

# A WELL-POSED SURFACE CURRENTS AND CHARGES SYSTEM FOR ELECTROMAGNETISM IN DIELECTRIC MEDIA

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**ABSTRACT.** The free space Maxwell dielectric problem can be reduced to a system of surface integral equations (SIE). A numerical formulation for the Maxwell dielectric problem using an SIE system presents two key advantages: first, the radiation condition at infinity is exactly satisfied, and second, there is no need to artificially define a truncated domain. Consequently, these SIE systems have generated much interest in physics, electrical engineering, and mathematics, and many SIE formulations have been proposed over time. In this article we introduce a new SIE formulation which is in the desirable operator form identity plus compact, is well-posed, and remains well-conditioned as the frequency tends to zero. The unknowns in the formulation are three dimensional vector fields on the boundary of the dielectric body. The SIE studied in this paper is derived from a formulation developed in earlier work by some of the authors [1]. Our initial formulation utilized linear constraints to obtain a uniquely solvable system for all frequencies. The new SIE introduced and analyzed in this article combines the integral equations from [1] with new constraints. We show that the new system is in the operator form identity plus compact in a particular functional space, and we prove well-posedness at all frequencies and low-frequency stability of the new SIE.

**1. Introduction.** Understanding the propagation of electromagnetic waves in three dimensional dielectric media is a fundamental problem which is central to a plethora of applications [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. For certain simple shaped dielectric bodies, the propagation process can be modeled by analytical techniques [3]. However, in general, numerical modeling [4, 8] of the Maxwell system [11, 12, 13] in unbounded three dimensional dielectric media is required. Direct simulation of the time-harmonic Maxwell system (using finite-difference/elements) requires truncation of the unbounded medium to a bounded region, and approximation of the associated Silver-Müller radiation condition. Fortunately, these two key physical aspects of the Maxwell system can be preserved in numerical modeling by first reformulating the model as an equivalent system of surface integral equations (SIE) defined on the boundary of the dielectric body and approximating the SIE system in a finite dimensional space.

There is a large literature spanning over a century on developing SIE reformulations of Maxwell equations [4, 5, 6, 11, 12, 13]. The extensive survey in [14] reviews a century of research on deriving and using such SIEs. The key emphasis in the survey in [14] is to highlight the lack of low-frequency stable SIE reformulation of the Maxwell equations that is also in the desirable second-kind (identity plus compact operator) form. The SIE formulation in [14] requires surface differential equations constraints. The main focus of our work is to develop a low-frequency stable and resonance-free well-posed system for all frequencies that involves only surface integral operators, and mathematically establish the robustness of the system with proofs in appropriate function spaces. For electromagnetism, appropriate function spaces play a crucial role [15] for simulating physically correct solutions.

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In previous work [1] we introduced an original SIE reformulation of the Maxwell dielectric system, which satisfies a few remarkable properties that, when taken together, are unique: first, this system is in the classical form identity plus compact. Second, it does not suffer from spurious eigenfrequencies. Third, it does not suffer from the low-frequency breakdown phenomenon: as the frequency tends to zero, this system remains uniformly well posed (see [16] for a simple account of the low-frequency breakdown). Fourth, the solution to this system is equal to the trace of the electric and magnetic fields on the dielectric. Fifth, applying numerical methods developed in [18], we achieved spectral numerical convergence of discrete solutions. Points one to four were proved in [1]. We explained in [1] that well-posedness at all frequencies is achieved when our “square system” is augmented by a linear surface integral based constraint. In this article we propose a new square system obtained by combining our initial system of SIEs with new constraints derived from those in [1]. We show that this leads to a linear system that enjoys all five properties listed above.

There have been (largely unsuccessful) attempts to achieve this goal: in [17], section 6 (see also [19]), linear constraints were added to “stabilize” a system similar to the one in [1]. We showed in [1] that adding constraints may fail for some particular (but reasonable) values of the frequency and dielectric parameters. In [1] we proposed to first multiply the constraint by a complex parameter  $\xi$  and to then add it to the system of linear equations. We showed in numerical simulations that the use of this parameter can prevent the emergence of additional singular cases but we did not provide a formal proof. In this present article we describe a new way of incorporating the constraints using the single layer potential for the Laplace equation. This adds to our Maxwell integral equation system a compact operator which is self adjoint and coercive on an adequate subspace.

This article is organized as follows: in Section 2 after introducing the Maxwell dielectric system, we recall the system of SIEs derived in [1]. We then discuss how several authors (including the co-authors of this paper) have attempted to incorporate linear constraints in the initial linear system in such a way to obtain an (unconstrained) well-posed integral equation system. Using an explicit example where the system derived in [1] becomes singular (previously, only numerical evidence was observed) we demonstrate that past attempts have not resolved the issue completely. To close this section, we introduce a novel system of integral equations obtained by incorporating constraints to the system from [1] by application of a compact operator which is self-adjoint and coercive on an appropriate subspace. Our main result is the proof of well-posedness for this novel system, which is stated in Section 3. The proof relies on introducing a particular function space in which tangential components of the unknown fields are required to be more regular than the normal components. We prove that the system is in the form identity plus compact. The associated operator can be expressed in the form  $I + M + \xi J$ , where  $M$  and  $J$  are compact and  $J$  is self-adjoint and positive on an appropriate subspace. We show that this operator has a continuous inverse for all sufficiently large  $\xi$ , and for all  $\xi$  outside a discrete set. This result is proved using a general functional analysis result which we state and prove in the appendix, and the properties of the well-known Müller system of integral equations for the dielectric problem c. (Müller’s system is known to be well-posed, but it becomes unbounded at low frequency [16]).

**2. Maxwell dielectric model and stable reformulations.** We consider the time-harmonic electromagnetic wave propagation model problem in three dimensional space

comprising a dielectric body whose geometry is given by a bounded domain  $D \subset \mathbb{R}^3$  with boundary  $\partial D$ . Let  $D^+ = \mathbb{R}^3 \setminus \overline{D}$ , and  $D^- = D$  be the exterior and interior of  $D$  respectively, and set  $\epsilon^\pm$  and  $\mu^\pm$  to be the permittivity and permeability constants in  $D^\pm$ . The interior permittivity  $\epsilon^-$  has positive real part and non-negative imaginary part while  $\epsilon^+$ ,  $\mu^+$ , and  $\mu^-$  are positive. Under this physically appropriate mild assumption on the dielectric medium, the time-harmonic Maxwell partial differential equation (PDE) model is resonance-free [22, Theorem 2.1]. We set

$$(2.1) \quad \epsilon(\mathbf{x}) = \epsilon^\pm, \quad \mu(\mathbf{x}) = \mu^\pm, \quad \text{for } \mathbf{x} \in D^\pm.$$

A time-harmonic incident electromagnetic field impinging on the dielectric body  $D$  induces an interior field with spatial components  $[\mathbf{E}^-, \mathbf{H}^-]$  in  $D^-$  and a scattered field with spatial components  $[\mathbf{E}^+, \mathbf{H}^+]$  in  $D^+$ . We set

$$(2.2) \quad \mathbf{E}(\mathbf{x}) = \mathbf{E}^\pm(\mathbf{x}), \quad \mathbf{H}(\mathbf{x}) = \mathbf{H}^\pm(\mathbf{x}), \quad \mathbf{x} \in D^\pm.$$

Then  $[\mathbf{E}, \mathbf{H}]$  satisfy the time-harmonic Maxwell equations [21, Page 253]

$$(2.3) \quad \mathbf{curl} \mathbf{E}(\mathbf{x}) - i\omega\mu(\mathbf{x})\mathbf{H}(\mathbf{x}) = \mathbf{0}, \quad \mathbf{curl} \mathbf{H}(\mathbf{x}) + i\omega\epsilon(\mathbf{x})\mathbf{E}(\mathbf{x}) = \mathbf{0}, \quad \mathbf{x} \in D^+, D^-,$$

and the Silver-Müller radiation condition

$$(2.4) \quad \lim_{|\mathbf{x}| \rightarrow \infty} \left[ \sqrt{\mu^+} \mathbf{H}(\mathbf{x}) \times \mathbf{x} - \sqrt{\epsilon^+} |\mathbf{x}| \mathbf{E}(\mathbf{x}) \right] = \mathbf{0}.$$

The incident electromagnetic field  $[\mathbf{E}_{\text{inc}}, \mathbf{H}_{\text{inc}}]$  is required to satisfy the Maxwell equations

$$(2.5) \quad \mathbf{curl} \mathbf{E}_{\text{inc}}(\mathbf{x}) - i\omega\mu^+ \mathbf{H}_{\text{inc}}(\mathbf{x}) = \mathbf{0}, \quad \mathbf{curl} \mathbf{H}_{\text{inc}}(\mathbf{x}) + i\omega\epsilon^+ \mathbf{E}_{\text{inc}}(\mathbf{x}) = \mathbf{0}, \quad \mathbf{x} \in \mathbb{R}^3 \setminus Q,$$

where  $Q \subset \mathbb{R}^3$  is a compact or empty set, bounded away from the dielectric body  $D$ . In practice, the dielectric body  $D$  is typically subject to excitation by an incident plane wave or a point source. It is convenient to define the total field

$$(2.6) \quad \mathbf{E}_{\text{tot}}(\mathbf{x}) = \mathbf{E}_{\text{tot}}^\pm(\mathbf{x}), \quad \mathbf{H}_{\text{tot}}(\mathbf{x}) = \mathbf{H}_{\text{tot}}^\pm(\mathbf{x}), \quad \mathbf{x} \in D^\pm,$$

which is related to the incident field and  $[\mathbf{E}, \mathbf{H}]$  by

$$(2.7) \quad \mathbf{E}_{\text{tot}}^- = \mathbf{E}^-, \quad \mathbf{E}_{\text{tot}}^+ = \mathbf{E}^+ + \mathbf{E}_{\text{inc}}, \quad \mathbf{H}_{\text{tot}}^- = \mathbf{H}^-, \quad \mathbf{H}_{\text{tot}}^+ = \mathbf{H}^+ + \mathbf{H}_{\text{inc}}.$$

We emphasize that inside  $D$  the induced field is the total field.

