

Character varieties of even classical pretzel knots

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Abstract

For each even classical pretzel knot $P(2k_1 + 1, 2k_2 + 1, 2k_3)$, we determine the character variety of irreducible $\mathrm{SL}(2, \mathbb{C})$ -representations, and clarify the steps of computing its A-polynomial.

Keywords: $\mathrm{SL}(2, \mathbb{C})$ -representation; character variety; even classical pretzel knot; A-polynomial

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1 Introduction

For a knot $K \subset S^3$, let $E_K = S^3 - N(K)$, with $N(K)$ a tubular neighborhood. The $\mathrm{SL}(2, \mathbb{C})$ -*representation variety* of K is the set $\mathcal{R}(K)$ consisting of representations $\rho : \pi_1(E_K) \rightarrow \mathrm{SL}(2, \mathbb{C})$, and the *character variety* of K is $\mathcal{X}(K) = \{\chi(\rho) : \rho \in \mathcal{R}(K)\}$, where the *character* of ρ is the function $\chi(\rho) : \pi_1(E_K) \rightarrow \mathbb{C}$ sending $x \in \pi_1(E_K)$ to $\mathrm{tr}(\rho(x))$.

The reason for calling $\mathcal{R}(K)$ and $\mathcal{X}(K)$ varieties is that they can be defined by a finite set of polynomial equations [5]. Denote the subset of $\mathcal{R}(K)$ consisting of irreducible representations by $\mathcal{R}^{\mathrm{irr}}(K)$, then up to conjugacy, each $\rho \in \mathcal{R}^{\mathrm{irr}}(K)$ is determined by $\chi(\rho)$. We mainly focus on $\mathcal{R}^{\mathrm{irr}}(K)$ and

$$\mathcal{X}^{\mathrm{irr}}(K) := \{\chi(\rho) : \rho \in \mathcal{R}^{\mathrm{irr}}(K)\};$$

reducible representations and their characters are easy to understand.

As seen in the literature, there seems to be difficulty in dealing with the representation/character variety of K when $\pi_1(E_K)$ is generated by at least three generators. Although an effective algorithm for finding the character

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variety of any finitely presented group has been developed in [1], systematically computing for a family at a time is another story. In [3], the author computed the character varieties for classical odd pretzel knots, which form a 3-parameter family.

In this paper, we contribute one more piece, by determining the irreducible character variety for each even classical pretzel knot:

Theorem 1.1. *The irreducible character variety of the even pretzel knot $P(2k_1 + 1, 2k_1 + 1, 2k_3)$ can be embedded in*

$$\{(t, s_1, s_2, s_3, \tau) \in \mathbb{C}^5 : \tau^2 - t(\sigma_1 + 2)\tau + t^2(\sigma_2 + 4) = 4 + \sigma_3 + 2\sigma_2 - \sigma_1^2\},$$

and is the disjoint union of four parts: $\mathcal{X}^{\text{irr}}(K) = \mathcal{X}_0 \sqcup \mathcal{X}_1 \sqcup \mathcal{X}_2 \sqcup \mathcal{X}_3$, where

- $\mathcal{X}_0 = \mathcal{X}_{0,1} \sqcup \mathcal{X}_{0,2}$, where $\mathcal{X}_{0,1}$ consists of $(0, s_1, s_2, s_3, \tau)$ with

$$\tau \neq 0, \quad \gamma_1 = -\beta_1, \quad \gamma_2 = -\beta_2, \quad \beta_3 = 0,$$

and $\mathcal{X}_{0,2}$ consists of $(0, 2 \cos \theta_1, 2 \cos \theta_2, 2 \cos \theta_3, 0)$ with

$$\cos(2k_1 + 1)\theta_1 = \cos(2k_2 + 1)\theta_2 = \cos(2k_3\theta_3) \neq -1;$$

- $\mathcal{X}_1 = \mathcal{X}_{1,1} \sqcup \mathcal{X}_{1,2} \sqcup \mathcal{X}_{1,3}$, where $\mathcal{X}_{1,3}$ consists of $(\pm 2, s_1, s_2, 2, \tau)$ with $\gamma_1 = \beta_1, \gamma_2 = \beta_2$, and for $j = 1, 2$, $\mathcal{X}_{1,j}$ consists of (t, s_1, s_2, s_3, τ) with

$$\gamma_{j\pm} = \beta_{j\pm}, \quad t^2 = s_j + 2 = s_{j+} + s_{j-};$$

- \mathcal{X}_2 consists of

$$\left(t, 2 \cos \frac{(2h_1 + 1)\pi}{2k_1 + 1}, 2 \cos \frac{(2h_2 + 1)\pi}{2k_2 + 1}, 2 \cos \frac{h_3\pi}{k_3}, \tau\right)$$

with $h_1 \in \{0, \dots, k_1\}$, $h_2 \in \{0, \dots, k_2\}$, $h_3 \in \{0, \dots, k_3 - 1\}$, so \mathcal{X}_2 is made up of $(k_1 + 1)(k_2 + 1)k_3$ conics;

- \mathcal{X}_3 consists of $(t, s_1, s_2, s_3, t\lambda)$ with

$$\begin{aligned} \sigma_1 + 2 - 2\lambda &\neq 0, & t &\neq 0, \\ (\lambda - 2 - s_j)\gamma_j &= (\sigma_1 - s_j - \lambda)\beta_j, & j &= 1, 2, \\ (\sigma_1 + 2 - 2\lambda)\alpha_3 &= (s_3^2 - s_3\lambda + \sigma_1 - 2)\beta_3. \end{aligned}$$

The dimensions are: $\dim \mathcal{X}_0 = \dim \mathcal{X}_1 = 0$, $\dim \mathcal{X}_2 = \dim \mathcal{X}_3 = 1$.

