A mathematical theory of imperfect communication: Energy efficiency considerations in multi-level coding

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Abstract

Is perfect error correction always worth the trouble? A framework is presented for the analysis of error detection and correction in multi-level systems of communication that takes into account degrees of freedom attended and ignored by different levels of analysis. It follows from this analysis that for a multi-level coding system, skipped or incomplete error correction at many levels can save energy and provide equally good results to perfect correction. This has relevance to approximate computing, and to questions of the robustness of machine learning applications. The finding also has significance in natural systems, such as neuronal signaling, vision, and molecular genetics, which are readily characterized as relying on multiple layers of inadequate error correction.

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1 Introduction

In a multi-level system of communication or computation, perfect error correction may be not be an efficient use of energy. This has implications for saving energy in computing, where perfect error correction has long been the norm. It also has consequences for how we understand communication in natural systems, that are expected to optimize for energy efficiency in the long run.

Many, if not most, forms of communication can be construed as using multiple levels of encoding and decoding. A note is typed into an email, encoded into a series of bytes, organized into packets of bits, and sent as current fluctuations to some other computer. On receipt the fluctuations become bits, then bytes, then letters on the recipient's screen. Not just email, but all computer communication is organized in this fashion, down to the intra-machine variety, and much software, too. Neural networks are readily understood as operating on a series of levels, with many inputs feeding a layer of nodes, whose outputs feed the next layer, and so on. It is equally common to find such arrangments in natural systems, since after all, software neural networks were modeled on the natural hardware. For example, both phonological (Liberman and Prince, 1977; Goldsmith, 1979) and visual (Marr, 1982) processing have long been understood to be arranged in tiers of functionally similar syntactic operations feeding the processors of the next higher tier.

However, since Shannon's establishment of communication theory (Shannon, 1948), there has been little consideration of ensembles of levels, together.¹ Obviously, many systems are composed of multiple levels of analysis, but information theorists typically take advantage of the independence of different levels by considering them in isolation. Lower levels of communication are considered features of the communication channel established between a sender and a receiver, only of concern to the extent they are a source of noise or uncertainty, but perhaps no further. Higher levels belong to a different analysis. This assumption of independence has been fruitful, but considering multiple levels together has value because in many cases their energy derives from the same source. They may run off a single battery, or a single stomach. Energy saved at one level may increase the energy available for processing at another. It behooves engineers of such systems to consider the ensemble in order to find opportunities for optimization. Students of natural systems will find the results of interest because they show how communication between individual organisms and individual cells should look in a world of energy constraints.

2 Development

We proceed by examining the flow of information through a complex network of independently functioning agents, or "nodes," paying particular attention to the implications of the Markov property to an individual node. Using the framework introduced there, we develop a method to compare the cost of error correction at different levels of analysis, and derive an expression to minimize the total energy used by an ensemble of levels. It is possible to draw conclusions about optimizing energy use under several different scenarios, despite not having exact knowledge of the parameters. The goal is to demonstrate that a wide range

¹There has been extensive examination of concatenated codes, where two encodings are combined into one (Dumer, 1998). While related to the current topic, these investigations are generally concerned with the combination of two distinct coding techniques to create a single encoding with enhanced properties. This is distinct from considering two different processors serially executing different encoding or decoding techniques.

of plausible assumptions about those parameters leads to similar results with significant consequences.

2.1 Conditionality and the Markov property

Consider some set of symbols $\mathscr{R} = \{r_1, r_2, r_3...\}$ each of which may be translated into a group of one or more symbols from $\mathscr{Q} = \{q_1, q_2, q_3...\}$ for transmission to some receiver. On receipt, the original members of \mathscr{R} are recreated from measurements of \mathscr{Q} via a process that reverses what came before. If *R* is a message made of $r \in \mathscr{R}$ and *Q* a message of $q \in \mathscr{Q}$ then together they can be arranged in a Markov chain:

$$R \to Q \to \text{channel} \to \hat{Q} \to \hat{R}$$

We banish the passive voice and consider two agents to accomplish the encoding and decoding, respectively:

$$R \to \text{Agent 1} \to Q \to \text{channel} \to \hat{Q} \to \text{Agent 2} \to \hat{R}$$

Agent 1 converts symbols of \mathscr{R} to symbols of \mathscr{Q} and sends them to Agent 2, who converts the received symbols of \mathscr{Q} back to symbols of \mathscr{R} . Think of the agents as special purpose devices, whose mechanism allows no discretion in how they operate. We will speak of an agent "perceiving" some set of symbols or "measuring" information, but mean only that its mechanism is designed to operate on those symbols, or that its (possibly imaginary) designer might be doing the perceiving or measuring at its inputs or outputs.

Note that such an agent may perceive symbols when another observer does not. For example, an agent might be a device built so that a set of eight voltage measurements constitutes a "symbol" for the purposes of its internal mechanism. Voltage levels in unconnected inputs to a circuit may float noisily, so depending on its design, it may perceive a series of eight-bit symbols if only one or even none of its inputs is actually connected to anything. An external observer may see it as disconnected from the world, but that may be irrelevant to its operation. In other words, we may speak of the subjectivity of observation, even of a primitive device.

Assume for the moment that Agent 1 encodes each symbol of \mathscr{R} into a two-symbol "word" composed of symbols of \mathscr{Q} . Consider an input alphabet $\{A, B, C, D\}$, to be encoded into two-symbol words from the alphabet $\{a, b\}$, as in Figure 1.

Because the positions of the two output letters are independently variable factors, each two-symbol word can be represented as a point in a two-dimensional Hamming, or phase, space. Three-letter words could similarly be described in three dimensions, four-letter words with four dimensions, and so on. This is the usual presentation of a Hamming space for a block code. Note that the dimensions of the code space can easily be construed to represent generalized degrees of freedom, each of which might represent letter position, but might also represent something else entirely, such as whether the symbol is printed in red or transmitted on an independent channel.

