

Blow-up phenomenon, ill-posedness and peakon solutions for the periodic Euler-Poincaré equations

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Abstract

In this paper we mainly investigate the initial value problem of the periodic Euler-Poincaré equations. We first present a new blow-up result to the system for a special class of smooth initial data by using the rotational invariant properties of the system. Then, we prove that the periodic Euler-Poincaré equations is ill-posed in critical Besov spaces by a contradiction argument. Finally, we verify the system possesses a class of peakon solutions in the sense of distributions.

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Contents

1	Introduction	2
2	Preliminaries	4
3	Blow up: $d = 2$	6

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4	Blow up: $d \geq 3$	9
5	Ill-posedness in critical Besov spaces	12
6	Periodic peakon solutions	14

1 Introduction

In this paper we consider the initial value problem for the following periodic Euler-Poincaré equations:

$$(1.1) \quad \begin{cases} m_t + u \cdot \nabla m + (\nabla u)^T m + m (\operatorname{div} u) = 0, \\ m|_{t=0}(x) = m_0(x), \\ m(t, x) = m(t, x + 1), \end{cases}$$

or in components,

$$(1.2) \quad \begin{cases} m_t^i + \sum_{j=1}^d u^j \partial_j m^i + \sum_{j=1}^d m^j \partial_i u^j + m_i \sum_{j=1}^d \partial_j u^j = 0, & i = 1, 2, \dots, d, \\ m^i|_{t=0}(x) = m_0^i(x). \\ m(t, x) = m(t, x + 1). \end{cases}$$

Here u is the velocity and $m = (1 - \Delta)u$ denotes the momentum, d is the spatial dimension. The Euler-Poincaré equations were first studied in [22, 23, 24] as a higher dimensional Camassa-Holm system for modeling and analyzing the nonlinear shallow water waves. Moreover, it can be viewed as the geodesic flow on $\mathcal{D}^1(M)$ with the right-invariant metric [22, 23]:

$$\langle u, v \rangle = \int_M uv + u_x v_x dx,$$

where M is a Riemannian manifold (In general, M is \mathbb{R}^d or \mathbb{T}^d). If $d = 1$ the Euler-Poincaré equations are reduced to the following famous Camassa-Holm equation [3]:

$$(1.3) \quad m_t + 2m_x u + m u_x = 0, \quad m = u - u_{xx}.$$

The Camassa-Holm equation was derived as a model for shallow water waves [3, 15]. It has been investigated extensively because of its great physical significance in the past two decades. The CH equation has a bi-Hamiltonian structure [6, 19] and is completely integrable [3, 8]. The solitary wave solutions of the CH equation were considered in [3, 4], where the authors showed that the CH equation possesses peakon solutions of the form $Ce^{-|x-Ct|}$. It is worth mentioning that the peakons are solitons and their shape is alike that of the travelling water waves of greatest height, arising as solutions to the

free-boundary problem for incompressible Euler equations over a flat bed (these being the governing equations for water waves), cf. the discussions in [9, 13, 14, 29]. Constantin and Strauss verified that the peakon solutions of the CH equation are orbitally stable in [17].

The local well-posedness for the CH equation was studied in [10, 12, 18, 28, 26]. Concretely, for initial profiles $u_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$, it was shown in [10, 12, 28] that the CH equation has a unique solution in $C([0, T]; H^s(\mathbb{R}))$. Moreover, the local well-posedness for the CH equation in Besov spaces $C([0, T]; B_{p,r}^s(\mathbb{R}))$ with $s > \max(\frac{3}{2}, 1 + \frac{1}{p})$ was proved in [18, 26]. The global existence of strong solutions were established in [7, 10, 12] under some sign conditions and it was shown in [7, 10, 12, 11] that the solutions will blow up in finite time when the slope of initial data was bounded by a negative quantity. The global weak solutions for the CH equation were studied in [16] and [30]. The global conservative and dissipative solutions of CH equation were presented in [1, 21] and [2], respectively.

Recently, the local well-posedness in $H^s(\mathbb{R}^d)$ with $s > 1 + \frac{d}{2}$ for the Euler-Poincaré equations was established in [5]. Moreover, the authors obtained a blow-up criteria, zero α limit and the Liouville type theorem. In [25], the authors constructed a special class of solutions ($m = \nabla\phi$) to the Euler-Poincaré equations which will blow up in finite time. The local well-posedness for the high-dimension CH equations in Besov spaces $C([0, T]; B_{p,r}^s(\mathbb{R}^d))$ with $s > \max(\frac{3}{2}, 1 + \frac{d}{p})$ or $s = 1 + \frac{d}{p}$, $r = 1$ was proved in [31].

The authors in [25] apply the symmetrical structure to reduce the higher dimension system into a one dimension equation, and then prove that this kind of special solutions to the Euler-Poincaré equations will blow up in finite time. In this paper we construct a new special class of strong solutions to the periodic Euler-Poincaré equations (1.1) which will also blow up in finite time. Our approach and the obtained result are quite different from the recent result in [25]. The main idea is that we use the rotational invariant properties of the system (1.1). If the initial data is constant along some vector field, and then the solution has the same property. This observation leads us to obtain a new blow-up result.

Recently, the authors in [20] prove ill-posedness of the Camassa-Holm type equations in the critical spaces. To our best knowledge, there is no any ill-posedness results for the Euler-Poincaré equations. Inspired by the works of [20, 27] about the Camassa-Holm equation and the Burgers equation, we use the contradiction argument to prove the system (1.1) is ill-posed in critical Besov spaces.

The remainder of the paper is organized as follows. In Section 2 we introduce some preliminaries which will be used in sequel. In Section 3 we prove the blow-up phenomenon of the system (1.1) with

$d = 2$. In Section 4 we generalize the obtained result from $d = 2$ to $d \geq 3$. In Section 5, we prove that the system (1.1) is ill-posed in critical Besov spaces. Section 6 is devoted to verifying that the system (1.1) possesses a class of peakon solutions.

2 Preliminaries

In this section, we recall some previous results for the Euler-Poincaré equations.

In [5], the authors write $(\nabla u)^T m$ in a tensor form:

$$\begin{aligned} \sum_{j=1}^d m^j \partial_i u^j &= \sum_{j=1}^d u^j \partial_i u^j - \sum_{j,k=1}^d \partial_k^2 u^j \partial_i u^j = \frac{1}{2} \partial_i |u|^2 - \sum_{j,k=1}^d \partial_k (\partial_i u^j \partial_k u^j) + \sum_{j,k=1}^d \partial_k \partial_i u^j \partial_k u^j \\ &= \sum_{j=1}^d \partial_j \left(\frac{1}{2} \delta_{ij} |u|^2 - \partial_i u \cdot \partial_j u + \frac{1}{2} \delta_{ij} |\nabla u|^2 \right). \end{aligned}$$

Denote the tensor $T^{ij} = m^i u^j + \frac{1}{2} \delta_{ij} |u|^2 - \partial_i u \cdot \partial_j u + \frac{1}{2} \delta_{ij} |\nabla u|^2$. Then the system (1.1) becomes

$$\partial_t m^i + \sum_{j=1}^d \partial_j T^{ij} = 0.$$

Let us recall the local well-posedness result and the blow-up criteria for the Euler-Poincaré equations.