The proof is given in Section 3. Based on this, in Section 4 we also present a method for computing the A-polynomial. The main line is parallel to that of [3], but now we simplify some key steps, and fix a few newly arising issues.

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2 Preliminary

For most part of this section, refer to [3] Section 2.

Let $\mathcal{M}(2, \mathbb{C})$ denote set of 2×2 matrices with entries in \mathbb{C} ; it is a 4-dimensional vector space over \mathbb{C} . Let $\mathcal{U}(2, \mathbb{C}) \subset \mathcal{M}(2, \mathbb{C})$ denote the subspace of upper-triangular matrices, and let $\mathcal{UT} = \mathcal{U}(2, \mathbb{C}) \cap \text{SL}(2, \mathbb{C})$. Let I denote the 2×2 identity matrix.

Given $t \in \mathbb{C}$ and $k \in \mathbb{Z}$, take a with $a + a^{-1} = t$ and put

$$\omega_k(t) = \begin{cases} (a^k - a^{-k})/(a - a^{-1}), & a \notin \{\pm 1\}, \\ ka^{k-1}, & a \in \{\pm 1\}; \end{cases}$$

note that the right-hand-side is unchanged when a is replaced by a^{-1} . It is easy to verify that for all $k \in \mathbb{Z}$,

$$\omega_k(t) + \omega_{-k}(t) = 0, \quad (1)$$

$$\omega_{k+1}(t) - t\omega_k(t) + \omega_{k-1}(t) = 0, \quad (2)$$

$$\omega_k(t)^2 - t\omega_k(t)\omega_{k-1}(t) + \omega_{k-1}(t)^2 = 1. \quad (3)$$

If $X \in \text{SL}(2, \mathbb{C})$ with $\text{tr}(X) = t$, then repeated applications of Cayley-Hamilton Theorem leads to

$$X^k = \omega_k(t)X - \omega_{k-1}(t)I \quad (4)$$

for all $k \in \mathbb{Z}$; in particular,

$$X^{-1} = tI - X. \quad (5)$$

Lemma 2.1. *For any $X, Y \in \text{SL}(2, \mathbb{C})$ with $\text{tr}(X) = t_1$, $\text{tr}(Y) = t_2$ and $\text{tr}(XY) = t_{12}$, one has*

$$XYX = t_{12}X - Y^{-1}, \quad (6)$$

$$XY + YX = (t_{12} - t_1t_2)I + t_2X + t_1Y. \quad (7)$$

We call a set $\{X_1, \dots, X_r\}$ *regular* if X_1, \dots, X_r do not have a common eigenvector. The following results are well-known; one can refer to [8] Section 2 and Section 5.

Lemma 2.2. *For any $X, Y \in \text{SL}(2, \mathbb{C}) - \{\pm I\}$, the following conditions are equivalent to each other:*

- (i) $\{X, Y\}$ is not regular;

- (ii) there exists $Z \in \mathrm{SL}(2, \mathbb{C})$ such that $ZXZ^{-1}, ZYZ^{-1} \in \mathcal{UT}$;
- (iii) I, X, Y, XY are linear dependent as elements of $\mathcal{M}(2, \mathbb{C})$.

Lemma 2.3. Given $t, t_{12}, t_{23}, t_{13}, t_{123} \in \mathbb{C}$, let

$$\begin{aligned}\nu_0 &= t^2(3 - t_{13} - t_{23} - t_{13}) + t_{12}^2 + t_{23}^2 + t_{13}^2 + t_{12}t_{23}t_{13} - 4, \\ \nu_1 &= t(t_{12} + t_{23} + t_{13}) - t^3.\end{aligned}$$

- (i) There exist $X_1, X_2, X_3 \in \mathrm{SL}(2, \mathbb{C})$ with $\mathrm{tr}(X_i) = t$, $\mathrm{tr}(X_i X_j) = t_{ij}$ for $1 \leq i < j \leq 3$ and $\mathrm{tr}(X_1 X_2 X_3) = t_{123}$ if and only if

$$t_{123}^2 - \nu_1 t_{123} + \nu_0 = 0. \tag{8}$$

- (ii) If (8) holds and $\{X_1, X_2, X_3\}$ is required to be regular, then the ordered triple (X_1, X_2, X_3) is unique up to simultaneous conjugacy.

3 Irreducible representations

In this section, we prove Theorem 1.1 through several lemmas and a corollary.

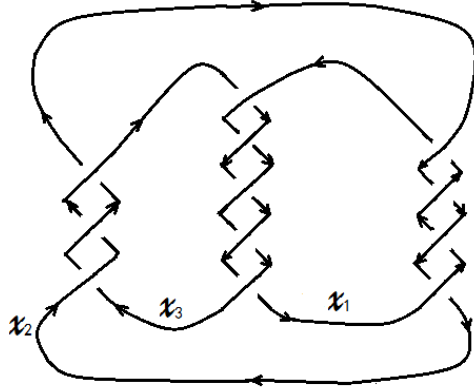


Figure 1: The pretzel knot $P(2k_1 + 1, 2k_2 + 1, 2k_3)$, with $k_1 = 1, k_2 = k_3 = 2$

Fix $K = P(2k_1 + 1, 2k_2 + 1, 2k_3)$ throughout this section. Let $x_1, x_2, x_3 \in \pi_1(E_K)$ be the elements represented by the arcs shown in Figure 1.