The tangential components of the total electric and magnetic fields (which are sometimes called the surface currents) are required to be continuous across the interface  $\partial D$ , leading to the interface conditions [21, Equation (5.6.66), Page 234]:

$$(2.8) \quad \mathbf{E}_{\text{tot}}^- \times \mathbf{n} = \mathbf{E}_{\text{tot}}^+ \times \mathbf{n}, \quad \mathbf{H}_{\text{tot}}^- \times \mathbf{n} = \mathbf{H}_{\text{tot}}^+ \times \mathbf{n}, \quad \text{on } \partial D,$$

where  $\mathbf{n}$  is the outward unit normal to  $\partial D$ . A consequence of (2.3), (2.5), and (2.8) is that the normal components of the fields (which are sometimes called the surface charges) satisfy the interface conditions:

$$(2.9) \quad \epsilon^- \mathbf{n} \cdot \mathbf{E}_{\text{tot}}^- = \epsilon^+ \mathbf{n} \cdot \mathbf{E}_{\text{tot}}^+, \quad \mu^- \mathbf{n} \cdot \mathbf{H}_{\text{tot}}^- = \mu^+ \mathbf{n} \cdot \mathbf{H}_{\text{tot}}^+ \quad \text{on } \partial D.$$

**2.1. A reformulated SIE system of the Maxwell model.** Denote by

$$(2.10) \quad G_\pm(\mathbf{x}, \mathbf{y}) = \frac{1}{4\pi} \frac{e^{ik_\pm |\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}$$

the free space Green's function for the Helmholtz operator  $\Delta + k_{\pm}^2$ , where  $k_{\pm} = \omega\sqrt{\epsilon^{\pm}\mu^{\pm}}$  is, respectively, the exterior and interior wavenumber. In [1] we proved that if the scattered field  $\mathbf{E}, \mathbf{H}$  satisfies the Maxwell PDE system (2.3)–(2.9) then the exterior traces  $\mathbf{e}, \mathbf{h}$  of the total fields on  $\partial D$  satisfy the system of integral equations,

$$(2.11) \quad \begin{aligned} & (\mathbf{e} \times \mathbf{n})(\mathbf{x}) - \frac{2}{\epsilon^+ + \epsilon^-} \left\{ - \int_{\partial D} \left[ \epsilon^+ \frac{\partial G_+}{\partial \mathbf{n}(\mathbf{x})} - \epsilon^- \frac{\partial G_-}{\partial \mathbf{n}(\mathbf{x})} \right] (\mathbf{e} \times \mathbf{n})(\mathbf{y}) \, ds(\mathbf{y}) \right. \\ & \quad + \int_{\partial D} [\mathbf{grad}_{\mathbf{x}}(\epsilon^+ G_+ - \epsilon^- G_-)] (\mathbf{e} \times \mathbf{n})(\mathbf{y}) \cdot [\mathbf{n}(\mathbf{x}) - \mathbf{n}(\mathbf{y})] \, ds(\mathbf{y}) \\ & \quad \left. - i\omega \int_{\partial D} [\epsilon^+ \mu^+ G_+ - \epsilon^- \mu^- G_-] (\mathbf{h} \times \mathbf{n})(\mathbf{y}) \times \mathbf{n}(\mathbf{x}) \, ds(\mathbf{y}) \right\} \\ & - \epsilon^+ \int_{\partial D} [\mathbf{grad}_{\mathbf{x}}(G_+ - G_-) \times \mathbf{n}(\mathbf{x})] (\mathbf{e} \cdot \mathbf{n})(\mathbf{y}) \, ds(\mathbf{y}) \Big\} = \frac{2\epsilon^+}{\epsilon^+ + \epsilon^-} (\mathbf{e}_{\text{inc}} \times \mathbf{n})(\mathbf{x}), \end{aligned}$$

$$(2.12) \quad \begin{aligned} & (\mathbf{e} \cdot \mathbf{n})(\mathbf{x}) - \frac{2\epsilon^-}{\epsilon^+ + \epsilon^-} \left\{ - i\omega \int_{\partial D} [\mu^+ G_+ - \mu^- G_-] (\mathbf{h} \times \mathbf{n})(\mathbf{y}) \cdot \mathbf{n}(\mathbf{x}) \, ds(\mathbf{y}) \right. \\ & \quad - \int_{\partial D} \left[ \frac{\partial G_+}{\partial \mathbf{n}(\mathbf{x})} - \frac{\epsilon^+}{\epsilon^-} \frac{\partial G_-}{\partial \mathbf{n}(\mathbf{x})} \right] (\mathbf{e} \cdot \mathbf{n})(\mathbf{y}) \, ds(\mathbf{y}) \\ & \quad \left. + \int_{\partial D} [\mathbf{grad}_{\mathbf{x}}(G_+ - G_-) \times \mathbf{n}(\mathbf{x})] \cdot (\mathbf{e} \times \mathbf{n})(\mathbf{y}) \, ds(\mathbf{y}) \right\} \\ & = \frac{2\epsilon^-}{\epsilon^+ + \epsilon^-} (\mathbf{e}_{\text{inc}} \cdot \mathbf{n})(\mathbf{x}), \end{aligned}$$

$$(2.13) \quad \begin{aligned} & (\mathbf{h} \times \mathbf{n})(\mathbf{x}) - \frac{2}{\mu^+ + \mu^-} \left\{ - \int_{\partial D} \left[ \mu^+ \frac{\partial G_+}{\partial \mathbf{n}(\mathbf{x})} - \mu^- \frac{\partial G_-}{\partial \mathbf{n}(\mathbf{x})} \right] (\mathbf{h} \times \mathbf{n})(\mathbf{y}) \, ds(\mathbf{y}) \right. \\ & \quad + \int_{\partial D} [\mathbf{grad}_{\mathbf{x}}(\mu^+ G_+ - \mu^- G_-)] (\mathbf{h} \times \mathbf{n})(\mathbf{y}) \cdot [\mathbf{n}(\mathbf{x}) - \mathbf{n}(\mathbf{y})] \, ds(\mathbf{y}) \\ & \quad \left. + i\omega \int_{\partial D} [\epsilon^+ \mu^+ G_+ - \epsilon^- \mu^- G_-] (\mathbf{e} \times \mathbf{n})(\mathbf{y}) \times \mathbf{n}(\mathbf{x}) \, ds(\mathbf{y}) \right\} \\ & - \mu^+ \int_{\partial D} [\mathbf{grad}_{\mathbf{x}}(G_+ - G_-) \times \mathbf{n}(\mathbf{x})] (\mathbf{h} \cdot \mathbf{n})(\mathbf{y}) \, ds(\mathbf{y}) \Big\} = \frac{2\mu^+}{\mu^+ + \mu^-} (\mathbf{h}_{\text{inc}} \times \mathbf{n})(\mathbf{x}), \end{aligned}$$

$$(2.14) \quad \begin{aligned} & (\mathbf{h} \cdot \mathbf{n})(\mathbf{x}) - \frac{2\mu^-}{\mu^+ + \mu^-} \left\{ i\omega \int_{\partial D} [\epsilon^+ G_+ - \epsilon^- G_-] (\mathbf{e} \times \mathbf{n})(\mathbf{y}) \cdot \mathbf{n}(\mathbf{x}) \, ds(\mathbf{y}) \right. \\ & \quad - \int_{\partial D} \left[ \frac{\partial G_+}{\partial \mathbf{n}(\mathbf{x})} - \frac{\mu^+}{\mu^-} \frac{\partial G_-}{\partial \mathbf{n}(\mathbf{x})} \right] (\mathbf{h} \cdot \mathbf{n})(\mathbf{y}) \, ds(\mathbf{y}) \\ & \quad \left. + \int_{\partial D} [\mathbf{grad}_{\mathbf{x}}(G_+ - G_-) \times \mathbf{n}(\mathbf{x})] \cdot (\mathbf{h} \times \mathbf{n})(\mathbf{y}) \, ds(\mathbf{y}) \right\} = \frac{2\mu^-}{\mu^+ + \mu^-} \mathbf{h}_{\text{inc}} \cdot \mathbf{n}(\mathbf{x}), \end{aligned}$$

where we use the abbreviation  $G_{\pm}$  for  $G_{\pm}(\mathbf{x}, \mathbf{y})$ , and  $(\mathbf{e}_{\text{inc}}, \mathbf{h}_{\text{inc}})$  is the trace of the incident field  $(\mathbf{E}_{\text{inc}}, \mathbf{H}_{\text{inc}})$  on  $\partial D$ . It is convenient to write this system in operator form. Setting

$$(2.15) \quad \mathbf{e}^i = \frac{2\epsilon^+}{\epsilon^+ + \epsilon^-} \mathbf{n} \times (\mathbf{e}_{\text{inc}} \times \mathbf{n}) + \frac{2\epsilon^-}{\epsilon^+ + \epsilon^-} \mathbf{n} (\mathbf{e}_{\text{inc}} \cdot \mathbf{n}),$$

$$(2.16) \quad \mathbf{h}^i = \frac{2\mu^+}{\mu^+ + \mu^-} \mathbf{n} \times (\mathbf{h}_{\text{inc}} \times \mathbf{n}) + \frac{2\mu^-}{\mu^+ + \mu^-} \mathbf{n} (\mathbf{h}_{\text{inc}} \cdot \mathbf{n}),$$

the system (2.11)–(2.14) is equivalent to

$$(2.17) \quad (\mathbb{I} + \mathbb{M})(\mathbf{e}, \mathbf{h}) = (\mathbf{e}^i, \mathbf{h}^i),$$

where  $\mathbb{I}$  is the identity operator and the explicit form of  $\mathbb{M}$  can be found in [1, Equation (3.17)] or directly inferred from (2.11)–(2.14).