Lemma 2.1. [5] *Let $u_0 \in H^k(\mathbb{R}^d)$ with $k > \frac{d}{2} + 3$. Then, there exists $T = T(\|u_0\|_{H^k(\mathbb{R}^d)})$ such that the Euler-Poincaré equations have a unique solution $u \in C([0, T]; H^k(\mathbb{R}^d))$.*

Lemma 2.2. [5] *Let $u_0 \in H^k(\mathbb{R}^d)$ with $k > \frac{d}{2} + 3$. Suppose that T^* is the lifespan of the solution u to the Euler-Poincaré equations. Then the solution blows up in finite time if and only if*

$$\int_0^{T^*} \|S(t)\|_{\dot{B}_{\infty, \infty}^0(\mathbb{R}^d)} dt = \infty,$$

where $S = (S^{ij})$ with $S^{ij} = \frac{1}{2} \partial_i u^j + \frac{1}{2} \partial_j u^i$.

Remark 2.3. *The embedding relation $L^\infty(\mathbb{R}^d) \hookrightarrow BMO(\mathbb{R}^d) \hookrightarrow \dot{B}_{\infty, \infty}^0(\mathbb{R}^d)$ implies that*

$$\|S(t)\|_{\dot{B}_{\infty, \infty}^0(\mathbb{R}^d)} \leq \|S\|_{L^\infty(\mathbb{R}^d)} \leq \|\nabla u\|_{L^\infty(\mathbb{R}^d)}.$$

Therefore we obtain the following criteria:

$$\limsup_{t \rightarrow T^*} \|u\|_{H^k(\mathbb{R}^d)} = \infty \quad \text{if and only if} \quad \int_0^{T^*} \|\nabla u\|_{L^\infty(\mathbb{R}^d)} dt = \infty.$$

For the periodic case, by virtue of the transport equation theory one can obtain the similar results as follows:

Lemma 2.4. *Let $u_0 \in H^k(\mathbb{T}^d)$ with $k > \frac{d}{2} + 3$. Then, there exists $T = T(\|u_0\|_{H^k(\mathbb{T}^d)})$ such that the system (1.1) has a unique solution $u \in C([0, T]; H^k(\mathbb{T}^d))$.*

Lemma 2.5. *Let $u_0 \in H^k(\mathbb{T}^d)$ with $k > \frac{d}{2} + 3$. Suppose that T^* is the lifespan of the solution u to (1.1). Then the solution blows up in finite time if and only if*

$$\int_0^{T^*} \|\nabla u\|_{L^\infty} dt = \infty.$$

Since the regularity index $k > \frac{d}{2} + 3$, it follows that the space $H^{k-2}(\mathbb{T}^d)$ is a Banach algebra, which ensures the local existence and uniqueness of the solution to a transport equation. Since the proof is similar to that of [5], we omit the details here.

Now, we introduce a crucial lemma which will be used to prove our main result.

Lemma 2.6. *Let $u_0 \in H^k(\mathbb{T}^d)$ with $k > \frac{d}{2} + 3$ and let u be the corresponding local solution to (1.1). If $\frac{\partial u^i}{\partial n} = 0$, $i = 1, 2, \dots, d$, then $\frac{\partial u^i}{\partial n} = 0$, $t \in [0, T]$, where $n = (n^1, n^2, \dots, n^d) \in \mathbb{R}^d$ is a constant vector field.*

Proof. Since $\frac{\partial f}{\partial n} = \sum_{i=1}^d n^i \partial_i f$, it follows that

$$\frac{\partial(fg)}{\partial n} = \frac{\partial f}{\partial n} g + f \frac{\partial g}{\partial n}.$$

Applying $\frac{\partial}{\partial n}$ to both sides of (1.1) in components, we have

$$(2.1) \quad \partial_n m^i + \sum_j^d (u^j \partial_{jn}^2 m^i + \partial_n u^j \partial_j m^i + \partial_{in}^2 u^j m^j + \partial_i u^j \partial_n m^j + \partial_n m^i \partial_j u^j + m^i \partial_{nj}^2 u^j) = 0.$$

Multiplying by $\partial_n m^i$ both sides of the above inequality and integrating over \mathbb{T}^d , we obtain

$$(2.2) \quad \begin{aligned} \frac{d}{dt} \int_{\mathbb{T}^d} |\partial_n m^i|^2 dx &= - \sum_j^d \int_{\mathbb{T}^d} (\partial_n u^j \partial_j m^i + \partial_{in}^2 u^j m^j + \partial_i u^j \partial_n m^j + m^i \partial_{nj}^2 u^j) \partial_n m^i dx \\ &\leq (\|\nabla m\|_{L^\infty} \|\partial_n u\|_{L^2} + 2\|m\|_{L^\infty} \|\partial_n \nabla u\|_{L^2} + \|\nabla u\|_{L^\infty} \|\partial_n m\|_{L^2}) \|\partial_n m\|_{L^2}. \end{aligned}$$

Since $m = (1 - \Delta)u$, it follows that

$$\|\partial_n u\|_{L^2} \leq \|\partial_n m\|_{L^2}, \quad \|\partial_n \nabla u\|_{L^2} \leq \|\partial_n m\|_{L^2}.$$

Plugging the above inequality into (2.2) yields that

$$(2.3) \quad \frac{d}{dt} \|\partial_n m\|_{L^2}^2 \leq (\|\nabla m\|_{L^\infty} + 2\|m\|_{L^\infty} + \|\nabla u\|_{L^\infty}) \|\partial_n m\|_{L^2}^2 \leq C \|u\|_{H^k} \|\partial_n m\|_{L^2}^2,$$

where we use the embedding relation $H^k \hookrightarrow W^{3,\infty}$ with $k > \frac{d}{2} + 1$. Taking advantage of Gronwall's inequality and the fact that $\partial_n m_0 = 0$, we see that

$$\|\partial_n m\|_{L^2} = 0.$$

□

Similar to the whole space case [5, 25], one can easily get the following lemma.

Lemma 2.7. *The system 1.1 has the following conservation law:*

$$(2.4) \quad H = \int_{\mathbb{T}^d} |u|^2 + |\nabla u|^2 dx.$$

For a function f on \mathbb{T}^d , we define its Fourier transform denoted by $\widehat{f}(\xi)$ as

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x) e^{-i\xi x} dx, \quad \xi \in \mathbb{Z}^d.$$

Let $\eta : \mathbb{R} \rightarrow [0, 1]$ be an even, smooth, non-negative and radially decreasing function which is supported in a ball $\{\xi : |\xi| \leq \frac{8}{5}\}$ and $\eta \equiv 1$ for $|\xi| \leq \frac{5}{4}$. Let $\varphi(\xi) \doteq \eta(\xi) - \eta(\frac{\xi}{2})$, and define the Littlewood-Paley operators Δ_j by

$$\widehat{\Delta_j f}(\xi) \doteq \varphi(2^{-j}\xi) \widehat{f}(\xi), \quad \widehat{\Delta_{-1} f}(\xi) \doteq \eta(\xi) \widehat{f}(\xi), \quad S_j f \doteq \sum_{j' < j} \Delta_{j'} f \quad \forall j \in \mathbb{Z},$$

We can then define the Besov space $B_{p,r}^s(\mathbb{T}^d)$ with norm

$$\|f\|_{B_{p,r}^s(\mathbb{T}^d)} = \|2^{js} \|\Delta_j f\|_{L^p(\mathbb{T}^d)}\|_{l^r}.$$

Notation. Since all function spaces in the following sections are over \mathbb{T}^d , for simplicity, we drop \mathbb{T}^d in the notation of function spaces if there is no ambiguity.

