Global solutions to the n-dimensional incompressible Oldroyd-B model without damping mechanism

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Abstract

The present work is dedicated to the global solutions to the incompressible Oldroyd-B model without damping on the stress tensor in $\mathbb{R}^n (n = 2, 3)$. This result allows to construct global solutions for a class of highly oscillating initial velocity. The proof uses the special structure of the system. Moreover, our theorem extends the previous result by Zhu [19] and covers the recent result by Chen and Hao [4].

Key Words: Global solutions; Oldroyd-B model; Besov space

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1. Introduction and the main result

In this paper, we mainly consider the incompressible Oldroyd-B model without damping mechanism which has the following form:

$$\begin{cases} \partial_t \tau + u \cdot \nabla \tau + F(\tau, \nabla u) - K_2 D(u) = 0, \\ \partial_t u + u \cdot \nabla u - \mu \Delta u + \nabla \Pi - K_1 \operatorname{div} \tau = 0, \\ \operatorname{div} u = 0, \\ (u, \tau)|_{t=0} = (u_0, \tau_0), \end{cases} \tag{1.1}$$

where $u = (u_1, u_2, \cdots, u_n)$ denotes the velocity, Π is the scalar pressure of fluid. $\tau = \tau_{i,j}$ is the non-Newtonian part of stress tensor which can be seen as a symmetric matrix here. $D(u)$ is the symmetric part of ∇u,

$$D(u) = \frac{1}{2}(\nabla u + (\nabla u)^T),$$

and F is a given bilinear form which can be chosen as

$$F(\tau, \nabla u) = \tau \Omega(u) - \Omega(u)\tau + b(D(u)\tau + \tau D(u)),$$

where b is a parameter in $[-1, 1]$, $\Omega(u)$ is the skew-symmetric part of ∇u, namely

$$\Omega(u) = \frac{1}{2}(\nabla u - (\nabla u)^T).$$

The coefficients μ, K_1, K_2 are assumed to be non-negative constants.

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In fact, the above system (1.1) is only the subsystem of the following full incompressible Oldroyd-B model:

$$\begin{cases} u_t + u \cdot \nabla u - \mu \Delta u + \nabla \Pi = K_1 \mathrm{div}\, \tau, \\ \tau_t + u \cdot \nabla \tau - \eta \Delta \tau + \beta \tau + F(\tau, \nabla u) = K_2 D(u), \\ \mathrm{div}\, u = 0, \\ (u, \tau)|_{t=0} = (u_0, \tau_0), \end{cases} \tag{1.2}$$

in which η and β are two non-negative constants.

The Oldroyd-B model describes the motion of some viscoelastic flows, for example, the system coupling fluids and polymers. It presents a typical constitutive law which does not obey the Newtonian law (a linear relationship between stress and the gradient of velocity in fluids). Such non-Newtonian property may arise from the memorability of some fluids. Formulations about viscoelastic flows of Oldroyd-B type are first introduced by Oldroyd [18] and are extensively discussed in [2].

About the derivation of the system (1.2), the interested readers can refer to [16], here we omit it. As one of the most popular constitutive laws, Oldroyd-B model of viscoelastic fluids has attracted many attentions and lots of excellent works have been done (see [3], [4], [6], [9], [10], [11], [12], [13], [14], [15], [19], [20]) and references therein. Guillopé and Saut [12], [13] got the local well-posedness with large initial data and global well-posedness provided that the coupling parameter and initial data are small enough. Lions and Masmoudi [17] got the global existence of weak solutions in the corotational case ($b = 0$). However, the case $b \neq 0$ is still not clear by now. In the framework of the near critical Besov spaces, Chemin and Masmoudi [3] first studied the local solutions and global small solutions of system (1.2) when $\mu > 0$, $K_1 > 0$, $K_2 > 0$, $\eta = 0$, $\beta > 0$. Zi, Fang and Zhang [20] improved the result obtained by Chemin and Masmoudi in [3] to the non-small coupling parameter case. Recently, Elgindi and Rousset [10] proved the global small solutions to (1.2) with $\mu = 0$, $K_1, K_2, \eta, \beta > 0$ in Sobolev space $H^s(\mathbb{R}^2), s > 2$. Moreover, if neglect the effect of the quadratic form $F(\tau, \nabla u)$ and let $\mu = 0, K_1 \geq 0, K_2 \in \mathbb{R}, \eta > 0, \beta \geq 0$, they also got the global solutions without any smallness imposed on the initial data in $\mathbb{R}^2$. Later on, Elgindi and Liu [9] consider the global well-posedness of system (1.2) in $\mathbb{R}^3$. When $\mu = 0, K_1 \geq 0, K_2 \in \mathbb{R}, \eta > 0, -\beta > 0$, they obtained the global small solutions in Sobolev spaces $H^s(\mathbb{R}^3)$, $s > 5/2$. Let us emphasis that the results obtained in [9], [10], [11], [17], [20] always require $\beta > 0$ in (1.2) (namely the system with damping) at least for non-trivial initial data. Thus, it's an interesting problem to study the global well-posedness when $\mu > 0, \eta = 0, \beta = 0, K_1 > 0, K_2 > 0$ in (1.2) in $\mathbb{R}^n (n = 2, 3)$. Most recently, Zhu [19] obtained the global small solutions to the three-dimensional incompressible Oldroyd-B model without damping on the stress tensor (i.e. $\beta = 0$), more precisely, the author in [19] proved the following theorem.

Theorem 1.1. *(see [19]) Let $n = 3$, μ, K_1, $K_2 > 0$. Suppose that $\text{div } u_0 = 0$, $(\tau_0)_{ij} = (\tau_0)_{ji}$ and initial data ε such that system (1.1) admits a unique global classical solution provided that*

$$\left\|\Lambda^{-1}u_0\right\|_{H^3} + \left\||\Lambda^{-1}\tau_0\right\|_{H^3} \leq \varepsilon,$$

where $\Lambda = (-\Delta)^{\frac{1}{2}}$.

However, the method used by Zhu in [19] is not valid for $n = 2$. Recently, Chen and Hao [4] generalized the result by Zhu in [19] to the critical Besov spaces. The aim of the present paper is to establish the global solutions of (1.1) with a class of highly oscillating initial velocity.

In all that follows, let $\mu = K_1 = K_2 = 1$ in (1.1), $\Lambda \overset{\text{def}}{=} (-\Delta)^{\frac{1}{2}}$, we shall denote the projector by $\mathbb{P} = \mathcal{I} - \mathcal{Q} \overset{\text{def}}{=} \mathcal{I} - \nabla\Delta^{-1}\text{div}$.

Now, we can state the main theorem of the present paper:

Theorem 1.2. *Let $n = 2, 3$ and*

$$2 \leq p \leq \min(4, 2n/(n-2)) \quad \text{and, additionally, } p \neq 4 \text{ if } n = 2.$$

For any $(u_0^\ell, \tau_0^\ell) \in \dot{B}_{2,1}^{\frac{n}{2}-1}(\mathbb{R}^n)$, $u_0^h \in \dot{B}_{p,1}^{\frac{n}{p}-1}(\mathbb{R}^n)$, $\tau_0^h \in \dot{B}_{p,1}^{\frac{n}{p}}(\mathbb{R}^n)$ with $\text{div } u_0 = 0$. If there exists a positive constant c_0 such that,

$$\left\|(u_0^\ell, \tau_0^\ell)\right\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \left\|u_0^h\right\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} + \left\|\tau_0^h\right\|_{\dot{B}_{p,1}^{\frac{n}{p}}} \leq c_0, \tag{1.3}$$

then the system (1.1) has a unique global solution (u, τ) so that for any $T > 0$

$$u^\ell \in C_b([0,T]; \dot{B}_{2,1}^{\frac{n}{2}-1}(\mathbb{R}^n)) \cap L^1([0,T]; \dot{B}_{2,1}^{\frac{n}{2}+1}(\mathbb{R}^n)),$$

$$\tau^\ell \in C_b([0,T]; \dot{B}_{2,1}^{\frac{n}{2}-1}(\mathbb{R}^n)), \quad (\Lambda^{-1}\mathbb{P}\text{div }\tau)^\ell \in L^1([0,T]; \dot{B}_{2,1}^{\frac{n}{2}+1}(\mathbb{R}^n)),$$

$$u^h \in C_b([0,T]; \dot{B}_{p,1}^{\frac{n}{p}-1}(\mathbb{R}^n)) \cap L^1([0,T]; \dot{B}_{p,1}^{\frac{n}{p}+1}(\mathbb{R}^n)),$$

$$\tau^h \in C_b([0,T]; \dot{B}_{p,1}^{\frac{n}{p}}(\mathbb{R}^n)), \quad (\Lambda^{-1}\mathbb{P}\text{div }\tau)^h \in L^1([0,T]; \dot{B}_{p,1}^{\frac{n}{p}}(\mathbb{R}^n)).$$

Remark 1.3. *By a similar argument as Zhu in [19], treating the nonlinear term to linear term, we can also get the global small solutions for the incompressible viscoelastic system with Hookean elasticity.*

Remark 1.4. *Most recently, Chen and Hao in [4] get the global well-posedness of (1.1) in $\mathbb{R}^n$ with $n \geq 2$. Compared with Chen and Hao in [4], the global solutions we constructed here allow the highly oscillating initial velocity. A typical example is*

$$u_0(x) = \sin\left(\frac{x_1}{\varepsilon}\right)\phi(x), \quad \phi(x) \in \mathcal{S}(\mathbb{R}^n), \quad p > n$$

which satisfies for any $\varepsilon > 0$

$$\left\|u_0^\ell\right\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \left\|u_0^h\right\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} \leq C\varepsilon^{1-\frac{n}{p}},$$

here C is a constant independent of ε (see [[5], Proposition 2.9]).