Similarly as in [3] Section 3.1, we can show: given $X_1, X_2, X_3 \in \mathrm{SL}(2, \mathbb{C})$, there exists a representation $\rho : \pi_1(E_K) \rightarrow \mathrm{SL}(2, \mathbb{C})$ with

$$X_1 = \rho(x_1), \quad X_2 = \rho(x_2), \quad X_3 = \rho(x_3^{-1})$$

if and only if

$$(X_2 X_3^{-1})^{k_1} X_2 (X_2 X_3^{-1})^{-k_1} = (X_3 X_1^{-1})^{k_2+1} X_1 (X_3 X_1^{-1})^{-(k_2+1)}, \quad (9)$$

$$(X_3 X_1^{-1})^{k_2} X_3 (X_3 X_1^{-1})^{-k_2} = (X_1 X_2^{-1})^{k_3} X_1^{-1} (X_1 X_2^{-1})^{-k_3}, \quad (10)$$

$$(X_1 X_2^{-1})^{k_3} X_2 (X_1 X_2^{-1})^{-k_3} = (X_2 X_3^{-1})^{k_1+1} X_3^{-1} (X_2 X_3^{-1})^{-(k_1+1)}. \quad (11)$$

Notation 3.1. To simplify the writing, by X_{j+} we mean X_{j+1} for $j \in \{1, 2\}$ and X_1 for $j = 3$; by X_{j-} we mean X_{j-1} for $j \in \{2, 3\}$ and X_3 for $j = 1$. Similarly for other situations.

Put

$$\begin{aligned} Y_j &= X_{j+} X_{j-}^{-1}, & j &= 1, 2, 3, \\ A_j &= Y_j^{k_j} X_{j+} Y_j^{-k_j} X_{j-}, & j &= 1, 2, \\ A_3 &= Y_3^{k_3} X_1^{-1} Y_3^{-k_3} X_1 = Y_3^{k_3} X_2^{-1} Y_3^{-k_3} X_2. \end{aligned}$$

Then (9)–(11) are equivalent to

$$A_1 = A_2 = A_3. \quad (12)$$

We only consider irreducible representations, so $\{X_1, X_2, X_3\}$ is assumed to be regular.

Suppose

$$\begin{aligned} \operatorname{tr}(X_1 X_2 X_3) &= r; & \operatorname{tr}(X_1) &= \operatorname{tr}(X_2) = \operatorname{tr}(X_3) = t = u + u^{-1}, \\ \operatorname{tr}(Y_j) &= s_j = v_j + v_j^{-1}, & j &= 1, 2, 3. \end{aligned}$$

Clearly,

$$\operatorname{tr}(X_{j+} X_{j-}) = \operatorname{tr}(X_{j+} (tI - X_{j-}^{-1})) = t^2 - s_j.$$

Let

$$\begin{aligned} \sigma_1 &= s_1 + s_2 + s_3, & \sigma_2 &= s_1 s_2 + s_2 s_3 + s_3 s_1, & \sigma_3 &= s_1 s_2 s_3, \\ \tau &= t^3 + t - r, \\ \delta &= 4 + \sigma_3 + 2\sigma_2 - \sigma_1^2, \\ \kappa &= \tau^2 - t(\sigma_1 + 2)\tau + t^2(\sigma_2 + 4). \end{aligned}$$

Then (8) can be rewritten as

$$\kappa = \delta. \quad (13)$$

For each j , denote

$$\alpha_j = \omega_{k_j-1}(s_j), \quad \beta_j = \omega_{k_j}(s_j), \quad \gamma_j = \omega_{k_j+1}(s_j).$$

Using (4) and (6), we compute

$$\begin{aligned} Y_j^{k_j} X_{j+} &= (\beta_j Y_j - \alpha_j I) X_{j+} = \beta_j X_{j+} X_{j-}^{-1} X_{j+} - \alpha_j X_{j+} \\ &= (s_j \beta_j - \alpha_j) X_{j+} - \beta_j X_{j-} \stackrel{(2)}{=} \gamma_j X_{j+} - \beta_j X_{j-}, \\ Y_j^{-k_j} X_{j-} &= (-\beta_j Y_j + \gamma_j I) X_{j-} = \gamma_j X_{j-} - \beta_j X_{j+}, \end{aligned}$$

so for $j = 1, 2$,

$$\begin{aligned} A_j &= (\gamma_j X_{j+} - \beta_j X_{j-})(\gamma_j X_{j-} - \beta_j X_{j+}) & (14) \\ &= \gamma_j^2 X_{j+} X_{j-} - \gamma_j \beta_j (X_{j+}^2 + X_{j-}^2) + \beta_j^2 X_{j-} X_{j+}, \\ &= (\gamma_j^2 - \beta_j^2) X_{j+} X_{j-} + t(\beta_j - \gamma_j) \beta_j (X_{j+} + X_{j-}) + (2\gamma_j - s_j \beta_j) \beta_j I; & (15) \end{aligned}$$

in the last line, we have used (4) and (7).