3 Blow up: $d = 2$

In order to explain our main idea to obtain the blow-up result, we first consider a simple case $d = 2$. Our main theorem in this section can be stated as follows.

Theorem 3.1. *Let $u_0 \in H^k(\mathbb{T}^2)$ with $k > 4$ and let u be the corresponding local solution to (1.1). Suppose that $\frac{\partial u_0^i}{\partial n} = 0$, $i = 1, 2$, where $n = (\cos \theta, \sin \theta)$, θ is a constant. If $\sin^2 \theta \partial_x u_0^1 - \sin \theta \cos \theta \partial_y u_0^1 - \sin \theta \cos \theta \partial_x u_0^2 + \cos^2 \theta \partial_y u_0^2 < -\sqrt{2} \|u_0\|_{H^1}$ for some $(x_0, y_0) \in \mathbb{T}^2$. Then the solution u blows up in finite time.*

Proof. Let $n^\perp = (-\sin \theta, \cos \theta)$ be the vertical vector of n . Define that

$$(3.1) \quad u^n = u \cdot n = \cos \theta u^1 + \sin \theta u^2, \quad u^{n^\perp} = u \cdot n^\perp = -\sin \theta u^1 + \cos \theta u^2,$$

and then

$$(3.2) \quad u^1 = \cos \theta u^n - \sin \theta u^{n^\perp}, \quad u^2 = \sin \theta u^n + \cos \theta u^{n^\perp}.$$

By directly calculating, we see that

$$(3.3) \quad \operatorname{div} u = \partial_x u^1 + \partial_y u^2 = \partial_x (\cos \theta u^n - \sin \theta u^{n^\perp}) + \partial_y (\sin \theta u^n + \cos \theta u^{n^\perp}) = \frac{\partial u^n}{\partial n} + \frac{\partial u^{n^\perp}}{\partial n^\perp},$$

$$(3.4) \quad u \cdot \nabla = u^1 \partial_x + u^2 \partial_y = (\cos \theta u^n - \sin \theta u^{n^\perp}) \partial_x + (\sin \theta u^n + \cos \theta u^{n^\perp}) \partial_y = u^n \frac{\partial}{\partial n} + u^{n^\perp} \frac{\partial}{\partial n^\perp}.$$

Multiplying by n^\perp both sides of (1.1), we obtain

$$(3.5) \quad \frac{\partial m^{n^\perp}}{\partial t} + u^n \frac{\partial m^{n^\perp}}{\partial n} + u^{n^\perp} \frac{\partial m^{n^\perp}}{\partial n^\perp} + m^{n^\perp} \left(\frac{\partial u^n}{\partial n} + \frac{\partial u^{n^\perp}}{\partial n^\perp} \right) + (\nabla u)^T m \cdot n^\perp = 0.$$

Since $\frac{\partial u_n^i}{\partial n} = 0$, by virtue of Lemma (2.6), we have $\frac{\partial u^i}{\partial n} = 0$ and

$$(3.6) \quad \frac{\partial m^{n^\perp}}{\partial t} + u^{n^\perp} \frac{\partial m^{n^\perp}}{\partial n^\perp} + m^{n^\perp} \frac{\partial u^{n^\perp}}{\partial n^\perp} + (\nabla u)^T m \cdot n^\perp = 0.$$

Next we deal with the term $(\nabla u)^T m \cdot n^\perp$ in components.

(3.7)

$$\begin{aligned} (\nabla u)^T m \cdot n^\perp &= -\sin \theta (m^1 \partial_x u^1 + m^2 \partial_x u^2) + \cos \theta (m^1 \partial_y u^1 + m^2 \partial_y u^2) \\ &= -\sin \theta \left\{ \frac{1}{2} \partial_x [(u^1)^2 - (\partial_x u^1)^2 + (\partial_y u^1)^2 + (u^2)^2 - (\partial_x u^2)^2 + (\partial_y u^2)^2] - \partial_y (\partial_y u^1 \partial_x u^1 + \partial_y u^2 \partial_x u^2) \right\} \\ &\quad + \cos \theta \left\{ \frac{1}{2} \partial_y [(u^1)^2 - (\partial_y u^1)^2 + (\partial_x u^1)^2 + (u^2)^2 - (\partial_y u^2)^2 + (\partial_x u^2)^2] - \partial_x (\partial_y u^1 \partial_x u^1 + \partial_y u^2 \partial_x u^2) \right\} \\ &= \frac{1}{2} \frac{\partial [(u^1)^2 + (u^2)^2]}{\partial n^\perp} + \left(\frac{\sin \theta}{2} \partial_x + \frac{\cos \theta}{2} \partial_y \right) [(\partial_x u^1)^2 + (\partial_x u^2)^2 - (\partial_y u^1)^2 + (\partial_y u^2)^2] \\ &\quad + (\sin \theta \partial_y - \cos \theta \partial_x) (\partial_y u^1 \partial_x u^1 + \partial_y u^2 \partial_x u^2) \\ &= \frac{1}{2} \frac{\partial (u^n)^2}{\partial n^\perp} + \frac{1}{2} \frac{\partial (u^{n^\perp})^2}{\partial n^\perp} + \left(\frac{\sin \theta}{2} \partial_x + \frac{\cos \theta}{2} \partial_y \right) [(\partial_x u^1)^2 + (\partial_x u^2)^2 - (\partial_y u^1)^2 + (\partial_y u^2)^2] \\ &\quad + (\sin \theta \partial_y - \cos \theta \partial_x) (\partial_y u^1 \partial_x u^1 + \partial_y u^2 \partial_x u^2), \end{aligned}$$

where we use the fact that $(u^1)^2 + (u^2)^2 = (u^n)^2 + (u^{n^\perp})^2$. Since $\partial_n u^1 = \partial_n u^2 = 0$, it follows that

$$(3.8) \quad \sin \theta \partial_x + \cos \theta \partial_y = (\cos^2 \theta - \sin^2 \theta) \frac{\partial}{\partial n^\perp}, \quad \sin \theta \partial_y - \cos \theta \partial_x = 2 \sin \theta \cos \theta \frac{\partial}{\partial n^\perp},$$

$$(3.9) \quad \begin{aligned} (\partial_x u^1)^2 + (\partial_x u^2)^2 &= (-\sin \theta \frac{\partial u^1}{\partial n^\perp})^2 + (-\sin \theta \frac{\partial u^2}{\partial n^\perp})^2 = \sin^2 \theta \left[\left(\frac{\partial u^n}{\partial n^\perp} \right)^2 + \left(\frac{\partial u^{n^\perp}}{\partial n^\perp} \right)^2 \right], \\ (\partial_y u^1)^2 + (\partial_y u^2)^2 &= (\cos \theta \frac{\partial u^1}{\partial n^\perp})^2 + (\cos \theta \frac{\partial u^2}{\partial n^\perp})^2 = \cos^2 \theta \left[\left(\frac{\partial u^n}{\partial n^\perp} \right)^2 + \left(\frac{\partial u^{n^\perp}}{\partial n^\perp} \right)^2 \right], \\ \partial_y u^1 \partial_x u^1 + \partial_y u^2 \partial_x u^2 &= -\cos \theta \sin \theta \left[\left(\frac{\partial u^1}{\partial n^\perp} \right)^2 + \left(\frac{\partial u^2}{\partial n^\perp} \right)^2 \right] = -\cos \theta \sin \theta \left[\left(\frac{\partial u^n}{\partial n^\perp} \right)^2 + \left(\frac{\partial u^{n^\perp}}{\partial n^\perp} \right)^2 \right]. \end{aligned}$$