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Remark 1.5. *Compared with the result obtained by Chemin and Masmoudi in [3], we also obtain the global small solutions, yet there is no damping mechanism.*

Remark 1.6. *Our methods can be used to other related models. Similar results for the compressible Oldroyd-B model will be given in a forthcoming paper.*

Scheme of the proof and organization of the paper. The main difficulty to the proof of Theorem 1.2 lies in the fact that there is no dissipation in stress tensor. Thus, we can not get directly any integration for stress tensor τ about time in the basic energy argument. Indeed, we also can not get any integration about time of u. One can see more detail in the derivation of (3.7) in the third section.

To exploit the dissipation of u and to find the partial dissipation hidden for τ, let us first study the linear system of (1.1) (without loss of generality, set $\mu = K_1 = K_2 = 1$).

Applying project operator $\mathbb{P}$ on both hand side of the first two equation in (1.1) gives

$$
\begin{cases}
\partial_t u - \Delta u - \mathbb{P}\mathrm{div}\,\tau = G_1, \\
\partial_t \mathbb{P}\mathrm{div}\,\tau - \Delta u = G_2.
\end{cases}
\tag{1.4}
$$

At the linear level, to weaken the effect of Δu appeared in the stress tensor equation, we introduce $\phi \stackrel{\text{def}}{=} \Lambda^{-1}\mathbb{P}\mathrm{div}\,\tau$ with $\Lambda \stackrel{\text{def}}{=} (-\Delta)^{\frac{1}{2}}$, a simple computation from (1.4) gives

$$
\begin{cases}
\partial_t \phi + \Lambda u = \Lambda^{-1} G_2, \\
\partial_t u - \Delta u - \Lambda \phi = G_1.
\end{cases}
\tag{1.5}
$$

The above system is similar to the linear system of the compressible Navier-Stokes equations [7]. In the following, we recall the analysis of the linearized system (1.5). Taking the Fourier transform with respect to x, System (1.5) translates into

$$
\frac{d}{dt}\begin{pmatrix} \hat{\phi} \\ \hat{u} \end{pmatrix} = A(\xi)\begin{pmatrix} \hat{\phi} \\ \hat{u} \end{pmatrix} + \begin{pmatrix} \widehat{\Lambda^{-1}G_2} \\ \widehat{G_1} \end{pmatrix} \quad \text{with} \quad A(\xi) \stackrel{\text{def}}{=} \begin{pmatrix} 0 & -|\xi| \\ |\xi| & -|\xi|^2 \end{pmatrix}.
\tag{1.6}
$$

- In the low frequency regime $|\xi| < 2$, $A(\xi)$ has two complex conjugated eigenvalues:

$$
\lambda_{\pm}(\xi) \stackrel{\text{def}}{=} -\frac{\xi^2}{2}\left(1 \pm i\sqrt{\frac{4 - \xi^2}{\xi^2}}\right)
$$

  which have real part $-\frac{\xi^2}{2}$, exactly as for the heat equation with diffusion $\frac{1}{2}$.

- In the high frequency regime $|\xi| > 2$, there are two distinct real eigenvalues:

$$
\lambda_{\pm}(\xi) \stackrel{\text{def}}{=} -\frac{\xi^2}{2}\left(1 \pm \sqrt{\frac{\xi^2 - 4}{\xi^2}}\right).
$$

  As $1 + \sqrt{\frac{\xi^2-4}{\xi^2}} \sim 2$ and $1 - \sqrt{\frac{\xi^2-4}{\xi^2}} \sim \frac{2}{\xi^2}$ for $\xi \to +\infty$, we can deduce that $\lambda_+(\xi) \sim -\xi^2$ and $\lambda_-(\xi) \sim -1$. In other words, a parabolic and a damped mode coexist.

Optimal a priori estimates may be easily derived by computing the explicit solution of (1.6) explicitly in the Fourier space.

In the second section, we shall collect some basic facts on Littlewood-Paley analysis and various product laws in Besov spaces. In Section 3, we will use three subsections to prove the main Theorem 1.2, we apply the Littlewood-Paley theory to get the basic energy estimates for (u, τ), and then by introducing a new quantity, we get the low frequency and high frequency of the solutions of $(a, \mathbb{P}\text{div } \tau)$ in the first subsection and the second subsection, respectively. Finally in the last subsection, we present the proof to the global well-posedness of Theorem 1.2 by standard continuous argument.

Notations : Let A, B be two operators, we denote $[A, B] = AB - BA$, the commutator between A and B. For $a \lesssim b$, we mean that there is a uniform constant C, which may be different on different lines, such that $a \leq Cb$. We shall denote by $\langle a, b \rangle$ the $L^2(\mathbb{R}^n)$ inner product of a and b.

2. Preliminaries

The Littlewood-Paley decomposition plays a central role in our analysis. To define it, fix some smooth radial non increasing function χ supported in the ball $B(0, \frac{4}{3})$ of $\mathbb{R}^n$, and with value 1 on, say, $B(0, \frac{3}{4})$, then set $\varphi(\xi) = \chi(\frac{\xi}{2}) - \chi(\xi)$. We have

$$\sum_{j \in \mathbb{Z}} \varphi(2^{-j} \cdot) = 1 \quad \text{in} \quad \mathbb{R}^n \setminus \{0\} \quad \text{and} \quad \text{Supp } \varphi \subset \left\{ \xi \in \mathbb{R}^n : \frac{3}{4} \leq |\xi| \leq \frac{8}{3} \right\}.$$

The homogeneous dyadic blocks $\dot{\Delta}_j$ are defined on tempered distributions by

$$\dot{\Delta}_j u \stackrel{\text{def}}{=} \varphi(2^{-j} D) u \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\varphi(2^{-j} \cdot) \mathcal{F} u).$$

In order to ensure that

$$u = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n), \tag{2.1}$$

we restrict our attention to those tempered distributions u such that

$$\lim_{k \to -\infty} \|\dot{S}_k u\|_{L^\infty} = 0, \tag{2.2}$$

where $\dot{S}_k u$ stands for the low frequency cut-off defined by $\dot{S}_k u \stackrel{\text{def}}{=} \chi(2^{-k} D) u$.

Definition 2.1. *For $s \in \mathbb{R}$, $1 \leq p \leq \infty$, the homogeneous Besov space $\dot{B}^s_{p,1} \stackrel{\text{def}}{=} \dot{B}^s_{p,1}(\mathbb{R}^n)$ is the set of tempered distributions u satisfying (2.2) and*

$$\|u\|_{\dot{B}^s_{p,1}} \stackrel{\text{def}}{=} \sum_{j \in \mathbb{Z}} 2^{js} \|\dot{\Delta}_j u\|_{L^p} < \infty. \tag{2.3}$$

Remark 2.2. *For $s \leq \frac{n}{p}$ (which is the only case we are concerned with in this paper), $\dot{B}^s_{p,1}$ is a Banach space which coincides with the completion for $\| \cdot \|_{\dot{B}^s_{p,1}}$ of the set $\mathcal{S}_0(\mathbb{R}^n)$ of Schwartz functions with Fourier transform supported away from the origin.*

In this paper, we frequently use the so-called "time-space" Besov spaces or Chemin-Lerner space first introduced by Chemin and Lerner [1].

Definition 2.3. *Let $s \in \mathbb{R}$ and $0 < T \leq +\infty$. We define*

$$\|u\|_{\widetilde{L}_T^q(\dot{B}_{p,1}^s)} \overset{\text{def}}{=} \sum_{j \in \mathbb{Z}} 2^{js} \left(\int_0^T \|\dot{\Delta}_j u(t)\|_{L^p}^q dt \right)^{\frac{1}{q}} \tag{2.4}$$

for $q, p \in [1, \infty)$ and with the standard modification for $p, q = \infty$.

By Minkowski's inequality, we have the following inclusions between the Chemin-Lerner space $\widetilde{L}_T^\lambda(\dot{B}_{p,1}^s)$ and the Bochner space $L_T^\lambda(\dot{B}_{p,1}^s)$:

$$\|u\|_{\widetilde{L}_T^\lambda(\dot{B}_{p,1}^s)} \leq \|u\|_{L_T^\lambda(\dot{B}_{p,1}^s)} \quad \text{if } \lambda \leq r, \quad \|u\|_{\widetilde{L}_T^\lambda(\dot{B}_{p,1}^s)} \geq \|u\|_{L_T^\lambda(\dot{B}_{p,1}^s)}, \quad \text{if } \lambda \geq r.$$

Restricting the above norms (2.3) and (2.4) to the low or high frequency parts of distributions will be crucial in our approach. For example, let us fix some integer j_0 (the value of which will follow from the proof of the main theorem) and set[1]

$$\|z^\ell\|_{\dot{B}_{p,1}^s} \overset{\text{def}}{=} \sum_{j \leq j_0} 2^{js} \|\dot{\Delta}_j z\|_{L^p} \quad \text{and} \quad \|z^h\|_{\dot{B}_{p,1}^s} \overset{\text{def}}{=} \sum_{j \geq j_0 - 2} 2^{js} \|\dot{\Delta}_j z\|_{L^p}, \tag{2.5}$$

$$\|z^\ell\|_{\widetilde{L}_T^\infty(\dot{B}_{p,1}^s)} \overset{\text{def}}{=} \sum_{j \leq j_0} 2^{js} \|\dot{\Delta}_j z\|_{L_T^\infty(L^p)} \quad \text{and} \quad \|z^h\|_{\widetilde{L}_T^\infty(\dot{B}_{p,1}^s)} \overset{\text{def}}{=} \sum_{j \geq j_0 - 2} 2^{js} \|\dot{\Delta}_j z\|_{L_T^\infty(L^p)}. \tag{2.6}$$

The following Bernstein's lemma will be repeatedly used throughout this paper.

