \quad \lim_{n\to\infty} z_{2n} = \infty.$$

Noting that (3.1) and taking limits on the both sides of three equations

$$x_{2n+1} = A + \frac{x_{2n-m}}{z_{2n}},\ y_{2n+1} = A + \frac{y_{2n-m}}{z_{2n}},\ z_{2n+1} = A + \frac{z_{2n-m}}{y_{2n}},$$

we have

$$\lim_{n\to\infty} x_{2n+1} = A, \quad \lim_{n\to\infty} y_{2n+1} = A, \quad \lim_{n\to\infty} z_{2n+1} = A.$$

**(ii):** Obviously, we have

$$\begin{aligned} x_1 &= A+\frac{x_{-m}}{z_0}>x_{-m}>\frac{1}{1-A}, \\ y_1 &= A+\frac{y_{-m}}{z_0}>y_{-m}>\frac{1}{1-A}, \\ z_1 &= A+\frac{z_{-m}}{y_0}>z_{-m}>\frac{1}{1-A}, \\ 0 &< x_2=A+\frac{x_{1-m}}{z_1}<A+\frac{1}{z_1}<A+(1-A)=1, \\ 0 &< y_2=A+\frac{y_{1-m}}{z_1}<A+\frac{1}{z_1}<A+(1-A)=1, \\ 0 &< z_2=A+\frac{z_{1-m}}{y_1}.<A+\frac{1}{y_1}<A+(1-A)=1. \end{aligned}$$

By induction for $n=1,2,...$, we obtain

$$\begin{aligned} x_{2n-1} &> \frac{1}{1-A},\ y_{2n-1}>\frac{1}{1-A},\ z_{2n-1}>\frac{1}{1-A}, \\ 0 &< x_{2n}<1, 0<y_{2n}<1, 0<z_{2n}<1. \end{aligned} \tag{3.2}$$

So, for $n\geq(m+2)/2$,

$$\begin{aligned} x_{2n+1} &= A+\frac{x_{2n-m}}{z_{2n}}>A+x_{2n-m}=2A+\frac{x_{(2n-2m)-1}}{z_{2n-(m+1)}} \\ &> 2A+x_{(2n-2m)-1}, \\ y_{2n+1} &= A+\frac{y_{2n-m}}{z_{2n}}>A+y_{2n-m}=2A+\frac{y_{(2n-2m)-1}}{z_{2n-(m+1)}} \\ &> 2A+y_{(2n-2m)-1}, \\ z_{2n+1} &= A+\frac{z_{2n-m}}{y_{2n}}>A+z_{2n-m}=2A+\frac{z_{(2n-2m)-1}}{y_{2n-(m+1)}} \\ &> 2A+z_{(2n-2m)-1}, \end{aligned}$$

from which we get

$$\lim_{n\to\infty}x_{2n+1}=\infty,\quad \lim_{n\to\infty}y_{2n+1}=\infty,\quad \lim_{n\to\infty}z_{2n+1}=\infty.$$

Noting that (3.2) and taking limits on the both sides of three equations

$$x_{2n}=A+\frac{x_{2n-(m+1)}}{z_{2n-1}},\ y_{2n+1}=A+\frac{y_{2n-(m+1)}}{z_{2n-1}},\ z_{2n+1}=A+\frac{z_{2n-(m+1)}}{y_{2n-1}},$$

we have

$$\lim_{n\to\infty}x_{2n}=A,\quad \lim_{n\to\infty}y_{2n}=A,\quad \lim_{n\to\infty}z_{2n+}=A.$$

THEOREM 3.3. *Suppose that $A=1$. Then every positive solution of system (1.4) is bounded and persists.*

**Proof.** Let $\{(x_n,y_n,z_n)\}$ be a positive solution of the system (1.4).

Obviously, $x_n>1$, $y_n>1$, $z_n>1$, for $n\geq 1$. So, we have

$$x_i,y_i,z_i\in\left[M,\frac{M}{M-1}\right],\ \ i=1,2,...,m+1,$$

where

$$M=\min\left\{\alpha,\frac{\beta}{\beta-1}\right\}>1,\alpha=\min_{1\leq i\leq m+1}\{x_i,y_i,z_i\},\beta=\max_{1\leq i\leq m+1}\{x_i,y_i,z_i\}.$$