Since

$$\begin{aligned} Y_3^{k_3} X_2^{-1} &= (\beta_3 Y_3 - \alpha_3 I) X_2^{-1} = t\beta_3 X_1 X_2^{-1} - \beta_3 X_1 - \alpha_3 X_2^{-1}, \\ Y_3^{-k_3} X_2 &= (\omega_{-k_3}(s_3) Y_3 - \omega_{-k_3-1}(s_3) I) X_2 \stackrel{(1)}{=} (\gamma_3 I - \beta_3 Y_3) X_2 = \gamma_3 X_2 - \beta_3 X_1, \end{aligned}$$

we have

$$\begin{aligned} A_3 &= (t\beta_3 X_1 X_2^{-1} - \beta_3 X_1 - \alpha_3 X_2^{-1})(\gamma_3 X_2 - \beta_3 X_1) \\ &= (\alpha_3 - \gamma_3) \beta_3 X_1 X_2 + t(\beta_3 - \alpha_3) \beta_3 (X_1 + X_2) + (\alpha_3^2 - \beta_3^2) I, & (16) \end{aligned}$$

where in the last line, (4)–(7) are applied.

The case when $t = 0$ turns out to require a special treatment. To this end, we cite the result of [2] Section 3, stated in a different form:

Lemma 3.2. *Suppose $t = 0$ and (13) is satisfied. Then (12) holds if and only one of the following cases occurs:*

- $\gamma_j = \beta_j$, $j = 1, 2$ and $v_3^{2k_3} = -1$;
- $\gamma_j = -\beta_j$, $j = 1, 2$ and $\beta_3 = 0$;
- $\delta = 0$, and there exists $\theta_j \in \mathbb{R}$ with $s_j = 2 \cos \theta_j$, $j = 1, 2, 3$ and $\cos(2k_1 + 1)\theta_1 = \cos(2k_2 + 1)\theta_2 = \cos(2k_3\theta_3) \neq \pm 1$.

Lemma 3.3. (a) For $j = 1, 2$, $A_j = I$ if and only if $\gamma_j = \beta_j$ or $X_{j+} = X_{j-}^{-1}$.
(b) If $\{X_1, X_2\}$ is regular, then $A_3 = I$ if and only if $\beta_3 = 0$.

Proof. (a) For $j = 1, 2$, from (14) we see that $A_j = I$ if and only if

$$\gamma_j X_{j+} - \beta_j X_{j-} = (\gamma_j X_{j-} - \beta_j X_{j+})^{-1} \stackrel{(5)}{=} t(\gamma_j - \beta_j)I - \gamma_j X_{j-} + \beta_j X_{j+},$$

which is equivalent to $(\gamma_j - \beta_j)(X_{j+} - X_{j-}^{-1}) = 0$.

(b) By (16) and the assumption on (X_1, X_2) , $A_3 = I$ if and only if

$$(\alpha_3 - \gamma_3)\beta_3 = t(\alpha_3 - \beta_3)\beta_3 = \alpha_3^2 - \beta_3^2 - 1 = 0.$$

Clearly this holds when $\beta_3 = 0$. Conversely, if $\beta_3 \neq 0$, then $\alpha_3 = \gamma_3$ so that $(v_3 - v_3^{-1})(v_3^{k_3} + v_3^{-k_3}) = 0$, and the last equality implies $s_3\alpha_3 = 2\beta_3$ so that $(v_3 - v_3^{-1})(v_3^{k_3-1} + v_3^{1-k_3}) = 0$; since clearly $v_3 \neq \pm 1$, this is a contradiction. \square

Corollary 3.4. If $t \neq 0$ and $\gamma_j = \beta_j$ for some $j \in \{1, 2\}$, then (12) holds if and only one of the following cases occurs:

- (i) $\gamma_1 = \beta_1, \gamma_2 \neq \beta_2, X_3 = X_1^{-1}, \beta_3 = 0$;
- (ii) $\gamma_1 \neq \beta_1, X_2 = X_3^{-1}, \gamma_2 = \beta_2, \beta_3 = 0$;
- (iii) $\gamma_1 = \beta_1, \gamma_2 = \beta_2, \{X_1, X_2\}$ is regular, $\beta_3 = 0$;
- (iv) $\gamma_1 = \beta_1, \gamma_2 = \beta_2, s_3 = 2, t \in \{\pm 2\}$ and $X_1 X_2 = X_2 X_1$.

Proof. Only case (iv) needs to be explained: if $\{X_1, X_2\}$ is not regular, then up to conjugacy we may assume $X_1 = \begin{pmatrix} u & 1 \\ 0 & u^{-1} \end{pmatrix}$, $X_2 = \begin{pmatrix} u^\epsilon & x \\ 0 & u^{-\epsilon} \end{pmatrix}$ with $\epsilon \in \{\pm 1\}$. From (16) we can deduce $u \in \{\pm 1\}$, $s_3 = 2$, and then clearly $X_1 X_2 = X_2 X_1$. \square

Lemma 3.5. Suppose $t \neq 0$, $\gamma_j \neq \beta_j$ for $j = 1, 2$, $\beta_3 \neq 0$, and (13) is satisfied. Let $\lambda = \tau/t$. Then (12) holds if and only

$$(\lambda - 2 - s_j)\gamma_j = (\sigma_1 - s_j - \lambda)\beta_j, \quad j = 1, 2, \quad (17)$$

$$(\sigma_1 + 2 - 2\lambda)\alpha_3 = (s_3^2 - s_3\lambda + \sigma_1 - 2)\beta_3. \quad (18)$$