Using the assumption that  $\partial D$  is smooth, it was explained in [1] that for any  $s \in \mathbb{R}$ , the operator  $\mathbb{M}$  is continuous from  $H^s(\partial D)^3 \times H^s(\partial D)^3$  to  $H^{s+1}(\partial D)^3 \times H^{s+1}(\partial D)^3$ . Consequently, the system (2.17) is of the form “identity plus compact”. For the mathematical analysis and the operator properties used in this article, it will be convenient to assume throughout the article that the dielectric body  $D$  is smooth. However, as in [1], most results in this article will still hold when  $D$  is a  $C^{1,\alpha}$  domain.

Many authors have derived systems of integral equations for the free space dielectric problem in this “identity plus compact” form. However they usually suffer from low-frequency break down, meaning that the norm of the inverse operator blows up as the frequency tends to zero [16]. One of the most interesting contributions of our system (2.17) is that  $\mathbb{I} + \mathbb{M}$  is norm convergent to an invertible operator as the frequency  $\omega$  tends to zero (see [1, Theorem B.1]). However, the operator  $\mathbb{I} + \mathbb{M}$  may be singular for some frequencies, as demonstrated in numerical simulations in [1] and demonstrated analytically in section 2.4.

**2.2. Constraints for all-frequency stabilization of the SIE (2.17).** Although the system (2.17) is always solvable, the solution may not be unique for some frequencies because  $\mathbb{I} + \mathbb{M}$  may have a nontrivial kernel. However, under the constraints

$$(2.18) \quad \operatorname{div}_{\partial D}(\mathbf{e} \times \mathbf{n}) = i\omega\mu^+ \mathbf{h} \cdot \mathbf{n}, \quad \operatorname{div}_{\partial D}(\mathbf{h} \times \mathbf{n}) = -i\omega\epsilon^+ \mathbf{e} \cdot \mathbf{n},$$

the solution to (2.17) is unique. This has been known for some time, see [11, chapter VI]. In this work we want to find a constraint in integral equation form, rather than PDE form. To that end, in [1] we defined operators

$$(2.19) \quad (\mathcal{S}_\pm w)(\mathbf{x}) = \int_{\partial D} G_\pm(\mathbf{x}, \mathbf{y}) w(\mathbf{y}) ds(\mathbf{y}), \quad \mathbf{x} \in \partial D,$$

$$(2.20) \quad (\mathcal{K}_\pm \mathbf{w})(\mathbf{x}) = \int_{\partial D} \left[ \mathbf{grad}_x G_\pm(\mathbf{x}, \mathbf{y}) \times \mathbf{n}(\mathbf{y}) \right] \cdot \mathbf{w}(\mathbf{y}) ds(\mathbf{y}), \quad \mathbf{x} \in \partial D,$$

for a scalar function  $w$  and a vector field  $\mathbf{w}$  on  $\partial D$ , and

$$(2.21) \quad J_1(\mathbf{e}, \mathbf{h})(\mathbf{x}) = (-i\omega\epsilon^+(\mathcal{S}_+ - \mathcal{S}_-)(\mathbf{e} \cdot \mathbf{n}) + (\mathcal{K}_+ - \mathcal{K}_-)\mathbf{h}),$$

$$(2.22) \quad J_2(\mathbf{e}, \mathbf{h})(\mathbf{x}) = (i\omega\mu^+(\mathcal{S}_+ - \mathcal{S}_-)(\mathbf{h} \cdot \mathbf{n}) + (\mathcal{K}_+ - \mathcal{K}_-)\mathbf{e}).$$

We proved in [1, Theorem 5.4] that the augmented system

$$(2.23) \quad (\mathbb{I} + \mathbb{M})(\mathbf{e}, \mathbf{h}) = (\mathbf{e}^i, \mathbf{h}^i), \quad J_1(\mathbf{e}, \mathbf{h}) = J_2(\mathbf{e}, \mathbf{h}) = 0,$$

has a unique solution for all frequencies  $\omega > 0$  and values of the electromagnetic parameters  $\epsilon, \mu$  considered in this paper.

**2.3. Stabilized SIE derived from the augmented system (2.23).** In applications, linear systems derived from formulation (2.23) by discretizing may be very large, and they may have to be solved a large number of times for different values of the frequency or of

the electromagnetic parameters. Consequently, imposing the two constraints from (2.23) may become expensive from a computational perspective for high frequencies and complex geometries.

There have been several attempts to find a system of linear integral equations which is equivalent to the augmented system (2.23). In [17, Section 6] (see also [19]), adding (arithmetically) conditions (2.21)–(2.22) to (2.12) and (2.14) was shown to be effective in a few examples. Numerical evidence in [1] demonstrated that this does not work for all frequencies, even in the simple case where  $\partial D$  is a sphere. In [1, Section 7] we demonstrated that adding a multiple of conditions (2.21)–(2.22) to the system (2.11)–(2.14) was effective for all frequencies (with the multiple depending on the frequency).

It is fruitful to recast this idea in terms of general operator theory. Let  $H$  be a separable complex Hilbert space and  $M$  and  $J$  two linear compact operators from  $H$  to  $H$ . Suppose that we know that for a given  $b \in H$  the constrained linear equation

$$(2.24) \quad (I + M)x = b, \quad Jx = 0$$

is uniquely solvable. It follows that  $(I + M + \xi J)x = b$  for any  $\xi$  in  $\mathbb{C}$ . Can we tell with certainty that for some  $\xi$  in  $\mathbb{C}$  the operator  $I + M + \xi J$  is invertible with bounded inverse?

If  $H$  is finite-dimensional, this relates to the so called theory of pencils, [20]. If  $H$  is finite-dimensional and  $J$  is regular, we write  $I + M + \xi J = J(J^{-1}(I + M) + \xi I)$ , which is regular, except for finitely many values of  $\xi$ . If  $J$  is singular and  $H$  is finite-dimensional then, using the determinant, we see that  $I + M + \xi J$  is either regular, except for finitely many values of  $\xi$ , or it is singular for all values of  $\xi$ . In the latter case  $I + M + \xi J$  is called a singular pencil.

Beside the case where  $H$  is finite-dimensional and  $J$  is regular, are there simple to use conditions on  $M$  and  $J$  that will guarantee that  $I + M + \xi J$  is not a singular pencil? We note that it is not sufficient that the intersection of the nullspaces of  $I + M$  and  $J$  is trivial. Here is an example in dimension 3, which can easily be generalized to any higher (including infinite) dimension. Set

$$I + M = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and let  $e_1, e_2, e_3$  be the natural basis of  $\mathbb{C}^3$ . Then  $e_3$  spans the nullspace of  $J$  and is not in the nullspace of  $I + M$ . Clearly,  $(I + M + \xi J)(e_1 - e_2 - \xi e_3) = 0$ , for all  $\xi \in \mathbb{C}$ .

We prove in Theorem 4.4 in the appendix that a sufficient condition for the pencil  $I + M + \xi J$  to be regular is to satisfy (i)  $M$  and  $J$  are compact; (ii) the intersection of the nullspaces of  $I + M$  and  $J$  is trivial; (iii)  $J$  is self-adjoint and non-negative; and (iv)  $N(J)$  is invariant under  $M$ . These strong conditions on  $J$  and on  $M$  are required. In Proposition 4.5 in the appendix we provide an example, valid in any separable Hilbert space, of two compact operators  $J$  and  $M$  such that  $J$  is injective and yet  $I + M + \xi J$  has a non-trivial nullspace, for all  $\xi \in \mathbb{C}$ . The following proposition explains the the numerical results in [1, Section 7], particularly the isolated peak in [1, Figure 7.2].

**Proposition 2.1.** *Let  $H$  be a separable Hilbert space and  $M$  and  $J$  two compact linear operators from  $H$  to  $H$ . Then  $I + M + \xi J$  is either singular for all values  $\xi \in \mathbb{C}$  or there is*

a discrete set  $Z$  such that  $I + M + \xi J$  is invertible for  $\xi \in \mathbb{C} \setminus Z$ .