Then, we obtain

$$\begin{aligned}
M &= 1+\frac{M}{M/(M-1)}\leq x_{m+2}=1+\frac{x_1}{z_{m+1}}\leq 1+\frac{M/(M-1)}{M}=\frac{M}{M-1},\\
M &= 1+\frac{M}{M/(M-1)}\leq y_{m+2}=1+\frac{y_1}{z_{m+1}}\leq 1+\frac{M/(M-1)}{M}=\frac{M}{M-1},\\
M &= 1+\frac{M}{M/(M-1)}\leq z_{m+2}=1+\frac{z_1}{y_{m+1}}\leq 1+\frac{M/(M-1)}{M}=\frac{M}{M-1}.
\end{aligned}$$

By induction, we get

$$x_i,y_i,z_i\in\left[M,\frac{M}{M-1}\right],\ \ i=1,2,....$$

Theorem 3.4. *Assume that $A=1$. Let $\{(x_n,y_n,z_n)\}$ be a positive solution of the system (1.4). Then, either $\{(x_n,y_n,z_n)\}$ consists of a single semicycle or $\{(x_n,y_n,z_n)\}$ oscillates about the equilibrium point $(\overline{x},\overline{y},\overline{z})=\left(\mu,\mu,\frac{\mu}{\mu-1}\right)$ with semicycles having at most $m$ terms.*

**Proof.** Suppose that $\{(x_n,y_n,z_n)\}$ has at least two semicycles Then, there exists $N\geq -m$ such that either $x_N<\overline{x}\leq x_{N+1}$ or $x_{N+1}<\overline{x}\leq x_N$ ($y_N<\overline{y}\leq y_{N+1}$ or $y_{N+1}<\overline{y}\leq y_N$ and $z_N<\overline{z}\leq z_{N+1}$ or $z_{N+1}<\overline{z}\leq z_N$). Firstly, we assume that the case $x_N<\overline{x}\leq x_{N+1}$, $y_N<\overline{y}\leq y_{N+1}$ and $z_N<\overline{z}\leq z_{N+1}$. Since the other case is similar, it will be omitted. Suppose that rhe positive semicycle beginning with the term $(x_{N+1},\ y_{N+1},\ z_{N+1})$ have $m$ terms. Then we have

$$\begin{aligned}
x_{N+1} &< \overline{x}=\mu\leq x_{N+m},\\
y_{N+1} &< \overline{y}=\mu\leq y_{N+m},\\
z_{N+1} &< \overline{z}=\frac{\mu}{\mu-1}\leq z_{N+m}.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
x_{N+m+1} &= 1+\frac{x_N}{z_{N+m}}<1+\frac{\overline{x}}{\overline{z}}=\mu,\\
y_{N+m+1} &= 1+\frac{y_N}{z_{N+m}}<1+\frac{\overline{y}}{\overline{z}}=\mu,\\
z_{N+m+1} &= 1+\frac{z_N}{y_{N+m}}<1+\frac{\overline{z}}{\overline{y}}=\frac{\mu}{\mu-1}.
\end{aligned}$$

This completes the proof.

Theorem 3.5. *Suppose that $A>1$. Then every positive solution of system (1.4) is bounded and persists.*

**Proof.** Let $\{(x_n,y_n,z_n)\}$ be a positive solution of the system (1.4).

Obviously, $x_n>A>1$, $y_n>A>1$, $z_n>A>1$, for $n\geq 1$. So, we have

$$x_i,y_i,z_i\in\left[M,\frac{M}{M-A}\right],\ \ i=1,2,...,m+1,$$

where

$$M=\min\left\{\alpha,\frac{\beta}{\beta-1}\right\}>1,\alpha=\min_{1\leq i\leq m+1}\left\{x_i,y_i,z_i\right\},\beta=\max_{1\leq i\leq m+1}\left\{x_i,y_i,z_i\right\}.$$