As a way to understand the virtues of a code, the concept of a code space is widely used. It is less often used to discuss transmission itself. But there are insights available in the discussion to follow by considering the agents involved in a transmission as translating a point in one multi-dimensional space to a point in some other multi-dimensional space. For convenience, we classify agents into "aggregators" where the number of degrees of



Figure 1: An encoding can be depicted as points in a code space: (a) is an example of how an alphabet of four symbols can be encoded in a two-symbol alphabet and (b) shows how they can each be described as a point in an abstract code space.

freedom in the input space is greater than in the output and "distributors" where the input degrees are fewer than the output.

Consider the information in a sequence R of ten three-letter words, received by an Agent 1, who will send them to an Agent 2, as above.

tap tap tap apt apt tap apt tap apt

This is a more complex code space than shown in Figure 1. If we consider each letter position to be independent, it has three dimensions, each of which seems to have two possible values, though only two of the eight possible combinations of letters seem to be in use.



Figure 2: A three-dimensional encoding. Only two of the possible eight code points, tap and apt are used in the example.

Since each of those combinations appears with a probability of $\frac{1}{2}$, the information carried by each word is $H(r) = -\sum_{r \in \mathscr{R}} \frac{1}{2} \log \frac{1}{2} = 1$ bit, and the total information in the sequence is $H(R) = 10 \times 1$ bits.² Agent 1 translates these words into individual letters, and notices that there are still 10 bits of information in its output, because the second and third letter of each word are uniquely identified by the first. Agent 2 agrees. We write H(Q|v) to imply the per-symbol information content of a message taking the degrees of freedom into account. If the letters of each word are indicated by $q_1q_2q_3$, we have:

 $H(Q|v) \equiv H(q_1) + H(q_2|q_1) + H(q_3|q_1q_2) = 10 + 0 + 0 = 10$ bits

²All logarithms in this article are assumed to be base 2.

A diagram of the exchange would show a distributor node feeding three output degrees of freedom to an aggregator expecting three inputs. Each perfectly agrees with its partner, as in Figure 3.

$$R \rightarrow \boxed{\text{Agent 1}} \rightarrow Q_2 \rightarrow \boxed{\text{channel}} \rightarrow \hat{Q}_1 \searrow \\ R \rightarrow \boxed{\text{Agent 1}} \rightarrow Q_2 \rightarrow \boxed{\text{channel}} \rightarrow \hat{Q}_2 \rightarrow \boxed{\text{Agent 2}} \rightarrow \hat{R} \\ \searrow Q_3 \rightarrow \boxed{\text{channel}} \rightarrow \hat{Q}_3 \nearrow$$

Figure 3: Agent 1 implicitly asserts that its outputs are conditional on each other, and Agent 2 implicitly agrees and simply reassembles what Agent 1 took apart.

Contrast that with another geometry, which accomplishes much the same thing, but over a more complicated network involving intermediate agents, perhaps created to improve the fidelity of transmission, as in Figure 4.

$$R \rightarrow \boxed{\text{Agent 1}} \rightarrow Q_1 \rightarrow \boxed{\text{Agent 1a}} \rightarrow \hat{Q}_1 \rightarrow Q_2 \rightarrow \boxed{\text{channel}} \rightarrow \boxed{\text{Agent 1b}} \rightarrow \hat{Q}_2 \rightarrow \boxed{\text{Agent 2}} \rightarrow \hat{R}$$
$$\searrow Q_3 \rightarrow \boxed{\text{channel}} \rightarrow \boxed{\text{Agent 1c}} \rightarrow \hat{Q}_3 \xrightarrow{\nearrow} \boxed{\text{channel}} \rightarrow \boxed{\text{Agent 1c}} \rightarrow \hat{Q}_3 \xrightarrow{\nearrow} \boxed{\text{channel}} \rightarrow \boxed{\text{Agent 1c}} \rightarrow \hat{Q}_3 \xrightarrow{\nearrow} \boxed{\text{channel}} \rightarrow \boxed{\text{Agent 1c}} \rightarrow \boxed{2} \xrightarrow{\nearrow} \boxed{2} \xrightarrow{\nearrow} \boxed{2} \xrightarrow{\nearrow} \boxed{2} \xrightarrow{\longrightarrow} \boxed{2} \xrightarrow$$

Figure 4: Agent 1 still asserts that its outputs are conditional on each other and Agent 2 asserts the same about its inputs, but the conditionality is irrelevant to Agent 1a in the middle, and its friends.

The Markov assumption implies the agents in the middle of Figure 4 are free to treat the input they receive as coming from a stochastic source of uncorrelated symbols, one at a time. Agent 1a sees tttaataata and concludes there are only two symbols, t and a, with one bit of information per symbol, giving ten bits total. An external observer can clearly see the conditionality, but there is nothing intrinsic to the message that demands Agent 1a care about it. Its two friends conclude the same about the sequences they observe. If H(Q)is the sum of the three agents' observations, then we have:

$$H(R) = H(Q|v) = 10 \text{ bits but } H(Q) = 30 \text{ bits}$$
(1)

The agents in the middle perceive more information than Agent 1 or Agent 2. This does not depend on the mutual isolation of the agents in the middle, but only on agents designed to attend or ignore conditionality. For example, one could consider the same example with only an Agent 1a in the middle, receiving and sending all the letters from Agent 1 to Agent 2, but not built to account for the conditionality of the letters:

$$R \to \text{Agent 1} \to Q \to \text{channel} \to \text{Agent 1a} \to \hat{Q} \to \text{Agent 2} \to \hat{R}$$

Here again, Agent 1 and Agent 2 agree that the message contains 10 bits of information. But Agent 1a, who knows nothing of the conditionality, sees a sequence of 30 symbols consisting of 3 different shapes occurring with equal frequency and concludes that $H(Q) = 30 \times 1/3 \log 1/3 \approx 15.8$ bits. Conditionality decreases information: $H(R|x) \le H(R)$ for any message R and any variable x. Thus, a communication node that ignores a conditioning variable x may perceive more information in a message than one that does not. A distributor node may embody an implicit assertion about the conditionality of its outputs that other agents in a network are free to ignore because such an assertion is not actually a part of the message. In the presence of such an implicit assertion, if the measurement of information is made by those agents that ignore it, they will perceive more information than the distributor. Conversely, an aggregator node's operation may embody an assertion of conditionality among its inputs, so will appear to decrease information, according to other nodes that ignore that assertion.