Lemma 2.4. *Let $\mathcal{B}$ be a ball and $\mathcal{C}$ a ring of $\mathbb{R}^n$. A constant C exists so that for any positive real number λ, any non-negative integer k, any smooth homogeneous function σ of degree m, and any couple of real numbers (p,q) with $1 \leq p \leq q \leq \infty$, there hold*

$$\text{Supp}\,\hat{u} \subset \lambda\mathcal{B} \Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^q} \leq C^{k+1} \lambda^{k+n(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^p},$$

$$\text{Supp}\,\hat{u} \subset \lambda\mathcal{C} \Rightarrow C^{-k-1}\lambda^k \|u\|_{L^p} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^p} \leq C^{k+1}\lambda^k \|u\|_{L^p},$$

$$\text{Supp}\,\hat{u} \subset \lambda\mathcal{C} \Rightarrow \|\sigma(D)u\|_{L^q} \leq C_{\sigma,m} \lambda^{m+n(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^p}.$$

Next we recall a few nonlinear estimates in Besov spaces which may be obtained by means of paradifferential calculus. Here, we recall the decomposition in the homogeneous context:

$$uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u,v), \tag{2.7}$$

where

$$\dot{T}_u v \overset{\text{def}}{=} \sum_{j \in \mathbb{Z}} \dot{S}_{j-1} u \dot{\Delta}_j v, \quad \dot{R}(u,v) \overset{\text{def}}{=} \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u \widetilde{\dot{\Delta}}_j v, \quad \text{and} \quad \widetilde{\dot{\Delta}}_j v \overset{\text{def}}{=} \sum_{|j-j'| \leq 1} \dot{\Delta}_{j'} v.$$

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[1]Note that for technical reasons, we need a small overlap between low and high frequency.

The paraproduct $\dot{T}$ and the remainder $\dot{R}$ operators satisfy the following continuous properties.

Lemma 2.5 ([1]). *For all $s \in \mathbb{R}$, $\sigma \geq 0$, and $1 \leq p, p_1, p_2 \leq \infty$, the paraproduct $\dot{T}$ is a bilinear, continuous operator from $\dot{B}^{-\sigma}_{p_1,1} \times \dot{B}^{s}_{p_2,1}$ to $\dot{B}^{s-\sigma}_{p,1}$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. The remainder $\dot{R}$ is bilinear continuous from $\dot{B}^{s_1}_{p_1,1} \times \dot{B}^{s_2}_{p_2,1}$ to $\dot{B}^{s_1+s_2}_{p,1}$ with $s_1 + s_2 > 0$, and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.*

Lemma 2.6. *Let $n = 2, 3$ and*

$$2 \leq p \leq \min(4, 2n/(n-2)) \quad \text{and, additionally, } p \neq 4 \text{ if } n = 2.$$

For any $(u, v) \in \dot{B}^{\frac{n}{p}-1}_{p,1} \cap \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)$, there holds

$$\|uv\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \lesssim \left(\|u\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} \|v\|_{\dot{B}^{\frac{n}{p}}_{p,1}} + \|u\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \|v\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} \right). \tag{2.8}$$

Proof. According to Bony's decomposition, we can write

$$uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u, v).$$

By Lemma 2.5, let $\frac{1}{p*} = \frac{1}{2} - \frac{1}{p}$, we have

$$\|\dot{T}_u v + \dot{R}(u,v)\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \lesssim \|u\|_{\dot{B}^{\frac{n}{p*}-1}_{p*,1}} \|v\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \lesssim \|u\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} \|v\|_{\dot{B}^{\frac{n}{p}}_{p,1}}.$$

Similarly, one can get

$$\|\dot{T}_v u\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \lesssim \|v\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} \|u\|_{\dot{B}^{\frac{n}{p}}_{p,1}}.$$

Thus, we complete the proof of this lemma. $\square$

We also need the following omitted proofs product law and commutator's estimates in Besov spaces.

Lemma 2.7. *Let $1 \leq p, q \leq \infty$, $s_1 \leq \frac{n}{q}$, $s_2 \leq n \min\{\frac{1}{p}, \frac{1}{q}\}$, and $s_1 + s_2 > n \max\{0, \frac{1}{p} + \frac{1}{q} - 1\}$. For $\forall (u, v) \in \dot{B}^{s_1}_{q,1}(\mathbb{R}^n) \times \dot{B}^{s_2}_{p,1}(\mathbb{R}^n)$, we have*

$$\|uv\|_{\dot{B}^{s_1+s_2-\frac{n}{q}}_{p,1}} \lesssim \|u\|_{\dot{B}^{s_1}_{q,1}} \|v\|_{\dot{B}^{s_2}_{p,1}}.$$

Lemma 2.8. *([1, Lemma 2.100]) Let $1 \leq p, q \leq \infty$, $-1 - n \min\{\frac{1}{p}, 1 - \frac{1}{p}\} < s \leq 1 + \frac{n}{p}$. For any $v \in \dot{B}^{s}_{p,1}(\mathbb{R}^n)$ and $\nabla u \in \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)$ with $\operatorname{div} u = 0$, there holds*

$$\|[u \cdot \nabla, \dot{\Delta}_j]v\|_{L^p} \lesssim d_j 2^{-js} \|\nabla u\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \|v\|_{\dot{B}^{s}_{p,1}}.$$

Lemma 2.9. *Let $n = 2, 3$ and*

$$2 \leq p \leq \min(4, 2n/(n-2)) \quad \text{and, additionally, } p \neq 4 \text{ if } n = 2.$$

For any $v^\ell \in \dot{B}^{\frac{n}{2}-1}_{2,1}(\mathbb{R}^n)$, $v^h \in \dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n)$, $\nabla u \in \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)$ with $\operatorname{div} u = 0$, there exists a constant C such that

$$\sum_{j \leq j_0} 2^{(\frac{n}{2}-1)j} \|[\dot{\Delta}_j, u \cdot \nabla]v\|_{L^2} \leq C \|\nabla u\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \left(\|v^\ell\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} + \|v^h\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} \right). \tag{2.9}$$

Proof. Using the notion of para-products, we can easily write

$$[\dot{\Delta}_j, u \cdot \nabla]v \overset{\text{def}}{=} I_j^1 + I_j^2 + I_j^3,$$

with

$$I_j^1 \overset{\text{def}}{=} \sum_{|k-j|\leq 2}[\dot{\Delta}_j, \dot{S}_{k-1}u \cdot \nabla]\dot{\Delta}_k v, \quad I_j^2 \overset{\text{def}}{=} \sum_{|k-j|\leq 2}[\dot{\Delta}_j, \dot{\Delta}_k u \cdot \nabla]\dot{S}_{k-1}v,$$

$$I_j^3 \overset{\text{def}}{=} \sum_{k\geq j-1}[\dot{\Delta}_j, \dot{\Delta}_k u \cdot \nabla]\widetilde{\dot{\Delta}}_k v, \qquad \widetilde{\dot{\Delta}}_k = \dot{\Delta}_{k-1} + \dot{\Delta}_k + \dot{\Delta}_{k+1}.$$

From the definition of Bony's decomposition, one can write I_j^1 into

$$I_j^1 = \sum_{|k-j|\leq 2}[\dot{\Delta}_j, \dot{S}_{k-1}u_m]\partial_m \dot{\Delta}_k v$$

$$=2^{jn} \sum_{|k-j|\leq 2}\int_{\mathbb{R}^n} h(2^j y)(\dot{S}_{k-1}u_m(x-y) - \dot{S}_{k-1}u_m(x))\partial_m\dot{\Delta}_k v(x-y)dy$$

$$=-2^{jn} \sum_{|k-j|\leq 2}\int_{\mathbb{R}^n} h(2^j y)\Big(\int_0^1 y\cdot\nabla\dot{S}_{k-1}u_m(x-\tau y)\,d\tau\Big)\partial_m\dot{\Delta}_k v(x-y)dy$$

from which and the Hölder inequality, we have

$$\|I_j^1\|_{L^2} \lesssim \sum_{|k-j|\leq 2}2^{-j}\|\nabla\dot{S}_{k-1}u\|_{L^\infty}\|\nabla\dot{\Delta}_k v\|_{L^2}$$

$$\lesssim \sum_{|k-j|\leq 2}2^{k-j}\|\nabla\dot{S}_{k-1}u\|_{L^\infty}\|\dot{\Delta}_k v\|_{L^2}. \tag{2.10}$$

As there is a small overlap between low and high frequency in the definition of (2.5), we can further deduce from (2.10) that

$$\sum_{j\leq j_0}2^{(\frac{n}{2}-1)j}\|I_j^1\|_{L^2} \leq C\|\nabla u\|_{L^\infty}\|v^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}. \tag{2.11}$$

Let us turn to the second term I_j^2. Using the fact that the support of $\dot{\Delta}_j(\dot{\Delta}_k u \cdot \nabla \dot{S}_{k-1}v)$ is restricted in an annulus, we can get similarly to I_j^1 that

$$\|I_j^2\|_{L^2} \lesssim \sum_{|k-j|\leq 2}2^{-j}\|\nabla\dot{S}_{k-1}v\|_{L^{\frac{2p}{p-2}}}\|\nabla\dot{\Delta}_k u\|_{L^p}$$

$$\lesssim \sum_{|k-j|\leq 2}2^{-j}\Big(\sum_{k'\leq k-2}2^{k'}2^{\frac{nk'}{p}}\|\dot{\Delta}_{k'}v\|_{L^2}\Big)\|\nabla\dot{\Delta}_k u\|_{L^p}$$

$$\lesssim \sum_{|k-j|\leq 2}(2^{(1-\frac{n}{2})k}d_k\|v^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}(2^{\frac{nk}{p}}\|\nabla\dot{\Delta}_k u\|_{L^p}),$$

which give rise to

$$\sum_{j\leq j_0}2^{(\frac{n}{2}-1)j}\|I_j^2\|_{L^2} \leq C\|\nabla u\|_{\dot{B}_{p,1}^{\frac{n}{p}}}\|v^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}. \tag{2.12}$$