Then, we obtain

$$\begin{aligned}
M &= A+\frac{M}{M/\left(M-A\right)}\leq x_{m+2}=1+\frac{x_1}{z_{m+1}}\leq 1+\frac{M/\left(M-A\right)}{M}=\frac{M}{M-A},\\
M &= A+\frac{M}{M/\left(M-A\right)}\leq y_{m+2}=1+\frac{y_1}{z_{m+1}}\leq 1+\frac{M/\left(M-A\right)}{M}=\frac{M}{M-A},\\
M &= A+\frac{M}{M/\left(M-A\right)}\leq z_{m+2}=1+\frac{z_1}{y_{m+1}}\leq 1+\frac{M/\left(M-A\right)}{M}=\frac{M}{M-A}.
\end{aligned}$$

By induction, we get

$$x_i,y_i,z_i\in\left[M,\frac{M}{M-A}\right],\ \ i=1,2,....$$

The proof is completed.

Before we give the following theorems about the stability of the equilibrium points, we consider the following transformation to build the corresponding linearized form of system (1.4) :

$$\begin{aligned}
&\left(x_n,x_{n-1},...,x_{n-m},y_n,y_{n-1},...,y_{n-m},z_n,z_{n-1},...,z_{n-m}\right)\\
\rightarrow\ &\left(f,f_1,...,f_m,g,g_1,...,g_m,h,h_1,...,h_m\right)
\end{aligned}$$

where

$$\begin{aligned}
f &= A+\frac{x_{n-m}}{z_n}\\
f_1 &= x_n\\
&\ \vdots\\
f_m &= x_{n-m}\\
g &= A+\frac{y_{n-m}}{z_n}\\
g_1 &= y_n\\
&\ \vdots\\
g_m &= y_{n-m}\\
h &= A+\frac{z_{n-m}}{y_n}\\
h_1 &= z_n\\
&\ \vdots\\
h_m &= z_{n-m}.
\end{aligned}$$

The Jacobian matrix about the equilibrium point $(\overline{x},\overline{y},\overline{z})$ under the above transformation is given by

$$B\left(\overline{x},\overline{y},\overline{z}\right)=\begin{pmatrix}
0 & \dots & 0 & \frac{1}{\overline{z}} & 0 & \dots & 0 & 0 & -\frac{\overline{x}}{\overline{z}^2} & \dots & 0 & 0\\
1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0\\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0\\
0 & \dots & 0 & 0 & 0 & \dots & 0 & \frac{1}{\overline{z}} & -\frac{\overline{y}}{\overline{z}^2} & \dots & 0 & 0\\
0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 & 0\\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & \dots & 0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 & 0\\
0 & \dots & 0 & 0 & -\frac{\overline{z}}{\overline{y}^2} & \dots & 0 & 0 & 0 & \dots & 0 & \frac{1}{\overline{y}}\\
0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0\\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 & 0
\end{pmatrix},$$

where $B=(b_{ij})$, $1\leq i,j\leq 3m+3$ is an $(3m+3)\times(3m+3)$ matrix.

Theorem 3.6. *If $A=1$, then the equilibrium point of system (1.4) is locally asymptotically stable.*

**Proof.** The linearized system of system (1.4) about the equilibrium point $(\overline{x},\overline{y},\overline{z})=\left(\mu,\mu,\frac{\mu}{\mu-1}\right)$ is

$$X_{n+1}=BX_n,$$

where $X_n=(x_n,x_{n-1},...,x_{n-m},y_n,y_{n-1},...,y_{n-m},z_n,z_{n-1},...,z_{n-m})^T$ and

$$B\left(\overline{x},\overline{y},\overline{z}\right)=\begin{pmatrix}
0 & \dots & 0 & \frac{\mu-1}{\mu} & 0 & \dots & 0 & 0 & -\frac{(\mu-1)^2}{\mu} & \dots & 0 & 0\\
1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0\\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0\\
0 & \dots & 0 & 0 & 0 & \dots & 0 & \frac{\mu-1}{\mu} & -\frac{(\mu-1)^2}{\mu} & \dots & 0 & 0\\
0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 & 0\\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & \dots & 0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 & 0\\
0 & \dots & 0 & 0 & -\frac{1}{\mu(\mu-1)^2} & \dots & 0 & 0 & 0 & \dots & 0 & \frac{1}{\mu}\\
0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0\\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 & 0
\end{pmatrix}.$$