In a complex network of independent communicating nodes, no individual node is forced to assume the perspective of any other node. Each may assume its inputs are uncorrelated stochastic sources, or it may assume correlations among them, or it may assume they are all transmitting nonsense. This is what independence means in the context of the Markov assumption.

Independent assumptions about conditionality are, of course, routine in practical communication systems. An SMTP server does not differentiate among the emails it handles according to the language they are written in, even though the letter frequencies are not the same. However, the spam filters that process those emails (or the people who read them) are specific to the language in use, and so incorporate the appropriate letter and word frequencies. The spam filter's estimation of the information content of any given email will thus be less than the estimation given by the SMTP server. The SMTP server, in turn, is probably communicating via ethernet, whose switches are completely ignorant of the internal structure of a MIME document. Thus the switches—or an observer assuming their limited perspective—perceive even more information than the server.

In each of these examples, each agent's operation is perfectly consistent, but the various agents can disagree with one another about the quantity of information flowing through. From a global perspective, perhaps the disagreement is merely an illusion created by agents ignoring important features of their input, but if one is pursuing insight into how best to engineer independently functioning agents, one must consider the limited perspective imposed by the Markov condition. In some respects, the situation is no different with thermodynamic entropy, where the definition of the relevant macrostates can differ among observers, even though they might be said to agree on the microstates (Jaynes, 1992). The motion of the same air molecules is relevant to both a pressure sensor and a bank teller awaiting a delivery via pneumatic tube, but they may not agree on which is the most relevant macrostate.

2.2 Energy use in error correction

We inquire into the energy use by nodes in some complex network. We first consider the transmission of discrete symbols to a single node, in the presence of noise. Encoding and decoding in this context is a well-studied subject. We do not examine the mechanics but rather the energy cost of achieving whatever rate of transmission is made possible by the channel capacity.

Decoding is often described as a single step, but there are three processes involved, and we can abstract them to create a general framework within which to analyze energy use. The first step for any agent in a network is to transform the input signal into a hypothesized output signal, to transform a point in an input code space to a point in an output space. This treatment has little to say about that step, which we take as given. The second step is comparing the hypothesis with expectations in some fashion, in order to detect errors. Conceptually, one can think of this as measuring the distance between a hypothesized point in the output space and members of the set of potentially valid outputs, as might be done with the distortion measure of rate-distortion theory. In practice, this step might involve comparing the output data with parity bits or Hamming data or data from some other forward error-correction (FEC) scheme, or it might be comparing output symbols with symbols from a dictionary available to the agent, or a template, or even some entirely different technique.

Having translated the symbols and found any errors, the third step is to do something about them: to correct them if the message has adequate redundancy, or ask for a retransmission if it does not. One might also signal an error, or simply give up if the errors cannot be corrected. The three steps then, are transformation, error detection, and error correction.

Consider the detection and correction steps of the process where some message Q expressed in symbols from alphabet \mathscr{Q} is translated into a message hypothesis \hat{R} , expressed in symbols of \mathscr{R} , and then corrected to produce message R. How much energy is used to convert \hat{R} into R?

We consider first the energy cost of checking a stream of symbols for errors.

Theorem 1. An efficient error-checking mechanism can do no better than an energy cost proportional to the number of bits of the signal it checks.

Proof. We consider how the most efficient possible error-checking mechanism would behave.

- 1. In order to be more efficient than spending the same energy on each received symbol, it must be able to exploit regularities in the signal to reduce the cost for some symbols. This implies that where there are no regularities and the input is merely random, the energy use would be maximized, and be proportional to the number of symbols.
- 2. One such regularity to be exploited might be when a subset of inputs *A* completely or partially determines the outcomes of another subset *B*. The total energy used by an efficient mechanism would be the energy to check errors in *A* plus an amount to check *B* that depends on the degree of dependence on *A*. Complete dependence means no energy need be expended checking *B* while complete independence means no energy is saved compared to the case of checking *B* alone.
- 3. An efficient mechanism need expend no energy to check the result of zero-probability events.

In their essentials, these are the three conditions for the uniqueness theorem of Khinchin, who proved that a measure satisfying those conditions would always be proportional to the entropy measure $H = -\sum_{s \in S} p_s \log p_s$ where p_s is the probability of observing symbol s in some alphabet S (Khinchin, 1957). The entropy calculated with the log base two is the number of bits in the message. \Box

Computation has thermodynamic consequences. Landauer (1961) showed that an irreversible process, such as the deletion of a bit, unavoidably costs some energy lost to heat. Transmission errors are random; there is no way to take advantage of the energy change during the instant some bit is accidentally flipped. Correction is thus an irreversible process and requires work (Bennett, 2003). One could imagine a system that stored the erroneous state to make the correction reversible, but ultimately that stored state is of no use and will be deleted, an irreversible process. Such a system is thus a way to delay the energy loss, not to avoid it. Practically speaking, the energy cost of correction could be as simple as the energy needed to erase a bit or as expensive as a request for retransmission, depending on context. We therefore model the correction step with another linear function, of the number of errors that require attention: the product of the total number of bad bits and the efficacy of identifying them.

(As an aside, we note that one need not take a position on the meaning of the analogy between signal entropy and thermodynamic entropy (Jaynes, 1957; Samardzija, 2007) to see that a reduction in signal entropy due to the correction of errors is accompanied by a proportional increase in heat energy, just as is the case with a reduction in thermodynamic entropy.)

