

**REMARKS ON ESTIMATES FOR
THE ADJOINT RESTRICTION OPERATOR TO CURVES
OVER THE SPHERE**

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ABSTRACT. Recently, two of the authors obtained estimates for the adjoint restriction operator to finite type curves with respect to general measures. Strikingly, it turns out that some of such estimates are sharp, especially when the measures are given by surface measures under certain condition. A typical example is the surface measure on the sphere. We demonstrate sharpness of such estimates by constructing an example and, also, discuss related estimates over different type of surfaces.

1. INTRODUCTION

Let $\gamma : I = [0, 1] \rightarrow \mathbb{R}^d$ be a smooth curve, and let the operator T_λ^γ be defined by

$$(1.1) \quad T_\lambda^\gamma f(x) = \int_I e^{i\lambda x \cdot \gamma(t)} f(t) dt.$$

The problem of characterizing p, q for which

$$(1.2) \quad \|T_\lambda^\gamma f\|_{L^q(\mathbb{R}^d)} \leq C\lambda^{-\frac{d}{q}} \|f\|_{L^p(I)}$$

holds has been studied by many authors and the estimates have been established up to the optimal range for a large class of curves. In particular, for the curves which satisfy the nonvanishing torsion condition

$$(1.3) \quad \det(\gamma'(t), \dots, \gamma^{(d)}(t)) \neq 0 \quad \text{on } I,$$

the estimates on the optimal range were obtained by Zygmund [31] and Drury [15] (also see [18, 27, 12]) and generalized to the variable curve cases by [21, 2, 4]. When the curves are degenerate, instead of the Lebesgue measure dt the affine arclength measure is used to recover the estimate in the optimal range. For more details regarding the restriction problems for the curves we refer the readers to [28, 16, 17, 26, 3, 4, 5, 14, 13, 30, 11] and the references therein.

In this note, we are concerned with the estimate for $T_\lambda^\gamma f$ over the sphere instead of \mathbb{R}^d . To be more precise, we consider the estimate

$$(1.4) \quad \|T_\lambda^\gamma f\|_{L^q(\mathbb{S}^{d-1})} \leq C\lambda^{-\frac{d-1}{q}} \|f\|_{L^p(I)}$$

and we investigate the optimal range of p and q for which the estimate (1.4) holds. The bound $\lambda^{-\frac{d-1}{q}}$ is the best possible one can expect (see Remark 3). By rescaling the estimate (1.4) is equivalent to $\|T_1^\gamma f\|_{L^q(\lambda\mathbb{S}^{d-1})} \leq C\|f\|_{L^p(I)}$.

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The estimate (1.4) can generally be regarded as an estimate for a special case of degenerate oscillatory integral operators (see [8]). When $d = 2$, Greenleaf and Seeger [19] proved that (1.4) holds if and only if $q \geq 3$ and $1/p + 2/q \leq 1$. The argument in [19] is based on kernel estimates for the oscillatory integral operators with the folding canonical relation. Also, Bennett, Carbery, Soria, and Vargas [9] obtained the same result via the weighted L^2 inequality for the Fourier extension operator defined by the circle. We further remark that Bennett and Seeger [8] obtained the optimal p, q range of the $L^p(\mathbb{S}^2) - L^q(\lambda\mathbb{S}^2)$ estimates for $\widehat{f d\sigma}$ with the spherical measure σ .

The following is our first result which gives the sharp p, q range for the estimate (1.4).

Theorem 1.1. *Let $d \geq 2$. If γ satisfies (1.3), then (1.4) holds provided that*

$$(1.5) \quad q > (d^2 + d)/2, \quad 1/p + (d^2 + d - 2)/2q < 1.$$

The result is sharp in that (1.4) fails if either

$$(1.6) \quad q < (d^2 + d)/2,$$

or

$$(1.7) \quad 1/p + (d^2 + d - 2)/2q > 1.$$

As is to be seen in its proof, the necessity part of Theorem 1.1 remains valid with \mathbb{S}^{d-1} replaced by any compact smooth hypersurface S as long as a tangent vector of γ is parallel to a normal vector to S at a point where the Gaussian curvature is nonvanishing.

For $d = 2$, Theorem 1.1 verifies again that sharpness of the aforementioned estimate by Greenleaf and Seeger [19] (as well as that in [9]). It is likely that the estimate continues to be true for the critical case $1/p + (d^2 + d - 2)/2q = 1$ or $q = (d^2 + d)/2$. On the other hand, it should be mentioned that (1.2) does not hold at the endpoint $q = (d^2 + d + 2)/2$ for nondegenerate curves. The failure can be shown by making use of the result in Arkhipov, Chubarikov, and Karatsuba [1] (also see [24]) when γ is the moment curve, and for the general nondegenerate curve γ it was shown by Ikromov [22]. But the weak type version of estimate (1.2) was established at the endpoint case $p = q = (d^2 + d + 2)/2$ by Bak, Oberlin and Seeger [4] for $d \geq 3$, while it fails for $d = 2$ as was shown by Beckner, Carbery, Semmes and Soria [7].

Remark 1. In fact, it was shown in [7] that the $L^{p,1}(\mathbb{S}^{d-1}) - L^{q,\infty}(\mathbb{R}^d)$ estimate for the extension operator $f \mapsto \widehat{f d\sigma}$ does not hold for $p = q = 2d/(d-1)$, but without difficulty their argument can be modified to show the failure of even the weaker $L^{p,1}(\mathbb{S}^{d-1}) - L^{2d/(d-1),\infty}(\mathbb{R}^d)$ estimate for any $p > 2d/(d-1)$. We provide a proof of this in Section 4.

The estimate for the restriction of $T_\chi^\gamma f$ to the sphere \mathbb{S}^{d-1} was earlier studied by Brandolini, Gigante, Greenleaf, Iosevich, Seeger, and Travaglini [10] but they considered simpler input function χ_I instead of general f , and they obtained the sharp decay rate of the Fourier transform of measures supported on curves. By contrast, Theorem 1.1 provides the maximal decay rate $(d-1)/q$ for general $f \in L^p$.

As mentioned in the above, for $d = 2$ the optimal result including the end line cases was obtained in [19] and [9]. For $d \geq 3$, the sufficiency part of Theorem 1.1 follows from Theorem 1.2 below which is a special case of [20, Theorem 1.1]. In [20], the estimates with respect to general α -dimensional measure (see Definition 3.1) were obtained and those results are sharp in that there are α -dimensional measures for which the estimate fails outside of the

asserted region. Clearly, since the surface measure is $(d - 1)$ -dimensional, from Theorem 3.2 ([20, Theorem 1.1] with $\alpha = d - 1$) we immediately have the following.

Theorem 1.2. *Let $d \geq 3$ and let S be a compact smooth hypersurface in \mathbb{R}^d . For γ satisfying (1.3), there exists $C > 0$ such that $\|T_\lambda^\gamma f\|_{L^q(S)} \leq C\lambda^{-(d-1)/q}\|f\|_{L^p(I)}$ holds if $q > (d^2 + d)/2$ and $1/p + (d^2 + d - 2)/2q < 1$.*

It is rather surprising that Theorem 1.2 gives the sharp results since the result in [20] does not rely on specific geometric properties of the associated measures but only on the dimensional condition of the measure. Thus our main contribution here is to show the failure of the estimate (1.4) for the cases (1.6) or (1.7). Necessity part of Theorem 1.1 can be generalized to the oscillatory integral operator \mathfrak{T}_λ defined by

$$(1.8) \quad \mathfrak{T}_\lambda f(y) = \int_I e^{i\lambda\Psi(y,t)} a(y,t) f(t) dt,$$

where $a \in C_0^\infty(\mathbb{R}^{d-1} \times \mathbb{R})$ is supported in a neighborhood of the origin and Ψ is a smooth real-valued function on the support of a .

Proposition 1.3. *For $d \geq 2$ let \mathfrak{T}_λ is given by (1.8). Suppose that $\partial_t \nabla_y \Psi(0,0) = 0$, and suppose that, for (y,t) contained in the support of a ,*

$$(1.9) \quad \det(\partial_t^2 \nabla_y \Psi(y,t), \dots, \partial_t^d \nabla_y \Psi(y,t)) \neq 0,$$

and

$$(1.10) \quad \det(\nabla_y \partial_t \nabla_y \Psi(y,t)) \neq 0.$$

Then the estimate $\|\mathfrak{T}_\lambda f\|_{L^q(\mathbb{R}^{d-1})} \leq C\lambda^{-\frac{d-1}{q}}\|f\|_{L^p(I)}$ fails if either (1.6) or (1.7) holds.

Hence, application of Proposition 1.3 to the setting of Theorem 1.1 shows the necessity part of Theorem 1.1 (see Section 2). It is plausible to expect that the estimate $\|\mathfrak{T}_\lambda f\|_q \leq C\lambda^{-\frac{d-1}{q}}\|f\|_p$ is true up to the critical cases $q = (d^2 + d)/2$ and $1/p + (d^2 + d - 2)/2q = 1$. However at the time of this writing, we do not know whether this is true or not.