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It is much more involved to handle the remainder term I_j^3. We split it into two terms: high frequency and low frequency

$$I_j^3 = \sum_{k \geq j-1} [\dot\Delta_j, \dot\Delta_k u \cdot \nabla]\widetilde{\dot\Delta_k} v$$

$$= \sum_{j-1 \leq k \leq j} [\dot\Delta_j, \dot\Delta_k u \cdot \nabla]\widetilde{\dot\Delta_k} v + \sum_{k > j}[\dot\Delta_j, \dot\Delta_k u \cdot \nabla]\widetilde{\dot\Delta_k} v. \tag{2.13}$$

Exact the same line as I_j^1, we can get

$$\sum_{j \leq j_0} 2^{(\frac{n}{2}-1)j}\Big\| \sum_{j-1 \leq k \leq j} [\dot\Delta_j, \dot\Delta_k u \cdot \nabla]\widetilde{\dot\Delta_k} v\Big\|_{L^2} \leq C\|\nabla u\|_{L^\infty}\|v^\ell\|_{\dot B_{2,1}^{\frac{n}{2}-1}}. \tag{2.14}$$

Due to lack of quasi-orthogonality, we divide the second term on the right hand side of (2.13) into two terms:

$$\sum_{k>j}[\dot\Delta_j, \dot\Delta_k u \cdot \nabla]\widetilde{\dot\Delta_k} v = \sum_{k>j}\dot\Delta_j(\dot\Delta_k u \cdot \nabla\widetilde{\dot\Delta_k} v) + \sum_{j<k,|k-j|\leq 3}\dot\Delta_k u \cdot \nabla\dot\Delta_j\widetilde{\dot\Delta_k} v$$

$$\stackrel{\text{def}}{=} I_j^{3,1} + I_j^{3,2}.$$

To bound $I_j^{3,1}$, we need to further write

$$\|I_j^{3,1}\|_{L^2} = \sum_{j<k\leq j_0}\|I_j^{3,1}\|_{L^2} + \sum_{j\leq j_0 < k}\|I_j^{3,1}\|_{L^2}. \tag{2.15}$$

Using the condition $\operatorname{div} u = 0$ and the Hölder inequality gives

$$\sum_{j<k\leq j_0}\|I_j^{3,1}\|_{L^2} \lesssim \sum_{j<k<j_0}\|\dot\Delta_j\partial_m(\dot\Delta_k u_m \cdot \widetilde{\dot\Delta_k} v)\|_{L^2}$$

$$\lesssim \sum_{j<k<j_0} 2^j\|\dot\Delta_k u \cdot \widetilde{\dot\Delta_k} v\|_{L^2}$$

$$\lesssim \sum_{j<k<j_0} 2^{j-k}\|\dot\Delta_k \nabla u\|_{L^\infty}\|\widetilde{\dot\Delta_k} v\|_{L^2}$$

$$\lesssim \|\nabla u\|_{L^\infty}\sum_{j<k<j_0} 2^{j-k}\|\widetilde{\dot\Delta_k} v\|_{L^2},$$

which implies

$$\sum_{j<k\leq j_0} 2^{(\frac{n}{2}-1)j}\|I_j^{3,1}\|_{L^2} \leq C\|\nabla u\|_{L^\infty}\|v^\ell\|_{\dot B_{2,1}^{\frac{n}{2}-1}}. \tag{2.16}$$

Similarly, the second term in (2.15) can be estimated as follow:

$$\sum_{j\leq j_0 < k} 2^{(\frac{n}{2}-1)j}\|I_j^{3,1}\|_{L^2} \lesssim \sum_{j\leq j_0 < k} 2^{(\frac{n}{2}-1)j}\|\dot\Delta_j\partial_m(\dot\Delta_k u_m \cdot \widetilde{\dot\Delta_k} v)\|_{L^2}$$

$$\lesssim \sum_{j\leq j_0 < k} 2^{(\frac{n}{2}-1)j}2^j\|\dot\Delta_k u \cdot \widetilde{\dot\Delta_k} v\|_{L^2}$$

$$\lesssim \sum_{j\leq j_0 < k} 2^{(\frac{n}{2}-1)j}2^{j-k}\|\dot\Delta_k \nabla u\|_{L^{\frac{2p}{p-2}}}\|\widetilde{\dot\Delta_k} v\|_{L^p}$$

$$\lesssim \sum_{j\leq j_0 < k} 2^{(\frac{n}{2}-1)j}2^{j-k}2^{(\frac{1}{p}-\frac{p-2}{2p})nk}\|\dot\Delta_k \nabla u\|_{L^p}\|\widetilde{\dot\Delta_k} v\|_{L^p}$$

$$\lesssim \|\nabla u\|_{\dot B_{p,1}^{\frac{n}{p}}}\|v^h\|_{\dot B_{p,1}^{\frac{n}{p}-1}}. \tag{2.17}$$

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In virtue of the embedding relation $\dot{B}_{p,1}^{\frac{n}{p}}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$, we get from (2.16) and (2.17) that

$$\sum_{j \leq j_0} 2^{(\frac{n}{2}-1)j} \|I_j^{3,1}\|_{L^2} \leq C \|\nabla u\|_{\dot{B}_{p,1}^{\frac{n}{p}}} \left(\|v^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} + \|v^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} \right). \tag{2.18}$$

Thanks to Lemma 2.4, we have

$$\sum_{j \leq j_0} 2^{(\frac{n}{2}-1)j} \|I_j^{3,2}\|_{L^2} \lesssim \sum_{j \leq j_0} 2^{(\frac{n}{2}-1)j} \sum_{j < k, |k-j| \leq 3} \|\dot{\Delta}_k u \cdot \nabla \dot{\Delta}_j \widetilde{\dot{\Delta}_k} v\|_{L^2}$$

$$\lesssim \sum_{j \leq j_0} 2^{(\frac{n}{2}-1)j} \sum_{|k-j| \leq 3} 2^{j-k} \|\nabla \dot{\Delta}_k u\|_{L^\infty} \|\dot{\Delta}_j \widetilde{\dot{\Delta}_k} v\|_{L^2}$$

$$\lesssim \|\nabla u\|_{L^\infty} \|v^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}. \tag{2.19}$$

Together with (2.14), (2.18), (2.19), we get from (2.13) that

$$\sum_{j \leq j_0} 2^{(\frac{n}{2}-1)j} \|I_j^3\|_{L^2} \leq C \|\nabla u\|_{\dot{B}_{p,1}^{\frac{n}{p}}} \left(\|v^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} + \|v^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} \right). \tag{2.20}$$

Thus, the estimate (2.9) can be obtained from the combinations of (2.11), (2.12), (2.20).

Consequently, we complete the proof of the lemma. $\qquad\square$

Corollary 2.10. *Under the assumption of Lemma 2.9, let $A(D)$ be a zero-order Fourier multiplier, by the same processes as the proof of Lemma 2.9, we can get the following two estimates hold:*

$$\sum_{j \leq j_0} 2^{(\frac{n}{2}-1)j} \left\| [\dot{\Delta}_j A(D), u \cdot \nabla] v \right\|_{L^2} \leq C \|\nabla u\|_{\dot{B}_{p,1}^{\frac{n}{p}}} \left(\|v^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|v^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} \right),$$

$$\sum_{j \leq j_0} 2^{(\frac{n}{2}-1)j} \left\| [\dot{\Delta}_j, u \cdot \nabla] A(D) v \right\|_{L^2} \leq C \|\nabla u\|_{\dot{B}_{p,1}^{\frac{n}{p}}} \left(\|v^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|v^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} \right).$$

Lemma 2.11. *([8, Lemma 6.1]) Let $A(D)$ be a zero-order Fourier multiplier. Let $j_0 \in \mathbb{Z}$, $s < 1$, $\sigma \in \mathbb{R}$, $1 \leq p_1, p_2 \leq \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then there exists a constant C depending only on j_0 and on the regularity parameters such that*

$$\left\| [\dot{S}_{j_0} A(D), T_u] v \right\|_{\dot{B}_{p,1}^{\sigma+s}} \leq C \|\nabla u\|_{\dot{B}_{p_1,1}^{s-1}} \|v\|_{\dot{B}_{p_2,1}^{\sigma}}$$

and, for $s = 1$,

$$\left\| [\dot{S}_{j_0} A(D), T_u] v \right\|_{\dot{B}_{p,1}^{\sigma+1}} \leq C \|\nabla u\|_{L^{p_1}} \|v\|_{\dot{B}_{p_2,1}^{\sigma}}.$$

3. The proof of the Theorem 1.2

According to the local well-posedness obtained by [3], [4], we can deduce similarly that there exists a positive time T so that the system (1.1) has a uniqueness local solution (u, τ) on $[0, T^*)$ such that for any $T < T^*$

$$u^\ell \in C_b([0,T]; \dot{B}_{2,1}^{\frac{n}{2}-1}(\mathbb{R}^n)) \cap L^1([0,T]; \dot{B}_{2,1}^{\frac{n}{2}+1}(\mathbb{R}^n)),$$

$$\tau^\ell \in C_b([0,T]; \dot{B}_{2,1}^{\frac{n}{2}-1}(\mathbb{R}^n)), \quad (\Lambda^{-1}\mathbb{P}\mathrm{div}\,\tau)^\ell \in L^1([0,T]; \dot{B}_{2,1}^{\frac{n}{2}+1}(\mathbb{R}^n)),$$

$$u^h \in C_b([0,T]; \dot{B}_{p,1}^{\frac{n}{p}-1}(\mathbb{R}^n)) \cap L^1([0,T]; \dot{B}_{p,1}^{\frac{n}{p}+1}(\mathbb{R}^n)),$$

$$\tau^h \in C_b([0,T]; \dot{B}_{p,1}^{\frac{n}{p}}(\mathbb{R}^n)), \quad (\Lambda^{-1}\mathbb{P}\mathrm{div}\,\tau)^h \in L^1([0,T]; \dot{B}_{p,1}^{\frac{n}{p}}(\mathbb{R}^n)). \tag{3.1}$$

We denote T^* to be the largest possible time such that there holds (3.1). Then, the proof of Theorem 1.2 is reduced to show that $T^* = \infty$ under the assumption of (1.3). In order to do so, we need to make a priori estimates for the smooth solution of system (1.1).