Let K_R be an estimate of the per-bit energy cost of assessing what the observations R should be, the detection step. We define a noise level (proportion of symbols transmitted incorrectly), $0 \le z < 1$, an efficacy function (the proportion of errors actually found as a function of the energy spent finding them), $0 \le \zeta(K_R) \le 1$, and a per-bit cost of repair, L_R . Using these, we can write an expression for the minimum work done in detection *and* correction during $\hat{R} \to R$:

$$E \ge K_R H(R) + L_R \zeta(K_R) z H(R) \tag{2}$$

If the error rate *z* is 10% and the efficacy of the error detection $\zeta(K_R)$ is 80%, then 8% of the symbols in *R* will need repair. Presumably, spending more energy per symbol in detection will bring ζ closer to its maximum and therefore require more energy for correction. We model the efficacy as a function of K_R with range [0, C], where *C* is the maximum efficacy permitted by the channel capacity, so $0 < C \le 1$. As an example, an inverse exponential like $\zeta = C(1 - e^{-K_R^2})$ captures the intuition that there is a point of diminishing returns, beyond which it costs significant amounts of energy to detect an increasingly small increment of errors, but we assume here only that the efficacy is a continuous and monotonic function of K_R .

2.2.1 Two levels of error correction

Consider now an arrangement of nodes where Q is transmitted via a noisy channel, and accuracy demands implementation of some system of error-checking, but there are multiple receivers in series.

$$Q \rightarrow \boxed{\text{Agent } 1} \rightarrow R \rightarrow \boxed{\text{Agent } 2} \rightarrow S \rightarrow \boxed{\text{Agent } 3} \rightarrow \dots$$

Each agent may have multiple inputs and outputs; what is shown is the path through these agents relevant for errors in the transmission of Q. Expanding the steps $Q \to R \to S$ we make a longer chain, where Q is translated to \hat{R} , using the information that encoded it $A_{\mathcal{RQ}_v}$, or some approximation. Additional data ε_R , received through some correction channel, is used to transform \hat{R} , producing R through the error detection and correction steps. This data could have arrived in the same channel as R, for example as parity bits, checksums, the extra bits added for a Hamming code, or some more elaborate FEC system still uninvented. It might also have been developed from other observations, experience, or prior arrangement. The steps look like Figure 5.

Figure 5: Transmission of Q is transformed to \hat{R} using $\mathbf{A}_{\mathscr{R}\mathscr{Q}_{v}}$, which corresponds to the dictionary or procedure originally used to encode R into Q. Information from the correction channel ε_{R} is then used to correct errors and turn \hat{R} into R. The detection step does not change the symbols, so is not represented.

For the moment we assume the only noise is in the transmission of Q, and our interest is in detecting and correcting the errors introduced there.

Consider the energy consumption of the two error detection and correction steps $\hat{R} \to R$ and $\hat{S} \to S$. For some error in the transmission of a symbol q, the error might be found and corrected at the first step, as it becomes a contribution to some r, or the second, as that r contributes to some s. We assume the system achieves the full channel capacity and all possible errors are corrected at one level or the other: $\zeta(K_R)$ of the errors are fixed at R, and $C - \zeta(K_R)$ are corrected at S so that C of the errors are corrected. We also assume for the moment that the transmission from R to S introduces no new errors. We can add to equation 2 and write an equation for the work done in error correction at the two levels, as a function of the energy invested in error correction at R:

$$E \ge K_R H(R) + z L_R H(R) \zeta(K_R) + K_S H(S) + z L_S H(S) (C - \zeta(K_R))$$
(3)

Agent 2 is independent of Agent 1. Since assumptions of conditionality can differ, the information as measured from the *S* perspective H(S) need not be the same as H(R). We define a ratio $\alpha \equiv H(S)/H(R)$ to compare the number of bits of information perceived by the two agents. Simplifying:

$$\frac{E}{H(R)} \ge K_R + zL_R\zeta(K_R) + K_S\alpha + zL_S\alpha(C - \zeta(K_R)).$$
(4)

This is an equation relating the energy used in error correction between two different levels of analysis, and can be used to explore the design space of energy trade-offs between one level and another by assuming different relations between K_R , K_S , L_R , L_S , and $\zeta(K_R)$. For example, we can differentiate with respect to K_R and set the derivative to zero to minimize the energy spent at R:

$$0 = 1 + z(L_R - \alpha L_S) \frac{\mathrm{d}\zeta}{\mathrm{d}K_R} + \alpha \frac{\mathrm{d}K_S}{\mathrm{d}K_R}.$$
(5)

Corollary 1. For $L_R/L_S > \alpha$ and K_S independent of K_R , there is no solution to equation 4.

Proof. Since ζ is monotonic increasing, and $\alpha L_S < L_R$, the second term on the right side of equation 5 is positive. With K_S is independent of K_R , the third term is zero, and thus the right side of the equation is positive and there is no value of K_R to solve equation 4, and therefore no positive value of K_R that will cost less energy than $K_R = 0$. \Box

This combination of parameters might correspond to two aggregator nodes in a row where the cost of error correction at the second level is comparable to the first. For these cases, it will always save energy to skip error correction at R in favor of S.

Corollary 2. For $L_R/L_S < \alpha$, any solution to equation 4 will occur at imperfect efficacy.

Proof. Note that $d\zeta/dK_R$ approaches or equals zero at high levels of efficacy, by the assumption of monotonicity. A solution to equation 4 thus occurs where $d\zeta/dK_R = 1/z(\alpha L_S - L_R)$. Unless $\alpha L_S \gg L_R$, $d\zeta/dK_R$ is not close to zero at the solution and thus ζ is not close to 1. \Box

If the agent at S perceives much more information than the one at R (α very large) or it is much more expensive to correct at S then at R, then it might be efficient to do complete error correction at R. Otherwise, so long as K_S is independent of K_R , it is likely that a solution will occur with Agent 1 operating at efficacy levels substantially lower than the maximum. Optimizing energy use would thus require what in isolation will appear to be sloppy error correction at the first level.