Finite type curves. Let us set $\mathcal{A} = \mathcal{A}(d) = \{\mathbf{a} = (\mathbf{a}_1, \dots, \mathbf{a}_d) : \mathbf{a}_i \in \mathbb{N}, i = 1, \dots, d, 1 \leq \mathbf{a}_1 < \dots < \mathbf{a}_d\}$ and $\|\mathbf{a}\|_1 = \mathbf{a}_1 + \dots + \mathbf{a}_d$. We recall the following from [20, Definition 1.2] (also see [12]).

Definition 1.1. Let $\gamma : I = [0, 1] \rightarrow \mathbb{R}^d$, $d \geq 2$ be a smooth curve. We say that γ is of finite type at $t \in I$ if there exists $\mathbf{a} = \mathbf{a}(t) \in \mathcal{A}$ such that

$$(1.11) \quad \det[\gamma^{(\mathbf{a}_1)}(t), \gamma^{(\mathbf{a}_2)}(t), \dots, \gamma^{(\mathbf{a}_d)}(t)] \neq 0.$$

Here the column vectors $\gamma^{(\mathbf{a}_i)}(t)$ are \mathbf{a}_i -th derivatives of γ . We say γ is of type $\mathbf{b} \in \mathcal{A}$ at t if the minimum of $\|\mathbf{a}(t)\|_1$ over all the possible choices of $\mathbf{a}(t)$ for which (1.11) holds is attained when $\mathbf{a}(t) = \mathbf{b}$. We also say that γ is of finite type if so is γ at every $t \in I$.

Theorem 1.4. *Let $d \geq 3$ and γ be of finite type. Suppose that γ is of type $\mathbf{a}(t)$ at t and $\|\mathbf{a}(t_0)\|_1 - \frac{d^2+d}{2} \geq 1$ for some $t_0 \in I$. Then, for p, q satisfying $q > \frac{d^2+d}{2}$ and $1/p + \max_{t \in I} \{\|\mathbf{a}(t)\|_1 - \mathbf{a}_1(t)\}/q \leq 1$,*

$$(1.12) \quad \|T_\lambda^\gamma f\|_{L^q(\mathbb{S}^{d-1})} \leq C\lambda^{-\frac{d-1}{q}}\|f\|_{L^{p,q}(I)}$$

holds. Furthermore (1.12) fails if $1/p + \max_{t \in I} \{\|\mathbf{a}(t)\|_1 - \mathbf{a}_1(t)\}/q > 1$.

Note that $\|\mathbf{a}\|_1 \geq \frac{d^2+d}{2}$ if $\mathbf{a} \in \mathcal{A}$. Thus, if γ does not satisfy the assumption of Theorem 1.4, $\|\mathbf{b}(t)\|_1 = \frac{d^2+d}{2}$ for all $t \in I$. This case was already considered in Theorem 1.2. Note that for $q \geq p$, (1.12) implies the strong type (p, q) estimate for p, q which satisfy $\frac{1}{p} + \max_{t \in I} \{\|\mathbf{b}(t)\|_1 - b_1(t)\}/q \leq 1$ by the inclusion $L^p \subset L^{p,q}$. In the case of $q < p$, the strong type estimate for $\frac{1}{p} + (\|\mathbf{b}\|_1 - b_1)\frac{1}{q} < 1$ follows by Hölder's inequality in the Lorentz space.

Theorem 1.4 is to be shown by considering the finite type curve as a union of small perturbation of monomial curves, which can be normalized into the curves contained in $\mathfrak{G}^{\mathbf{a}}(\epsilon)$ (see (3.1)). If ϵ is small enough, the torsion of curves in $\mathfrak{G}^{\mathbf{a}}(\epsilon)$ can be controlled uniformly and vanish only at the origin. By dyadic decomposition away from the origin, we can apply Theorem 3.2 for the curves on each dyadic interval via rescaling. For the purpose we will consider $L^p - L^q$ estimate for T_λ^γ with respect to general α -dimensional measures, which was considered in [20] (also see [25, 23, 6] for the Stein-Tomas restriction theorem with respect to general measures).

Hyperplane. As is to be seen later, in Theorem 1.1, i.e. the case of \mathbb{S}^{d-1} , the sharpness of the range of p, q is shown by making use of the fact that for any tangent vector γ' to γ there is a normal vector to the sphere which is parallel to γ' . However, this is not the case for hyperplanes, so it is natural to expect that a weak type version of (1.4) generically holds on a wider range of p, q than that in Theorem 1.1. In the following we provide a complete characterization of p, q for which a weak type version of (1.4) holds.

Proposition 1.5. *Let $d \geq 3$, $\gamma : I \rightarrow \mathbb{R}^d$ be of finite type, and let H be a hyperplane with a normal vector \mathbf{n} . Suppose that γ is of type $\mathbf{b}(t) = (\mathbf{b}_1(t), \dots, \mathbf{b}_d(t)) \in \mathcal{A}(d)$ at each $t \in I$. Let $\omega(t)$ be the minimum of $\mathbf{a}_1 + \dots + \mathbf{a}_{d-1}$ while $\mathbf{a}_i \in \{\mathbf{b}_1(t), \dots, \mathbf{b}_d(t)\}$ and the vectors $\mathbf{n}, \gamma^{(\mathbf{a}_1)}(t), \dots, \gamma^{(\mathbf{a}_{d-1})}(t)$ are linearly independent and set*

$$\omega_* = \max_{t \in I} \omega(t).$$

Then, the estimate

$$(1.13) \quad \|T_\lambda^\gamma f\|_{L^q(H)} \lesssim \lambda^{-(d-1)/q} \|f\|_{L^{p,q}}$$

holds if and only if $q > d(d-1)/2 + 1$ and $1/p + \omega_/q \leq 1$.*

The necessity of the condition $1/p + \omega_*/q \leq 1$ can be shown by using a Knapp type example (for example, see the proof of Proposition 3.1). When $q < p$, the failure of the estimate $\|T_\lambda^\gamma f\|_{L^q(H)} \lesssim \lambda^{-(d-1)/q} \|f\|_{L^p}$ with the critical p, q satisfying $1/p + \omega_*/q = 1$ was shown in [30, Section 5].

In order to compare Proposition 1.5 with Theorem 1.1, we consider the case of a nondegenerate curve γ . In this case ω_* takes its value in $[d(d-1)/2, d(d-1)/2 + 1]$. If $\omega_* = d(d+1)/2 - 1$, the range of p, q in Proposition 1.5 becomes the smallest but it properly contains the range p, q in Theorem 1.1. So (1.13) holds for p, q which are contained in a wider range than that of (1.5). This explains how the curvature of the surface plays a significant role even in the nondegenerate case. On the other hand, if $\omega_* = d(d-1)/2$, we get the largest range of p, q which coincides with that of the adjoint restriction estimate to the nondegenerate curves in \mathbb{R}^{d-1} .

Remark 2. The result in [20] (Theorem 3.2) also shows that $\|T_\lambda^\gamma f\|_{L^q(S)} \leq C\lambda^{-\frac{k}{q}} \|f\|_{L^p(I)}$ holds for any k -dimensional compact submanifold S for $k \geq 2$ whenever $1/p + (2d - k + 1)k/2q < 1$ and $q > (2d - k + 1)k/2 + 1$. In Section 4, we show that the condition $q \geq$

$(2d - k + 1)k/2 + 1$ is generally necessary by constructing a k -dimensional submanifold S for which $\|T_\lambda^\gamma f\|_{L^q(S)} \leq C\lambda^{-\frac{k}{q}}\|f\|_{L^p(I)}$ fails if $q < (2d - k + 1)k/2 + 1$.

Remark 3. The decay rate $\lambda^{-(d-1)/q}$ in (1.4) is optimal for any smooth hypersurface S . We consider a ball $B(x_0, \lambda^{-1})$ such that $|S \cap B(x_0, \lambda^{-1})| > C\lambda^{-(d-1)}$. Let us take $f(t) = \chi_{[0, \epsilon_0]}(t)e^{-i\lambda x_0 \cdot \gamma(t)}$. With a small enough $\epsilon_0 > 0$, $|T_\lambda^\gamma f(x)| \gtrsim 1$ if $x \in B(x_0, \lambda^{-1})$. Thus we see $\|T_\lambda^\gamma f\|_{L^q(S \cap B(x_0, \lambda^{-1}))} \geq C\lambda^{-(d-1)/q}$. This shows the optimality of the bound.

Outline of the paper. In Section 2, we make observations regarding geometric properties of the phase function, and we prove Theorem 1.1 and Proposition 1.3 by randomization argument based on Khintchine's inequality and by adapting the Knapp type example. The proofs of Theorem 1.4 and Proposition 1.5 are given in Section 3. In Section 4 we provide details concerning Remark 1 and the example mentioned in Remark 2.

Finally, for $A, B > 0$ we write $A \lesssim B$ if $A \leq CB$ for a constant C . Also the constant C may differ at each occurrence.

2. PROOF OF THEOREM 1.1 AND PROPOSITION 1.3

We first prove Proposition 1.3 by using a randomization argument for (1.6) and modifying the Knapp example for (1.7). Then, we use Proposition 1.3 to show the necessity part of Theorem 1.1.