*Proof.* Suppose there is a  $\xi_0 \in \mathbb{C}$  such that  $I + M + \xi_0 J$  is invertible. Write

$$I + M + \xi J = (I + M + \xi_0 J)(I + (\xi - \xi_0)(I + M + \xi_0 J)^{-1}J).$$

We then notice that  $(I + M + \xi_0 J)^{-1}J$  is a compact operator, so according to basic functional analysis theory,  $I + (\xi - \xi_0)(I + M + \xi_0 J)^{-1}J$  is invertible except possibly for a  $\xi - \xi_0$  in a discrete subset of  $\mathbb{C}$ .  $\square$

**2.4. An explicit sphere case where equations (2.11-2.14) are singular.** To find a simple case where equations (2.11)–(2.14) are singular, we set  $\mathbf{e} \times \mathbf{n}$  and  $\mathbf{h} \times \mathbf{n}$  to be zero in (2.11)–(2.14) to obtain the reduced homogeneous system

$$(2.25) \quad \int_{\partial D} [\mathbf{grad}_{\mathbf{x}}(G_+ - G_-) \times \mathbf{n}(\mathbf{x})] (\mathbf{e} \cdot \mathbf{n})(\mathbf{y}) ds(\mathbf{y}) = \mathbf{0},$$

$$(2.26) \quad (\mathbf{e} \cdot \mathbf{n})(\mathbf{x}) + \frac{2\epsilon^-}{\epsilon^+ + \epsilon^-} \int_{\partial D} \left[ \frac{\partial G_+}{\partial \mathbf{n}(\mathbf{x})} - \frac{\epsilon^+}{\epsilon^-} \frac{\partial G_-}{\partial \mathbf{n}(\mathbf{x})} \right] (\mathbf{e} \cdot \mathbf{n})(\mathbf{y}) ds(\mathbf{y}) = 0,$$

$$(2.27) \quad \int_{\partial D} [\mathbf{grad}_{\mathbf{x}}(G_+ - G_-) \times \mathbf{n}(\mathbf{x})] (\mathbf{h} \cdot \mathbf{n})(\mathbf{y}) ds(\mathbf{y}) = 0$$

$$(2.28) \quad (\mathbf{h} \cdot \mathbf{n})(\mathbf{x}) + \frac{2\mu^-}{\mu^+ + \mu^-} \int_{\partial D} \left[ \frac{\partial G_+}{\partial \mathbf{n}(\mathbf{x})} - \frac{\mu^+}{\mu^-} \frac{\partial G_-}{\partial \mathbf{n}(\mathbf{x})} \right] (\mathbf{h} \cdot \mathbf{n})(\mathbf{y}) ds(\mathbf{y}) = 0.$$

Our example uses the following technical lemma.

**Lemma 2.2.** For any complex number  $k$ , and  $\mathbf{x} \in \mathbb{R}^3$ , let  $v(\mathbf{x}) = \int_S \frac{1}{4\pi} \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} ds(\mathbf{y})$ , where  $S$  denotes the unit sphere in  $\mathbb{R}^3$  centered at the origin. Let  $r = |\mathbf{x}|$ . Then

$$v(\mathbf{x}) = \begin{cases} \frac{e^{ikr} \sin k}{r \frac{k}{k}}, & \text{if } r \geq 1, \\ e^{ik} \frac{\sin k}{k}, & \text{if } r = 1, \\ \frac{\sin kr e^{ik}}{r \frac{k}{k}}, & \text{if } r \leq 1. \end{cases}$$

*Proof.* Let  $R$  be a rotation of  $\mathbb{R}^3$  with axis passing through the origin. Then

$$v(R\mathbf{x}) = \int_S \frac{1}{4\pi} \frac{e^{ik|R\mathbf{x}-\mathbf{y}|}}{|R\mathbf{x}-\mathbf{y}|} ds(\mathbf{y}) = \int_S \frac{1}{4\pi} \frac{e^{ik|\mathbf{x}-R^T\mathbf{y}|}}{|\mathbf{x}-R^T\mathbf{y}|} ds(\mathbf{y}) = v(\mathbf{x}).$$

Thus  $v$  is a radial function and we can write  $v(\mathbf{x}) = \tilde{v}(r)$ . Away from  $S$  and the origin,  $v$  satisfies the differential equation  $(\Delta + k^2)v = 0$ , thus  $\tilde{v}$  satisfies for  $r$  in  $(0, 1)$  and  $r$  in  $(1, \infty)$ ,

$$\partial_r^2 \tilde{v} + 2r^{-1} \partial_r \tilde{v} + k^2 \tilde{v} = 0.$$

If we set  $f(r) = r\tilde{v}(r)$ ,  $f$  satisfies  $\partial_r^2 f + k^2 f = 0$ , thus  $f(r)$  must be a linear combination of  $e^{ikr}$  and  $e^{-ikr}$ . It follows that  $\tilde{v}(r)$  is a linear combination of  $\frac{e^{ikr}}{r}$  and  $\frac{e^{-ikr}}{r}$  in  $(0, 1)$  and in

$(1, \infty)$ . Since  $\tilde{v}$  is continuous at zero,  $\tilde{v}(r)$  must be a multiple of  $\frac{\sin kr}{r}$  for  $r$  in  $[0, 1]$ . From the definition of  $v$  we have  $\tilde{v}(0) = v((0, 0, 0)) = e^{ik}$ , thus  $\tilde{v}(r) = \frac{\sin kr}{r} \frac{e^{ik}}{k}$ , if  $0 \leq r \leq 1$ .

Next we use the fact that  $v$  is continuous across  $S$  and that the normal derivative of  $v$  satisfies the jump condition  $(\partial_r v)^+ - (\partial_r v)^- = -1$ , to find the expression of  $v$  as function of  $r$  outside  $S$ .  $\square$

We now use Lemma 2.2 to construct a solution of (2.25)–(2.28). First we define

$$w(\mathbf{x}) = \begin{cases} \frac{e^{ik_- r} \sin k_-}{r k_-}, & \text{if } r > 1, \\ \frac{\sin k_+ r e^{ik_+}}{r k_+}, & \text{if } r < 1. \end{cases}$$

Across  $S$  we see that

$$(2.29)^+ (\partial_r w)^+ - \epsilon^- (\partial_r w)^- = \epsilon^+ \sin k_- e^{ik_-} \left( i - \frac{1}{k_-} \right) - \epsilon^- e^{ik_+} \left( \cos k_+ - \frac{\sin k_+}{k_+} \right).$$

There are infinitely many values of  $\epsilon^+$ ,  $\epsilon^-$ ,  $k_+$ ,  $k_-$  for which the right hand side of (2.29) is zero. These values can be found numerically. For example, when  $\epsilon^+ = 1$ ,  $\epsilon^- = 6$  one can find a zero of (2.29) at

$$k_+ = 0.76345236818 \text{ (to 11 d.p.)} \quad k_- = 1.83536815862 \text{ (to 11 d.p.)}$$

For such zeros of (2.29), if  $\mathbf{e} \cdot \mathbf{n} = 1$ ,  $\mathbf{h} \cdot \mathbf{n} = 0$  then (2.27) and (2.28) are trivially satisfied. Using Lemma 2.2 we obtain

$$w(\mathbf{x}) = \begin{cases} \int_S \frac{1}{4\pi} \frac{e^{ik_- |\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} ds(\mathbf{y}), & \text{if } r > 1, \\ \int_S \frac{1}{4\pi} \frac{e^{ik_+ |\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} ds(\mathbf{y}), & \text{if } r < 1. \end{cases}$$

Consequently, using (2.29)

$$(2.30) \quad \begin{aligned} & 1 + \frac{2\epsilon^-}{\epsilon^+ + \epsilon^-} \int_S \left[ \frac{\partial G_+}{\partial \mathbf{n}(\mathbf{x})} - \frac{\epsilon^+}{\epsilon^-} \frac{\partial G_-}{\partial \mathbf{n}(\mathbf{x})} \right] ds(\mathbf{y}) \\ &= -\frac{2\epsilon^-}{\epsilon^+ + \epsilon^-} \left( \epsilon^+ \sin k_- e^{ik_-} \left( i - \frac{1}{k_-} \right) - \epsilon^- e^{ik_+} \left( \cos k_+ - \frac{\sin k_+}{k_+} \right) \right), \end{aligned}$$

and hence (2.26) is satisfied for the above specific choice of dielectric parameters and  $\mathbf{e} \cdot \mathbf{n} = 1$ . Further, for  $\mathbf{x} \in S$ ,

$$\int_S (G_+ - G_-) 1 ds(\mathbf{y}) = (w)^- - (w)^+,$$

which is constant due to Lemma 2.2, so that its surface gradient is zero on  $S$ , and thus (2.25) is also satisfied.

In conclusion if  $\partial D = S$  we have found that the system (2.11-2.14) is singular for these values of  $\epsilon^+$ ,  $\epsilon^-$ ,  $k_+$ ,  $k_-$ .



**2.5. A new way of incorporating constraints (2.18) into (2.11–2.14).** In order to establish a well-posed SIE system for all frequencies, we consider a new approach consisting of incorporating the constraints (2.18) in surface integral form. We recall the single layer potential for the Laplacian

$$(2.31) \quad \mathbf{S}w(\mathbf{x}) = \frac{1}{4\pi} \int_{\partial D} \frac{1}{|\mathbf{x} - \mathbf{y}|} w(\mathbf{y}) ds(\mathbf{y})$$

and the associated gradient vector potential

$$(2.32) \quad \mathbf{D}w(\mathbf{x}) = \frac{1}{4\pi} \int_{\partial D} \left[ \mathbf{grad}_x \left( \frac{1}{|\mathbf{x} - \mathbf{y}|} \right) \times \mathbf{n}(\mathbf{y}) \right] \cdot w(\mathbf{y}) ds(\mathbf{y}).$$

The operator  $\mathbf{S}$  has the following properties [2, 21].