Let $\lambda_1, \lambda_2, ..., \lambda_{3m+3}$ denote the $3m+3$ eigenvalues of the matrix $B$ and $D = diag\,(d_1, d_2, ..., d_{3m+3})$ be a diagonal matrix, where $d_1 = d_{m+2} = d_{2m+3} = 1$, $d_{1+k} = d_{m+2+k} = d_{2m+3+k} = 1 - k\varepsilon$, $1 \le k \le m$ and

$$0 < \varepsilon < \left\{ \frac{\mu^2 - 2\mu + 2}{m\mu}, \frac{\mu^2 - 2\mu + 2}{m\mu\,(\mu - 1)} \right\}.$$

Obviously, $D$ is invertible. Computing matrix $DBD^{-1}$, we have that

$$DBD^{-1}$$

is equal to

$$\left( \begin{array}{cccccccccccc}
0 & \dots & 0 & \frac{\mu-1}{\mu}\frac{d_1}{d_{m+1}} & 0 & \dots & 0 & 0 & -\frac{(\mu-1)^2}{\mu}\frac{d_1}{d_{2m+3}} & \dots & 0 & 0 \\
\frac{d_2}{d_1} & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & \frac{d_{m+1}}{d_m} & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & \dots & 0 & 0 & 0 & \dots & 0 & \frac{\mu-1}{\mu}\frac{d_{m+2}}{d_{2m+2}} & -\frac{(\mu-1)^2}{\mu}\frac{d_{m+2}}{d_{2m+3}} & \dots & 0 & 0 \\
0 & \dots & 0 & 0 & \frac{d_{m+3}}{d_{m+2}} & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & 0 & \dots & \frac{d_{2m+2}}{d_{2m+1}} & 0 & 0 & \dots & 0 & 0 \\
0 & \dots & 0 & 0 & -\frac{1}{\mu(\mu-1)^2}\frac{d_{2m+3}}{d_{m+2}} & \dots & 0 & 0 & 0 & \dots & 0 & \frac{1}{\mu}\frac{d_{2m+3}}{d_{3m+3}} \\
0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \frac{d_{2m+4}}{d_{2m+3}} & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & \frac{d_{3m+3}}{d_{3m+2}} & 0
\end{array} \right).$$

The three chains of inequalities

$$\begin{aligned}
1 &= d_1 > d_2 > ... > d_m > d_{m+1} > 0, \\
1 &= d_{m+2} > d_{m+3} > ... > d_{2m+1} > d_{2m+2} > 0, \\
1 &= d_{2m+3} > d_{2m+4} > ... > d_{3m+2} > d_{3m+3} > 0,
\end{aligned}$$

imply that

$$\begin{aligned}
d_2 d_1^{-1} &< 1,\ d_3 d_2^{-1} < 1, ..., d_{m+1} d_m^{-1} < 1, \\
d_{m+3} d_{m+2}^{-1} &< 1,\ d_{m+4} d_{m+3}^{-1} < 1, ..., d_{2m+2} d_{2m+1}^{-1} < 1, \\
d_{2m+4} d_{2m+3}^{-1} &< 1,\ d_{2m+5} d_{2m+4}^{-1} < 1, ..., d_{3m+3} d_{3m+2}^{-1} < 1.
\end{aligned}$$