Plugging (3.8) and (3.9) into (3.7) yields that

$$(3.10) \quad (\nabla u)^T m \cdot n^\perp = \frac{1}{2} \frac{\partial (u^n)^2}{\partial n^\perp} + \frac{1}{2} \frac{\partial (u^{n^\perp})^2}{\partial n^\perp} - \frac{1}{2} \frac{\partial}{\partial n^\perp} \left[\left(\frac{\partial u^n}{\partial n^\perp} \right)^2 + \left(\frac{\partial u^{n^\perp}}{\partial n^\perp} \right)^2 \right].$$

For simplicity, we denote that $v = \partial_{n^\perp} u^{n^\perp}$, $w = \partial_{n^\perp} u^n$. Using (3.6) and (3.10) we get

$$(3.11) \quad \frac{\partial m^{n^\perp}}{\partial t} + u^{n^\perp} \frac{\partial m^{n^\perp}}{\partial n^\perp} + m^{n^\perp} \frac{\partial u^{n^\perp}}{\partial n^\perp} + \frac{1}{2} \frac{\partial (u^n)^2}{\partial n^\perp} + \frac{1}{2} \frac{\partial (u^{n^\perp})^2}{\partial n^\perp} - \frac{1}{2} \frac{\partial}{\partial n^\perp} (v^2 + w^2) = 0.$$

Since $m = (1 - \Delta)u = (1 - \partial_x^2 - \partial_y^2)u = (1 - \partial_{n^\perp}^2)u$, it follows that

$$(3.12) \quad (1 - \partial_{n^\perp}^2) \left(\frac{\partial u^{n^\perp}}{\partial t} + u^{n^\perp} \frac{\partial u^{n^\perp}}{\partial n^\perp} \right) = \frac{\partial m^{n^\perp}}{\partial t} + u^{n^\perp} \frac{\partial m^{n^\perp}}{\partial n^\perp} + 3m^{n^\perp} \frac{\partial u^{n^\perp}}{\partial n^\perp} - 3u^{n^\perp} \frac{\partial u^{n^\perp}}{\partial n^\perp}.$$

Combining (3.12) with (3.11) yields that

$$(3.13) \quad \frac{\partial u^{n^\perp}}{\partial t} + u^{n^\perp} \frac{\partial u^{n^\perp}}{\partial n^\perp} = -\partial_{n^\perp} (1 - \partial_{n^\perp}^2)^{-1} \left[(u^{n^\perp})^2 + \frac{1}{2} (u^n)^2 + \frac{1}{2} v^2 - \frac{1}{2} w^2 \right].$$

Applying $\frac{\partial}{\partial n^\perp}$ on both sides of the above inequality, we have

$$(3.14) \quad \begin{aligned} \partial_t v + u^{n^\perp} \partial_{n^\perp} v &= -v^2 - \partial_{n^\perp}^2 (1 - \partial_{n^\perp}^2)^{-1} f = -v^2 + f - (1 - \partial_{n^\perp}^2)^{-1} f \\ &= -\frac{1}{2} (v^2 + w^2) + (u^{n^\perp})^2 + \frac{1}{2} (u^n)^2 - (1 - \partial_{n^\perp}^2)^{-1} f, \end{aligned}$$

where $f = (u^{n^\perp})^2 + \frac{1}{2} (u^n)^2 + \frac{1}{2} v^2 - \frac{1}{2} w^2$.

Define the characteristics $\Phi(t, x, y)$ as the solution of

$$(3.15) \quad \begin{cases} \frac{d\Phi(t, x, y)}{dt} = u(t, \Phi(t, x, y)), \\ \Phi(0, x, y) = (x, y). \end{cases}$$

It is easy to check that

$$\frac{d}{dt} v(t, \Phi(t, x, y)) = v_t + u \nabla v = \partial_t v + u^{n^\perp} \partial_{n^\perp} v,$$

which leads to

$$(3.16) \quad \frac{d}{dt} v(t, \Phi(t, x, y)) = -\frac{1}{2} (v^2 + w^2)(t, \Phi(t, x, y)) + (u^{n^\perp})^2 + \frac{1}{2} (u^n)^2 - (1 - \partial_{n^\perp}^2)^{-1} f.$$

Since $\|u\|_{H^1} = \|u_0\|_{H^1}$, it follows that

$$(3.17) \quad -(1 - \partial_{n^\perp}^2)^{-1} f \leq \frac{1}{2} (1 - \partial_{n^\perp}^2)^{-1} w^2 \leq \frac{1}{2} \|u_0\|_{H^1}^2.$$

By Sobolev's embedding, we see that $\|u^n\|_{L^\infty}^2 + \|u^{n^\perp}\|_{L^\infty}^2 \leq \frac{1}{2} (\|\frac{\partial u^n}{\partial n^\perp}\|_{L^2}^2 + \|\frac{\partial u^{n^\perp}}{\partial n^\perp}\|_{L^2}^2) \leq \frac{1}{2} \|u\|_{H^1}^2 = \frac{1}{2} \|u_0\|_{H^1}^2$. Using the fact that $-\frac{1}{2} w^2 \leq 0$ and the above estimate, we have

$$(3.18) \quad \frac{dg(t)}{dt} \leq -\frac{1}{2} g^2(t) + \|u_0\|_{H^1}^2 \leq -\frac{1}{2} (g(t) + \sqrt{2} \|u_0\|_{H^1}) (g(t) - \sqrt{2} \|u_0\|_{H^1}),$$

where $g(t) = v(t, \Phi(t, x_0, y_0))$. The assumption on the initial datum and the definition of v guarantee that $g(0) < -\sqrt{2} \|u_0\|_{H^1}$. By virtue of the continuity argument, we see that $g(t) < -\sqrt{2} \|u_0\|_{H^1}$ for all $t \in [0, T^*)$. From the above inequality, we obtain that

$$(3.19) \quad -\frac{d}{dt} \left(\frac{1}{g(t) - \sqrt{2} \|u_0\|_{H^1}} \right) \leq -\left(1 + \frac{2\sqrt{2} \|u_0\|_{H^1}}{g(t) - \sqrt{2} \|u_0\|_{H^1}} \right),$$

which leads to

$$(3.20) \quad \frac{g(0) + \sqrt{2}\|u_0\|_{H^1}}{g(0) - \sqrt{2}\|u_0\|_{H^1}} e^{\sqrt{2}\|u_0\|_{H^1}t} - 1 \leq \frac{2\sqrt{2}\|u_0\|_{H^1}}{g(t) - \sqrt{2}\|u_0\|_{H^1}} \leq 0.$$

Since $0 < \frac{g(0) + \sqrt{2}\|u_0\|_{H^1}}{g(0) - \sqrt{2}\|u_0\|_{H^1}} < 1$, then there exists

$$0 < T \leq \frac{\sqrt{2}}{2\|u_0\|_{H^1}} \ln \frac{g(0) - \sqrt{2}\|u_0\|_{H^1}}{g(0) + \sqrt{2}\|u_0\|_{H^1}},$$

such that $\lim_{t \rightarrow T} g(t) = -\infty$. By the definition of $g(t)$, we see that $|g(t)| \leq \|v\|_{L^\infty} \leq \|\frac{\partial u^{n-1}}{\partial n^\perp}\|_{L^\infty} \leq \|\nabla u\|_{L^\infty}$, which implies that the solution blows up in finite time. \square

Remark 3.2. *If a non-zero function f satisfies that $\partial_n f = 0$, then the H^1 norm of f in \mathbb{R}^2 is infinity i.e. $\|f\|_{H^1(\mathbb{R}^2)} = \infty$. Thus, the argument in the above theorem only holds true in the periodic case. For the whole space, one should use the other function space to deal with this problem.*

4 Blow up: $d \geq 3$

Now we turn our attention to the higher dimension ($d \geq 3$) case. Let's first introduce some notations.