**Proposition 2.3.** *For any  $s \in \mathbb{R}$ , the operator  $\mathbf{S} : H^s(\partial D) \rightarrow H^{s+1}(\partial D)$  is continuous and invertible with continuous inverse. Furthermore, the operator  $\mathbf{S} : L^2(\partial D) \rightarrow L^2(\partial D)$  is self adjoint and coercive.*

**Lemma 2.4.** *Let  $\mathbf{e}, \mathbf{h} \in L^2(\partial D)^3$ . The constraints (2.18) are equivalent to*

$$(2.33) \quad (-i\omega\epsilon^+ \mathbf{S}(\mathbf{e} \cdot \mathbf{n}) + \mathbf{D}\mathbf{h}) = 0,$$

$$(2.34) \quad (i\omega\mu^+ \mathbf{S}(\mathbf{h} \cdot \mathbf{n}) + \mathbf{D}\mathbf{e}) = 0.$$

*Proof.* Applying Green's theorem on the closed surface  $\partial D$  establishes that (2.33)–(2.34) are equivalent to

$$\mathbf{S}(\operatorname{div}_{\partial D}(\mathbf{e} \times \mathbf{n}) - i\omega\mu^+ \mathbf{h} \cdot \mathbf{n}) = 0,$$

$$\mathbf{S}(\operatorname{div}_{\partial D}(\mathbf{h} \times \mathbf{n}) + i\omega\epsilon^+ \mathbf{e} \cdot \mathbf{n}) = 0.$$

The result then follows by Proposition 2.3. □

We now define the operator  $\mathbb{J}$  by setting

$$(2.35) \quad \mathbb{J}(\mathbf{e}, \mathbf{h})(\mathbf{x}) = \left( \mathbf{n}(\mathbf{x})(\omega^2\epsilon^+ \mathbf{S}(\mathbf{e} \cdot \mathbf{n}) + i\omega\mathbf{D}\mathbf{h}), \mathbf{n}(\mathbf{x})(\omega^2\mu^+ \mathbf{S}(\mathbf{h} \cdot \mathbf{n}) - i\omega\mathbf{D}\mathbf{e}) \right).$$

The constrained problem (2.23) implies that for any  $\xi \in \mathbb{C}$ ,

$$(2.36) \quad (\mathbb{I} + \mathbb{M} + \xi\mathbb{J})(\mathbf{e}, \mathbf{h}) = (\mathbf{e}^i, \mathbf{h}^i).$$

This formulation does not suffer from low-frequency breakdown. Indeed,  $\mathbb{I} + \mathbb{M}$  converges in operator norm to an invertible linear operator as the frequency  $\omega \rightarrow 0$  [1, Appendix B] and clearly  $\mathbb{J}$  converges in operator norm to zero as  $\omega \rightarrow 0$ .

**3. Unique solvability of The SIE system (2.36).** To prove that equation (2.36) is well-posed (for a range of  $\xi$  that we specify later) we introduce the space

$$X = \{(\mathbf{e}, \mathbf{h}) : \mathbf{e} \times \mathbf{n}, \mathbf{h} \times \mathbf{n} \in H^1(\partial D)^3, \mathbf{e} \cdot \mathbf{n}, \mathbf{h} \cdot \mathbf{n} \in L^2(\partial D)\}$$

with norm  $\|\cdot\|_X$  defined by

$$\|(\mathbf{e}, \mathbf{h})\|_X^2 = \|\mathbf{e} \times \mathbf{n}\|_{H^1(\partial D)^3}^2 + \|\mathbf{h} \times \mathbf{n}\|_{H^1(\partial D)^3}^2 + \|\mathbf{e} \cdot \mathbf{n}\|_{L^2(\partial D)}^2 + \|\mathbf{h} \cdot \mathbf{n}\|_{L^2(\partial D)}^2.$$

and associated induced inner product from each term. It is clear that  $X$  is a Hilbert space.

**Proposition 3.1.** *The linear operator  $\mathbb{M} : X \rightarrow X$  is compact.*

*Proof.* Recall that  $\mathbb{M}$  was defined through the system (2.11)–(2.14). We first examine the terms from (2.11),

$$\begin{aligned} & \int_{\partial D} \left[ \epsilon^+ \frac{\partial G_+}{\partial \mathbf{n}(\mathbf{x})} - \epsilon^- \frac{\partial G_-}{\partial \mathbf{n}(\mathbf{x})} \right] (\mathbf{e} \times \mathbf{n})(\mathbf{y}) \, ds(\mathbf{y}), \\ & \int_{\partial D} [\mathbf{grad}_{\mathbf{x}}(\epsilon^+ G_+ - \epsilon^- G_-)] (\mathbf{e} \times \mathbf{n})(\mathbf{y}) \cdot [\mathbf{n}(\mathbf{x}) - \mathbf{n}(\mathbf{y})] \, ds(\mathbf{y}), \\ & i\omega \int_{\partial D} [\epsilon^+ \mu^+ G_+ - \epsilon^- \mu^- G_-] (\mathbf{h} \times \mathbf{n})(\mathbf{y}) \times \mathbf{n}(\mathbf{x}) \, ds(\mathbf{y}). \end{aligned}$$

It is clear from integral operator theory [5, 6, 12] that if  $(\mathbf{e}, \mathbf{h}) \in X$  then each of these three terms is in  $H^2(\partial D)^3$ , and the associated linear operators are bounded. That is, the terms depend continuously on the  $H^1$  norms of  $\mathbf{e} \times \mathbf{n}$  and  $\mathbf{h} \times \mathbf{n}$ . Next we examine the term

$$(3.1) \quad \int_{\partial D} [\mathbf{grad}_{\mathbf{x}}(G_+ - G_-) \times \mathbf{n}(\mathbf{x})] (\mathbf{e} \cdot \mathbf{n})(\mathbf{y}) \, ds(\mathbf{y})$$

in (2.11). Using the Taylor series of the exponential,

$$\mathbf{grad}_{\mathbf{x}}(G_+ - G_-) = \left( \frac{(ik_-)^2 - (ik_+)^2}{8\pi} + O(|\mathbf{x} - \mathbf{y}|) \right) \mathbf{grad}_{\mathbf{x}}|\mathbf{x} - \mathbf{y}|,$$

from integral operator theory [5, 6, 12] the term (3.1) is in  $H^2(\partial D)^3$  if  $(\mathbf{e}, \mathbf{h}) \in X$  and depends continuously on the  $L^2$  norm of  $\mathbf{e} \cdot \mathbf{n}$ .

We now write the terms in (2.12) involved in defining  $\mathbb{M}$ ,

$$\begin{aligned} & \int_{\partial D} [\mu^+ G_+ - \mu^- G_-] (\mathbf{h} \times \mathbf{n})(\mathbf{y}) \cdot \mathbf{n}(\mathbf{x}) \, ds(\mathbf{y}), \\ & \int_{\partial D} \left[ \frac{\partial G_+}{\partial \mathbf{n}(\mathbf{x})} - \frac{\epsilon^+}{\epsilon^-} \frac{\partial G_-}{\partial \mathbf{n}(\mathbf{x})} \right] (\mathbf{e} \cdot \mathbf{n})(\mathbf{y}) \, ds(\mathbf{y}), \\ & \int_{\partial D} [\mathbf{grad}_{\mathbf{x}}(G_+ - G_-) \times \mathbf{n}(\mathbf{x})] \cdot (\mathbf{e} \times \mathbf{n})(\mathbf{y}) \, ds(\mathbf{y}). \end{aligned}$$

It is clear that each of these terms is in  $H^1(\partial D)$  and that they depend continuously on the  $H^1(\partial D)^3$  norms of  $\mathbf{e} \times \mathbf{n}$  and  $\mathbf{h} \times \mathbf{n}$  and on the  $L^2$  norm of  $\mathbf{e} \cdot \mathbf{n}$ .

The analysis of the terms in (2.13) and (2.14) is similar. □

**Proposition 3.2.** *The linear operator  $\mathbb{J} : X \rightarrow X$  is compact.*

*Proof.* We see from definition (2.32) that  $\mathbf{D}\mathbf{w}$  depends on  $\mathbf{w}$  only through its tangential part. Thus since  $\mathbf{h} \times \mathbf{n} \in H^1(\partial D)^3$ ,  $\mathbf{D}\mathbf{h} \in H^1(\partial D)$  and depends continuously on the  $H^1(\partial D)^3$  norm of  $\mathbf{h} \times \mathbf{n}$ .  $\mathbf{S}(\mathbf{e} \cdot \mathbf{n})$  is also in  $H^1(\partial D)$  and it depends continuously on the  $L^2(\partial D)$  norm of  $\mathbf{e} \cdot \mathbf{n}$ .