Also,

$$\begin{aligned}\left(\frac{\mu-1}{\mu}\right)d_1d_{m+1}^{-1}+\left(-\frac{(\mu-1)^2}{\mu}\right)d_1d_{2m+3}^{-1} &= \left(\frac{\mu-1}{\mu}\right)\left(\frac{1}{1-m\varepsilon}\right)-\frac{(\mu-1)^2}{\mu}\\ &< \left(\frac{\mu-1}{\mu}\right)\left(\frac{1}{1-m\varepsilon}\right)-\left(\frac{(\mu-1)^2}{\mu}\right)\left(\frac{1}{1-m\varepsilon}\right)\\ &= \left(\frac{-\mu^2+3\mu-2}{\mu}\right)\left(\frac{1}{1-m\varepsilon}\right)<1,\end{aligned}$$

$$\begin{aligned}\left(\frac{\mu-1}{\mu}\right)d_{m+2}d_{2m+2}^{-1}+\left(-\frac{(\mu-1)^2}{\mu}\right)d_{m+2}d_{2m+3}^{-1} &= \left(\frac{\mu-1}{\mu}\right)\left(\frac{1}{1-m\varepsilon}\right)-\frac{(\mu-1)^2}{\mu}\\ &= \left(\frac{-\mu^2+3\mu-2}{\mu}\right)\left(\frac{1}{1-m\varepsilon}\right)<1,\end{aligned}$$

$$\begin{aligned}\left(-\frac{1}{\mu(\mu-1)^2}\right)d_{2m+3}d_{m+2}^{-1}+\left(\frac{1}{\mu}\right)d_{2m+3}d_{3m+3}^{-1} &= \left(-\frac{1}{\mu(\mu-1)^2}\right)+\left(\frac{1}{\mu}\right)\left(\frac{1}{1-m\varepsilon}\right)\\ &< \left(-\frac{1}{\mu(\mu-1)^2}\right)\left(\frac{1}{1-m\varepsilon}\right)+\left(\frac{1}{\mu}\right)\left(\frac{1}{1-m\varepsilon}\right)\\ &= \left(\frac{\mu-2}{\mu(\mu-1)}\right)\left(\frac{1}{1-m\varepsilon}\right)<1.\end{aligned}$$

Since $B$ has the same eigenvalues as $DBD^{-1}=E=(e_{ij})$, we obtain that

$$\begin{aligned}\max_{1\le i\le 3m+3}|\lambda_i| &\le \left\|DBD^{-1}\right\|_\infty\\ &= \max_{1\le i\le 3m+3}\left\{\sum_{j=1}^{3m+3}|e_{ij}|\right\}\\ &= \max\left\{\begin{array}{c} d_2d_1^{-1},\ d_3d_2^{-1},\ldots,d_{m+1}d_m^{-1},\\ d_{m+3}d_{m+2}^{-1},\ d_{m+4}d_{m+3}^{-1},\ldots,d_{2m+2}d_{2m+1}^{-1},\\ d_{2m+4}d_{2m+3}^{-1},\ d_{2m+5}d_{2m+4}^{-1},\ldots,d_{3m+3}d_{3m+2}^{-1},\\ \left(\frac{\mu-1}{\mu}\right)d_1d_{m+1}^{-1}-\left(\frac{(\mu-1)^2}{\mu}\right)d_1d_{2m+3}^{-1},\\ \left(\frac{\mu-1}{\mu}\right)d_{m+2}d_{2m+2}^{-1}-\left(\frac{(\mu-1)^2}{\mu}\right)d_{m+2}d_{2m+3}^{-1},\\ \left(-\frac{1}{\mu(\mu-1)^2}\right)d_{2m+3}d_{m+2}^{-1}+\left(\frac{1}{\mu}\right)d_{2m+3}d_{3m+3}^{-1}\end{array}\right\}\\ &< 1.\end{aligned}$$

This implies that the equilibrium point of system (1.4) is locally asymptotically stable.

Theorem 3.7. *If $A>1$, then the equilibrium point of system (1.4) is locally asymptotically stable.*

**Proof.** The linearized system of system (1.4) about the equilibrium point $(\overline{x},\overline{y},\overline{z})=(A+1,A+1,A+1)$ is

$$X_{n+1}=BX_n,$$

where $X_n = (x_n, x_{n-1}, ..., x_{n-m}, y_n, y_{n-1}, ..., y_{n-m}, z_n, z_{n-1}, ..., z_{n-m})^T$ and

$$B(\overline{x}, \overline{y}, \overline{z}) = \begin{pmatrix} 0 & \dots & 0 & c^{-1} & 0 & \dots & 0 & 0 & -c^{-1} & \dots & 0 & 0 \\ 1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & c^{-1} & -c^{-1} & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 & -c^{-1} & \dots & 0 & 0 & 0 & \dots & 0 & c^{-1} \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix},$$

where $c = A + 1$.