It is plausible that K_S might not be independent of K_R , in which case there may be a non-zero solution to equation 4. Perhaps a certain amount of energy spent checking for errors at R would mean spending less at S to achieve the same result. We model K_S as the sum of b_S , a component independent of K_R , and another component that is a function of K_R . This function starts at some level $k_S(0)$, the energy spent if no correction is done at R, and declines to reach or approach zero for large values of K_R :

$$K_S = b_S + k_S(K_R). \tag{6}$$

If the $k_S(K_R)$ decreases from $k_S(0)$ to zero, then for some or all of its domain, its derivative must be negative. Substituting into equation 5 and moving to the other side of the equation, the b_S term will disappear in the differentiation, leaving:

$$-\alpha \frac{\mathrm{d}k_S}{\mathrm{d}K_R} = 1 + z(L_R - \alpha L_S) \frac{\mathrm{d}\zeta}{\mathrm{d}K_R}.$$
(7)

There are too many unknowns in this equation to say much about it, but some observations are possible. For example, for values of α close to one, there may be a solution if the derivative of k_S is close to minus one, indicating that it might not matter whether correction happens at *R* or *S*, which is intuitively sensible. Further, if the cost of repair is substantially higher at *S* than *R* ($L_R/L_S < \alpha$), there may be a substantial range of K_R values in which to find a minimum.

For values of α much smaller than one, if the value of k_S declines abruptly at any point as K_R increases, the left side of equation 7 will be large and make it more likely that a plausible selection of parameters would provide a solution to the equation, where a nonzero value for K_R would minimize energy use. For example, this could be the case if noise above a certain level precluded efficient decoding at *S* entirely and required a request for retransmission. Alternatively, if k_S has only a gentle dependence on K_R , a solution would be less likely for $\alpha < 1$.

Corollary 3. When $L_R/L_S > \alpha$, and there is dependence between K_S and K_R , any solution to equation 4 will occur at imperfect efficacy.

Proof. Assume a non-zero solution to equation 4 when $L_R/L_S > \alpha$. The condition of perfect error correction at *R* would imply that no further increase in energy spent on correction at that level would reduce the cost of checking errors at *S*, thus that dk_S/dK_R must be at or approaching zero. But we have already observed that at perfect efficacy, the derivative of $\zeta(K_R)$ will approach zero, so at the solution:

$$-\frac{\mathrm{d}k_S}{\mathrm{d}K_R}>\frac{1}{\alpha}.$$

Where there is a solution to equation 4, the correction by Agent 1 would be considered imperfect in isolation, even under the assumption of maximum error correction at *S*. \Box

We have assumed no noise in the $R \rightarrow S$ step. Were we to reverse that assumption, the correction system at *S* would still have to check all the bits, though it would be more expensive to correct the larger number of incorrect bits. In other words, noise would simply add a term to the right side of equation 3 proportional to H(S). This quantity would have no dependence on K_R and so the term would disappear in the differentiation step. Noise may also reduce the value of α , making it less likely to be worth doing error correction at *R*. In the case of noisy transmission where K_S is dependent on K_R , noise will appear to reduce the efficacy of the correction at *R*, leading to a lower dz/dK_R , and making it more likely that there is a non-zero solution to equation 7.

2.2.2 Energy use and sparsity

We digress briefly to consider K_R . In the case of two aggregators in a row, in $Q \to R \to S$, it is possible to express R and S using the symbols of \mathscr{Q} . This is the case, for example, when letters are assembled into words and then sentences, or sounds assembled into phonemes and then words. In such a case, the code space of R has $N_{\mathscr{Q}}^{V_{\mathscr{R}}}$ code points, and the code space of S has $N_{\mathscr{Q}}^{V_{\mathscr{R}} \times V_{\mathscr{S}}}$ points. Thus the number of possible code points increases while the information content in the message—the number of valid code points—decreases. This creates a sparser code space with a lower density of valid points.

We assume that the energy cost of comparing two values is proportional to the number of bits of information that must be compared. In a computer's circuitry, a comparison is typically done by adding two bits with a zero result indicating equality. Since the equality condition does not include the bit values (which are destroyed in the process) a comparison of two bits is generally an irreversible operation and the energy cost is therefore a straightforward consequence of Landauer's principle (Landauer, 1961; Bennett, 2003). There is ongoing research on reversible comparators, that function by recording the inputs so the operation can be rewound and thus presumably avoid the thermodynamic implications of irreversibility (e.g. Harith and Vasanthanayaki, 2017). However, though these approaches create energy savings, they are still not free and retain a dependency on the number of bits compared. As with error correction above, the state saved in order to create reversibility is of questionable value. The method may only be a way to delay the heat loss, not prevent it.

Proposition 1. For some arbitrary code space of N points, containing both valid and invalid points, the maximum number of bits necessary to uniquely specify a distance between any two points is given by log N.

The least efficient method for locating points in space is simply to enumerate them, which requires $\log N$ bits to specify. One might also think of it as arranging them in a one-dimensional code space. A coordinate space of dimension greater than one will allow a more efficient method. One can also observe that the maximum Hamming distance between two code points can be expressed with $\log \log N$ bits, quite a bit smaller. For many codes, this will be the important distance measure.

This is of interest because the task of finding a correspondence between a hypothesis point in code space and one of the valid code points is that of comparing all the distances between the hypothesis and the set of valid code points, to find the smallest. It is challenging to generalize over all possible physical manifestations of the abstractions of a code space, but given all these assumptions, we assert the following proposition.

Proposition 2. The energy spent detecting errors is positively related to the density of valid code points.