A similar argument applies to the second component of  $\mathbb{J}$ . Thus  $\mathbb{J} : X \rightarrow X$  is compact. □

Let us now write a reduced system in  $\mathbf{e} \cdot \mathbf{n}$ ,  $\mathbf{h} \cdot \mathbf{n}$  derived from (2.36) where we set  $\mathbf{e} \times \mathbf{n} = \mathbf{h} \times \mathbf{n} = 0$ ,

$$\begin{aligned} (\mathbf{e} \cdot \mathbf{n})(\mathbf{x}) + \frac{2\epsilon^-}{\epsilon^+ + \epsilon^-} \int_{\partial D} \left[ \frac{\partial G_+}{\partial \mathbf{n}(\mathbf{x})} - \frac{\epsilon^+}{\epsilon^-} \frac{\partial G_-}{\partial \mathbf{n}(\mathbf{x})} \right] (\mathbf{e} \cdot \mathbf{n})(\mathbf{y}) \, ds(\mathbf{y}) + \xi \omega^2 \epsilon^+ \mathbf{S}(\mathbf{e} \cdot \mathbf{n}) &= f, \\ (\mathbf{h} \cdot \mathbf{n})(\mathbf{x}) + \frac{2\mu^-}{\mu^+ + \mu^-} \int_{\partial D} \left[ \frac{\partial G_+}{\partial \mathbf{n}(\mathbf{x})} - \frac{\mu^+}{\mu^-} \frac{\partial G_-}{\partial \mathbf{n}(\mathbf{x})} \right] (\mathbf{h} \cdot \mathbf{n})(\mathbf{y}) \, ds(\mathbf{y}) + \xi \omega^2 \mu^+ \mathbf{S}(\mathbf{h} \cdot \mathbf{n}) &= g, \end{aligned}$$

for two given forcing terms  $f$  and  $g$ . It is helpful to write this system in operator form as

$$(3.2) \quad (\mathbf{e} \cdot \mathbf{n}) + \mathbf{K}_1(\mathbf{e} \cdot \mathbf{n}) + \xi \omega^2 \epsilon^+ \mathbf{S}(\mathbf{e} \cdot \mathbf{n}) = f,$$

$$(3.3) \quad (\mathbf{h} \cdot \mathbf{n}) + \mathbf{K}_2(\mathbf{h} \cdot \mathbf{n}) + \xi \omega^2 \mu^+ \mathbf{S}(\mathbf{h} \cdot \mathbf{n}) = g,$$

where

$$\begin{aligned} \mathbf{K}_1 w(\mathbf{x}) &= \frac{2\epsilon^-}{\epsilon^+ + \epsilon^-} \int_{\partial D} \left[ \frac{\partial G_+}{\partial \mathbf{n}(\mathbf{x})} - \frac{\epsilon^+}{\epsilon^-} \frac{\partial G_-}{\partial \mathbf{n}(\mathbf{x})} \right] w(\mathbf{y}) \, ds(\mathbf{y}), \\ \mathbf{K}_2 w(\mathbf{x}) &= \frac{2\mu^-}{\mu^+ + \mu^-} \int_{\partial D} \left[ \frac{\partial G_+}{\partial \mathbf{n}(\mathbf{x})} - \frac{\mu^+}{\mu^-} \frac{\partial G_-}{\partial \mathbf{n}(\mathbf{x})} \right] w(\mathbf{y}) \, ds(\mathbf{y}). \end{aligned}$$

**Proposition 3.3.** *There is a  $\xi_0 > 0$  such that for all  $\xi > \xi_0$ , equations (3.2) and (3.3) are well posed for  $f, g \in L^2(\partial D)$ . In addition, there is a constant  $C$  such that for all  $\xi > \xi_0$ ,*

$$\|\mathbf{e} \cdot \mathbf{n}\|_{L^2(\partial D)} \leq C \|f\|_{L^2(\partial D)}, \quad \|\mathbf{h} \cdot \mathbf{n}\|_{L^2(\partial D)} \leq C \|g\|_{L^2(\partial D)}.$$

*Proof.* The operator  $\mathbf{K}_1 : L^2(\partial D) \rightarrow L^2(\partial D)$  is compact[5, 6, 12]. Since  $\mathbf{S}$  satisfies the properties stated in Proposition 2.3, we can apply Theorem 4.2. A similar argument establishes well posedness of (3.3) and the second inequality.  $\square$

**Theorem 3.4.** *The linear operator  $\mathbb{I} + \mathbb{M} + \xi \mathbb{J} : X \rightarrow X$  is invertible for all  $\xi > \xi_0$ , and also for all  $\xi \in \mathbb{C} \setminus Y$  where  $Y$  is a discrete set. The corresponding inverse operator is uniformly bounded for all  $\xi > \xi_0$ .*

*Proof.* To prove that  $\mathbb{I} + \mathbb{M} + \xi \mathbb{J}$  is invertible for all  $\xi$  sufficiently large, since  $\mathbb{M}$  and  $\mathbb{J}$  are compact operators, we can argue by contradiction by assuming that there is a sequence  $(\xi_p) \in \mathbb{R}$  with  $\xi_p \rightarrow \infty$  as  $p \rightarrow \infty$ , and a sequence  $(\mathbf{e}_p, \mathbf{h}_p) \in X$  such that

$$(3.4) \quad \|\mathbf{e}_p \times \mathbf{n}\|_{H^1(\partial D)}^2 + \|\mathbf{h}_p \times \mathbf{n}\|_{H^1(\partial D)}^2 + \|\mathbf{e}_p \cdot \mathbf{n}\|_{L^2(\partial D)}^2 + \|\mathbf{h}_p \cdot \mathbf{n}\|_{L^2(\partial D)}^2 = 1,$$

and

$$(3.5) \quad (\mathbb{I} + \mathbb{M} + \xi_p \mathbb{J})(\mathbf{e}_p, \mathbf{h}_p) = 0.$$

We first write the equation for  $(\mathbf{e}_p, \mathbf{h}_p)$  corresponding to (3.5) in the first normal component,

$$\begin{aligned}
(\mathbf{e}_p \cdot \mathbf{n})(\mathbf{x}) - \frac{2\epsilon^-}{\epsilon^+ + \epsilon^-} & \left\{ -i\omega \int_{\partial D} [\mu^+ G_+ - \mu^- G_-] (\mathbf{h}_p \times \mathbf{n})(\mathbf{y}) \cdot \mathbf{n}(\mathbf{x}) ds(\mathbf{y}) \right. \\
& - \int_{\partial D} \left[ \frac{\partial G_+}{\partial \mathbf{n}(\mathbf{x})} - \frac{\epsilon^+}{\epsilon^-} \frac{\partial G_-}{\partial \mathbf{n}(\mathbf{x})} \right] (\mathbf{e}_p \cdot \mathbf{n})(\mathbf{y}) ds(\mathbf{y}) \\
& \left. + \int_{\partial D} [\mathbf{grad}_x(G_+ - G_-) \times \mathbf{n}(\mathbf{x})] \cdot (\mathbf{e}_p \times \mathbf{n})(\mathbf{y}) ds(\mathbf{y}) \right\} \\
(3.6) \quad & + \xi_p \omega \mathbf{S}(\omega \epsilon^+ \mathbf{e}_p \cdot \mathbf{n} - i \operatorname{div}_{\partial D}(\mathbf{h}_p \times \mathbf{n})) = 0,
\end{aligned}$$

where Green's theorem was used to introduce the term  $\operatorname{div}_{\partial D}(\mathbf{h}_p \times \mathbf{n})$ . It follows from Proposition 2.3 that

$$(3.7) \quad \|\operatorname{div}_{\partial D}(\mathbf{h}_p \times \mathbf{n}) + i\omega \epsilon^+ \mathbf{e}_p \cdot \mathbf{n}\|_{H^{-1}(\partial D)} = O(\xi_p^{-1}).$$

We now rearrange terms in the row of equation (3.5) corresponding in the first tangential components to obtain

$$\begin{aligned}
(\mathbf{e} \times \mathbf{n})(\mathbf{x}) - \frac{2}{\epsilon^+ + \epsilon^-} & \left\{ - \int_{\partial D} \left[ \epsilon^+ \frac{\partial G_+}{\partial \mathbf{n}(\mathbf{x})} - \epsilon^- \frac{\partial G_-}{\partial \mathbf{n}(\mathbf{x})} \right] (\mathbf{e} \times \mathbf{n})(\mathbf{y}) ds(\mathbf{y}) \right. \\
& + \int_{\partial D} [\mathbf{grad}_x(\epsilon^+ G_+ - \epsilon^- G_-)] (\mathbf{e} \times \mathbf{n})(\mathbf{y}) \cdot [\mathbf{n}(\mathbf{x}) - \mathbf{n}(\mathbf{y})] ds(\mathbf{y}) \\
& - i\omega \int_{\partial D} [\epsilon^+ \mu^+ G_+ - \epsilon^- \mu^- G_-] (\mathbf{h} \times \mathbf{n})(\mathbf{y}) \times \mathbf{n}(\mathbf{x}) ds(\mathbf{y}) \\
& \left. - \frac{i}{\omega} \int_{\partial D} [\mathbf{grad}_x(G_+ - G_-) \times \mathbf{n}(\mathbf{x})] (\operatorname{div}_{\partial D} \mathbf{h} \times \mathbf{n})(\mathbf{y}) ds(\mathbf{y}) \right\} \\
(3.8) \quad & = \frac{2}{\epsilon^+ + \epsilon^-} \frac{i}{\omega} \int_{\partial D} [\mathbf{grad}_x(G_+ - G_-) \times \mathbf{n}(\mathbf{x})] (\operatorname{div}_{\partial D} \mathbf{h} \times \mathbf{n} + i\omega \epsilon^+ \mathbf{e}_p \cdot \mathbf{n})(\mathbf{y}) ds(\mathbf{y}),
\end{aligned}$$