Let

$$A = (A_1, A_2, \dots, A_d) = \{(a^{ij})\},$$

where A_j is the column vector of the matrix of A . Denote that

$$\bar{u} = u^A = Au, \quad \bar{\nabla} f = (A\nabla)f,$$

or in components

$$\bar{u}^i = \sum_j a^{ij} u^j, \quad \bar{\partial}_i = \sum_j a^{ij} \partial_j.$$

Our main result in this section can be stated as follows.

Theorem 4.1. *Assume that A is an orthogonal matrix. Let $u_0 \in H^k(\mathbb{T}^d)$ with $k > \frac{d}{2} + 3$ and let u be the corresponding local solution to (1.1). Suppose that $\bar{\partial}_i u_0 = 0$, $i = 2, 3, \dots, d$. If there exists some $x_0 \in \mathbb{T}^d$ such that $\bar{\partial}_1 \bar{u}_0^1 < -\sqrt{2}\|u_0\|_{H^1}$. Then the solution u blows up in finite time.*

Proof. Since A is an orthogonal matrix, it follows that

$$(4.1) \quad \operatorname{div} u = \sum_{i=1}^d \partial_i u^i = \sum_{i,j=1}^d a^{ji} \bar{\partial}_j u^i = \sum_{j=1}^d \bar{\partial}_j \bar{u}^j = \bar{\operatorname{div}} \bar{u},$$

$$(4.2) \quad u \cdot \nabla = \sum_{i=1}^d u^i \partial_i = \sum_{i,j=1}^d a^{ji} \bar{u}^j \bar{\partial}_i = \sum_{j=1}^d \bar{u}^j \bar{\partial}_j = \bar{u} \cdot \bar{\nabla},$$

Applying A to both sides of (1.1) and using the above equality, we see that

$$(4.3) \quad \bar{m}_t + \bar{u} \cdot \bar{\nabla} \bar{m} + A \cdot [(\nabla u)^T m] + \bar{m} (\bar{div} \bar{u}) = 0.$$

According to Lemma 2.6 and the assumption $\bar{\partial}_i u_0 = 0$, $i = 2, 3, \dots, d$, we get $\bar{\partial}_i u = 0$, $i = 2, 3, \dots, d$.

From (4.3), we see that

$$(4.4) \quad \bar{m}_t + \bar{u}^1 \cdot \bar{\partial}_1 \bar{m} + A \cdot [(\nabla u)^T m] + \bar{m} (\bar{\partial}_1 \bar{u}^1) = 0.$$

Now we consider the term $A \cdot [(\nabla u)^T m]$ in components. By directly calculating, we deduce that

$$(4.5) \quad \begin{aligned} (A \cdot [(\nabla u)^T m])^i &= \sum_{j,k=1}^d a^{ij} m^k \partial_j u^k = \sum_{j,k=1}^d a^{ij} \partial_k \left(\frac{1}{2} \delta_{jk} |u|^2 - \partial_j u \cdot \partial_k u + \frac{1}{2} \delta_{jk} |\nabla u|^2 \right) \\ &= \frac{1}{2} \sum_{j=1}^d a^{ij} \partial_j (|u|^2 + |\nabla u|^2) - \sum_{j,k=1}^d a^{ij} \partial_k (\partial_j u \cdot \partial_k u) \\ &= \frac{1}{2} \bar{\partial}_i (|u|^2 + |\nabla u|^2) - \sum_{j,k=1}^d a^{ij} \partial_k (\partial_j u \cdot \partial_k u). \end{aligned}$$

Since A is an orthogonal matrix, it follows that

$$(4.6) \quad \begin{aligned} |u|^2 &= u^T \cdot u = u^T A^T A u = (A u)^T (A u) = |\bar{u}|^2, \\ |\nabla u|^2 &= (\nabla u)^T \cdot \nabla u = (\nabla u)^T A^T A \nabla u = |\bar{\nabla} \bar{u}|^2 = |\bar{\partial}_1 \bar{u}|^2. \end{aligned}$$

By directly calculating, we see that

$$(4.7) \quad \partial_k (\partial_j u \cdot \partial_k u) = \sum_{l=1}^d a^{lk} \bar{\partial}_l \left[\sum_{i=1}^d \left(\sum_{m=1}^d a^{jm} \bar{\partial}_m \bar{u}^i \right) \left(\sum_{n=1}^d a^{kn} \bar{\partial}_n \bar{u}^i \right) \right] = a^{1k} a^{j1} a^{k1} \bar{\partial}_1 (|\bar{\partial}_1 \bar{u}|^2).$$

Combining (4.6) and (4.7) with (4.5) yields that

$$(4.8) \quad (A \cdot [(\nabla u)^T m])^1 = \frac{1}{2} \bar{\partial}_1 (|\bar{u}|^2 + |\bar{\partial}_1 \bar{u}|^2) - \sum_{j,k=1}^d a^{1j} a^{1k} a^{j1} a^{k1} \bar{\partial}_1 (|\bar{\partial}_1 \bar{u}|^2) = \frac{1}{2} \bar{\partial}_1 (|\bar{u}|^2 - |\bar{\partial}_1 \bar{u}|^2),$$

where we use the fact that $\sum_k a^{1k} a^{k1} = 1$. From (4.4) and (4.8) we deduce that

$$(4.9) \quad \bar{m}_t^1 + \bar{u}^1 \cdot \bar{\partial}_1 \bar{m}^1 + \frac{1}{2} \bar{\partial}_1 (|\bar{u}|^2 - |\bar{\partial}_1 \bar{u}|^2) + \bar{m}^1 (\bar{\partial}_1 \bar{u}^1) = 0.$$

Since $\bar{m}^1 = (1 - \bar{\partial}_1^2) \bar{u}^1$, it follows that

$$(4.10) \quad (1 - \bar{\partial}_1^2) (\bar{u}_t^1 + \bar{u}^1 \bar{\partial}_1 \bar{u}^1) = \bar{m}_t^1 + \bar{u}^1 \bar{\partial}_1 \bar{m}^1 + 3\bar{m}^1 (\bar{\partial}_1 \bar{u}^1) - 3\bar{u}^1 (\bar{\partial}_1 \bar{u}^1).$$

Plugging (4.10) into (4.9) yields that

$$(4.11) \quad \bar{u}_t^1 + \bar{u}^1 \bar{\partial}_1 \bar{u}^1 = -\bar{\partial}_1 (1 - \bar{\partial}_1^2)^{-1} \left[\frac{1}{2} (\bar{u}^1)^2 + \frac{1}{2} |\bar{u}|^2 - \frac{1}{2} |\bar{\partial}_1 \bar{u}|^2 + (\bar{\partial}_1 \bar{u})^2 \right].$$