For a space with a minimum code distance d, one seeks the valid point whose distance from the hypothesized point is smaller than d, so one need only compare the highest-order bits. For some alphabet of symbols $r \in \mathscr{R}$, there will be hypothesized symbols \hat{r} from a decoding, and valid symbols r. Given a probability distribution $P(\hat{r})$ for the hypothesized points and a "true" probability distribution P(r) for the symbols of message R, the Kullback-Leibler divergence can be construed as indicating the reduction in number of bits necessary to identify a point using the wrong probability distribution. In parallel fashion for this case, it can be used as a measure of the reduction in number of bits necessary to be compared to find the valid point less than d from the hypothesis. Let N_{comp} be the number of bits necessary to compare to find the distance less than d:

$$N_{comp} \le \sum_{r \in \mathscr{R}} P(\hat{r}) \log P(\hat{r}) - P(r) \log \frac{P(r)}{P(\hat{r})}$$
(8)

$$=H(\hat{R})-D_{KL}(R\parallel\hat{R}). \tag{9}$$

The divergence between the distribution of symbols P(r) and the distribution of hypotheses $P(\hat{r})$ can also be regarded as a measure of the effective sparsity of code points. Since some of the \hat{r} will be symbols whose probability is zero, in effect it is measuring sparsity by estimating the likelihood that a hypothesis hits a valid point. Thus as the sparsity increases, the number of bits that must be compared decreases, providing a demonstration of Proposition 2.

One might also suggest that the difficulty of finding a minimum distance is related to the probability that any given \hat{r} is ambiguous: equidistant, or nearly so, from two valid points. But the ambiguous points exist close to the midpoint between valid code points, at or near a hypersurface squeezed between hyperspheres of radius greater than d, by the definition of d. Thus the number of ambiguous points will be proportional to d^{N-1} , the "area" of a hypersurface, while the unambiguous points will be in the interior of those hyperspheres, and be of a number proportional to d^N . An increase in d means the proportion of ambiguous points will decline, and thus the average number of bits that must be compared to find a minimum will also decline.

Proposition 2 implies that when considering errors in the transmission $Q \rightarrow R \rightarrow S$ for aggregators *R* and *S*, not only are there fewer bits to check in *S* than in *R*, but it is likely to be less expensive to detect the problems. Stated in terms of equation 4, if *S* is a sparser

code space then *R*, then not only do we have $\alpha < 1$ but also $K_S < K_R$, making a non-zero minimum even less likely.

2.2.3 Multiple levels

The findings for a two-level system can easily be extended to an arbitrary number of levels using an inductive argument. Consider the sequence $Q \rightarrow R \rightarrow S \rightarrow T$. We can regard *S* and *T* as a single level while considering whether to do error correction at *R* or at that second level. Once decided, we can use the same procedure to decide how much energy to invest in *S* or *T*.

2.2.4 Continuous systems

Continuous valued systems may also have systems of error correction. Indeed, this is the definition of servo control. Such continuous systems can be layered, like a discrete system. For example, the autopilot on a ship might have a servo that controls the position of the rudder, and another servo "above" that one, using the rudder position to control the ship's heading. Just as one might ask about the necessity of error correction in the discrete case, one might ask whether one can trade off precision in error correction at one level or another and still reach the destination. (One also does not want to capsize or run aground along the way, but these are separate considerations, to be put aside for this discussion of energy.)

As with the discrete case, error correction for a continuous signal consists of two steps: measuring the error and doing something about it. A continuous signal X can be measured with differential entropy integrated over its support, $h(X) = -\int_X p(x) \log p(x) dx$, where p(x) is the probability density function of the signal. Like discrete entropy, there is a uniqueness theorem for differential entropy, so by a parallel argument to Theorem 1, the energy spent in error measurement can be no less than proportional to the information in the signal (Kotz, 1966, pp10-11) because a measurement that could do better would itself be another, different, measurement of entropy and thus violate the uniqueness of the information measure.

For a continuous signal transmitted through a channel with a Gaussian source of noise, an elementary theorem of information theory shows that the differential entropy in the resulting signal \hat{R} is the sum of the mutual information between the output \hat{R} and the source *R* and the entropy contributed by the noise *Z* (Cover and Thomas, 2006, chapter 9):

$$I(R;\hat{R}) + h(Z) = h(\hat{R}).$$
(10)

We can therefore represent noise as a proportion of h(R): $z \equiv h(Z)/h(\hat{R})$.

Beneath the abstractions of information theory, the quantities are representations of physical quantities and effects. The signals in question might be mechanical or electrical in nature, or even something else, but they are representations of real physical processes. The process of acquiring and removing errors in some quantity can be usefully compared to the process of isothermally re-compressing a gas that was allowed to expand freely (Bennett, 2003, especially the discussion related to figure 1). Because no energy was stored from the acquisition of the errors, there is none available to compress them away. Thus, by the same arguments made by Landauer and Bennett (Landauer, 1961), the process of error correction in a continuous system is irreversible, and therefore requires work to accomplish.

Like the detection step, one can go a step further, and use the uniqueness theorem to say that the work required can be no less than proportional to the differential information. Again, were it otherwise, one could use the process of correction to create a different entropy measure, and thus violate the uniqueness theorem.

Thus for a continuous process being measured at some node *R*, we can write an equation for the energy used to correct errors exactly analogous to equation 2, using analogous definitions of per-bit energy cost K_R and efficacy as a function of the energy spent finding errors $\zeta(K_R)$:

$$E \ge K_R h(R) + L_R \zeta(K_R) z h(R). \tag{11}$$

For a two-stage system like the autopilot, with measurements at nodes R and S, we can write a two-level equation virtually identical to equation 4. It can be used to optimize energy use in the same way as the discrete case, with the same conclusions.

2.2.5 Conclusion

To summarize:

- For $\alpha < 1$, if K_R and K_S are independent and L_R and L_S of comparable size, it will save energy if Agent 1 does no error correction at all, so long as Agent 2 can function at the resulting error rate.
- For two aggregators in a row, the code space becomes larger as the information in a message decreases. Not only is there less information to correct $\alpha < 1$, but the errors become less expensive to detect: $K_S < K_R$.
- Where it is possible to do so, error correction in a series of aggregators should be delayed to the end.
- If K_S is dependent on K_R and monotonically increasing, then investing energy at R is efficient only if the decline in the energy necessary at S is steep.
- Even when it is efficient to put energy into correction at *R*, such as when $\alpha > 1$, it is unlikely to be worth correcting 100%.
- A system that is not energy-constrained can still benefit from these results. Error correction is not instantaneous, so skimping can save time.

Obviously, factors such as accuracy and reliability are important to a communication system, so it may not always be feasible to seek the least possible use of energy for some system. However, these are the directions that energy efficiency would suggest.