Proof. By (15), for $j = 1, 2$,

$$\begin{aligned} \text{tr}(A_j) &= (\gamma_j^2 - \beta_j^2)(t^2 - s_j) + t(\beta_j - \gamma_j)\beta_j \cdot 2t + (2\gamma_j - s_j\beta_j)\beta_j \cdot 2 \\ &= 2 - (s_j + 2 - t^2)(\gamma_j - \beta_j)^2, \end{aligned} \quad (19)$$

$$\text{tr}(A_j X_{j\pm}^{-1}) = (\gamma_j^2 - \beta_j^2)t + t(\beta_j - \gamma_j)\beta_j(s_j + 2) + (2\gamma_j - s_j\beta_j)\beta_j t = t, \quad (20)$$

and since $\text{tr}(X_{j+}X_j - X_j^{-1}) = \text{tr}(X_{j+}X_j - (tI - X_j)) = \tau - t(1 + s_j)$, we have

$$\begin{aligned}\text{tr}(A_j X_j^{-1}) &= (\gamma_j^2 - \beta_j^2)(\tau - t(1 + s_j)) + t(\beta_j - \gamma_j)(\sigma_1 - s_j) + (2\gamma_j - s_j\beta_j)\beta_j t \\ &= t(\gamma_j - \beta_j)((\lambda - 2 - s_j)\gamma_j - (\sigma_1 - s_j - \lambda)\beta_j) + t.\end{aligned}\quad (21)$$

By (16),

$$\begin{aligned}\text{tr}(A_3) &= (\alpha_3 - \gamma_3)\beta_3(t^2 - s_3) + t(\beta_3 - \alpha_3)\beta_3 \cdot 2t + (\alpha_3^2 - \beta_3^2) \cdot 2 \\ &= 2 + (s_3 - 2)(s_3 + 2 - t^2)\beta_3^2,\end{aligned}\quad (22)$$

$$\text{tr}(A_3 X_{3\pm}^{-1}) = (\alpha_3 - \gamma_3)\beta_3 t + t(\beta_3 - \alpha_3)\beta_3(2 + s_3) + (\alpha_3^2 - \beta_3^2)t = t, \quad (23)$$

$$\begin{aligned}\text{tr}(A_3 X_3^{-1}) &= (\alpha_3 - \gamma_3)\beta_3 t(\lambda - 1 - s_3) + t(\beta_3 - \alpha_3)\beta_3(\sigma_1 - s_3) + (\alpha_3^2 - \beta_3^2)t \\ &= t((s_3^2 - s_3\lambda + \sigma_1 - 2)\beta_3 - (\sigma_1 + 2 - 2\lambda)\alpha_3)\beta_3 + t.\end{aligned}\quad (24)$$

If (12) holds, then

$$\text{tr}(A_1) = \text{tr}(A_2) = \text{tr}(A_3), \quad (25)$$

$$\text{tr}(A_1 X_j^{-1}) = \text{tr}(A_2 X_j^{-1}) = \text{tr}(A_3 X_j^{-1}), \quad j = 1, 2, 3. \quad (26)$$

Due to the assumptions $t \neq 0$, $\gamma_1 \neq \beta_1$, $\gamma_2 \neq \beta_2$, and $\beta_3 \neq 0$, from (20), (21), (23), (24) we see that (26) is equivalent to (17) and (18).

Now suppose (17) and (18) hold (so that (26) is satisfied).

For $j = 1, 2$,

$$(\sigma_1 + 2 - 2\lambda)\gamma_j = (\sigma_1 - s_j - \lambda)(\gamma_j - \beta_j), \quad (27)$$

$$(\sigma_1 + 2 - 2\lambda)\beta_j = (\lambda - 2 - s_j)(\gamma_j - \beta_j), \quad (28)$$

hence

$$\begin{aligned}(\sigma_1 + 2 - 2\lambda)^2 &\stackrel{(3)}{=} (\sigma_1 + 2 - 2\lambda)^2(\gamma_j^2 - s_j\gamma_j\beta_j + \beta_j^2) \\ &= ((\sigma_1 - s_j - \lambda)^2 - s_j(\sigma_1 - s_j - \lambda)(\lambda - 2 - s_j) + (\lambda - 2 - s_j)^2) \cdot (\gamma_j - \beta_j)^2 \\ &= (t^{-2}\kappa(s_j + 2) - \delta)(\gamma_j - \beta_j)^2,\end{aligned}$$

where in the last line we use $s_j^3 - \sigma_1 s_j^2 + \sigma_2 s_j - \sigma_3 = 0$. By (13),

$$(\sigma_1 + 2 - 2\lambda)^2 = t^{-2}\kappa(s_j + 2 - t^2)(\gamma_j - \beta_j)^2.$$

If $\sigma_1 + 2 - 2\lambda = 0$, then (27) and (28) imply $s_1 = s_2 = \lambda - 2$, $s_3 = 2$, so that $\delta = 0$ and $\kappa = 4t^2$, contradicting (13) and the assumption $t \neq 0$. Hence $\sigma_1 + 2 - 2\lambda \neq 0$, so that

$$s_j \neq t^2 - 2, \quad j = 1, 2. \quad (29)$$

Consequently,

$$\operatorname{tr}(A_j) - 2 = -(s_j + 2 - t^2)(\gamma_j - \beta_j)^2 = -t^2\kappa^{-1}(\sigma_1 + 2 - 2\lambda)^2.$$