3.1. The low frequency estimates of the solutions

Applying $\dot{\Delta}_j \mathbb{P}$ to the second equation in (1.1) and using a standard commutator's process give

$$\partial_t \dot{\Delta}_j u + u \cdot \nabla \dot{\Delta}_j u - \Delta \dot{\Delta}_j u - \dot{\Delta}_j \mathbb{P} \mathrm{div}\, \tau = [u \cdot \nabla, \dot{\Delta}_j \mathbb{P}] u. \tag{3.2}$$

Similarly, from the first equation in (1.1), we have

$$\partial_t \dot{\Delta}_j \tau + u \cdot \nabla \dot{\Delta}_j \tau + \dot{\Delta}_j F(\tau, \nabla u) - \dot{\Delta}_j D(u) = [u \cdot \nabla, \dot{\Delta}_j] \tau. \tag{3.3}$$

Taking L^2 inner product with $\dot{\Delta}_j u$ on both hand side of (3.2) and using the fact that $\langle u \cdot \nabla \dot{\Delta}_j u, \dot{\Delta}_j u \rangle = 0$ give

$$\frac{1}{2} \frac{d}{dt} \|\dot{\Delta}_j u\|_{L^2}^2 + c_1 \|\nabla \dot{\Delta}_j u\|_{L^2}^2 = \langle \dot{\Delta}_j \mathbb{P} \mathrm{div}\, \tau, \dot{\Delta}_j u \rangle + \langle [u \cdot \nabla, \dot{\Delta}_j \mathbb{P}] u, \dot{\Delta}_j u \rangle. \tag{3.4}$$

Similarly, taking L^2 inner product with $\dot{\Delta}_j \tau$ on both hand side of (3.3) and using the fact that $\langle u \cdot \nabla \dot{\Delta}_j \tau, \dot{\Delta}_j \tau \rangle = 0$, we can get

$$\frac{1}{2} \frac{d}{dt} \|\dot{\Delta}_j \tau\|_{L^2}^2 = \langle \dot{\Delta}_j D(u), \dot{\Delta}_j \tau \rangle + \langle [u \cdot \nabla, \dot{\Delta}_j] \tau, \dot{\Delta}_j \tau \rangle - \langle \dot{\Delta}_j F(\tau, \nabla u), \dot{\Delta}_j \tau \rangle. \tag{3.5}$$

It's not difficult to check

$$\langle \dot{\Delta}_j \mathbb{P} \mathrm{div}\, \tau, \dot{\Delta}_j u \rangle + \langle \dot{\Delta}_j D(u), \dot{\Delta}_j \tau \rangle = 0.$$

Thus, summing up (3.4), (3.5) and using the above fact we have

$$\frac{1}{2} \frac{d}{dt} (\|\dot{\Delta}_j u\|_{L^2}^2 + \|\dot{\Delta}_j \tau\|_{L^2}^2) + c_1 2^{2j} \|\dot{\Delta}_j u\|_{L^2}^2$$
$$\lesssim |\langle [u \cdot \nabla, \dot{\Delta}_j \mathbb{P}] u, \dot{\Delta}_j u \rangle| + |\langle [u \cdot \nabla, \dot{\Delta}_j] \tau, \dot{\Delta}_j \tau \rangle| + |\langle \dot{\Delta}_j F(\tau, \nabla u), \dot{\Delta}_j \tau \rangle| \tag{3.6}$$

in which we have used the following Bernstein's inequality: there exists a positive constant c_1 so that

$$- \int_{\mathbb{R}^n} \Delta \dot{\Delta}_j u \cdot \dot{\Delta}_j u \, dx \geq c_1 2^{2j} \|\dot{\Delta}_j u\|_{L^2}^2.$$

Due to lack of full dissipation for stress tensor τ in (1.1), thus, we have to give up the dissipation for u also at present. In the following, we will get back the full dissipation of velocity and the partial dissipation of stress tensor by introducing a new quantity.

Employing the Hölder inequality to (3.6), integrating the resultant inequality from 0 to t, and multiplying by $2^{j(\frac{n}{2}-1)}$, we can get by summing up about $j \leq j_0$ that

$$\|(u^\ell, \tau^\ell)\|_{\widetilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})} \lesssim \|(u_0^\ell, \tau_0^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \sum_{j \leq j_0} 2^{(\frac{n}{2}-1)j} \|[u \cdot \nabla, \dot{\Delta}_j \mathbb{P}] u\|_{L_t^1(L^2)}$$

$$+ \sum_{j \leq j_0} 2^{(\frac{n}{2}-1)j} \|[u \cdot \nabla, \dot{\Delta}_j] \tau\|_{L_t^1(L^2)} + \int_0^t \|(F(\tau, \nabla u))^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} \, ds. \tag{3.7}$$

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It follows from Lemma 2.9 that

$$\sum_{j\leq j_0} 2^{(\frac{n}{2}-1)j}\|[u\cdot\nabla,\dot{\Delta}_j\mathbb{P}]u\|_{L^1_t(L^2)}$$

$$\lesssim \int_0^t \|\nabla u\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\left(\|u^\ell\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}+\|u^h\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\right)ds$$

$$\lesssim \int_0^t \left(\|u^\ell\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}+\|u^h\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\right)\left(\|u^\ell\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}+\|u^h\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\right)ds \qquad (3.8)$$

and

$$\sum_{j\leq j_0} 2^{(\frac{n}{2}-1)j}\|[u\cdot\nabla,\dot{\Delta}_j]\tau\|_{L^1_t(L^2)}$$

$$\lesssim \int_0^t \|\nabla u\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\left(\|\tau^\ell\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}+\|\tau^h\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\right)ds$$

$$\lesssim \int_0^t \left(\|\tau^\ell\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}+\|\tau^h\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\right)\left(\|u^\ell\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}+\|u^h\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\right)ds. \qquad (3.9)$$

In order to estimate the last term in (3.7), we first use the Bony decomposition to write

$$\dot{S}_{j_0+1}(\tau\nabla u) = \dot{S}_{j_0+1}\left(\dot{T}_\tau\nabla u + \dot{R}(\tau,\nabla u)\right) + \dot{T}_{\nabla u}\dot{S}_{j_0+1}\tau + [\dot{S}_{j_0+1},T_{\nabla u}]\tau. \qquad (3.10)$$

By virtue of Lemma 2.5, we obtain

$$\|\dot{T}_{\nabla u}\dot{S}_{j_0+1}\tau\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \lesssim \|\nabla u\|_{L^\infty}\|\tau^\ell\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \lesssim \|\tau^\ell\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\|\nabla u\|_{\dot{B}^{\frac{n}{p}}_{p,1}}, \qquad (3.11)$$

and for $\frac{1}{p*}=\frac{1}{2}-\frac{1}{p}$

$$\|\dot{S}_{j_0+1}\left(T_\tau\nabla u + \dot{R}(\tau,\nabla u)\right)\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \lesssim \|\tau\|_{\dot{B}^{-1+\frac{n}{p*}}_{p*,1}}\|\nabla u\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \lesssim \|\tau\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}}\|\nabla u\|_{\dot{B}^{\frac{n}{p}}_{p,1}}. \qquad (3.12)$$

By Lemma 2.11, we have

$$\|[\dot{S}_{j_0+1},\dot{T}_{\nabla u}]\tau\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \lesssim \|\nabla^2 u\|_{\dot{B}^{\frac{n}{p*}-1}_{p*,1}}\|\tau\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} \lesssim \|\tau\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\|\nabla u\|_{\dot{B}^{\frac{n}{p}}_{p,1}}. \qquad (3.13)$$

Combining with (3.10)–(3.13) implies

$$\int_0^t \|(F(\tau,\nabla u))^\ell\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}ds \lesssim \int_0^t \|\nabla u\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\left(\|\tau^\ell\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}+\|\tau^h\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\right)ds$$

$$\lesssim \int_0^t \left(\|\tau^\ell\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}+\|\tau^h\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\right)\left(\|u^\ell\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}+\|u^h\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\right)ds. \qquad (3.14)$$

Taking estimates (3.9) and (3.14) into (3.7) gives

$$\|(u^\ell,\tau^\ell)\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n}{2}-1}_{2,1})} \lesssim \|(u_0^\ell,\tau_0^\ell)\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}$$

$$+\int_0^t \left(\|(u^\ell,\tau^\ell)\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}+\|u^h\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}+\|\tau^h\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\right)\left(\|u^\ell\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}+\|u^h\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\right)ds. \qquad (3.15)$$

In the above low frequency arguments, we do not get any integration in time for u, τ. Next, we shall use the special structure of (1.1) to obtain the smoothing effect of u and partial smoothing effect of τ.