2.1. Proof of Proposition 1.3. Since $\partial_t \nabla_y \Psi(0, 0) = 0$ and $\det \nabla_y \partial_t \nabla_y \Psi \neq 0$ on the support of a , by the implicit function theorem, there exists a neighborhood $U \times V \subset \mathbb{R}^{d-1} \times \mathbb{R}$ of $(0, 0)$ and a C^1 function $g: V \rightarrow \mathbb{R}^{d-1}$ such that $g(0) = 0$, and $\partial_t \nabla_y \Psi(g(t), t) = 0$ for all $t \in V$. For a fixed $t_k \in V \cap \text{supp } a$, let us set $y_k = g(t_k) \in U \cap \text{supp } a$.

By the Taylor expansion of Ψ at y_k and then at t_k , we have

$$\begin{aligned} \Psi(y, t) &= \Psi(y_k, t) + \langle \nabla_y \Psi(y_k, t), y - y_k \rangle + \langle \partial_t^2 \nabla_y \Psi(y_k, t) \frac{(t - t_k)^2}{2!}, y - y_k \rangle + \\ &\quad \cdots + \langle \partial_t^d \nabla_y \Psi(y_k, t) \frac{(t - t_k)^d}{d!}, y - y_k \rangle + O(|y - y_k|^2 + |y - y_k||t - t_k|^{d+1}), \end{aligned}$$

where the first order term vanishes because of $\partial_t \nabla_y \Psi(y_k, t_k) = 0$. Let us set

$$\gamma_\circ(t) = (t^2/2!, \dots, t^d/d!).$$

Discarding harmless factors $\Psi(y_k, t)$ and $\langle \nabla_y \Psi(y_k, t), y - y_k \rangle$, we may assume that

$$(2.1) \quad \Psi(y, t) = \langle \mathcal{M}(t_k) \gamma_\circ(t - t_k), y - y_k \rangle + O(|y - y_k|^2 + |y - y_k||t - t_k|^{d+1}).$$

Here $\mathcal{M}(t_k)$ is the matrix of which j -th column vector is given by $\partial_t^{j+1} \nabla_y \Psi(y_k, t_k)$, $1 \leq j \leq d - 1$. By the assumption (1.9), $\mathcal{M}(t_k)$ is nonsingular on the support of a .

Let us fix $\delta > 0$ such that $[0, \delta] \subset V \cap \text{supp } a$, and take $\lambda > 0$ such that $\lambda^{-1/(2d)} < \delta$ and $\delta\lambda^{1/(2d)} =: \ell \in \mathbb{N}$. We decompose the interval $[0, \delta]$ into intervals $I_k = [t_{k-1}, t_k]$, $1 \leq k \leq \ell$, of length $|I_k| \sim \lambda^{-1/(2d)}$ such that $[0, \delta] = \bigcup_{1 \leq k \leq \ell} I_k$. On each interval I_k , we observe the following.

Lemma 2.1. *Let $\rho = 1/(2d)$. Consider a rectangle $\mathcal{R} \subset \mathbb{R}^{d-1}$ defined by*

$$\mathcal{R} = \{(x_2, \dots, x_d) : |x_j| \leq c\lambda^{-1+j\rho}, 2 \leq j \leq d\}$$

with a small constant $c > 0$. For each interval I_k , let \mathcal{P}_k be the parallelepiped defined by

$$\mathcal{P}_k = \{y : \mathcal{M}^T(t_k)(y - y_k) \in \mathcal{R}\},$$

where $y_k = g(t_k)$ and $\mathcal{M}^T(t_k)$ is the transpose of the matrix of $\mathcal{M}(t_k)$. If c is sufficiently small, then $|\Psi(y, t)| \leq \lambda^{-1}$ for $y \in \mathcal{P}_k$ and $t \in I_k$.

Proof. Since $|t - t_k| \lesssim \lambda^{-\rho}$, we have, for $y \in \mathcal{P}_k$,

$$(2.2) \quad |\langle \mathcal{M}(t_k)\gamma_\circ(t - t_k), y - y_k \rangle| = |\langle \gamma_\circ(t - t_k), \mathcal{M}^T(t_k)(y - y_k) \rangle| \lesssim (d-1)c\lambda^{-1}.$$

If we set $\|v(t_k)\| = \max_i |v_i(t_k)|$ for the column vectors $v_i(t_k)$ of $\mathcal{M}^{-T}(t_k)$, then $|y - y_k| \leq (d-1)c\|v(t_k)\|\lambda^{-1+d\rho}$ for $y \in \mathcal{P}_k$. Hence, we obtain

$$|y - y_k|^2 \lesssim c^2\lambda^{-2+2d\rho} = c^2\lambda^{-1} \quad \text{and} \quad |y - y_k||t - t_k|^{d+1} \lesssim c\lambda^{-1-\rho} \ll c\lambda^{-1}.$$

Thus, by (2.1), (2.2), and the above we see $|\Psi(y, t)| \leq \lambda^{-1}$ for a sufficiently small $c > 0$. \square

To prove Proposition 1.3 we need to show that the estimate

$$(2.3) \quad \|\mathfrak{I}_\lambda f\|_{L^q(\mathbb{R}^{d-1})} \leq C\lambda^{-\frac{d-1}{q}} \|f\|_{L^p(I)}$$

implies

$$(2.4) \quad q \geq (d^2 + d)/2,$$

$$(2.5) \quad 1/p + (d^2 + d - 2)/2q \leq 1.$$

Proof of (2.3) \Rightarrow (2.4). Let $\{\epsilon_k\}_{k=0}^\ell$ be independent random variables having the values ± 1 with equal probability. We set

$$f(t) = \sum_{k=0}^{\ell} \epsilon_k \chi_{I_k}(t)$$

and consider the expectation $\mathbb{E}(\|\sum_k \epsilon_k \mathfrak{I}_\lambda \chi_{I_k}\|_{L^q}^q)$. By Fubini's theorem and Khintchine's inequality, we get, for $1 < q < \infty$,

$$(2.6) \quad \mathbb{E}(\|\sum_k \epsilon_k \mathfrak{I}_\lambda \chi_{I_k}\|_{L^q}^q) = \int \mathbb{E}(|\sum_k \epsilon_k \mathfrak{I}_\lambda \chi_{I_k}(y)|^q) dy \sim \int \left(\sum_k |\mathfrak{I}_\lambda \chi_{I_k}(y)|^2 \right)^{\frac{q}{2}} dy.$$

By Lemma 2.1 we have $|\lambda\Psi(y, t)| \leq 1$ for $y \in \mathcal{P}_k$ and $t \in I_k$. It is easy to see

$$|\mathfrak{I}_\lambda \chi_{I_k}|^2 \gtrsim |I_k|^2 \chi_{\mathcal{P}_k} \sim \lambda^{-1/d} \chi_{\mathcal{P}_k}.$$

Thus, it follows that

$$\int \left(\sum_k |\mathfrak{I}_\lambda \chi_{I_k}(y)|^2 \right)^{\frac{q}{2}} dy \gtrsim \lambda^{-\frac{q}{2d}} \int |\sum_k \chi_{\mathcal{P}_k}|^{\frac{q}{2}} dy \gtrsim \lambda^{-\frac{q}{2d}} \int \sum_k \chi_{\mathcal{P}_k} dy = \lambda^{-\frac{q}{2d}} \sum_k |\mathcal{P}_k|.$$

For the second inequality, we use the fact that $q \geq 2$. Combining this with (2.6) and using (2.3), we see that

$$(2.7) \quad \lambda^{-\frac{q}{2d}} \sum_{k=0}^{\ell} |\mathcal{P}_k| \lesssim \left\| \sum_{k=0}^{\ell} |\mathfrak{I}_\lambda \chi_{I_k}|^2 \right\|_{L^{\frac{q}{2}}}^{\frac{q}{2}} \sim \mathbb{E}(\|\sum_k \epsilon_k \mathfrak{I}_\lambda \chi_{I_k}\|_{L^q}^q) \lesssim \lambda^{-(d-1)} \delta^{\frac{q}{p}}.$$

From the definition of \mathcal{P}_k in Lemma 2.1, it follows that $|\mathcal{P}_k| \sim \lambda^{-(d-1)+(\frac{d^2+d}{2}-1)\cdot\frac{1}{2d}}$. Since $\ell \sim \delta\lambda^{\frac{1}{2d}}$, we have

$$\delta\lambda^{-\frac{1}{2d}\left(q-\frac{d^2+d}{2}\right)} \lesssim \delta^{\frac{q}{p}}.$$

For a fixed constant $\delta > 0$, we see that (2.4) is necessary by letting $\lambda \rightarrow \infty$. \square

Proof of (2.3) \Rightarrow (2.5). Let $J \subset [0, \delta]$ be an interval of length $|J| = \lambda^{-1/(2d)}$. By Lemma 2.1, we can find a parallelepiped \mathcal{P} such that $|\psi(y, t)| \leq \lambda^{-1}$ for $y \in \mathcal{P}$, $t \in J$. If we set $f = \chi_J$, it follows that

$$\|\mathfrak{T}_\lambda f\|_{L^q(\mathcal{P})} \geq C\lambda^{-\frac{1}{2d}}|\mathcal{P}|^{1/q} \geq \lambda^{-\frac{1}{2d}}\left(\lambda^{-(d-1)+\frac{d^2+d-2}{2}\cdot\frac{1}{2d}}\right)^{\frac{1}{q}}.$$

By (2.3), we obtain $\lambda^{-\frac{1}{2d}}\lambda^{-\frac{d-1}{q}+\frac{d^2+d-2}{2}\cdot\frac{1}{2dq}} \lesssim \lambda^{-\frac{d-1}{q}}\lambda^{-\frac{1}{2dp}}$. Thus we get (2.5) by letting $\lambda \rightarrow \infty$. \square

Proof of Theorem 1.1. The sufficiency part follows from Theorem 1.2. To prove the necessity part it is enough to show that the estimate for $T_\lambda^\gamma f$ over \mathbb{S}^{d-1} can be reformulated to an estimate for \mathfrak{T}_λ (see (1.8)) while the phase function Ψ satisfies the hypotheses (1.9) and (1.10) in Proposition 1.3.