which is  $O(\xi_p^{-1})$  in  $H^1$  norm thanks to the smoothing property of  $\mathbf{grad}_x(G_+ - G_-)$  mentioned in the proof of Proposition 3.1. Similar considerations will lead to

$$\begin{aligned}
(\mathbf{h} \times \mathbf{n})(\mathbf{x}) - \frac{2}{\mu^+ + \mu^-} & \left\{ - \int_{\partial D} \left[ \mu^+ \frac{\partial G_+}{\partial \mathbf{n}(\mathbf{x})} - \mu^- \frac{\partial G_-}{\partial \mathbf{n}(\mathbf{x})} \right] (\mathbf{h} \times \mathbf{n})(\mathbf{y}) ds(\mathbf{y}) \right. \\
& + \int_{\partial D} [\mathbf{grad}_x(\mu^+ G_+ - \mu^- G_-)] (\mathbf{h} \times \mathbf{n})(\mathbf{y}) \cdot [\mathbf{n}(\mathbf{x}) - \mathbf{n}(\mathbf{y})] ds(\mathbf{y}) \\
& + i\omega \int_{\partial D} [\epsilon^+ \mu^+ G_+ - \epsilon^- \mu^- G_-] (\mathbf{e} \times \mathbf{n})(\mathbf{y}) \times \mathbf{n}(\mathbf{x}) ds(\mathbf{y}) \\
(3.9) \quad & \left. + \frac{i}{\omega} \int_{\partial D} [\mathbf{grad}_x(G_+ - G_-) \times \mathbf{n}(\mathbf{x})] (\operatorname{div}_{\partial D} \mathbf{e} \times \mathbf{n})(\mathbf{y}) ds(\mathbf{y}) \right\} = O(\xi_p^{-1}),
\end{aligned}$$

in  $H^1$  norm. We now recognize that the left hand side of (3.8-3.9) is in the standard Müller equation form for the unknown  $(\mathbf{e}_p \times \mathbf{n}, \mathbf{h}_p \times \mathbf{n})$ . That equation is well posed for all  $\omega > 0$ , and all values of  $\epsilon^\pm$  and  $\mu^\pm$  considered in this paper, as proved in [11], chapter VI. We have thus found that  $\mathbf{e}_p \times \mathbf{n}$  and  $\mathbf{h}_p \times \mathbf{n}$  are  $O(\xi_p^{-1})$ , in  $H^1$  norm. It follows that  $\operatorname{div}_{\partial D}(\mathbf{h}_p \times \mathbf{n})$  is  $O(\xi_p^{-1})$ , in  $L^2$  norm, and because of (3.7),  $\mathbf{e}_p \cdot \mathbf{n}$  is  $O(\xi_p^{-1})$ , in  $H^{-1}$  norm.

Taking the inner product of the terms in equation (3.6) with  $\mathbf{e}_p \cdot \mathbf{n}$  on  $\partial D$  yields

$$\begin{aligned}
& \langle \mathbf{e}_p \cdot \mathbf{n}, (I + \mathbf{K}_1 + \xi \omega^2 \epsilon^+ \mathbf{S}) \mathbf{e}_p \cdot \mathbf{n} \rangle = \\
& \frac{2\epsilon^-}{\epsilon^+ + \epsilon^-} \langle \mathbf{e}_p \cdot \mathbf{n}, -i\omega \int_{\partial D} [\mu^+ G_+ - \mu^- G_-] (\mathbf{h}_p \times \mathbf{n})(\mathbf{y}) \cdot \mathbf{n}(\mathbf{x}) ds(\mathbf{y}) \rangle \\
& + \frac{2\epsilon^-}{\epsilon^+ + \epsilon^-} \langle \mathbf{e}_p \cdot \mathbf{n}, \int_{\partial D} [\mathbf{grad}_x(G_+ - G_-) \times \mathbf{n}(\mathbf{x})] \cdot (\mathbf{e}_p \times \mathbf{n})(\mathbf{y}) ds(\mathbf{y}) \rangle \\
(3.10) \quad & -i\xi_p \omega \langle \mathbf{e}_p \cdot \mathbf{n}, \mathbf{S} \operatorname{div}_{\partial D}(\mathbf{h}_p \times \mathbf{n}) \rangle,
\end{aligned}$$

where  $\mathbf{K}_1$  is as in (3.2). The first two terms of the right hand side of (3.10) are clearly of order  $O(\xi_p^{-1})$  in  $H^1$  norm since  $\mathbf{e}_p \times \mathbf{n}$  and  $\mathbf{h}_p \times \mathbf{n}$  are  $O(\xi_p^{-1})$  in  $H^1$  norm. For the third term, using that  $\mathbf{S}$  is self adjoint,

$$\xi_p \omega \langle \mathbf{e}_p \cdot \mathbf{n}, \mathbf{S} \operatorname{div}_{\partial D}(\mathbf{h}_p \times \mathbf{n}) \rangle = \xi_p \omega \langle \mathbf{S}(\mathbf{e}_p \cdot \mathbf{n}), \operatorname{div}_{\partial D}(\mathbf{h}_p \times \mathbf{n}) \rangle,$$

and since  $\mathbf{e}_p \cdot \mathbf{n}$  is  $O(\xi_p^{-1})$  in  $H^{-1}$  norm we have  $\mathbf{S}(\mathbf{e}_p \cdot \mathbf{n})$  is  $O(\xi_p^{-1})$  in  $L^2$  norm. Finally, since  $\operatorname{div}_{\partial D}(\mathbf{h}_p \times \mathbf{n})$  is  $O(\xi_p^{-1})$  in  $L^2$  norm we deduce that  $\xi_p \omega \langle \mathbf{S}(\mathbf{e}_p \cdot \mathbf{n}), \operatorname{div}_{\partial D}(\mathbf{h}_p \times \mathbf{n}) \rangle$  is of order  $O(\xi_p^{-1})$ . In summary, the right hand side of (3.10) is of order  $O(\xi_p^{-1})$ . However, Theorem 4.2 establishes that the real part of the left hand side of (3.10) is bounded below by  $\frac{1}{2} \|\mathbf{e}_p \cdot \mathbf{n}\|_{L^2(\partial D)}^2$  for all  $p$  sufficiently large, so that  $\mathbf{e}_p \cdot \mathbf{n}$  is of order  $O(\xi_p^{-1/2})$  in the  $L^2$  norm. We have thus proved that  $(\mathbf{e}, \mathbf{h})$  converges strongly to zero in  $X$ , which contradicts (3.4).

To show the uniform bound for the operator  $(\mathbb{I} + \mathbb{M} + \xi \mathbb{J})^{-1}$  for all  $\xi$  sufficiently large, we can repeat the same argument by contradiction by assuming (3.4) and in place of (3.5) we assume

$$\lim_{p \rightarrow \infty} (\mathbb{I} + \mathbb{M} + \xi_p \mathbb{J})(\mathbf{e}_p, \mathbf{h}_p) = 0.$$

Finally, Proposition 2.1 proves that  $\mathbb{I} + \mathbb{M} + \xi \mathbb{J}$  is invertible for all  $\xi \in \mathbb{C} \setminus Y$  where  $Y$  is a discrete set.  $\square$

We now remark that in practice the forcing term in equation (2.36) is smooth since it is derived from an incident field. It is thus useful to study the regularity properties of equation (2.36). To that effect we define for  $s > 0$  the functional space

$$X^s = \{(\mathbf{e}, \mathbf{h}) : \mathbf{e} \times \mathbf{n}, \mathbf{h} \times \mathbf{n} \in H^{s+1}(\partial D)^3, \mathbf{e} \cdot \mathbf{n}, \mathbf{h} \cdot \mathbf{n} \in H^s(\partial D)\}$$

with the norm  $\|\cdot\|_{X^s}$  defined by

$$\|(\mathbf{e}, \mathbf{h})\|_{X^s}^2 = \|\mathbf{e} \times \mathbf{n}\|_{H^{s+1}(\partial D)^3}^2 + \|\mathbf{h} \times \mathbf{n}\|_{H^{s+1}(\partial D)^3}^2 + \|\mathbf{e} \cdot \mathbf{n}\|_{H^s(\partial D)}^2 + \|\mathbf{h} \cdot \mathbf{n}\|_{H^s(\partial D)}^2.$$

Identical arguments to those used in the proofs of Propositions 3.1 and 3.2 can be repeated to show that  $\mathbb{M}, \mathbb{J} : X^s \rightarrow X^s$  are compact. In fact  $\mathbb{I} + \mathbb{M} + \xi \mathbb{J} : X^s \rightarrow X^s$  is invertible with continuous inverse for the same values of  $\xi$  as in the statement of Theorem 3.4. These results are summarized in the next theorem.