Applying project operator $\mathbb{P}$ on both hand side of the first two equation in (1.1) gives

$$\begin{cases} \partial_t u + \mathbb{P}(u \cdot \nabla u) - \Delta u - \mathbb{P}\mathrm{div}\, \tau = 0, \\ \partial_t \mathbb{P}\mathrm{div}\, \tau + \mathbb{P}\mathrm{div}\, (u \cdot \nabla \tau) - \Delta u + \mathbb{P}\mathrm{div}\, (F(\tau, \nabla u)) = 0. \end{cases} \tag{3.16}$$

Define

$$\phi = \Lambda^{-1}\mathbb{P}\mathrm{div}\, \tau \quad \text{and} \quad w = \Lambda\phi - u,$$

we can get by a simple computation from (3.16) that

$$\begin{cases} \partial_t \phi + u \cdot \nabla \phi + \Lambda u = f, \\ \partial_t u + u \cdot \nabla u - \Delta u - \Lambda\phi = g, \\ \partial_t w + u \cdot \nabla w + \Lambda\phi = G, \end{cases} \tag{3.17}$$

in which

$$f \overset{\text{def}}{=} -[\Lambda^{-1}\mathbb{P}\mathrm{div}, u \cdot \nabla]\tau - \Lambda^{-1}\mathbb{P}\mathrm{div}\, (F(\tau, \nabla u)),$$
$$g \overset{\text{def}}{=} -[\mathbb{P}, u \cdot \nabla]u, \qquad G \overset{\text{def}}{=} -[\Lambda, u \cdot \nabla]\phi + \Lambda f - g.$$

As discussed in the first section, we will set our energy estimates about (3.17) in low frequency and high frequency respectively. Applying $\dot{\Delta}_j$ to the first equation in (3.17) gives

$$\partial_t \dot{\Delta}_j \phi + u \cdot \nabla \dot{\Delta}_j \phi + \dot{\Delta}_j \Lambda u = -[\dot{\Delta}_j, u \cdot \nabla]\phi + \dot{\Delta}_j f. \tag{3.18}$$

Taking L^2 inner product of $\dot{\Delta}_j \phi$ with (3.18) and using integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\dot{\Delta}_j \phi\|_{L^2}^2 + \int_{\mathbb{R}^n} \dot{\Delta}_j \Lambda\phi \cdot \dot{\Delta}_j u\, dx = -\int_{\mathbb{R}^n} [\dot{\Delta}_j, u \cdot \nabla]\phi \cdot \dot{\Delta}_j \phi\, dx + \int_{\mathbb{R}^n} \dot{\Delta}_j f \cdot \dot{\Delta}_j \phi\, dx. \tag{3.19}$$

Similarly, we have

$$\frac{1}{2}\frac{d}{dt}\|\dot{\Delta}_j u\|_{L^2}^2 + 2^{2j}\|\dot{\Delta}_j u\|_{L^2}^2 - \int_{\mathbb{R}^n} \dot{\Delta}_j \Lambda\phi \cdot \dot{\Delta}_j u\, dx$$
$$= -\int_{\mathbb{R}^n} [\dot{\Delta}_j, u \cdot \nabla]u \cdot \dot{\Delta}_j u\, dx + \int_{\mathbb{R}^n} \dot{\Delta}_j g \cdot \dot{\Delta}_j u\, dx, \tag{3.20}$$

$$\frac{1}{2}\frac{d}{dt}\|\dot{\Delta}_j w\|_{L^2}^2 + \|\dot{\Delta}_j \Lambda\phi\|_{L^2}^2 - \int_{\mathbb{R}^n} \dot{\Delta}_j \Lambda\phi \cdot \dot{\Delta}_j u\, dx$$
$$= -\int_{\mathbb{R}^n} [\dot{\Delta}_j, u \cdot \nabla]w \cdot \dot{\Delta}_j w\, dx + \int_{\mathbb{R}^n} \dot{\Delta}_j G \cdot \dot{\Delta}_j w\, dx, \tag{3.21}$$

in which we have used the following fact:

$$\int_{\mathbb{R}^n} \dot{\Delta}_j \Lambda\phi \cdot \dot{\Delta}_j w\, dx = \int_{\mathbb{R}^n} \dot{\Delta}_j \Lambda\phi \cdot (\dot{\Delta}_j \Lambda\phi - \dot{\Delta}_j u)\, dx$$
$$= \|\dot{\Delta}_j \Lambda\phi\|_{L^2}^2 - \int_{\mathbb{R}^n} \dot{\Delta}_j \Lambda\phi \cdot \dot{\Delta}_j u\, dx.$$

Let $0 < \eta < 1$ be a small constant which will be determined later on. Summing up (3.19)–(3.21) and using the Hölder inequality and Berntein's lemma, we have

$$\frac{1}{2}\frac{d}{dt}(\|\dot\Delta_j\phi\|_{L^2}^2 + (1-\eta)\|\dot\Delta_j u\|_{L^2}^2 + \eta\|\dot\Delta_j w\|_{L^2}^2) + (1-\eta)2^{2j}\|\dot\Delta_j u\|_{L^2}^2 + \eta 2^{2j}\|\dot\Delta_j\phi\|_{L^2}^2$$
$$\lesssim \|\dot\Delta_j\phi\|_{L^2}(\|[\dot\Delta_j, u\cdot\nabla]\phi\|_{L^2} + \|\dot\Delta_j f\|_{L^2}) + \|\dot\Delta_j u\|_{L^2}(\|[\dot\Delta_j, u\cdot\nabla]u\|_{L^2} + \|\dot\Delta_j g\|_{L^2})$$
$$+ \|\dot\Delta_j w\|_{L^2}(\|[\dot\Delta_j, u\cdot\nabla]w\|_{L^2} + \|\dot\Delta_j G\|_{L^2}). \tag{3.22}$$

For any $j \le j_0$, we can find an $\eta > 0$ small enough such that

$$\|\dot\Delta_j\phi\|_{L^2}^2 + (1-\eta)\|\dot\Delta_j u\|_{L^2}^2 + \eta\|\dot\Delta_j w\|_{L^2}^2 \ge \frac{1}{C}(\|\dot\Delta_j\phi\|_{L^2}^2 + \|\dot\Delta_j u\|_{L^2}^2). \tag{3.23}$$

From (3.22), one can deduce that

$$\frac{1}{2}\frac{d}{dt}(\|\dot\Delta_j\phi\|_{L^2}^2 + \|\dot\Delta_j u\|_{L^2}^2) + 2^{2j}(\|\dot\Delta_j\phi\|_{L^2}^2 + \|\dot\Delta_j u\|_{L^2}^2)$$
$$\lesssim \|\dot\Delta_j\phi\|_{L^2}(\|[\dot\Delta_j, u\cdot\nabla]\phi\|_{L^2} + \|\dot\Delta_j f\|_{L^2}) + \|\dot\Delta_j u\|_{L^2}(\|[\dot\Delta_j, u\cdot\nabla]u\|_{L^2} + \|\dot\Delta_j g\|_{L^2})$$
$$+ \|\dot\Delta_j w\|_{L^2}(\|[\dot\Delta_j, u\cdot\nabla]w\|_{L^2} + \|\dot\Delta_j G\|_{L^2}). \tag{3.24}$$

By the definition of the Besov space, we can further get

$$\|(u^\ell, \phi^\ell)\|_{\widetilde{L}_t^\infty(\dot B_{2,1}^{\frac{n}{2}-1})} + \int_0^t \|(u^\ell, \phi^\ell)\|_{\dot B_{2,1}^{\frac{n}{2}+1}}\, ds$$
$$\lesssim \|(u_0^\ell, \phi_0^\ell)\|_{\dot B_{2,1}^{\frac{n}{2}-1}} + \int_0^t \|(f, G, g)^\ell\|_{\dot B_{2,1}^{\frac{n}{2}-1}}\, ds$$
$$+ \int_0^t \sum_{j\le j_0} 2^{(\frac{n}{2}-1)j}(\|[\dot\Delta_j, u\cdot\nabla]\phi\|_{L^2} + \|[\dot\Delta_j, u\cdot\nabla]u\|_{L^2} + \|[\dot\Delta_j, u\cdot\nabla]\Lambda\phi\|_{L^2})\, ds$$
$$\lesssim \|(u_0^\ell, \phi_0^\ell)\|_{\dot B_{2,1}^{\frac{n}{2}-1}} + \int_0^t (\|f^\ell\|_{\dot B_{2,1}^{\frac{n}{2}-1}} + \|g^\ell\|_{\dot B_{2,1}^{\frac{n}{2}-1}} + \|([\Lambda, u\cdot\nabla]\phi)^\ell\|_{\dot B_{2,1}^{\frac{n}{2}-1}})\, ds$$
$$+ \int_0^t \sum_{j\le j_0} 2^{(\frac{n}{2}-1)j}(\|[\dot\Delta_j, u\cdot\nabla]\phi\|_{L^2} + \|[\dot\Delta_j, u\cdot\nabla]u\|_{L^2} + \|[\dot\Delta_j, u\cdot\nabla]\Lambda\phi\|_{L^2})\, ds. \tag{3.25}$$

Next, we give the estimates to the terms in the righthand side of the above inequality. A simple computation implies

$$\dot\Delta_j([\Lambda^{-1}\mathbb{P}\text{div}, u\cdot\nabla]\tau) = \dot\Delta_j(\Lambda^{-1}\mathbb{P}\text{div}(u\cdot\nabla\tau) - \dot\Delta_j(u\cdot\nabla\Lambda^{-1}\mathbb{P}\text{div}\,\tau)$$
$$= [\dot\Delta_j\Lambda^{-1}\mathbb{P}\text{div}, u\cdot\nabla]\tau - [\dot\Delta_j, u\cdot\nabla]\Lambda^{-1}\mathbb{P}\text{div}\,\tau. \tag{3.26}$$