For a given γ , we write $\gamma(t) = (\gamma_1(t), \gamma_*(t)) \in \mathbb{R} \times \mathbb{R}^{d-1}$. Since γ satisfies (1.3), we have $\gamma'(0) \neq 0$. By rotation we may assume that γ satisfies (1.3),

$$(2.8) \quad \gamma'_1(0) \neq 0, \text{ and } \gamma'_*(0) = 0.$$

Then we consider the part of \mathbb{S}^{d-1} near $-e_1$. That is to say, $\mathbb{S}^{d-1} \cap B(-e_1, \epsilon_0)$ for some small $\epsilon_0 > 0$. Then, we can parametrize $\mathbb{S}^{d-1} \cap B(-e_1, \epsilon_0)$ with a smooth function ϕ such that $y \mapsto (\phi(y) - 1, y)$ for $y = (y_2, \dots, y_d) \in \mathbb{R}^{d-1}$ near the origin, $\phi(0) = 0$, $\nabla_y \phi(0) = 0$, and

$$(2.9) \quad \det H\phi = \det \left(\frac{\partial^2 \phi}{\partial y_i \partial y_j} \right)_{2 \leq i, j \leq d} \neq 0$$

near 0. Here H denotes the Hessian matrix. Then, discarding the harmless constant -1 , it suffices to consider an oscillatory integral operator

$$(2.10) \quad \mathfrak{T}_\lambda f(y) = \int_I e^{i\lambda\psi(y, t)} a(y, t) f(t) dt,$$

where $\psi(y, t) = (\phi(y), y) \cdot \gamma(t)$ for $(y, t) \in \mathbb{R}^{d-1} \times \mathbb{R}$ and a is a smooth cutoff function which is supported in a small enough neighborhood of the origin. Thus it remains to check ψ satisfies (1.9) and (1.10) near the origin.

Since $\partial_t \nabla_y \psi(0, 0) = \nabla_y \phi(0) \gamma'_1(0) + \gamma'_*(0) = 0$, it remains to check that ψ satisfies (1.9) and (1.10) on the support of a . Because $\det \nabla_y \partial_t \nabla_y \psi(0, 0) = \gamma'_1(0) \det H\phi(0) \neq 0$, (1.10) follows by continuity provided that the support of a is small enough. It remains to check that ψ satisfies (1.9) on the support of a . By the implicit function theorem, there exist neighborhoods $U \subset \mathbb{R}^{d-1}$ and $V \subset \mathbb{R}$ of $(0, 0)$, and $g \in C^1(V)$ such that $g(0) = 0$, $g(V) \subset U$, and

$$(2.11) \quad \partial_t \nabla_y \psi(g(t), t) = 0 \text{ for all } t \in V.$$

As observed in the proof of Proposition 1.3, it is enough to show that ψ satisfies (1.9) for $y = g(t)$. By (2.11), we have $\partial_t \nabla_y \psi(g(t), t) = \gamma'_1(t) \nabla_y \phi(g(t)) + \gamma'_*(t) = 0$ and $\gamma'_1(t) \neq 0$. For

(1.9), we observe that

$$\partial_t^{j+1} \nabla_y \psi(g(t), t) = \partial_t^{j+1} \left(\gamma_1(t) \nabla_y \phi(y) + \gamma_*(t) \right) \Big|_{y=g(t)} = -\frac{\gamma_1^{(j+1)}(t)}{\gamma_1'(t)} \gamma_*'(t) + \gamma_*^{(j+1)}(t).$$

Using this we have

$$\begin{aligned} & \frac{1}{\gamma_1'(t)} \det(\gamma'(t), \dots, \gamma^{(d)}(t)) \\ &= \det \begin{pmatrix} 1 & \gamma_1''(t)/\gamma_1'(t) & \cdots & \gamma_1^{(d)}(t)/\gamma_1'(t) \\ \gamma_*'(t) & \gamma_*''(t) & \cdots & \gamma_*^{(d)}(t) \end{pmatrix} \\ &= \det \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \gamma_*'(t) & \gamma_*''(t) - \frac{\gamma_1''(t)}{\gamma_1'(t)} \gamma_*'(t) & \cdots & \gamma_*^{(d)}(t) - \frac{\gamma_1^{(d)}(t)}{\gamma_1'(t)} \gamma_*'(t) \end{pmatrix} \\ &= \det(\partial_t^2 \nabla_y \psi(g(t), t), \dots, \partial_t^d \nabla_y \psi(g(t), t)). \end{aligned}$$

Therefore (1.9) holds since γ is nondegenerate on I . \square

3. PROOF OF THEOREM 1.4 AND PROPOSITION 1.5

We first prove Theorem 1.4.

If γ is a finite type curve, after finite decomposition, translation (also subtracting a harmless constant) and rescaling, we may regard the curve as the one given by a small perturbation of a monomial curve. Thus we are naturally led to consider the class of curve $\mathfrak{G}^{\mathbf{a}}(\epsilon)$ which is defined as follows: For $\epsilon > 0$ and $\mathbf{a} \in \mathcal{A}$,

$$(3.1) \quad \mathfrak{G}^{\mathbf{a}}(\epsilon) = \{ \gamma \in C^\infty(I) : \gamma(t) = (t^{\mathbf{a}_1} \varphi_1(t), \dots, t^{\mathbf{a}_d} \varphi_d(t)), \|\varphi_i - 1/(\mathbf{a}_i!)\|_{C^{\mathbf{a}_d+1}(I)} \leq \epsilon \}.$$

In order to prove Theorem 1.4 it is enough to show the desired estimate with $\gamma \in \mathfrak{G}^{\mathbf{a}}(\epsilon)$ while the surface measure is replaced with the $(d-1)$ -dimensional measure (see Definition 3.1).

This type of reduction from finite type to almost monomial type already appeared in [20, Section 3], so we shall be brief. We set $[a, b]^* = [a, b]$ if $a < b$, or $[a, b]^* = [b, a]$ if $a > b$. Suppose γ is of type $\mathbf{a}(t)$ at t and let us set

$$M_t = [\gamma^{(\mathbf{a}_1(t))}(t), \dots, \gamma^{(\mathbf{a}_d(t))}(t)], \quad D_t^u = (u^{\mathbf{a}_1(t)} e_1, \dots, u^{\mathbf{a}_d(t)} e_d).$$

Then, by Taylor's theorem, it is not difficult to see that there exists $\delta > 0$ such that, if $[t_0, t_0 + u]^* \subset I$ and $|u| < \delta$,

$$(3.2) \quad \gamma(ut + t_0) - \gamma(t_0) = M_{t_0} D_{t_0}^u (t^{\mathbf{a}_1(t_0)} \varphi_1(ut), \dots, t^{\mathbf{a}_d(t_0)} \varphi_d(ut)), \quad t \in I,$$

where φ_i are smooth functions satisfying $\varphi_i(ut) = 1/(\mathbf{a}_j(t_0)!) + O(\delta)$. Thus, for any $\epsilon > 0$, there exists $\delta = \delta(\epsilon, t_0)$ such that

$$\gamma_{t_0}^u(t) := (M_{t_0} D_{t_0}^u)^{-1} (\gamma(ut + t_0) - \gamma(t_0)) \in \mathfrak{G}^{\mathbf{a}(t_0)}(\epsilon)$$

whenever $|u| < \delta$ and $[t_0, t_0 + u]^* \subset I$. See [20, Lemma 3.1, Lemma 3.3] for details. Suppose now that $\epsilon > 0$ be fixed. Since I is compact, we can decompose I into finitely many intervals $I_\ell = [t_\ell, t_\ell + u_\ell]^*$ such that $\gamma_{t_\ell}^{u_\ell}(t) \in \mathfrak{G}^{\mathbf{a}(t_\ell)}(\epsilon)$. Recalling $d\sigma$ denotes the surface measure on \mathbb{S}^{d-1} , we define a positive measure $d\sigma_\ell$ defined by

$$\int F(x) d\sigma_\ell(x) := \int F((M_{t_\ell} D_{t_\ell}^{u_\ell})^T x) d\sigma(x), \quad {}^1 F \in C_c(\mathbb{R}^d),$$

¹The definition can be justified via the Riesz representation theorem. See [20, pp. 257–258].

which is clearly a $(d-1)$ -dimensional measure. By making change of variables, we see that

$$\begin{aligned} \|T_\lambda^\gamma f\|_{L^q(\mathbb{S}^{d-1})} &\leq \sum_\ell \left\| \int_{[t_\ell, t_\ell + u_\ell]^*} e^{i\lambda x \cdot \gamma(t)} f(t) dt \right\|_{L^q(\mathbb{S}^{d-1})} \\ &= \sum_\ell \left\| \int_I e^{i\lambda(M_{t_\ell} D_{t_\ell}^{u_\ell})^T x \cdot \gamma_{t_\ell}^{u_\ell}(t)} f_{u_\ell}(t) dt \right\|_{L^q(\mathbb{S}^{d-1})} = \sum_\ell \left\| T_\lambda^{\gamma_{t_\ell}^{u_\ell}} f_{u_\ell} \right\|_{L^q(d\sigma_\ell)}, \end{aligned}$$

where $\mathbf{a}(t_\ell) = (\mathbf{a}_1(t_\ell), \dots, \mathbf{a}_d(t_\ell))$, $f_{u_\ell}(t) = u_\ell f(u_\ell t + t_\ell)$. Since there are only finitely many ℓ , so the proof of Theorem 1.4 reduces to showing that, for each ℓ ,

$$(3.3) \quad \left\| T_\lambda^{\gamma_{t_\ell}^{u_\ell}} g \right\|_{L^q(d\sigma_\ell)} \leq \lambda^{-(d-1)/q} \|g\|_{L^{p,q}(I)}$$

holds whenever $q > d(d+1)/2$ and $1/p + \max_{t \in I} \{\|\mathbf{a}(t)\|_1 - \mathbf{a}_1(t)\}/q \leq 1$. For this purpose, we actually prove more than what we need by replacing the $(d-1)$ -dimensional measure σ_ℓ by a general α -dimensional measure. We basically follow the argument in [20].