Applying $\overline{\partial}_1$ to both sides of the above inequality, we have

$$(4.12) \quad \begin{aligned} \overline{\partial}_1 \overline{u}_t^1 + \overline{u}^1 \overline{\partial}_1 \overline{u}^1 &= -(\overline{\partial}_1 \overline{u}^1)^2 - \overline{\partial}_1^2 (1 - \overline{\partial}_1^2)^{-1} f = -(\overline{\partial}_1 \overline{u}^1)^2 + f - (1 - \overline{\partial}_1^2)^{-1} f \\ &= -\frac{1}{2} |\overline{\partial}_1 \overline{u}^1|^2 + \frac{1}{2} (\overline{u}^1)^2 + \frac{1}{2} |\overline{u}^1|^2 - (1 - \overline{\partial}_1^2)^{-1} f, \end{aligned}$$

where $f = \frac{1}{2} (\overline{u}^1)^2 + \frac{1}{2} |\overline{u}^1|^2 - \frac{1}{2} |\overline{\partial}_1 \overline{u}^1|^2 + (\overline{\partial}_1 \overline{u}^1)^2$. Define the characteristics $\Phi(t, x) \in \mathbb{R}^d$ as the solution of

$$(4.13) \quad \begin{cases} \frac{d\Phi(t, x)}{dt} = u(t, \Phi(t, x)), \\ \Phi(0, x) = x. \end{cases}$$

Then we deduce that

$$\frac{d}{dt} v(t, \Phi(t, x)) = v_t + u \nabla v = \partial_t v + \overline{u}^1 \overline{\partial}_1 v,$$

which leads to

$$(4.14) \quad \frac{d}{dt} (\overline{\partial}_1 \overline{u}^1 \circ \Phi) = -\frac{1}{2} |\overline{\partial}_1 \overline{u} \circ \Phi|^2 + \frac{1}{2} (\overline{u}^1 \circ \Phi)^2 + \frac{1}{2} |\overline{u} \circ \Phi|^2 - (1 - \overline{\partial}_1^2)^{-1} f \circ \Phi.$$

By virtue of the conservation law $\|u\|_{H^1} = \|u_0\|_{H^1}$, we obtain

$$(4.15) \quad -(1 - \overline{\partial}_1^2)^{-1} f \leq \frac{1}{2} (1 - \overline{\partial}_1^2)^{-1} |\overline{\partial}_1 \overline{u}^1|^2 \leq \frac{1}{2} \|u_0\|_{H^1}^2.$$

Taking advantage of Sobolev's embedding, we see that $\|\overline{u}\|_{L^\infty}^2 \leq \frac{1}{2} \|\overline{\partial}_1 \overline{u}\|_{L^2}^2 \leq \frac{1}{2} \|u\|_{H^1}^2 = \frac{1}{2} \|u_0\|_{H^1}^2$.

Using the fact that $-\frac{1}{2} |\overline{\partial}_1 \overline{u}^1|^2 \leq -\frac{1}{2} (\overline{\partial}_1 \overline{u}^1)^2$ and the above estimate, we have

$$(4.16) \quad \frac{dh(t)}{dt} \leq -\frac{1}{2} h^2(t) + \|u_0\|_{H^1}^2 \leq -\frac{1}{2} (h(t) + \sqrt{2} \|u_0\|_{H^1}) (h(t) - \sqrt{2} \|u_0\|_{H^1}),$$

where $h(t) = \overline{\partial}_1 \overline{u}^1 \circ \Phi(t, x_0)$. Note that $h(0) < -\sqrt{2} \|u_0\|_{H^1}$. The continuity argument ensures that $h(t) < -\sqrt{2} \|u_0\|_{H^1}$ for all $t \in [0, T^*)$. By the same token, we can solve the inequality (4.16) and deduce that

$$(4.17) \quad \frac{h(0) + \sqrt{2} \|u_0\|_{H^1}}{h(0) - \sqrt{2} \|u_0\|_{H^1}} e^{\sqrt{2} \|u_0\|_{H^1} t} - 1 \leq \frac{2\sqrt{2} \|u_0\|_{H^1}}{h(t) - \sqrt{2} \|u_0\|_{H^1}} \leq 0.$$

Thanks to $0 < \frac{h(0) + \sqrt{2} \|u_0\|_{H^1}}{h(0) - \sqrt{2} \|u_0\|_{H^1}} < 1$, we can find

$$(4.18) \quad 0 < T \leq \frac{\sqrt{2}}{2 \|u_0\|_{H^1}} \ln \frac{h(0) - \sqrt{2} \|u_0\|_{H^1}}{h(0) + \sqrt{2} \|u_0\|_{H^1}},$$

such that $\lim_{t \rightarrow T} h(t) = -\infty$. Since $|h(t)| \leq \|\overline{\partial}_1 \overline{u}^1\|_{L^\infty} \leq \|\nabla u\|_{L^\infty}$, it follows that the solution blows up in T . \square

5 Ill-posedness in critical Besov spaces

In this section, we are going to prove the norm inflation of the system (1.1) in $B_{p,r}^{1+\frac{d}{p}}$ with $d \leq p \leq +\infty$ and $1 < r \leq +\infty$. Our main result can be stated as follows:

Theorem 5.1. *Let $d \leq p \leq +\infty$ and $1 < r \leq +\infty$. For any $\varepsilon > 0$, there exists $u_0 \in H^\infty(\mathbb{T}^d)$, such that the following holds:*

- (1) $\|u_0\|_{B_{p,r}^{1+\frac{d}{p}}(\mathbb{T}^d)} \leq \varepsilon$;
- (2) *There exists a unique solution $u \in C([0, T]; H^\infty(\mathbb{T}^d))$ of the system (1.1) with maximal $T < \varepsilon$;*
- (3) $\limsup_{t \rightarrow T^-} \|u\|_{B_{p,r}^{1+\frac{d}{p}}(\mathbb{T}^d)} \geq \limsup_{t \rightarrow T^-} \|u\|_{B_{\infty,\infty}^1(\mathbb{T}^d)} = \infty$.

Remark 5.2. (i) *In the case $d = 2$, taking $p = r = 2$, we see that our theorem implies the ill-posedness in the critical Sobolev space H^2 .*

(ii) *If $d \geq 3$, we don't know whether the system (1.1) is well-posed or ill-posed in the critical Sobolev space $H^{1+\frac{d}{2}}$.*

In order to prove Theorem 5.1, we need some useful lemmas.