Let $\lambda_1, \lambda_2, ..., \lambda_{3m+3}$ denote the $3m+3$ eigenvalues of the matrix $B$ and $D = diag\,(d_1, d_2, ..., d_{3m+3})$ be a diagonal matrix, where $d_1 = d_{m+2} = d_{2m+3} = 1$, $d_{1+k} = d_{m+2+k} = d_{2m+3+k} = 1 - k\varepsilon$, $1 \le k \le m$ and

$$0 < \varepsilon < \left\{ \frac{1}{m}, \frac{c-2}{cm} \right\}.$$

Obviously, $D$ is invertible. Computing matrix $DBD^{-1}$, we have that

$$DBD^{-1}$$

$$= \begin{pmatrix} 0 & \dots & 0 & \frac{c^{-1}d_1}{d_{m+1}} & 0 & \dots & 0 & 0 & \frac{-c^{-1}d_1}{d_{2m+3}} & \dots & 0 & 0 \\ \frac{d_2}{d_1} & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \frac{d_{m+1}}{d_m} & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & \frac{c^{-1}d_{m+2}}{d_{2m+2}} & \frac{-c^{-1}d_{m+2}}{d_{2m+3}} & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 & \frac{d_{m+3}}{d_{m+2}} & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & \frac{d_{2m+2}}{d_{2m+1}} & 0 & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 & \frac{-c^{-1}d_{2m+3}}{d_{m+2}} & \dots & 0 & 0 & 0 & \dots & 0 & \frac{c^{-1}d_{2m+3}}{d_{3m+3}} \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \frac{d_{2m+4}}{d_{2m+3}} & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & \frac{d_{3m+3}}{d_{3m+2}} & 0 \end{pmatrix}.$$

The three chains of inequalities

$$\begin{aligned} 1 &= d_1 > d_2 > ... > d_m > d_{m+1} > 0, \\ 1 &= d_{m+2} > d_{m+3} > ... > d_{2m+1} > d_{2m+2} > 0, \\ 1 &= d_{2m+3} > d_{2m+4} > ... > d_{3m+2} > d_{3m+3} > 0, \end{aligned}$$

imply that

$$\begin{aligned} d_2 d_1^{-1} &< 1,\ d_3 d_2^{-1} < 1,...,d_{m+1} d_m^{-1} < 1, \\ d_{m+3} d_{m+2}^{-1} &< 1,\ d_{m+4} d_{m+3}^{-1} < 1,...,d_{2m+2} d_{2m+1}^{-1} < 1, \\ d_{2m+4} d_{2m+3}^{-1} &< 1,\ d_{2m+5} d_{2m+4}^{-1} < 1,...,d_{3m+3} d_{3m+2}^{-1} < 1. \end{aligned}$$

Also,

$$\begin{aligned} c^{-1} d_1 d_{m+1}^{-1} + c^{-1} d_1 d_{2m+3}^{-1} &= c^{-1}\left(\frac{1}{1-m\varepsilon} + 1\right) \\ &< c^{-1}\frac{2}{1-m\varepsilon} < 1, \end{aligned}$$

$$\begin{aligned} c^{-1} d_{m+2} d_{2m+2}^{-1} + c^{-1} d_{m+2} d_{2m+3}^{-1} &= c^{-1}\left(\frac{1}{1-m\varepsilon} + 1\right) \\ &< c^{-1}\frac{2}{1-m\varepsilon} < 1, \end{aligned}$$

$$\begin{aligned} c^{-1} d_{2m+3} d_{m+2}^{-1} + c^{-1} d_{2m+3} d_{3m+3}^{-1} &= c^{-1}\left(1 + \frac{1}{1-m\varepsilon}\right) \\ &< c^{-1}\frac{2}{1-m\varepsilon} < 1. \end{aligned}$$