Definition 3.1. Let $\alpha \in (0, d]$ and by $B(x, r)$ we denote the ball centered at x of radius r . Suppose that μ is a positive Borel regular measure with compact support such that

$$(3.4) \quad \mu(B(x, r)) \leq C_\mu r^\alpha \text{ for } (x, r) \in \mathbb{R}^d \times \mathbb{R}_+$$

with $C_\mu > 0$ independent of x, r . Then we say μ is α -dimensional.

For $\nu \in \mathbb{R}$ we denote by $\lceil \nu \rceil$ the smallest integer which is not less than ν . For $(\mathbf{a}, \alpha) \in \mathcal{A} \times (0, d]$ we set

$$\kappa(\mathbf{a}, \alpha) := (\alpha + 1 - \lceil \alpha \rceil) \mathbf{a}_{d - \lceil \alpha \rceil + 1} + \sum_{i=d - \lceil \alpha \rceil + 2}^d \mathbf{a}_i, \quad \beta(\alpha) := \kappa((1, 2, \dots, d), \alpha).$$

Thus $\kappa(\mathbf{a}, \alpha) \geq \beta(\alpha)$ and $\kappa(\mathbf{a}, \alpha) = \beta(\alpha)$ if and only if $\mathbf{a} = (1, 2, \dots, d)$. We also note that $\kappa(\mathbf{a}, \alpha) > \beta(\alpha)$ implies $\|\mathbf{a}\|_1 > d(d+1)/2$.

Proposition 3.1. Let $d \geq 3$. Let $\gamma \in \mathfrak{G}^{\mathbf{a}}(\epsilon)$ for some $\mathbf{a} \in \mathcal{A}$ with $\|\mathbf{a}\|_1 - d(d+1)/2 \geq 1$. Suppose that μ is a compactly supported positive Borel measure satisfying (3.4) with $\alpha \in [d-1, d]$. If $\epsilon > 0$ is sufficiently small, then

$$(3.5) \quad \|T_\lambda^\gamma f\|_{L^q(d\mu)} \lesssim \lambda^{-\alpha/q} \|f\|_{L^{p,q}(I)}$$

holds for $1/p + \kappa(\mathbf{a}, \alpha)/q \leq 1$ and $q > \beta(\alpha) + 1$. Moreover, if $1/p + \kappa(\mathbf{a}, \alpha)/q > 1$, there is a measure μ which satisfies (3.4) but the estimate (3.5) fails.

In fact, Proposition 3.1 continues to hold for $\alpha \in (0, d-1)$ under some additional conditions on p, q as explained after the statement of Theorem 3.2.

Proof. We decompose $T_\lambda^\gamma f = \sum_{\ell=0}^\infty T_\ell f$ where $T_\ell f$ is defined by

$$(3.6) \quad T_\ell f(x) = \int_{[2^{-\ell-1}, 2^{-\ell}]} e^{i\lambda x \cdot \gamma(t)} f(t) dt.$$

For $h > 0$, let us define $\mathfrak{D}_h^{\mathbf{a}} = (h^{\mathbf{a}_1} e_1, \dots, h^{\mathbf{a}_d} e_d)$. For each fixed ℓ , we define a positive Borel measure μ_ℓ by setting

$$\int F(x) d\mu_\ell(x) = 2^{-\ell \kappa(\mathbf{a}, \alpha)} \int F(\mathfrak{D}_{2^{-\ell}}^{\mathbf{a}} x) d\mu(x), \quad F \in C_c(\mathbb{R}^d).$$

We now show that μ_ℓ satisfies (3.4). Note that the set $\mathcal{R} = \{y : \mathfrak{D}_{2^{-\ell}}^{\mathbf{a}} y \in B(x, r)\}$, which is contained in a rectangle of dimensions $C2^{\ell \mathbf{a}_1} r \times C2^{\ell \mathbf{a}_2} r \times \cdots \times C2^{\ell \mathbf{a}_d} r$, can be covered by as many as $O(\prod_{i=d+2-\lceil \alpha \rceil}^d 2^{\ell(\mathbf{a}_i - \mathbf{a}_{d+1-\lceil \alpha \rceil})})$ cubes of side length $2^{\ell \mathbf{a}_{d+1-\lceil \alpha \rceil}} r$. Thus, applying (3.4) to each of these cubes, we see that

$$\begin{aligned} \mu_\ell(B(x, r)) &\lesssim 2^{-\ell \kappa(\mathbf{a}, \alpha)} \int \chi_{B(x, r)}(\mathfrak{D}_{2^{-\ell}}^{\mathbf{a}} y) d\mu(y) \lesssim 2^{-\ell \kappa(\mathbf{a}, \alpha)} \mu(\mathcal{R}) \\ &\lesssim 2^{-\ell \kappa(\mathbf{a}, \alpha)} 2^{\ell((1-\lceil \alpha \rceil) \mathbf{a}_{d+1-\lceil \alpha \rceil} + \sum_{i=d+2-\lceil \alpha \rceil}^d \mathbf{a}_i)} (2^{\ell \mathbf{a}_{d+1-\lceil \alpha \rceil}} r)^\alpha \lesssim r^\alpha. \end{aligned}$$

Therefore μ_ℓ satisfies (3.4). Also we consider γ_ℓ and f_ℓ which are defined by

$$\gamma_\ell(t) := \mathfrak{D}_{2^{-\ell}}^{\mathbf{a}} \gamma(2^{-\ell} t) = (t^{\mathbf{a}_1} \varphi_1(2^{-\ell} t), \dots, t^{\mathbf{a}_d} \varphi_d(2^{-\ell} t)), \quad f_\ell(t) = 2^{-\ell} f(2^{-\ell} t),$$

respectively. Then, by scaling $t \rightarrow 2^{-\ell} t$ we have that

$$\begin{aligned} \|T_\ell f\|_{L^q(d\mu)}^q &= \int \left| \int_{[1/2, 1]} e^{i\lambda \mathfrak{D}_{2^{-\ell}}^{\mathbf{a}} x \cdot \gamma_\ell(t)} f_\ell(t) dt \right|^q d\mu(x) \\ (3.7) \quad &= 2^{\ell \kappa(\mathbf{a}, \alpha)} \int \left| \int_{[1/2, 1]} e^{i\lambda x \cdot \gamma_\ell(t)} f_\ell(t) dt \right|^q d\mu_\ell(x). \end{aligned}$$

We now use the following to get a bound for each T_ℓ , which is a special case of Theorem 1.1 in [20].

Theorem 3.2 (Theorem 1.1 in [20]). *Let $d \geq 3$ and $\alpha \in [d-1, d]$. Suppose that γ satisfies (1.3) and μ is α -dimensional. Then, for p, q satisfying*

$$(3.8) \quad 1/p + \beta(\alpha)/q < 1, \quad q > \beta(\alpha) + 1,$$

we have the estimate $\|T_\lambda^\gamma f\|_{L^q(d\mu)} \lesssim \lambda^{-\alpha/q} \|f\|_{L^p(I)}$.

Actually the estimate is valid on a wider range of α , p , and q but this is not relevant to our purpose. The additional restriction $d/q \leq (1 - 1/p)$ and $q \geq 2d$ which is in Theorem 1.1 in [20] is not necessary here because $d \geq 3$ and $\alpha \in [d-1, d]$.