Moreover,

$$\begin{aligned} 1 &\stackrel{(3)}{=} \alpha_3^2 - s_3\alpha_3\beta_3 + \beta_3^2 \stackrel{(18)}{=} \beta_3^2 \left(\left(\frac{\sigma_1 + 2 - 2\lambda}{\sigma_1 + 2 - 2\lambda} \right)^2 - s_3 \frac{\sigma_1 + 2 - 2\lambda}{\sigma_1 + 2 - 2\lambda} + 1 \right) \\ &= \left(\frac{\beta_3}{\sigma_1 + 2 - 2\lambda} \right)^2 \frac{\kappa}{t^2} (4 - 2t^2 + t^2s_3 - s_3^3), \end{aligned}$$

so that $\operatorname{tr}(A_3) - 2 = -t^2\kappa^{-1}(\sigma_1 + 2 - 2\lambda)^2$. This establishes (25), which together with (22), (19), (29) implies

$$s_3 \notin \{2, t^2 - 2\}. \quad (30)$$

The proof will be complete once $I, X_1^{-1}, X_2^{-1}, X_3^{-1}$ are shown to form a basis of $\mathcal{M}(2, \mathbb{C})$; then (25), (26) will imply that $\operatorname{tr}(A_1Z) = \operatorname{tr}(A_2Z) = \operatorname{tr}(A_3Z)$ for all $Z \in \mathcal{M}(2, \mathbb{C})$, forcing $A_1 = A_2 = A_3$.

Assume on the contrary that $I, X_1^{-1}, X_2^{-1}, X_3^{-1}$ are linearly dependent, so are I, X_1, X_2, X_3 . Note that I, X_1, X_2 are linearly independent: otherwise $X_1 = X_2$ or $X_1 = X_2^{-1}$, which would respectively imply $s_3 = 2$ or $s_3 = t^2 - 2$, either contradicting (30). So we have $X_3 = aI + bX_1 + cX_2$ for some a, b, c . Then $t = \operatorname{tr}(X_3) = 2a + t(b + c)$, implying

$$\bar{X}_3 = b\bar{X}_1 + c\bar{X}_2, \quad \text{with} \quad \bar{X}_j = X_j - \frac{t}{2}I.$$

The following can be computed using (7):

$$\begin{aligned} \bar{X}_1\bar{X}_2 + \bar{X}_2\bar{X}_1 &= pI, \quad \text{with} \quad p = \frac{1}{2}t^2 - s_3, \\ \bar{X}_j^2 &= dI, \quad \text{with} \quad d = \frac{1}{4}t^2 - 1. \end{aligned}$$

Consequently, $\operatorname{tr}(\bar{X}_1\bar{X}_2) = p$, $\operatorname{tr}(\bar{X}_1^2) = \operatorname{tr}(\bar{X}_2^2) = 2d$, and $\operatorname{tr}(\bar{X}_1^2\bar{X}_2) = \operatorname{tr}(\bar{X}_1\bar{X}_2^2) = 0$. We can rewrite (15) as

$$\begin{aligned} A_1 &= (\gamma_1^2 - \beta_1^2)\bar{X}_2\bar{X}_3 + \frac{t}{2}(\gamma_1 - \beta_1)^2(\bar{X}_2 + \bar{X}_3) + f_1I \\ &= g_1\bar{X}_1\bar{X}_2 + \frac{t}{2}(\gamma_1 - \beta_1)^2(b\bar{X}_1 + (c+1)\bar{X}_2) + h_1I, \\ A_2 &= (\gamma_2^2 - \beta_2^2)\bar{X}_3\bar{X}_1 + \frac{t}{2}(\gamma_2 - \beta_2)^2(\bar{X}_3 + \bar{X}_1) + f_2I \\ &= g_2\bar{X}_1\bar{X}_2 + \frac{t}{2}(\gamma_2 - \beta_2)^2((b+1)\bar{X}_1 + c\bar{X}_2) + h_2I, \end{aligned}$$

and re-write (16) as

$$A_3 = g_3 \overline{X}_1 \overline{X}_2 + \frac{t}{2} (\beta_3^2 - \gamma_3 \beta_3) (\overline{X}_1 + \overline{X}_2) + g_3 I,$$

where the f_i 's, g_i 's, h_i 's are coefficients that are irrelevant.

By (25), (26), $\text{tr}(A_1 \overline{X}_\ell) = \text{tr}(A_2 \overline{X}_\ell) = \text{tr}(A_3 \overline{X}_\ell)$, $\ell = 1, 2$. Hence

$$\begin{aligned} (\gamma_1 - \beta_1)^2 (2db + p(c + 1)) &= (\gamma_2 - \beta_2)^2 (2d(b + 1) + pc) = (\beta_3^2 - \gamma_3 \beta_3) (2d + p), \\ (\gamma_1 - \beta_1)^2 (pb + 2d(c + 1)) &= (\gamma_2 - \beta_2)^2 (p(b + 1) + 2dc) = (\beta_3^2 - \gamma_3 \beta_3) (2d + p), \end{aligned}$$

which lead to

$$\begin{aligned} 2db + p(c + 1) &= pb + 2d(c + 1), \\ 2d(b + 1) + pc &= p(b + 1) + 2dc. \end{aligned}$$

These force $p = 2d$, so that $s_3 = 2$. But this contradicts (30). \square

4 On the A-polynomial

For background on A-polynomial, see [4, 10].