As the Fourier multiplier $\Lambda^{-1}\mathbb{P}\text{div}$ is of degree 0, thus, from (3.26) and Corollary 2.10, we have

$$\|([\Lambda^{-1}\mathbb{P}\text{div}, u\cdot\nabla]\tau)^\ell\|_{\dot B_{2,1}^{\frac{n}{2}-1}}$$
$$\lesssim (\|\tau^\ell\|_{\dot B_{2,1}^{\frac{n}{2}-1}} + \|\tau^h\|_{\dot B_{p,1}^{\frac{n}{p}-1}})\|\nabla u\|_{\dot B_{p,1}^{\frac{n}{p}}}$$
$$\lesssim (\|\tau^\ell\|_{\dot B_{2,1}^{\frac{n}{2}-1}} + \|\tau^h\|_{\dot B_{p,1}^{\frac{n}{p}}})(\|u^\ell\|_{\dot B_{2,1}^{\frac{n}{2}+1}} + \|u^h\|_{\dot B_{p,1}^{\frac{n}{p}+1}}). \tag{3.27}$$

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The term $\|(F(\tau,\nabla u))^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}$ can be dealt with the same method as (3.14), as a result, we can get

$$\|f^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} \lesssim (\|\tau^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|\tau^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}})(\|u^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|u^h\|_{\dot{B}_{p,1}^{\frac{n}{p}+1}}). \tag{3.28}$$

Thanks to Corollary 2.10, we obtain

$$\|g^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} \lesssim \|([\mathbb{P}, u\cdot\nabla]u)^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} \lesssim \|u\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}}\|u\|_{\dot{B}_{p,1}^{\frac{n}{p}+1}}$$

$$\lesssim (\|u^\ell\|_{\dot{B}_{2,1}^{-1+\frac{n}{2}}} + \|u^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}})(\|u^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|u^h\|_{\dot{B}_{p,1}^{\frac{n}{p}+1}}). \tag{3.29}$$

We get by a similar derivation of (3.10)–(3.13) that

$$\|([\Lambda, u\cdot\nabla]\phi)^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} \lesssim \|(\Lambda u\cdot\nabla\phi)^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}$$

$$\lesssim (\|\phi^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|\phi^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}})(\|u^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|u^h\|_{\dot{B}_{p,1}^{\frac{n}{p}+1}}). \tag{3.30}$$

By using Lemma 2.8, we can get

$$\int_0^t \sum_{j\leq j_0} 2^{(\frac{n}{2}-1)j}(\|[\dot{\Delta}_j, u\cdot\nabla]\phi\|_{L^2} + \|[\dot{\Delta}_j, u\cdot\nabla]u\|_{L^2} + \|[\dot{\Delta}_j, u\cdot\nabla]\Lambda\phi\|_{L^2})\,ds$$

$$\lesssim \int_0^t (\|\phi^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|\phi^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}})(\|u^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|u^h\|_{\dot{B}_{p,1}^{\frac{n}{p}+1}})\,ds$$

$$+ \int_0^t (\|u^\ell\|_{\dot{B}_{2,1}^{-1+\frac{n}{2}}} + \|u^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}})(\|u^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|u^h\|_{\dot{B}_{p,1}^{\frac{n}{p}+1}})\,ds. \tag{3.31}$$

Inserting (3.28), (3.29), (3.30) and (3.31) into (3.25) gives

$$\|(u^\ell,\phi^\ell)\|_{\widetilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})} + \int_0^t \|(u^\ell,\phi^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}}\,ds$$

$$\lesssim \|(u_0^\ell,\phi_0^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}$$

$$+ \int_0^t (\|(u^\ell,\tau^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|u^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} + \|\tau^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}})(\|u^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|u^h\|_{\dot{B}_{p,1}^{\frac{n}{p}+1}})\,ds. \tag{3.32}$$

Combining with (3.15) and (3.32), we can get

$$\|(u^\ell,\tau^\ell)\|_{\widetilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})} + \|\tau^\ell\|_{\widetilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}})} + \int_0^t \|u^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}}\,ds + \int_0^t \|(\Lambda^{-1}\mathbb{P}\mathrm{div}\,\tau)^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}}\,ds$$

$$\lesssim \|(u_0^\ell,\tau_0^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}$$

$$+ \int_0^t (\|(u^\ell,\tau^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|u^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} + \|\tau^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}})(\|u^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|u^h\|_{\dot{B}_{p,1}^{\frac{n}{p}+1}})\,ds. \tag{3.33}$$

3.2. The high frequency estimates of the solutions

In the following, we are concerned with the estimates for the high frequency part of the solution. We shall find the damping effect of ϕ and smoothing effect of u in the high frequency part.

Let $\Gamma = u - \Lambda^{-1}\phi$, we can get by a simple computation from (3.17) that

$$
\begin{cases}
\partial_t \phi + u \cdot \nabla \phi + \phi = f - \Lambda \Gamma, \\
\partial_t \Gamma - \Delta \Gamma = \Gamma + \Lambda^{-1}\phi - \mathbb{P}(u \cdot \nabla u) - \Lambda^{-1}(u \cdot \nabla \phi) - \Lambda^{-1} f.
\end{cases}
\tag{3.34}
$$

Applying $\dot\Delta_j$ to the first equation in (3.34) and taking L^2 inner product with $|\dot\Delta_j\phi|^{p-2}\dot\Delta_j\phi$, using integrating by part and the Hölder inequality, we thus get for all $t \geq 0$,

$$
\|\dot\Delta_j\phi(t)\|_{L^p} + \int_0^t \|\dot\Delta_j\phi\|_{L^p}\, ds
$$

$$
\leq \|\dot\Delta_j\phi_0\|_{L^p} + \int_0^t (\|[\dot\Delta_j, u \cdot \nabla]\phi\|_{L^p} + \|\dot\Delta_j f\|_{L^p} + \|\dot\Delta_j(\Lambda\Gamma)\|_{L^p})\, ds
\tag{3.35}
$$

from which and the definition of Besovs spaces that

$$
\|\phi^h\|_{\widetilde{L}_t^\infty(\dot B_{p,1}^{\frac{n}{p}})} + \|\phi^h\|_{L_t^1(\dot B_{p,1}^{\frac{n}{p}})} \lesssim \|\phi_0^h\|_{\dot B_{p,1}^{\frac{n}{p}}} + \sum_{j \geq j_0} 2^{\frac{nj}{p}} \|[\dot\Delta_j, u \cdot \nabla]\phi\|_{L_t^1(L^p)}
$$

$$
+ \|f^h\|_{L_t^1(\dot B_{p,1}^{\frac{n}{p}})} + \|\Gamma^h\|_{L_t^1(\dot B_{p,1}^{\frac{n}{p}+1})}.
\tag{3.36}
$$

Similarly, we get the high frequency of Γ that

$$
\|\Gamma^h\|_{\widetilde{L}_t^\infty(\dot B_{p,1}^{\frac{n}{p}-1})} + \|\Gamma^h\|_{L_t^1(\dot B_{p,1}^{\frac{n}{p}+1})}
$$

$$
\lesssim \|\Gamma_0^h\|_{\dot B_{p,1}^{\frac{n}{p}-1}} + \|\Gamma^h\|_{L_t^1(\dot B_{p,1}^{\frac{n}{p}-1})} + \|(\Lambda^{-1}\phi)^h\|_{L_t^1(\dot B_{p,1}^{\frac{n}{p}-1})}
$$

$$
+ \|(\mathbb{P}(u \cdot \nabla u))^h\|_{L_t^1(\dot B_{p,1}^{\frac{n}{p}-1})} + \|(\Lambda^{-1}f)^h\|_{L_t^1(\dot B_{p,1}^{\frac{n}{p}-1})} + \|(\Lambda^{-1}(u \cdot \nabla \phi))^h\|_{L_t^1(\dot B_{p,1}^{\frac{n}{p}-1})}
$$

$$
\lesssim \|\Gamma_0^h\|_{\dot B_{p,1}^{\frac{n}{p}-1}} + 2^{-2j_0}\|\Gamma^h\|_{L_t^1(\dot B_{p,1}^{\frac{n}{p}+1})} + 2^{-2j_0}\|\phi^h\|_{L_t^1(\dot B_{p,1}^{\frac{n}{p}})}
$$

$$
+ \|(\mathbb{P}(u \cdot \nabla u))^h\|_{L_t^1(\dot B_{p,1}^{\frac{n}{p}-1})} + \|f^h\|_{L_t^1(\dot B_{p,1}^{\frac{n}{p}-1})} + \|(\Lambda^{-1}(u \cdot \nabla \phi))^h\|_{L_t^1(\dot B_{p,1}^{\frac{n}{p}-1})}.
\tag{3.37}
$$

Combining with (3.36) and (3.37), one can deduce from $u = \Gamma + \Lambda^{-1}\phi$ that

$$
\|u^h\|_{\widetilde{L}_t^\infty(\dot B_{p,1}^{\frac{n}{p}-1})} + \|\phi^h\|_{\widetilde{L}_t^\infty(\dot B_{p,1}^{\frac{n}{p}})} + \|\phi^h\|_{L_t^1(\dot B_{p,1}^{\frac{n}{p}})} + \|u^h\|_{L_t^1(\dot B_{p,1}^{\frac{n}{p}+1})}
$$

$$
\lesssim \|u_0^h\|_{\dot B_{p,1}^{\frac{n}{p}-1}} + \|\phi_0^h\|_{\dot B_{p,1}^{\frac{n}{p}}} + \|f^h\|_{L_t^1(\dot B_{p,1}^{\frac{n}{p}})} + \sum_{j \geq j_0} 2^{\frac{nj}{p}} \|[\dot\Delta_j, u \cdot \nabla]\phi\|_{L_t^1(L^p)}
$$

$$
+ \|(\mathbb{P}(u \cdot \nabla u))^h\|_{L_t^1(\dot B_{p,1}^{\frac{n}{p}-1})} + \|(\Lambda^{-1}(u \cdot \nabla \phi))^h\|_{L_t^1(\dot B_{p,1}^{\frac{n}{p}-1})}.
\tag{3.38}
$$