The bound $\|T_\lambda^\gamma\|_{p \rightarrow q}$ in Theorem 3.2 is stable under small smooth perturbation of γ . Since $\gamma_\ell \in \mathfrak{G}^{\mathbf{a}}(\epsilon)$ (in fact, $\gamma_\ell(t) \in \mathfrak{G}^{\mathbf{a}}(C2^{-\ell}\epsilon)$ for some constant $C > 0$), the torsion τ_ℓ of γ_ℓ at t is $|\tau_\ell(t)| \sim t^{\|\mathbf{a}\|_1 - d(d+1)/2}$ where the implicit constant is independent of ℓ (see Lemma 3.4 in [20]). Thus, choosing a sufficiently small $\epsilon > 0$, we see that γ_ℓ is a small smooth perturbation of the curve $(\frac{t^{\mathbf{a}_1}}{\mathbf{a}_1!}, \dots, \frac{t^{\mathbf{a}_d}}{\mathbf{a}_d!})$, which is nondegenerate on the interval $[1/2, 1]$. Recalling that μ_ℓ satisfies (3.4) with $\mu = \mu_\ell$, we apply Theorem 3.2 to (3.7) and obtain, for p, q satisfying (3.8),

$$(3.9) \quad \|T_\ell f\|_{L^q(d\mu)}^q \lesssim 2^{\ell q(\frac{1}{p} + \frac{\kappa(\mathbf{a}, \alpha)}{q} - 1)} \lambda^{-\alpha} \|f\|_p^q.$$

Using this, we can get a weak type estimate for T_λ^γ on the critical line $1/p + \kappa(\mathbf{a}, \alpha)/q = 1$. With an integer N which is to be chosen later, we consider

$$\mu(\{x : |T_\lambda^\gamma f(x)| > \delta\}) \leq \mu\left(\left\{x : \left| \sum_{\ell=-\infty}^N T_\ell f(x) \right| > \frac{\delta}{2}\right\}\right) + \mu\left(\left\{x : \left| \sum_{\ell=N+1}^{\infty} T_\ell f(x) \right| > \frac{\delta}{2}\right\}\right).$$

Here we trivially extend T_ℓ to $\ell = -1, -2, \dots$ by setting $T_\ell = 0$. Now, fixing p, q satisfying $1/p + \kappa(\mathbf{a}, \alpha)/q = 1$ and $q > \beta(\alpha) + 1$, we show the estimate (3.5). We choose $1 \leq q_1, q_2 \leq \infty$ such that (3.8) holds with $(p, q) = (p, q_i)$, $i = 1, 2$, $1/p + \kappa(\mathbf{a}, \alpha)/q_1 > 1$, and $1/p + \kappa(\mathbf{a}, \alpha)/q_2 < 1$. Such choices are possible since $\kappa(\mathbf{a}, \alpha) > \beta(\alpha)$. Since $1/p + \kappa(\mathbf{a}, \alpha)/q_2 - 1 <$

$0 < 1/p + \kappa(\mathbf{a}, \alpha)/q_1 - 1$, by Chebyshev's inequality and Minkowski's inequality, and then making use of (3.9), we have

$$\begin{aligned} \mu(\{x : |T_\lambda^\gamma f(x)| > \delta\}) &\lesssim \delta^{-q_1} \left(\sum_{\ell=-\infty}^N \|T_\ell f\|_{L^{q_1}(d\mu)} \right)^{q_1} + \delta^{-q_2} \left(\sum_{\ell=N+1}^{\infty} \|T_\ell f\|_{L^{q_2}(d\mu)} \right)^{q_2} \\ &\lesssim \delta^{-q_1} \left(2^{N(1/p + \kappa(\mathbf{a}, \alpha)/q_1 - 1)} \lambda^{-\alpha/q_1} \|f\|_p \right)^{q_1} + \delta^{-q_2} \left(2^{N(1/p + \kappa(\mathbf{a}, \alpha)/q_2 - 1)} \lambda^{-\alpha/q_2} \|f\|_p \right)^{q_2}. \end{aligned}$$

Taking N such that $2^N \sim \delta^{-p'} \|f\|_p^{p'}$, we get $\mu(\{x : |T_\lambda^\gamma f(x)| > \delta\}) \lesssim \delta^{-q} \lambda^{-\alpha} \|f\|_p^q$ for p, q satisfying $1/p + \kappa(\mathbf{a}, \alpha)/q = 1$, and hence T_λ^γ is of weak type (p, q) . By real interpolation along the resulting estimates and Hölder's inequality, we get (3.5) for $1/p + \kappa(\mathbf{a}, \alpha)/q \leq 1$.

Now we show that the condition $1/p + \kappa(\mathbf{a}, \alpha)/q \leq 1$ is necessary for (3.5). Let us consider the measure $d\mu$ which is defined by

$$d\mu(x) = \chi_{B(0,1)} \prod_{i=1}^{d-\lceil\alpha\rceil} d\delta(x_i) |x_{d-\lceil\alpha\rceil+1}|^{\alpha-\lceil\alpha\rceil} dx_{d-\lceil\alpha\rceil+1} dx_{d-\lceil\alpha\rceil+2} \cdots dx_d$$

Here $d\delta$ is the one dimensional Dirac measure. It is easy to check that μ satisfies (3.4). If we take $f(t) = \chi_{[0, \lambda^{-\rho}]}(t)$ for some $\rho > 0$, then $|T_\lambda^\gamma f(x)| \geq C\lambda^{-\rho}$ whenever $x \in \mathcal{R}_\mathbf{a} = \{x \in \mathbb{R}^d : |x_i| \leq c\lambda^{-1+\rho\mathbf{a}_i}\}$ for a small $c > 0$. Since $\|f\|_{p,q} \sim \|f\|_p = \lambda^{-\rho/p}$ and

$$\mu(\mathcal{R}_\mathbf{a}) \sim (\lambda^{-1+\rho\mathbf{a}_{d-\lceil\alpha\rceil+1}})^{\alpha-\lceil\alpha\rceil+1} \lambda^{1-\lceil\alpha\rceil+\rho \sum_{i=d-\lceil\alpha\rceil+2}^d \mathbf{a}_i} = \lambda^{-\alpha+\rho\kappa(\mathbf{a}, \alpha)},$$

the estimate (3.5) implies $\lambda^{-\rho} (\lambda^{-\alpha} \lambda^{\rho\kappa(\mathbf{a}, \alpha)})^{1/q} \leq C\lambda^{-\alpha/q} \lambda^{-\rho/p}$. Taking λ which tends to ∞ gives the desired condition $1/p + \kappa(\mathbf{a}, \alpha)/q \leq 1$. \square

Proof of (3.3). To begin with we recall that $\gamma_{t_\ell}^{u_\ell} \in \mathfrak{G}^{\mathbf{a}(t_\ell)}(\epsilon)$. It is obvious that σ_ℓ satisfies (3.4) with $\alpha = \lceil\alpha\rceil = d-1$. So, we have $\beta(d-1) = d(d+1)/2 - 1$ and $\kappa(\mathbf{a}(t_\ell), d-1) = \|\mathbf{a}(t_\ell)\|_1 - \mathbf{a}_1(t_\ell)$. We consider the two cases: **(A)** $\|\mathbf{a}(t_\ell)\|_1 - d(d+1)/2 \geq 1$ and **(B)** $\|\mathbf{a}(t_\ell)\|_1 - d(d+1)/2 < 1$, separately.

If $\|\mathbf{a}(t_\ell)\|_1 - d(d+1)/2 \geq 1$, applying Proposition 3.1, we obtain (3.3) for $q > d(d+1)/2$ and $1/p + (\|\mathbf{a}(t_\ell)\|_1 - \mathbf{a}_1(t_\ell))/q \leq 1$. If $\|\mathbf{a}(t_\ell)\|_1 - d(d+1)/2 < 1$, then $\|\mathbf{a}(t_\ell)\|_1 = d(d+1)/2$, i.e., $\mathbf{a}(t_\ell) = (1, 2, \dots, d)$. Thus, the curve $\gamma_{t_\ell}^{u_\ell}$ is now nondegenerate, that is to say, $\gamma_{t_\ell}^{u_\ell}$ satisfies (1.3) with $\gamma = \gamma_{t_\ell}^{u_\ell}$. Regarding this case, we may directly apply Theorem 3.2 to get the strong type $L^p - L^q(d\sigma_\ell)$ estimate for $T_\lambda^{\gamma_{t_\ell}^{u_\ell}}$ provided that $q > d(d+1)/2$ and $1/p + (d(d+1) - 2)/(2q) < 1$. Now we note that $(d(d+1) - 2)/2 < \max_\ell \{\|\mathbf{a}(t_\ell)\|_1 - \mathbf{a}_1(t_\ell)\}$ because $\|\mathbf{a}(t_0)\|_1 - d(d+1)/2 \geq 1$ for some $t_0 \in I$. This is clear since $\mathbf{a}_i(t_\ell) \geq i$, $i = 1, \dots, d$, and $\mathbf{a}_i(t_\ell) > i_0$ for some $1 \leq i_0 \leq d$. Therefore, combining the estimates for the cases **(A)** and **(B)** we get (3.3) whenever $q > d(d+1)/2$ and $1/p + \max_\ell \{\|\mathbf{a}(t_\ell)\|_1 - \mathbf{a}_1(t_\ell)\}/q \leq 1$. This completes the proof. \square

This shows the sufficiency part Theorem 1.4 and we now turn to proof of the necessity part of Theorem 1.4, which is slightly more involved since we need to deal with higher order derivatives.