Lemma 5.3. [20] *Let $T > 0$. Assume that $A(t) \in C^1[0, T]$, $A(t) > 0$ and there exists a constant B such that*

$$\frac{d}{dt}A(t) \leq BA(t) \ln(2 + A(t)), \quad \forall t \in [0, T].$$

Then we have

$$A(t) \leq (2 + A(0))e^{Bt}, \quad \forall t \in [0, T].$$

Lemma 5.4. [20] *Assume that $u \in H^{1+\frac{d}{2}+\varepsilon}$ with $\varepsilon > 0$. We have*

$$\|\nabla u\|_{L^\infty} \leq \frac{C}{\varepsilon} \|u\|_{B_{\infty,\infty}^1} \ln\left(2 + \frac{\|u\|_{H^{1+\frac{d}{2}+\varepsilon}}}{\|u\|_{B_{\infty,\infty}^1}}\right)$$

Proof of Theorem 5.1: From now on, we turn our attention to prove Theorem 5.1. Fix $d \leq p \leq \infty$, $1 < r \leq \infty$. We define the periodic function $h(x_1)$ as follows:

$$W(x_1) = - \sum_{k \geq 2} \frac{1}{k^{1+\frac{d}{p}} \ln^{\frac{2}{1+r}} k} \sin(kx_1).$$

Directly calculating, we have

$$\Delta_j W(x_1) = - \sum_{2^j < k < 2^{j+2}} \varphi\left(\frac{k}{2^j} \vec{e}_1\right) \frac{1}{k^{1+\frac{d}{p}} \ln^{\frac{2}{1+r}} k} \sin(kx_1),$$

where $\vec{e}_1 = (1, 0, 0, \dots, 0)$. On the other hand,

$$\|\sin(kx_1)\|_{L^p(\mathbb{T}^d)} = \left(\int_{\mathbb{T}^d} |\sin(kx_1)|^p dx \right)^{\frac{1}{p}} = (2\pi)^{\frac{d-1}{p}} \left(\frac{1}{k} \int_0^{2k\pi} |\sin(y)|^p dy \right)^{\frac{1}{p}} = C_{p,d},$$

where $C_{p,d}$ is a constant independent on k , which implies that $\|\Delta_j W(x_1)\|_{L^p(\mathbb{T}^d)} \sim 2^{-j(1+\frac{d}{p})} j^{-\frac{2}{1+r}}$, and thus

$$\|W(x_1)\|_{B_{p,q}^{1+\frac{d}{p}}(\mathbb{T}^d)} \sim \left\| \frac{1}{j^{\frac{2}{1+r}}} \right\|_{l^q}.$$

From this we see that $h \in B_{p,r}^{1+\frac{d}{p}}$ but $h \notin B_{p,1}^{1+\frac{d}{p}}$. Since $\frac{d}{p} \leq 1$ and $r > 1$, it follows that

$$W'(0) = - \sum_{k \geq 2} \frac{1}{k^{\frac{d}{p}} \ln^{\frac{2}{1+r}} k} = -\infty.$$

For any $\varepsilon > 0$, let

$$u_{0,\varepsilon}^1 = \frac{\varepsilon S_N W(x_1)}{\|W\|_{B_{p,r}^{1+\frac{d}{p}}}}, \quad u_0^2 = u_0^3 = \dots = u_0^n = \varepsilon f(x_1),$$

where N is large enough such that $\partial_1 u_{0,\varepsilon}^1(0) \sim \varepsilon^{-10}$ and f is arbitrary smooth periodic function such that $\|f\|_{B_{p,r}^{1+\frac{d}{p}}} < 1$. (For example, one can choose $f = \frac{\sin(x_1)}{\|\sin(x_1)\|_{B_{p,r}^{1+\frac{d}{p}}}}$). Then $u_{0,\varepsilon} \in H^\infty$ and $\|u_0\|_{B_{p,r}^{1+\frac{d}{p}}} \leq \varepsilon$. By virtue of Lemma 2.4, there exists a T_ε such that the system (1.1) has a unique solution $u_\varepsilon \in C([0, T_\varepsilon]; H^\infty)$. Taking $A = Id$ and $x_0 = 0$ in Theorem 4.1, from the estimate (4.18), we see that the maximal time

$$T_\varepsilon \leq \frac{\sqrt{2}}{2\|u_{0,\varepsilon}\|_{H^1}} \ln \frac{\partial_1 u_{0,\varepsilon}^1(0) - \sqrt{2}\|u_{0,\varepsilon}\|_{H^1}}{\partial_1 u_{0,\varepsilon}^1(0) + \sqrt{2}\|u_{0,\varepsilon}\|_{H^1}} \leq C\varepsilon^{10}.$$

It suffices to show that

$$(5.1) \quad \limsup_{t \rightarrow T_\varepsilon} \|u_\varepsilon\|_{B_{\infty,\infty}^1(\mathbb{T}^d)} = \infty.$$

Suppose that (5.1) fails, we can find a M_ε such that

$$\sup_{0 < t < T_\varepsilon} \|u_\varepsilon\|_{B_{\infty,\infty}^1} \leq M_\varepsilon.$$

Let $k = 2 + [\frac{d}{2}] > 1 + \frac{d}{2}$. Taking advantage of the energy estimate and Lemma 5.4, we obtain

$$(5.2) \quad \frac{d}{dt} \|u_\varepsilon\|_{H^k}^2 \leq \|\nabla u\|_{L^\infty} \|u\|_{H^k}^2 \leq C_k \|u_\varepsilon\|_{B_{\infty,\infty}^1} \|u\|_{H^k}^2 (\ln(2 + \frac{\|u\|_{H^k}}{\|u_\varepsilon\|_{B_{\infty,\infty}^1}})) \leq C_k M_\varepsilon \|u\|_{H^k}^2 \ln(2 + \|u\|_{H^k}^2).$$

By virtue of Lemma 5.3, we deduce that $\sup_{t \in [0, T_\varepsilon]} \|u_\varepsilon\|_{H^k} < +\infty$ which means that u_ε will not blow up in T_ε . This contradicts with Theorem 4.1.

6 Periodic peakon solutions

In this section we verify that the system (1.1) possesses a special class of periodic peakon solutions as follows:

$$(6.1) \quad u^i = M\Phi(a \cdot x - Ct) = M\Phi\left(\sum_{j=1}^d a^j x^j - Ct\right), i = 1, 2, \dots, d,$$

where $\Phi(z) = \frac{\cosh(\frac{1}{2}-z)}{\sinh(\frac{1}{2})}$ is a periodic function with $z \in [0,1]$, M, C are constants and $a = (a^1, a^2, \dots, a^d)$ is a vector such that $C = \frac{\cosh(\frac{1}{2})}{\sinh(\frac{1}{2})} M \sum_{j=1}^d a^j$ and $|a| = 1$.

Firstly, we recall the definition for weak solution of the system (1.1).

Definition 6.1. [5] $u \in L^\infty(0, T; H^1(\mathbb{T}^d))$ is a weak solution of the system (1.1) with initial data $u_0 \in H^1(\mathbb{T}^d)$ if the following equation holds for all vector field $\phi(t, x)$ such that $\phi \in C^1([0, T]; C^\infty(\mathbb{T}^d))$ and $\Phi(T, x) = 0$ for all $x \in \mathbb{T}^d$,

$$(6.2) \quad \int_0^T \int_{\mathbb{T}^d} (u \cdot \phi_t + \nabla u : \nabla \phi_t) dx dt + \int_{\mathbb{T}^d} (u_0 \cdot \phi(0, x) + \nabla u_0 : \nabla \phi(0, x)) dx \\ + \int_0^T \int_{\mathbb{T}^d} T^a : \nabla \phi dx dt + \sum_{i,j,k=1}^d \int_0^T \int_{\mathbb{T}^d} u^j \partial_k u^i \partial_{jk} \phi^i dx dt = 0,$$

where $T^a = u \otimes u + \nabla u (\nabla u)^T - (\nabla u)^T (\nabla u) + \frac{1}{2}(|u|^2 + |\nabla u|^2) Id$ is the symmetric part of the tensor T .