**Theorem 3.5.** *Suppose that  $(\mathbf{e}^i, \mathbf{h}^i) \in X^s$ . Then for all  $\xi \geq \xi_0$ , or  $\xi \in \mathbb{C} \setminus Y$  for some discrete set  $Y$ , equation (2.36) is uniquely solvable. The solution  $(\mathbf{e}, \mathbf{h}) \in X^s$ , and  $\|(\mathbf{e}, \mathbf{h})\|_{X^s}$  is linearly bounded by  $\|(\mathbf{e}^i, \mathbf{h}^i)\|_{X^s}$ . The bound is uniform for all  $\xi$  such that  $\xi \geq \xi_0$ .*

#### 4. Appendix.

**Lemma 4.1.** *Let  $H$  be a separable Hilbert space and  $\{e_j : j \in \mathbb{N}\}$  be a Hilbert basis of  $H$ . Let  $P_m$  be the orthogonal projection onto  $\text{span}\{e_1, \dots, e_m\}$ , and  $Q_m = I - P_m$ . Let  $K : H \rightarrow H$  be a linear compact operator. Then  $Q_m K$  and  $K Q_m$  converge to zero in operator norm, as  $m \rightarrow \infty$ .*

*Proof.* Arguing by contradiction, suppose that  $Q_m K$  does not converge to zero in operator norm. Then there is a positive  $\alpha$  and a sequence  $(x_m) \in H$  such that  $\|x_m\| = 1$  and  $\|Q_m K x_m\| \geq \alpha$ . Since  $K$  is compact, there is a subsequence  $(K x_{m_p})$  of  $(K x_m)$  which converges to  $Kx$ , for some  $x$  in  $H$ . Clearly,  $Q_{m_p} K x$  converges to zero. Now

$$\|Q_{m_p} K x_{m_p}\| \leq \|Q_{m_p} K x\| + \|Q_{m_p}(K x_{m_p} - Kx)\| \leq \|Q_{m_p} K x\| + \|K x_{m_p} - Kx\|.$$

Thus  $Q_{m_p} K x_{m_p}$  converges to zero as  $m_p \rightarrow \infty$ , which is a contradiction. Note that  $Q_m K^* = Q_m^* K^* = (K Q_m)^*$ , using  $Q_m^* = Q_m$ . Since in the result above  $K$  is arbitrary, we deduce that  $Q_m K^*$  also converges to zero.  $\square$

**Theorem 4.2.** *Let  $H$  be a separable Hilbert space and  $M, J : H \rightarrow H$  be two linear and compact operators. Assume that  $J$  is injective, self adjoint, and positive. Then there is a positive  $\xi_0$  such that  $I + M + \xi J$  is invertible for all  $\xi > \xi_0$  and  $\|(I + M + \xi J)^{-1}\|$  is uniformly bounded. In addition  $\xi_0$  can be chosen such that for all  $\xi > \xi_0$  and  $x$  in  $H$ ,*

$$\text{Re} \langle (I + M + \xi J)x, x \rangle \geq \frac{1}{2} \|x\|^2.$$

*Proof.* Let  $\lambda_1 \geq \lambda_2 \geq \dots \lambda_j \geq \dots > 0$  be the eigenvalues of  $J$ . Let  $e_1, \dots, e_j, \dots$  be associated eigenvectors with norm 1. They form an orthonormal Hilbert basis of  $H$ , since  $J$  is injective.

Let  $P_m$  be the orthonormal projection on  $\text{span}\{e_1, \dots, e_m\}$  and  $Q_m = I - P_m$ . By Lemma 4.1, we can pick  $m$  large enough such that  $\|Q_m M\| < \frac{1}{4}$ . For  $\xi > 0$  we write

$$\text{Re} \langle (I + M + \xi J)x, x \rangle = t_1 + t_2 + t_3 + t_4,$$

with

$$\begin{aligned} t_1 &= \|x\|^2, \\ t_2 &= \xi \text{Re} \langle P_m J x, P_m x \rangle + \text{Re} \langle P_m M x, P_m x \rangle, \\ t_3 &= \xi \text{Re} \langle Q_m J x, Q_m x \rangle, \\ t_4 &= \text{Re} \langle Q_m M x, Q_m x \rangle. \end{aligned}$$

As  $J$  commutes with  $P_m$  and  $Q_m$

$$\begin{aligned} t_2 &= \xi \langle J P_m x, P_m x \rangle + \text{Re} \langle P_m M x, P_m x \rangle, \\ t_3 &= \xi \langle J Q_m x, Q_m x \rangle. \end{aligned}$$

As  $J$  is injective, self-adjoint, coercive, and the range of  $P_m$  is finite dimensional, there is a positive  $C$  such that  $\langle JP_mx, P_mx \rangle \geq C\|P_mx\|^2$ . It follows that

$$\begin{aligned} t_2 &\geq \xi C\|P_mx\|^2 - \|M\|\|x\|\|P_mx\| \\ &\geq \xi C\|P_mx\|^2 - \frac{1}{2}\frac{1}{\beta}\|M\|^2\|P_mx\|^2 - \frac{1}{2}\beta\|x\|^2, \end{aligned}$$

for any  $\beta > 0$ . We note that  $t_3 \geq 0$  and  $t_4 \geq -\frac{1}{4}\|x\|^2$ . We set  $\beta = \frac{1}{2}$ ,  $\xi_0 = \frac{\|M\|^2}{C}$  to find, for any  $\xi \geq \xi_0$

$$\operatorname{Re} \langle (I + M + \xi J)x, x \rangle \geq \frac{1}{2}\|x\|^2.$$

By the Lax Milgram theorem, this estimate shows that  $(I + M + \xi J)$  is invertible for all  $\xi \geq \xi_0$ . As  $\operatorname{Re} \langle (I + M + \xi J)(I + M + \xi J)^{-1}x, (I + M + \xi J)^{-1}x \rangle$  is greater or equal than  $\frac{1}{2}\|(I + M + \xi J)^{-1}x\|^2$  and less or equal than  $\|x\|\|(I + M + \xi J)^{-1}x\|$  for all  $x$ , it follows that the norm of  $(I + M + \xi J)^{-1}$  is less or equal than 2.  $\square$

**Proposition 4.3.** *Let  $H$  be a separable Hilbert space and  $M, J : H \rightarrow H$  be two linear and compact operators. Assume that  $J$  is injective, self-adjoint, and positive. Then  $I + M + \xi J$  is invertible, with continuous inverse, except possibly for  $\xi$  in a discrete set of  $\mathbb{C}$ .*

*Proof.* The result follows from Proposition 2.1 and Theorem 4.2.  $\square$

**Theorem 4.4.** *Let  $H$  be a separable Hilbert space, and  $M, J : H \rightarrow H$  be two linear compact operators. Assume  $J$  is self-adjoint and non-negative,  $N(J)$  is invariant under  $M$ , and the intersection of  $N(J)$  and  $N(I + M)$  is trivial. Then  $I + M + \xi J$  is invertible for all real  $\xi > \xi_0$ , for some constant  $\xi_0$  and for all  $\xi \in \mathbb{C} \setminus Y$  for some discrete subset  $Y \subseteq \mathbb{C}$ .*

*Proof.* Let  $P$  denote the orthogonal projection on  $N(J)$ . The operator  $I + M + \xi J$  can be re-written in blocks

$$I + M + \xi J = \begin{pmatrix} (I - P)(I + M + \xi J)(I - P) & +PM(I - P) \\ +P(I + M)P & \end{pmatrix}.$$

We note that due to the assumptions on  $M$  and  $P$ ,  $P(I + M)P$  is an invertible operator from  $N(J)$  to  $N(J)$ , with continuous inverse. Since we can apply Theorem 4.2 to the block  $(I - P)(I + M + \xi J)(I - P)$ , the result is proved.  $\square$

**Proposition 4.5.** *Let  $H$  be a separable Hilbert space. There exist two compact linear operators  $M, J : H \rightarrow H$  such that  $J$  is injective and  $I + M + \xi J$  is singular for all  $\xi \in \mathbb{C}$ .*

*Proof.* Let  $H$  be a separable Hilbert space over  $\mathbb{C}$  of infinite dimension with Hilbert basis  $\{e_1, e_2, \dots\}$ . We can define a compact linear operator  $M$  by setting  $Me_1 = -e_1$ , and  $Me_k = 0$ , for all  $k \geq 2$ . Now we define a compact linear operator  $J$  by setting  $Je_k = \frac{e_{k+1}}{k+1}$

for all  $k$ . Since for all  $x$  in  $H$ ,  $\|Jx\|^2 = \sum_{k=1}^{\infty} \frac{|\langle x, e_k \rangle|^2}{(k+1)^2}$ ,  $J$  is injective.

Now let  $\xi \in \mathbb{C}$  and set

$$u = \sum_{k=1}^{\infty} \frac{(-\xi)^{k-1} e_k}{k!}.$$

Note that  $\|u\|^2 \geq 1$ . Moreover,

$$(I + M)u = \sum_{k=2}^{\infty} \frac{(-\xi)^{k-1} e_k}{k!} = \sum_{k=1}^{\infty} (-1)^k \frac{\xi^k e_{k+1}}{(k+1)!},$$

and

$$\xi Ju = \xi \sum_{k=1}^{\infty} \frac{(-\xi)^{k-1} J e_k}{k!} = - \sum_{k=1}^{\infty} (-1)^k \frac{\xi^k e_{k+1}}{(k+1)!},$$

Thus  $(I + M + \xi J)u = 0$ , for all  $\xi \in \mathbb{C}$ . □

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