Proof of the necessity part of Theorem 1.4. We show the condition $1/p + \max_{t \in I} \{\|\mathbf{a}(t)\|_1 - \mathbf{a}_1(t)\}/q \leq 1$ is necessary for (1.12). Let $t_0 \in I$ be the point where γ is of type \mathbf{a} at t_0 and $\|\mathbf{a}\|_1 - \mathbf{a}_1 = \max_{t \in I} \{\|\mathbf{a}(t)\|_1 - \mathbf{a}_1(t)\}$. It suffices to show that (1.12) implies $1/p + (\|\mathbf{a}\|_1 - \mathbf{a}_1)/q \leq 1$ provided f is supported in $[t_0, t_0 \pm \epsilon_0]^* \subset I$. We only consider the case

$[t_0, t_0 + \epsilon_0] \subset I$, and the other case can be handled similarly. From Taylor's expansion we have (see [20, Section 3]) that, for $t \in [0, \epsilon_0]$,

$$(3.10) \quad \begin{aligned} \gamma(t + t_0) - \gamma(t_0) &= \gamma^{(\mathbf{a}_1)}(t_0) \frac{t^{\mathbf{a}_1}}{\mathbf{a}_1!} (1 + O(t)) + \\ &\quad \gamma^{(\mathbf{a}_2)}(t_0) \frac{t^{\mathbf{a}_2}}{\mathbf{a}_2!} (1 + O(t)) + \cdots + \gamma^{(\mathbf{a}_d)}(t_0) \frac{t^{\mathbf{a}_d}}{\mathbf{a}_d!} (1 + O(t)). \end{aligned}$$

Since $\gamma^{(\mathbf{a}_1)}(t_0), \dots, \gamma^{(\mathbf{a}_d)}(t_0)$ are linearly independent, we can choose orthonormal vectors $\mathbf{v}_1, \dots, \mathbf{v}_{d-1}$ one after another such that, for $i = 1, \dots, d-1$,

$$\mathbf{v}_i \perp \text{span}\{\gamma^{(\mathbf{a}_1)}(t_0), \dots, \gamma^{(\mathbf{a}_{d-i})}(t_0)\}, \quad \mathbf{v}_{i+1} \in \text{span}\{\gamma^{(\mathbf{a}_1)}(t_0), \dots, \gamma^{(\mathbf{a}_{d-i})}(t_0)\}.$$

Additionally, let \mathbf{v}_d be the unit vector such that $\mathbf{v}_d \perp \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_{d-1}\}$. For $\mathbf{y} = (\mathbf{y}_1, \dots, \mathbf{y}_{d-1})$ we parametrize the part of \mathbb{S}^{d-1} near $-\mathbf{v}_d$ by $\mathbf{y} \in \mathbb{R}^{d-1} \mapsto \mathbf{y}_1 \mathbf{v}_1 + \cdots + \mathbf{y}_{d-1} \mathbf{v}_{d-1} + (\phi(\mathbf{y}) - 1) \mathbf{v}_d$ such that $\phi(0) = \nabla \phi(0) = 0$. In fact, $\phi(\mathbf{y}) = 1 - \sqrt{1 - |\mathbf{y}|^2}$. Thus, the measure on \mathbb{S}^{d-1} is given by $d\mu = (1 + |\nabla \phi(\mathbf{y})|^2)^{1/2} d\mathbf{y}$.

For some small enough $\epsilon_0 > 0$ let us set

$$\overline{T}_\lambda f(\mathbf{y}) = \chi_{B(0, \epsilon_0)}(\mathbf{y}) \int e^{i\lambda \Phi(\mathbf{y}, t)} f(t) \chi_{[0, \epsilon_0]}(t) dt,$$

where $\Phi(\mathbf{y}, t) = (\sum_{i=1}^{d-1} \mathbf{y}_i \mathbf{v}_i + (\phi(\mathbf{y}) - 1) \mathbf{v}_d) \cdot (\gamma(t + t_0) - \gamma(t_0))$. Subtracting harmless factors, it is sufficient to consider, instead of T_λ^γ , the operator \overline{T}_λ and to show the estimate

$$(3.11) \quad \|\overline{T}_\lambda f\|_{L^q} \leq C \lambda^{-\frac{d-1}{q}} \|f\|_{L^{p,q}}$$

implies $1/p + (\|\mathbf{a}\|_1 - \mathbf{a}_1)/q \leq 1$. With a small enough $c > 0$ and $0 < \rho < (2\mathbf{a}_d - \mathbf{a}_1)^{-1}$ let us set

$$\mathcal{R}_\mathbf{a} = \{\mathbf{y} \in \mathbb{R}^{d-1} : |\mathbf{y}_i| \leq c \lambda^{-1 + \rho \mathbf{a}_{d+1-i}}, 1 \leq i \leq d-1\}.$$

We now recall that $a_1 < \cdots < a_d$. By the choice of $\mathbf{v}_1, \dots, \mathbf{v}_{d-1}$ and using (3.10) and $\phi(\mathbf{y}) = O(|\mathbf{y}|^2)$ we notice that, for $t \in [0, \lambda^{-\rho}]$ and $\mathbf{y} \in \mathcal{R}_\mathbf{a}$,

$$\begin{aligned} \Phi(\mathbf{y}, t) &= \sum_{i=1}^{d-1} \mathbf{y}_i \mathbf{v}_i \cdot \left(\sum_{j=d+1-i}^d \gamma^{(\mathbf{a}_j)}(t_0) \frac{t^{\mathbf{a}_j}}{\mathbf{a}_j!} (1 + O(t)) \right) + O(|\mathbf{y}|^2 |t|^{\mathbf{a}_1}) \\ &= \sum_{i=1}^{d-1} O(|\mathbf{y}_i| |t|^{\mathbf{a}_{d+1-i}}) + O(|\mathbf{y}|^2 |t|^{\mathbf{a}_1}) = O(c \lambda^{-1}). \end{aligned}$$

Taking sufficiently small $c > 0$, we have $|\Phi(\mathbf{y}, t)| \leq 10^{-2} \lambda^{-1}$ if $\mathbf{y} \in \mathcal{R}_\mathbf{a}$ and $t \in [0, \lambda^{-\rho}]$. Therefore, if we take $f = \chi_{[0, \lambda^{-\rho}]}$ with a large λ , we see that $|\overline{T}_\lambda f| \gtrsim \lambda^{-\rho}$ on $\mathcal{R}_\mathbf{a}$. Since $|\mathcal{R}_\mathbf{a}| \sim \lambda^{-(d-1) + \rho(\|\mathbf{a}\|_1 - \mathbf{a}_1)}$, the estimate (3.11) implies

$$\lambda^{-\rho} \lambda^{-\frac{d-1}{q} + (\|\mathbf{a}\|_1 - \mathbf{a}_1) \frac{\rho}{q}} \leq C \lambda^{-\frac{d-1}{q}} \lambda^{-\frac{\rho}{p}}.$$

Thus, taking $\lambda \rightarrow \infty$ we see that $1/p + (\|\mathbf{a}\|_1 - \mathbf{a}_1)/q \leq 1$ is necessary for (3.11). This completes the proof. \square

We prove Proposition 1.5 by making use of Proposition 3.1.

Proof of Proposition 1.5. We may assume $H = \{x \in \mathbb{R}^d : x \cdot \mathbf{n} = 0\}$ for a nonzero vector $\mathbf{n} = (n_1, \dots, n_d)$. We may assume that $n_k \neq 0$ for some k .² Then H is parametrized by

²The choice of k is not important since the type of curve does not change under affine transformation.

$x_k = \mathbf{h} \cdot \bar{x}$, where each element h_i of \mathbf{h} is given by $h_i = -n_i/n_k$, $1 \leq i \neq k \leq d$ and $\bar{x} = (x_1, \dots, \widetilde{x_k}, \dots, x_d) \in \mathbb{R}^{d-1}$. Here $\widetilde{x_k}$ means the omission of the k -th element x_k .

Hence, we have

$$(3.12) \quad x \cdot \gamma(t) = \bar{x} \cdot \gamma_{\mathbf{h}}(t), \quad \gamma_{\mathbf{h}}(t) := \overline{\gamma(t)} + \gamma_k(t)\mathbf{h}.$$

To prove the sufficiency part of Proposition 1.5, it suffices to show that

$$(3.13) \quad \|T_{\lambda}^{\gamma_{\mathbf{h}}} f\|_{L^q(\mathbb{R}^{d-1})} \lesssim \lambda^{-(d-1)/q} \|f\|_{L^{p,q}(I)}$$

holds if $q > \frac{d(d-1)}{2} + 1$ and $1/p + \max_{t \in I} \omega(t)/q \leq 1$.

For $t \in I$, we may assume that $\omega(t)$ attains its minimum at $\mathbf{a}(t) = (\mathbf{a}_1(t), \dots, \mathbf{a}_{d-1}(t))$. Then we have

$$\det \left[\gamma_{\mathbf{h}}^{(\mathbf{a}_1)}(t), \quad \dots, \quad \gamma_{\mathbf{h}}^{(\mathbf{a}_{d-1})}(t) \right] = \frac{(-1)^{k+1}}{n_k} \det \left[\mathbf{n}, \quad \gamma^{(\mathbf{a}_1)}(t), \quad \dots, \quad \gamma^{(\mathbf{a}_{d-1})}(t) \right] \neq 0,$$

where we make use of elementary row operations since $n_k \neq 0$. Hence $\gamma_{\mathbf{h}}$ is of type $\mathbf{a}(t) = (\mathbf{a}_1, \dots, \mathbf{a}_{d-1})$.