With the aid of Lemmas 2.7 and 2.8, we have

$$
\|f^h\|_{L_t^1(\dot B_{p,1}^{\frac{n}{p}})} \lesssim \int_0^t \|[\Lambda^{-1}\mathbb{P}\mathrm{div}, u \cdot \nabla]\tau\|_{\dot B_{p,1}^{\frac{n}{p}}} + \|\Lambda^{-1}\mathbb{P}\mathrm{div}\,(F(\tau, \nabla u))\|_{\dot B_{p,1}^{\frac{n}{p}}}\, ds
$$

$$
\lesssim \int_0^t \|\tau\|_{\dot B_{p,1}^{\frac{n}{p}}} \|\nabla u\|_{\dot B_{p,1}^{\frac{n}{p}}}\, ds
$$

$$
\lesssim \int_0^t (\|\tau^\ell\|_{\dot B_{2,1}^{-1+\frac{n}{2}}} + \|\tau^h\|_{\dot B_{p,1}^{\frac{n}{p}}})(\|u^\ell\|_{\dot B_{2,1}^{\frac{n}{2}+1}} + \|u^h\|_{\dot B_{p,1}^{\frac{n}{p}+1}})\, ds.
\tag{3.39}
$$

Similarly,

$$\sum_{j \geq j_0} 2^{\frac{nj}{p}} \|[\dot{\Delta}_j, u \cdot \nabla]\phi\|_{L_t^1(L^p)} \lesssim \int_0^t \|\nabla u\|_{\dot{B}_{p,1}^{\frac{n}{p}}} \|\phi\|_{\dot{B}_{p,1}^{\frac{n}{p}}} \, ds$$

$$\lesssim \int_0^t (\|\tau^\ell\|_{\dot{B}_{2,1}^{-1+\frac{n}{2}}} + \|\tau^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}})(\|u^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|u^h\|_{\dot{B}_{p,1}^{\frac{n}{p}+1}}) \, ds,$$

$$\|(\mathbb{P}(u \cdot \nabla u))^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}-1})} \lesssim \int_0^t \|u\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} \|\nabla u\|_{\dot{B}_{p,1}^{\frac{n}{p}}} \, ds$$

$$\lesssim \int_0^t (\|u^\ell\|_{\dot{B}_{2,1}^{-1+\frac{n}{2}}} + \|u^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}})(\|u^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|u^h\|_{\dot{B}_{p,1}^{\frac{n}{p}+1}}) \, ds. \quad (3.40)$$

By Lemma 2.7, one has

$$\|(\Lambda^{-1}(u \cdot \nabla \phi))^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}-1})} \lesssim \int_0^t \|\Lambda^{-1}\mathrm{div}\,(u\phi)\|_{\dot{B}_{p,1}^{\frac{n}{p}}} \, ds \lesssim \int_0^t \|u\|_{\dot{B}_{p,1}^{\frac{n}{p}}} \|\phi\|_{\dot{B}_{p,1}^{\frac{n}{p}}} \, ds$$

$$\lesssim \int_0^t (\|u\|_{\dot{B}_{p,1}^{\frac{n}{p}}}^2 + \|\phi^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}}}^2 + \|\phi^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}}^2) \, ds$$

$$\lesssim \int_0^t (\|u\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} \|u\|_{\dot{B}_{p,1}^{\frac{n}{p}+1}} + \|\phi^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} \|\phi^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|\phi^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}}^2) \, ds$$

$$\lesssim \int_0^t (\|u^\ell\|_{\dot{B}_{2,1}^{-1+\frac{n}{2}}} + \|u^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}})(\|u^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|u^h\|_{\dot{B}_{p,1}^{\frac{n}{p}+1}}) \, ds$$

$$+ \int_0^t (\|\tau^\ell\|_{\dot{B}_{2,1}^{-1+\frac{n}{2}}} + \|\tau^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}})(\|\phi^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|\phi^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}}) \, ds. \quad (3.41)$$

Plugging (3.39)–(3.41) into (3.38) implies

$$\|u^h\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n}{p}-1})} + \|\phi^h\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n}{p}})} + \|\phi^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}})} + \|u^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}+1})}$$

$$\lesssim \|u_0^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} + \|\phi_0^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}}$$

$$+ \int_0^t (\|\tau^\ell\|_{\dot{B}_{2,1}^{-1+\frac{n}{2}}} + \|\tau^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}})(\|\phi^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|\phi^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}}) \, ds$$

$$+ \int_0^t (\|(u^\ell, \tau^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|u^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} + \|\tau^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}})(\|u^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|u^h\|_{\dot{B}_{p,1}^{\frac{n}{p}+1}}) \, ds. \quad (3.42)$$

From the first equation in (1.1), we can get similarly to (3.36) that

$$\|\tau^h\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n}{p}})} \lesssim \|\tau_0^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}} + \int_0^t \|u^h\|_{\dot{B}_{p,1}^{\frac{n}{p}+1}} \, ds + \int_0^t \|\nabla u\|_{\dot{B}_{p,1}^{\frac{n}{p}}} \|\tau\|_{\dot{B}_{p,1}^{\frac{n}{p}}} \, ds. \quad (3.43)$$

Together with (3.42) and (3.43), one has

$$\|u^h\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n}{p}-1})} + \|\tau^h\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n}{p}})} + \|(\Lambda^{-1}\mathbb{P}\mathrm{div}\,\tau)^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}})} + \|u^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}+1})}$$

$$\lesssim \|u_0^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} + \|\tau_0^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}}$$

$$+ \int_0^t (\|\tau^\ell\|_{\dot{B}_{2,1}^{-1+\frac{n}{2}}} + \|\tau^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}})(\|\phi^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|\phi^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}}) \, ds$$

$$+ \int_0^t (\|(u^\ell, \tau^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|u^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} + \|\tau^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}})(\|u^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|u^h\|_{\dot{B}_{p,1}^{\frac{n}{p}+1}}) \, ds. \quad (3.44)$$

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3.3. Complete the proof of our main Theorem 1.2

Now, we can complete the proof of our main Theorem 1.2 by the continuous arguments. Denote

$$
X(t) \stackrel{\text{def}}{=} \|(u^\ell, \tau^\ell)\|_{\widetilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})} + \|u^\ell\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}+1})} + \|(\Lambda^{-1}\mathbb{P}\mathrm{div}\,\tau)^\ell\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}+1})}
$$
$$
+ \|u^h\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n}{p}-1})} + \|\tau^h\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n}{p}})} + \|(\Lambda^{-1}\mathbb{P}\mathrm{div}\,\tau)^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}})} + \|u^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}+1})}.
$$

Combining with (3.33) and (3.44), we can get

$$
\|(u^\ell, \tau^\ell)\|_{\widetilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})} + \|u^\ell\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}+1})} + \|(\Lambda^{-1}\mathbb{P}\mathrm{div}\,\tau)^\ell\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}+1})}
$$
$$
+ \|u^h\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n}{p}-1})} + \|\tau^h\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n}{p}})} + \|(\Lambda^{-1}\mathbb{P}\mathrm{div}\,\tau)^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}})} + \|u^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}+1})}
$$
$$
\lesssim \|(u_0^\ell, \tau_0^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|u_0^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} + \|\tau_0^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}}
$$
$$
+ \int_0^t (\|\tau^\ell\|_{\dot{B}_{2,1}^{-1+\frac{n}{2}}} + \|\tau^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}})(\|(\Lambda^{-1}\mathbb{P}\mathrm{div}\,\tau)^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|(\Lambda^{-1}\mathbb{P}\mathrm{div}\,\tau)^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}})\,ds
$$
$$
+ \int_0^t (\|(u^\ell, \tau^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|u^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} + \|\tau^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}})(\|u^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|u^h\|_{\dot{B}_{p,1}^{\frac{n}{p}+1}})\,ds. \tag{3.45}
$$

From (3.45) and the Gronwall inequality, we have

$$
X(t) \leq C e^{CX(t)}(\|(u_0^\ell, \tau_0^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|u_0^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} + \|\tau_0^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}}). \tag{3.46}
$$

Now let δ be a positive constant, which will be determined later on. For any $T^\flat \in [0, T^*)$, we define

$$
T^{**} \stackrel{\text{def}}{=} \sup \left\{ t \in [0, T^\flat) : X(t) \leq \delta \right\}.
$$

From (3.46), we have for any $t \in [0, T^{**})$ there holds

$$
X(t) \leq C_1 e^{C_1 \delta}(\|(u_0^\ell, \tau_0^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|u_0^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} + \|\tau_0^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}}). \tag{3.47}
$$

Choosing $\delta < \frac{1}{4C_1}$ fixed and then letting

$$
\|(u_0^\ell, \tau_0^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|u_0^h\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} + \|\tau_0^h\|_{\dot{B}_{p,1}^{\frac{n}{p}}} < \frac{1}{8C_1},
$$

we can get from (3.47) that

$$
X(t) \leq \frac{\delta}{2}, \quad \forall t \in [0, T^{**}],
$$

this contradicts with the definition of T^{**}, thus we conclude that $T^{**} = T^*$. Consequently, we complete the proof of Theorem 1.2 by standard continuation argument. $\square$

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