From (6.1), we see that

$$(6.3) \quad \partial_k u^i = -a^k M \frac{\sinh(\frac{1}{2} - a \cdot x + Ct + l)}{\sinh(\frac{1}{2})}, \quad \text{if } a \cdot x - Ct \in (l, l+1),$$

$$(6.4) \quad \partial_k^2 u^i = -(a^k)^2 M \frac{\cosh(\frac{1}{2} - a \cdot x + Ct + l)}{\sinh(\frac{1}{2})}, \quad \text{if } a \cdot x - Ct \in (l, l+1).$$

Note that $\frac{\Phi}{2}$ is the Green function of $1 - \partial_{zz}$. By virtue of the above equality, we obtain that $(1 - \Delta)u^i = 2M\delta(a \cdot x - Ct)$ where δ is the Dirac function. Taking advantage of integration by parts, we have

$$(6.5) \quad \int_0^T \int_{\mathbb{T}^d} (u \cdot \phi_t + \nabla u : \nabla \phi_t) dx dt = 2 \sum_{i=1}^d \int_0^T \int_{\mathbb{T}^d} M \delta(a \cdot x - Ct) \cdot \phi_t^i dx dt,$$

$$(6.6) \quad \int_{\mathbb{T}^d} (u_0 \cdot \phi(0, x) + \nabla u_0 : \nabla \phi(0, x)) dx = 2 \sum_{i=1}^d \int_{\mathbb{T}^d} M \delta(a \cdot x) \cdot \phi^i(0, x) dx.$$

Using the fact that $\sinh^2 z = \cosh^2 z - 1$ and (6.3), we deduce that

$$(6.7) \quad [u \otimes u + (\nabla u)(\nabla u)^T - (\nabla u)^T (\nabla u)] : \nabla \phi = \sum_{i,k}^d [u^i u^k + \sum_j^d (\partial_j u^i \partial_j u^k - \partial_i u^j \partial_k u^j)] \partial_i \phi^k$$

$$= M^2 \sum_{i,k=1}^d (2\Phi^2(a \cdot x - Ct) - 1 - d\Phi^2(a \cdot x - Ct)a^i a^k - da^i a^k) \partial_i \phi^k.$$

Since Φ is periodic function, it follows that

$$(6.8) \quad \int_0^T \int_{\mathbb{T}^d} T^a : \nabla \phi dx dt = dM^2 \sum_{i,k=1}^d \int_0^T \int_{\mathbb{T}^d} \Phi^2(a \cdot x - Ct) (\delta_{ik} - a^i a^k) \partial_i \phi^k dx dt \\ + 2M^2 \sum_{i,k=1}^d \int_0^T \int_{\mathbb{T}^d} \Phi^2(a \cdot x - Ct) \partial_i \phi^k dx dt.$$

Taking advantage of integration by parts, we have

$$(6.9) \quad \sum_{i,k=1}^d \int_0^T \int_{\mathbb{T}^d} \Phi^2(a \cdot x - Ct) (\delta_{ik} - a^i a^k) \partial_i \phi^k dx dt \\ = - \sum_{i,k=1}^d \int_0^T \int_{\mathbb{T}^d} (a^i \delta_{ik} - (a^i)^2 a^k) \Phi(a \cdot x - Ct) \Phi'(a \cdot x - Ct) \phi^k dx dt = 0.$$

$$(6.10) \quad \sum_{k=1}^d \int_0^T \int_{\mathbb{T}^d} u^j \partial_k u^i \partial_j \phi^i dx dt = - \sum_{k=1}^d \int_0^T \int_{\mathbb{T}^d} (u^j \partial_k^2 u^i + \partial_k u^j \partial_k u^i) (\partial_j \phi^i) dx dt \\ - M^2 \int_0^T \int_{\mathbb{T}^d} [\Phi(a \cdot x - Ct) \Phi''(a \cdot x - Ct) \partial_j \phi^i + \Phi^2(a \cdot x - Ct) \partial_j \phi^i] dx dt.$$

Combining (6.8), (6.9) and (6.10), we see that

$$(6.11) \quad \int_0^T \int_{\mathbb{T}^d} T^a : \nabla \phi dx dt + \sum_{i,j,k=1}^d \int_0^T \int_{\mathbb{T}^d} u^j \partial_k u^i \partial_j \phi^i dx dt \\ = 2M^2 \sum_{i,k=1}^d \int_0^T \int_{\mathbb{T}^d} \Phi(a \cdot x - Ct) \delta(a \cdot x - Ct) \partial_i \phi^k dx dt.$$

Let $y = a \cdot x$ and $dx = dy dz$. By virtue of the properties of the Dirac function, we deduce that

$$(6.12) \quad \int_0^T \int_{\mathbb{T}^d} \delta(a \cdot x - Ct) \cdot \phi_t^i dx dt = \int_0^T \int_{\Sigma} \phi_t^i(t, Ct, z) dz,$$

$$(6.13) \quad \int_{\mathbb{T}^d} \delta(a \cdot x) \cdot \phi^i(0, x) dx = \int_{\Sigma} \phi^i(0, 0, z) dz,$$

$$(6.14) \quad \int_0^T \int_{\mathbb{T}^d} \Phi(a \cdot x - Ct) \delta(a \cdot x - Ct) \partial_i \phi^k dx dt = \frac{\cosh(\frac{1}{2})}{\sinh(\frac{1}{2})} \int_0^T \int_{\Sigma} \partial_k \phi^i(t, Ct, z) dz.$$

Note that $\frac{\cosh(\frac{1}{2})}{\sinh(\frac{1}{2})} M(\sum_j^d a^j) = C$. Since

$$\frac{d}{dt}(\phi^i(t, Ct, z)) = \phi_t^i(t, Ct, z) + C \partial_y \phi^i(t, Ct, z),$$

it follows that

$$(6.15) \quad M \int_0^T \int_{\Sigma} \phi_t^i(t, Ct, z) dz + M \int_{\Sigma} \phi^i(0, 0, z) dz + M^2 \sum_{k=1}^d \frac{\cosh(\frac{1}{2})}{\sinh(\frac{1}{2})} \int_0^T \int_{\Sigma} \partial_k \phi^i(t, Ct, z) dz = 0.$$

From (6.5) and (6.11), we deduce that

$$(6.16) \quad \int_0^T \int_{\mathbb{T}^d} (u \cdot \phi_t + \nabla u : \nabla \phi_t) dx dt + \int_{\mathbb{T}^d} (u_0 \cdot \phi(0, x) + \nabla u_0 : \nabla \phi(0, x)) dx \\ + \int_0^T \int_{\mathbb{T}^d} T^a : \nabla \phi dx dt + \sum_{i,j,k=1}^d \int_0^T \int_{\mathbb{T}^d} u^j \partial_k u^i \partial_{jk} \phi^i dx dt = 0,$$

which implies that (5.1) is the weak solution of the system (1.1).

Remark 6.2. *In the whole space, the Euler-Poincaré equations possess a special class of peakon solutions as follows:*

$$(6.17) \quad u^i = M e^{-|a \cdot x - Ct|} = M e^{-|\sum_{j=1}^d a^j x^j - Ct|}, i = 1, 2, \dots, d.$$

One can verify that the above solutions are weak solutions of the Euler-Poincaré equations in the sense of distributions by the above similar argument.

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