3 Discussion

The claim in Section 2.1 that assumptions of conditionality can affect the measure of information across a complex communication network tugs at connections to the very heart of probability theory. If one agent in a network can choose to ignore conditionality that another relies on, what does a probability estimate mean? How reliable can it be?

The question of whether probabilities are objective measures of the world or subjective evaluations of experience is one that cannot be settled here. However, for the designer of some device, the important question is not philosophical; it is about what he or she can anticipate happening *at the inputs of that device*. A designer of an ethernet port, a

device through which many different kinds of data may pass, is well justified in assuming that all byte values are equally likely. The designer of translation software to process data that arrives via that ethernet port is equally well justified in making a completely different evaluation of the likelihood of different byte values. A molecule of DNA might encode any number of different peptide chains, but the enzymes awaiting the output of a particular segment to assemble it into some protein need not be so inclusive in their expectations.

Non-conservation of information is not a novel concept. After all, rate distortion theory, part of Shannon's original paper, is meant to address exactly the case where information is not preserved across some transformation. More recently, the original statement of the "Information Bottleneck" by Tishby *et al.* extends the classical rate distortion formulation to use a third source to choose a distortion measure (Tishby et al., 2000). The authors couch the information change as "lossy compression" but the theory's subsequent application to neural networks supports viewing such networks as a way to reduce information, as a collection of aggregators would do (Shwartz-Ziv and Tishby, 2017). The goals of that work—clarifying the internal mechanics of a neural network—are more or less orthogonal to the concerns of energy use here. The approaches are broadly compatible, though the information bottleneck theory is constrained to use aggregator nodes in an orderly enough arrangement that an entire layer of such nodes can be considered to be a single functional unit.

Concern with the internal mechanics of a complex network is important in a way that has not yet been addressed. After all, the discussion in Section 2.2 is merely a claim that one might trade off error correction between levels, which differs significantly from showing that such trade-offs might be possible given demands for accuracy and reliability. The approach of Section 2.1 provides some direction.

As we have seen, one can regard a node in some communication network as transforming points from one multi-dimensional input message space to a different multi-dimensional output message space. One can regard a collection of nodes as acting in a similar fashion on a collection of inputs, and thus the question of feasibility can be addressed topologically. The claim that some system of communication is resilient to errors is merely a claim that the input point that produces some output is surrounded in input space by a region of points that produce the same result. A large and convex region of input space corresponding to some point (or small region) of output space will indicate that error correction may be unimportant. Conversely, if a point in output space corresponds to a collection of small, disjoint, or non-convex regions in the input space, then perfect error correction may be vital to its correct function.

This is not a trivial point, of course. To consider one class of complex system, the shape of the input space to some neural network is notoriously opaque. Szegedy *et al.* observed that neural networks can show remarkable sensitivity to what appear to be insignificant changes in their input (Szegedy et al., 2014), precisely the issue under consideration here. This has resulted in a substantial body of research into generating adversarial examples to test the robustness of neural networks and to inform approaches to learning via "generative adversarial networks" (GANs) (Goodfellow et al., 2015; Warde-Farley and Goodfellow, 2016; Sharif et al., 2019). Some of this research has a topological cast. Dube has presented ways to characterize the topology of the input spaces as a source of insight into how adversarial examples are enabled (Dube, 2018). Gilmer *et al.* argue that the shape may be determined by the data itself and the high dimensionality of vision datasets (Gilmer et al., 2018). At least one line of research into GANs seems to have had promising results by explicitly re-

garding the input vector as being composed of signal and noise and seeking to shape the spaces "perceived" by the hidden layers accordingly, though the approach there is not explicitly topological (Chen et al., 2016). It is possible that the topological simplicity of the input space may be not only an indication of resilience to error, but to a more general sense of reliability in the presence of novel inputs, too (Dube, 2018). Generating adversarial examples could be used to characterize the topology of neural networks for just this purpose.

3.1 Energy use in computing

Energy use in computing has become an increasingly important issue, powered by two converging but independent forces: the advent of tremendously effective, but tremendously compute-intensive machine learning applications and the advancing demands for both performance and battery life in mobile devices. In an architecture of multiple levels of communication, it is clear from the analysis presented here that it is often inefficient to insist on complete error correction at any individual level.

As we have seen, this has implications for machine learning since nodes in a neural network are aggregators, producing a single output from multiple inputs. Such networks consist of many layers of such nodes, so one might predict that error correction—and thus precision of calculation—in neural networks may not be important, *pace* the adversarial research described above. Google's experience with implementation of its TensorFlow computing software and hardware confirms the point. In that case, the ever-increasing electricity usage of their translation software led Google to use quantization and low-precision libraries in the implementation of the software (Abadi et al., 2015) and to develop approximate hardware Tensor Processing Units (Jouppi et al., 2017). By shortchanging the error checking in the aggregator nodes of the network, energy savings and performance enhancement resulted with no loss of accuracy in the ultimate results.

More generally, advances in approximate computing are ongoing, though remain without a comprehensive theoretical basis (Xu et al., 2016; Mittal, 2016). Including Google's work, several promising avenues of inquiry in the field lead down the path indicated by the analysis developed here. Leem et al. provide a framework for approximate computing that distinguishes between the accuracy needed for execution control and the lower degree of accuracy needed for calculations in certain classes of algorithms, such as K-means clustering, loopy belief propagation, and Bayesian inference networks (Leem et al., 2010). Each of the specific algorithms considered, just like neural networks, consist of multiple applications of aggregator nodes and thus yields to the analysis offered here. Similarly, Samadi et al. proposed Paraprox, software that encompasses a method of finding patterns in application programs that lend themselves to an approximate approach and then instructing a compiler accordingly (Samadi et al., 2014). Four of the six patterns it can identify (there named Reduction, Scan, Stencil, and Partition) are arrangements of aggregator nodes and the other two (Map and Scatter/Gather) could be, depending on the structure of the function being mapped or gathered. Even more aptly, Shanbhag et al. present a framework for approximate computing directly inspired by information theoretic concepts (Shanbhag et al., 2019). The architecture creates "fusion blocks" in a variety of geometries to reconcile results from low-accuracy processors with shadow results from high-accuracy (but lowprecision) processors. Essentially, the authors have created artificial aggregator nodes atop their approximate processors, and show that indeed the error correction can be delayed until the fusion block and that the resulting system is more efficient when it is. In a discussion of loosening synchronization requirements in parallel computing, Rinard suggests that computations that "combine multiple contributions to a composite result" that another phase of computation consumers—aggregator nodes again—would be most resilient to the resulting errors (Rinard, 2012). All of these examples are confirmation of the points made here.