For a knot $K \subset S^3$, choose a meridian-longitude pair $(\mathfrak{m}, \mathfrak{l})$ of K . Let

$$\mathcal{R}_U(K) = \{\rho \in \mathcal{R}(K) : \rho(\mathfrak{m}), \rho(\mathfrak{l}) \in \mathcal{UT}\},$$

and define $\xi = (\xi_1, \xi_2) : \mathcal{R}_U(K) \rightarrow \mathbb{C}^2$ by setting $\xi_1(\rho)$ (resp. $\xi_2(\rho)$) to be the upper-left entry of $\rho(\mathfrak{m})$ (resp. $\rho(\mathfrak{l})$). As a known fact, each component of the Zariski closure \mathcal{V} of $\text{Im}(\xi)$ has dimension 0 or 1. The defining polynomial of the 1-dimensional part of \mathcal{V} is called the *A-polynomial* A_K . The importance of this two-variable polynomial has at least two aspects. As shown in [4], “boundary slopes are boundary slopes”, meaning that the slope of each side of the Newton polygon of A_K equals the boundary slope of an incompressible surface in E_K . The *AJ conjecture*, formulated in [6], asserts a close relation between the recurrence polynomial of the colored Jones polynomials and A_K (see also [7, 9]); it is till now still widely open and attracts much attention.

Remark 4.1. The A-polynomial is notoriously difficult to compute. One reason is that $\rho(\mathfrak{l})$ is usually quite complicated, and no general law has been found to simplify the expression.

For the knot at hand, $K = P(2k_1 + 1, 2k_2 + 1, 2k_3)$, take $\mathfrak{m} = x_1$. The corresponding longitude can be found to be

$$\mathfrak{l} = (x_1 x_2^{-1})^{-k_3} (x_3^{-1} x_1^{-1})^{k_2} (x_2 x_3)^{-k_1 - 1} (x_1 x_2^{-1})^{k_3} (x_2 x_3)^{-k_1} (x_3^{-1} x_1^{-1})^{k_2 + 1}.$$

Let ρ be a representation of $\pi_1(E_K)$ as in Section 3. Then

$$\rho(\mathfrak{l}) = Y_3^{-k_3} Y_2^{k_2} Y_1^{-k_1-1} Y_3^{k_3} Y_1^{-k_1} Y_2^{k_2+1}.$$

Suppose $\rho \in \mathcal{R}_U(K)$, and let u, w denote the upper-left entries of $X_1, \rho(\mathfrak{l})$, respectively, so that $\text{tr}(\rho(\mathfrak{l})) = w + w^{-1}$. It is easy to see that

$$\rho(\mathfrak{l}) = \frac{w - w^{-1}}{u - u^{-1}} X_1 + \frac{uw^{-1} - wu^{-1}}{u - u^{-1}} I. \quad (31)$$

Let $B_1 = Y_3^{-k_3-1} Y_2^{k_2}$, $B_2 = Y_1^{-k_1-1} Y_3^{k_3}$, $B_3 = Y_1^{-k_1} Y_2^{k_2+1}$, then $B_3 = B_2 B_1$ and $B_1 B_2 B_3 = Y_3^{-1} \rho(\mathfrak{l})$. Hence

$$\begin{aligned} \text{tr}(B_3) &= \text{tr}(B_1 B_2) = \text{tr}(Y_3^{-1} \rho(\mathfrak{l}) B_3^{-1}) \\ &\stackrel{(31)}{=} \frac{w - w^{-1}}{u - u^{-1}} \text{tr}(X_2 B_3^{-1}) + \frac{uw^{-1} - wu^{-1}}{u - u^{-1}} \text{tr}(X_2 X_1^{-1} B_3^{-1}) \\ &= \frac{w - w^{-1}}{u - u^{-1}} \text{tr}(B_3 X_2^{-1}) + \frac{uw^{-1} - wu^{-1}}{u - u^{-1}} \text{tr}(B_3); \end{aligned} \quad (32)$$

in the last line we have used (9) to deduce

$$B_3 X_1 = X_2 B_3, \quad (33)$$

so that $\text{tr}(X_2 X_1^{-1} B_3^{-1}) = \text{tr}(B_3^{-1}) = \text{tr}(B_3)$. Rewrite (32) as

$$(w + 1) \text{tr}(B_3 X_2^{-1}) = (u + u^{-1} w) \text{tr}(B_3). \quad (34)$$

Similarly as in [3], we focus on the ‘‘hard’’ part \overline{A}_K of A_K which is contributed by \mathcal{X}_3 . Since

$$B_3 = \beta_1 \beta_2 Y_1 - \beta_1 \gamma_2 Y_3^{-1} - \gamma_1 \beta_2 I + \gamma_1 \gamma_2 Y_2,$$