Since $B$ has the same eigenvalues as $DBD^{-1} = E = (e_{ij})$, we obtain that

$$\begin{aligned} \max_{1\le i\le 3m+3} |\lambda_i| &\le \left\| DBD^{-1} \right\|_\infty \\ &= \max_{1\le i\le 3m+3} \left\{ \sum_{j=1}^{3m+3} |e_{ij}| \right\} \\ &= \max \left\{ \begin{array}{c} d_2 d_1^{-1},\ d_3 d_2^{-1},...,d_{m+1} d_m^{-1}, \\ d_{m+3} d_{m+2}^{-1},\ d_{m+4} d_{m+3}^{-1},...,d_{2m+2} d_{2m+1}^{-1}, \\ d_{2m+4} d_{2m+3}^{-1},\ d_{2m+5} d_{2m+4}^{-1},...,d_{3m+3} d_{3m+2}^{-1}, \\ c^{-1} d_1 d_{m+1}^{-1} + c^{-1} d_1 d_{2m+3}^{-1}, \\ c^{-1} d_{m+2} d_{2m+2}^{-1} + c^{-1} d_{m+2} d_{2m+3}^{-1}, \\ c^{-1} d_{2m+3} d_{m+2}^{-1} + c^{-1} d_{2m+3} d_{3m+3}^{-1} \end{array} \right\} \\ &< 1. \end{aligned}$$

This implies that the equilibrium point of system (1.4) is locally asymptotically stable.

THEOREM 3.8. *Assume that $A > 1$. Then, the positive equilibrium point of system (1.4) is globally asymptotically stable.*

**Proof.** Using Theorem (3.5), we have

$$\begin{aligned} L_1 &= \lim_{n\to\infty} \sup x_n,\ \ L_2 = \lim_{n\to\infty} \sup y_n,\ \ L_3 = \lim_{n\to\infty} \sup z_n, \\ m_1 &= \lim_{n\to\infty} \inf x_n,\ m_2 = \lim_{n\to\infty} \inf y_n,\ \ m_3 = \lim_{n\to\infty} \inf z_n. \end{aligned} \tag{3.3}$$

Then, from (1.4) and (3.3) we have

$$\begin{aligned} L_1 &\leq A + \frac{L_1}{m_3},\ \ L_2 \leq A + \frac{L_2}{m_3},\ \ L_3 \leq A + \frac{L_3}{m_2}, \\ m_1 &\geq A + \frac{m_1}{L_3},\ \ m_2 \geq A + \frac{m_2}{L_3},\ \ m_3 \geq A + \frac{m_3}{L_2}. \end{aligned} \tag{3.4}$$

Relations (3.4) imply that

$$AL_2 + m_3 \leq m_3 L_2 \leq Am_3 + L_2,\ \ AL_3 + m_2 \leq m_2 L_3 \leq Am_2 + L_3,$$

from which we have

$$(A-1)(L_2 - m_3) \leq 0,\ \ \ (A-1)(L_3 - m_2) \leq 0.$$

Since $A > 1$, we get

$$L_2 \leq m_3 \leq L_3,\ \ L_3 \leq m_2 \leq L_2,$$

from which

$$L_2 = L_3 = m_2 = m_3. \tag{3.5}$$

Moreover, from (3.4) it follows that

$$L_1 m_3 \leq Am_3 + L_1,\ \ m_1 L_3 \leq AL_3 + m_1,$$

from which

$$L_1 (m_3 - 1) \leq Am_3,\ \ AL_3 \leq m_1 (L_3 - 1).$$

Using (3.5), we have

$$L_1 (L_3 - 1) \leq m_1 (L_3 - 1),$$

from which

$$L_1 \leq m_1.$$

Since $x_n$ is bounded, it implys that

$$L_1 = m_1.$$

Hence, every positive solution $\{(x_n, y_n, z_n)\}$ of system (1.4) tends to the positive equilibrium system (1.4). So, the proof is completed.

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