3.2 Communication in natural systems

In addition to the consequences for approximate computing, one might frame the results here to say that adding an *ad hoc* layer of analysis may save energy over an investment in better error correction.

Resource constraints are an important source of evolutionary selection pressures. Brains, for example, are expensive organs to support (Robin, 1973; Aiello and Wheeler, 1995), so strategies to minimize this energy use are important to an organism's fitness. As a consequence, it is unsurprising to find natural systems using multiple levels of analysis and apparently inadequate error correction. There is empirical support for both. Clark (2013) reviews a great deal of support for multiple levels in cognition, and there is also evidence for the inadequacy of error correction in natural systems where such systems have been identified. For example, the behavior of retinal cells is often not adequate to disambiguate luminance values (Purves et al., 2004) and memory cues can aid phonological segmentation, but may still not be adequate to eliminate uncertainty (Gow and Zoll, 2002). Reproduction of DNA is a similar case, where one finds multiple levels of repair implemented in a cell (Fleck and Nielsen, 2004; Fijalkowska et al., 2012; Ganai and Johansson, 2016). However, the error correction in some levels can be artificially improved, implying that the natural state at those levels could be considered inadequate in isolation (e.g. Sivaramakrishnan et al., 2017; Ye et al., 2018).

If a multi-layer system of communication can save energy by delaying error correction, it follows that for a system plastic enough, it may cost less energy to create a new layer than to get the error correction right. To reduce uncertainty in detection, evolution might equally well lead to higher resolution retinas or higher levels of processing (Sgouros, 2005a). Such effects might be ontogenetic as well as phylogenetic. For example, learned pattern recognition can be a way to introduce a new level to some analysis, thereby reducing the energy or time needed for processing. Indeed, pattern matching ability is associated with improving facility in reading (Blank et al., 1968), in arithmetic (Koontz and Berch, 1996), and in musical sight-reading (Waters et al., 1997).

The information with which some mechanism transforms received information at the point of decoding—the $A_{\mathcal{RP}}$ and $A_{\mathcal{PR}}$ in Figure 5, representing the dictionaries and the instructions for using them—is also a source of interest, as important to the quality of transmission as the message itself. Computer communication is carefully regulated by several different standards committees³ to make sure receivers understand exactly how to decode the messages senders present. The mechanism by which this compatibility is created in natural systems remains the subject of research. It is clear, for example, that proper decoding of DNA requires compatible concentrations and varieties of non-coding RNA present in a cell (Collins et al., 2011) as well as reconcilible methylation patterns (Zemach et al., 2010), but the mechanisms of creating compatibility remain cloudy. Without the oversight of standards committees, a propensity to create new layers through experience will eventu-

³e.g. Ethernet is controlled by the IEEE standards committee, TCP by the Internet Engineering Task Force (IETF), USB by the International Electrotechnical Commission (IEC), and so on.

ally create mismatch between any two communicating natural systems, even if they derive from identical sources, because experiences differ. Furthermore, if the error correction information does not travel with the message, it may also vary between sender and receiver (Sgouros, 2005b), such as when a message arrives before a newly reissued code book. As a consequence, perfect transmission of a message from one natural system to another may never happen.

A node in a communication network with significant convex regions of its input space corresponding to points or regions in a sparse output space is not only one that is resilient to transmission noise, but is also likely to be able to assign an output to a novel input with confidence. Exploration of the energy demands of hypothesis formation under these conditions is ongoing, but the fundamental point is that a system with forgiving input and output topologies can receive most messages, even if the results are not quite what the sender intended. Fortunately, imperfect transmission is often good enough, and there is an added benefit.

It has long been thought that creativity requires some kind of randomness (e.g. Hofstadter, 1996; Marshall, 2002). In her writing about the possibility of creativity in artificial intelligence, Boden (2004) developed a sophisticated taxonomy of randomness, differentiating between truly random ("A-random"), random relative to expectations ("E-random") and random relative to a specific observer's expectations ("R-random"), and made the point that a convincing model of creativity need only fulfill the last. Consider an imperfect transmission. If a sender sends a message unanticipated by some receiver and that receiver interprets it in a manner impossible for the sender, then together they have created something that neither one could have made alone. In information theoretic terms, though there may be a substantial amount of mutual information in utterance and interpretation there is also a good deal besides. Both parties have contributed information to a result that neither controls. By failing to achieve perfect transmission of meaning, they have achieved Boden's R-randomness without random numbers, quantum fluctuations, or magic.

Ultimately the creativity that most needs explanation is not the well of inspiration for artists but the creativity of a bird building a nest from unfamiliar materials, a cell growing in a changing environment, or just of a person walking down a busy street. The important mystery is the everyday creativity required to make one's way in a complex world and to use language in the face of what Moravcsik (1998) called a "constant barrage of small semantic emergencies." The mathematics of energy efficiency and error correction holds true at every level, from the interaction between two people down to the communication between two neurons, or from a mother cell to its children. Creativity born of miscommunication may be a source for the variation on which natural selection acts, and opens up for consideration the ways in which it may not be strictly random.

The findings also allow us to see that the absence of creativity in machines is more about engineering precedent and high functioning standards committees than fundamental principle. Re-engineering computers to accommodate less-than-perfect transmission of messages could not only be a way to save energy and create more robust computing, but in the long term, could open the door to truly creative machines.

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