we have

$$\begin{aligned}
\mathrm{tr}(B_3 X_2^{-1}) &= t(\beta_1 \beta_2 - \beta_1 \gamma_2 - \gamma_1 \beta_2 + \gamma_1 \gamma_2 (s_1 + s_2 + 1 - \lambda)) \\
&= t((\sigma_1 - \lambda - s_3) \gamma_1 \gamma_2 + (\gamma_1 - \beta_1)(\gamma_2 - \beta_2)) \\
&\stackrel{(17)}{=} \frac{t\beta_1\beta_2}{(\lambda - 2 - s_1)(\lambda - 2 - s_2)} \left(\prod_{j=1}^3 (\sigma_1 - \lambda - s_j) + (\sigma_1 + 2 - 2\lambda)^2 \right) \\
&\stackrel{(13)}{=} \frac{t\beta_1\beta_2}{(\lambda - 2 - s_1)(\lambda - 2 - s_2)} (\sigma_1 + 2 - \lambda - t^2) \kappa, \\
\mathrm{tr}(B_3 X_1) &= t(\beta_1 \beta_2 (\lambda - 1 - s_3) - \beta_1 \gamma_2 - \gamma_1 \beta_2 + \gamma_1 \gamma_2) \\
&= t((\lambda - 2 - s_3) \beta_1 \beta_2 + (\gamma_1 - \beta_1)(\gamma_2 - \beta_2)) \\
&\stackrel{(17)}{=} \frac{t\beta_1\beta_2}{(\lambda - 2 - s_1)(\lambda - 2 - s_2)} \left(\prod_{j=1}^3 (\lambda - 2 - s_j) + (\sigma_1 + 2 - 2\lambda)^2 \right) \\
&\stackrel{(13)}{=} \frac{t\beta_1\beta_2}{(\lambda - 2 - s_1)(\lambda - 2 - s_2)} (\lambda - t^2) \kappa,
\end{aligned}$$

and then

$$\begin{aligned}
\mathrm{tr}(B_3) &\stackrel{(5)}{=} t^{-1}(\mathrm{tr}(B_3 X_1^{-1}) + \mathrm{tr}(B_3 X_1)) \\
&\stackrel{(33)}{=} t^{-1}(\mathrm{tr}(B_3 X_2^{-1}) + \mathrm{tr}(B_3 X_1)) \\
&= \frac{\beta_1 \beta_2}{(\lambda - 2 - s_1)(\lambda - 2 - s_2)} (\sigma_1 + 2 - 2t^2) \kappa.
\end{aligned}$$

Thus (34) becomes

$$(w + 1)t(\sigma_1 + 2 - \lambda - t^2) = (u + u^{-1}w)(\sigma_1 + 2 - 2t^2).$$

Then \overline{A}_K , as a polynomial in u, w , can be obtained by computing the multi-variable resultant of the following (remembering $t = u + u^{-1}$):

$$(\lambda - 2 - s_j) \gamma_j = (\sigma_1 - s_j - \lambda) \beta_j, \quad j = 1, 2, \quad (35)$$

$$(\sigma_1 + 2 - 2\lambda) \alpha_3 = (s_3^2 - s_3 \lambda + \sigma_1 - 2) \beta_3, \quad (36)$$

$$t^2(\lambda^2 - (\sigma_1 + 2)\lambda + \sigma_2 + 4) = 4 + \sigma_3 + 2\sigma_2 - \sigma_1^2, \quad (37)$$

$$(w + 1)t(\sigma_1 + 2 - \lambda - t^2) = (u + u^{-1}w)(\sigma_1 + 2 - 2t^2). \quad (38)$$

References

- [1] C. Ashley, J.-P. Burelle, S. Lawton, *Rank 1 character varieties of finitely presented groups*. Geometriae Dedicata 192 (2018), no. 1, 1–19.
<https://link.springer.com/article/10.1007/s10711-017-0281-6>

- [2] H.-M. Chen, *Trace-free $SL(2, \mathbb{C})$ -representations of Montesinos links*. J. Knot Theor. Ramif. 27 (2018), no. 8, 1850050 (10 pages).
<https://www.worldscientific.com/doi/abs/10.1142/S0218216518500505>
- [3] H.-M. Chen, *Character varieties of odd classical pretzel knots*. Int. J. Math. 29 (2018), no. 9, 1850060 (15 pages).
<https://www.worldscientific.com/doi/abs/10.1142/S0129167X1850060X>
- [4] D. Cooper, M. Culler, H. Gillet, D. D. Long, P. B. Shalen, *Plane curves associated to character varieties of 3-manifolds*. Invent. Math. 118 (1994), 47–84.
<https://link.springer.com/article/10.1007%2FBF01231526?LI=true>
- [5] M. Culler, P.B. Shalen, *Varieties of group representations and splittings of 3-manifolds*. Ann. Math. 117 (1983), no. 1, 109–146.
https://www.jstor.org/stable/2006973?seq=1#metadata_info_tab_contents
- [6] S. Garoufalidis, *On the characteristic and deformation varieties of a knot*. Proceedings of the Casson Fest, Geom. Topol. Monogr., vol. 7, Geom. Topol. Publ., Coventry, 2004, 291–309 (electronic).
<https://msp.org/gtm/2004/07/gtm-2004-07-012p.pdf>
- [7] R. Gelca, *On the relations between the A -polynomial and the Jones polynomial*. Proc. Amer. Math. Soc. 130 (2002), no. 4, 1235–1241.
<https://www.ams.org/journals/proc/2002-130-04/S0002-9939-01-06157-3/S0002-9939-01-06157-3.pdf>
- [8] W.M. Goldman, *Trace coordinates on Fricke spaces of some simple hyperbolic surfaces*. arXiv:0901.1404. <https://arxiv.org/abs/0901.1404>
- [9] T.T.Q. Lê, *The colored Jones polynomial and the A -polynomial of knots*. Adv. Math. 207 (2006), no. 2, 782–804.
<https://www.sciencedirect.com/science/article/pii/S0001870806000156>
- [10] D.D. Long, A.W. Reid, *Integral points on character variety*. Math. Ann. 325 (2003), 299–321.
<https://link.springer.com/article/10.1007%2Fs00208-002-0380-y?LI=true>