As in the proof of Theorem 1.4, we may work with a function f supported in sufficiently a small neighborhood of t . Thus, if $\|\mathbf{a}(t)\|_1 > d(d-1)/2$, we note that $\kappa(\mathbf{a}(t), d-1) = \|\mathbf{a}(t)\|_1$ and $\beta(d-1) = d(d-1)/2$. Hence, by Proposition 3.1 with μ which is the Lebesgue measure in \mathbb{R}^{d-1} , we get (3.13) for $q > d(d-1)/2 + 1$ and $1/p + \|\mathbf{a}(t)\|_1/q \leq 1$ provided f is supported in a small neighborhood of t . On the other hand, if $\|\mathbf{a}(t)\|_1 = d(d-1)/2$, the curve $\gamma_{\mathbf{h}}$ is a nondegenerate (in \mathbb{R}^{d-1}) near the point t . In this case, the desired estimate follows by the typical Fourier restriction estimates for nondegenerate curves in \mathbb{R}^{d-1} (see [15, 2, 4] for example). Thus we get (3.13) for $q > d(d-1)/2 + 1$ and $1/p + d(d-1)/2q \leq 1$ whenever f is supported near the point t . Since I is compact, combining those two types of local results we obtain (3.13) for $q > d(d-1)/2 + 1$ and $1/p + \max_{t \in I} \|\mathbf{a}(t)\|_1/q \leq 1$.

This range is optimal because the conditions $q > d(d-1)/2 + 1$ and $1/p + \max_{t \in I} \|\mathbf{a}(t)\|_1/q \leq 1$ are necessary for (3.13). The first one is obvious because we can not have (3.13) for $q \leq d(d-1)/2 + 1$ in \mathbb{R}^{d-1} even for the nondegenerate curve as is mentioned in the introduction. The necessity of the second condition can be shown by following the argument in the proof of the necessity part of Theorem 1.4. \square

Remark 4. The projection of a nondegenerate polynomial curve in \mathbb{R}^d to $(d-1)$ -dimensional hyperplane can be seen as a degenerate polynomial curve in \mathbb{R}^{d-1} . So, Proposition 1.5 also can be deduced from the Fourier restriction theorem for polynomial curves with affine arclength measure (see [28, 29, 12, 16, 17, 3, 30]).

4. DETAILS ON REMARKS

4.1. Proof of Remark 2. Let S be a k -dimensional surface in \mathbb{R}^d . Also let $\gamma(t) = (P_1(t), \dots, P_d(t))$ for polynomials P_i of degree i . Thus, γ satisfies (1.3). For $l = d - k$, we parametrize S by $y = (y_1, \dots, y_k) \mapsto (\phi_1(y), \dots, \phi_l(y), y)$. We intend to find ϕ_1, \dots, ϕ_l such that the phase function $\psi(y, t) = (\phi_1(y), \dots, \phi_l(y), y) \cdot \gamma(t)$ satisfies

$$(4.1) \quad \partial_t \nabla_y \psi(y, t) = \dots = \partial_t^l \nabla_y \psi(y, t) = 0,$$

and

$$(4.2) \quad \det(\partial_t^{l+1}\nabla_y\psi, \dots, \partial_t^d\nabla_y\psi)(y, t) \neq 0$$

when $y = g(t)$ for some $g(t)$.

Let us write $\gamma = (\gamma_a, \gamma_b) \in \mathbb{R}^l \times \mathbb{R}^k$ and we set

$$\begin{aligned} A_1(t) &= (\gamma'_a, \dots, \gamma_a^{(l)})(t), \quad A_2(t) = (\gamma_a^{(l+1)}, \dots, \gamma_a^{(d)})(t), \\ B_1(t) &= (\gamma'_b, \dots, \gamma_b^{(l)})(t), \quad B_2(t) = (\gamma_b^{(l+1)}, \dots, \gamma_b^{(d)})(t). \end{aligned}$$

Since γ is nondegenerate, by changing coordinates we may assume that $A_1(t)$ is invertible. Now we note that

$$(4.3) \quad \psi(y, t) = (\phi_1, \dots, \phi_l) \cdot \gamma_a(t) + y \cdot \gamma_b(t)$$

and

$$(4.4) \quad (\partial_t \nabla \psi, \dots, \partial_t^l \nabla \psi)(g(t), t) = (\nabla_y \phi_1, \dots, \nabla_y \phi_l)(g(t))A_1(t) + B_1(t).$$

Thus (4.1) follows if

$$(4.5) \quad (\nabla_y \phi_1, \dots, \nabla_y \phi_l)(g(t)) = -B_1(t)A_1^{-1}(t)$$

To obtain ϕ_1, \dots, ϕ_l satisfying (4.5) for some g , we simply take $g(t) = (t, \dots, t)$ and set

$$\phi_j(y_1, \dots, y_k) = \sum_{i=1}^k \int_0^{y_i} [-B_1(t)A_1^{-1}(t)]_{ij} dt,$$

where $[M]_{ij}$ denotes the (i, j) -th element of the matrix M . Then (4.1) clearly holds.

Now we show that (4.2) holds with our choices of ϕ_1, \dots, ϕ_l and g . From (4.3) it follows that $(\partial_t^{l+1}\nabla_y\psi, \dots, \partial_t^d\nabla_y\psi)(g(t), t) = (\nabla_y\phi_1, \dots, \nabla_y\phi_l)(g(t))A_2(t) + B_2(t)$. Hence, using (4.5), we see that

$$(\partial_t^{l+1}\nabla_y\psi, \dots, \partial_t^d\nabla_y\psi)(g(t), t) = B_2(t) - B_1(t)A_1^{-1}(t)A_2(t).$$

We recall the identity concerning the determinant of block matrix

$$\det \begin{pmatrix} B_2(t) & B_1(t) \\ A_2(t) & A_1(t) \end{pmatrix} = \det \left(B_2(t) - B_1(t)A_1^{-1}(t)A_2(t) \right) \det A_1(t).$$

Since γ is nondegenerate, the determinant in the left-hand side is nonzero. Recall $A_1(t)$ is invertible and therefore (4.2) holds.

Once we have (4.1) and (4.2) for some g , we can repeat the same argument as in the proof of Proposition 1.3. In fact, as before we partition $I = [0, 1]$ such that $I = \cup_m I_m$ and $I_m = [t_m, t_{m+1}]$ of length $\sim \lambda^{-1/(2d)}$. Let $\mathcal{M}(t_m)$ be the $k \times k$ matrix whose j -th column vector is $\partial_t^{d-k+j}\nabla_y\psi(g(t_m), t_m)$. For the rectangle $\overline{\mathcal{R}}$ which is given by

$$\overline{\mathcal{R}} = \{(x_{d-k+1}, \dots, x_d) \in \mathbb{R}^k : |x_j| \leq c\lambda^{-1+j\rho}, \quad d-k+1 \leq j \leq d\},$$

we consider the parallelepiped defined by

$$\overline{\mathcal{P}}_m = \{y \in \mathbb{R}^k : \mathcal{M}^T(t_m)(y - g(t_m)) \in \overline{\mathcal{R}}\}.$$

By the same argument as in the proof of Lemma 2.1 to $\overline{\mathcal{P}}_m$ (instead of \mathcal{P}_k), one can easily see $|\psi(y, t)| \leq \lambda^{-1}$ whenever $y \in \overline{\mathcal{P}}_m$ and $t \in I_m$. Then, we repeat the argument in the proof of Proposition 1.3. The only difference is that the size of \mathcal{P}_k is now replaced by $|\overline{\mathcal{P}}_m| =$

$\lambda^{-k + (\frac{d^2+d}{2} - \frac{(d-k)^2 + (d-k)}{2}) \frac{1}{2d}}$. Using this for (2.7), we see that the estimate $\|T_\lambda^\gamma f\|_{L^q(S)} \leq C\lambda^{-\frac{k}{q}} \|f\|_{L^p(I)}$ implies that

$$\lambda^{-\frac{q}{2d}} \sum_m \overset{\sim \lambda^{1/2d}}{|\overline{\mathcal{P}}_m|} \lesssim \lambda^{-k} \|f\|_{L^p(I)}^q.$$

This yields $\lambda^{-\frac{q}{2d}} \lambda^{\frac{1}{2d}} \lambda^{-k + \frac{1}{2d}(\frac{d^2+d}{2} - \frac{(d-k)^2 + (d-k)}{2})} \lesssim \lambda^{-k}$. Hence, by letting $\lambda \rightarrow \infty$ it follows that the condition $q \geq (2d - k + 1)k/2 + 1$ is necessary. \square

4.2. Failure of $L^{p,1}(\mathbb{S}^{d-1}) - L^{2d/(d-1),\infty}(\mathbb{R}^d)$ for $\widehat{f d\sigma}$. We now shows the failure of $L^{p,1}(\mathbb{S}^{d-1}) - L^{2d/(d-1),\infty}(\mathbb{R}^d)$ of $f \mapsto \widehat{f d\sigma}$ for any $p > 2d/(d-1)$. This improves results in [7] where the estimate $L^{p,1}(\mathbb{S}^{d-1}) - L^{2d/(d-1),\infty}(\mathbb{R}^d)$, $p = 2d/(d-1)$ fails.

We take a small $\delta > 0$ and decompose \mathbb{S}^{d-1} into spherical caps U_j of diameter δ . Let \mathcal{T}_j be the tube centered at 0 which is dual to U_j with the short axes of size $c\delta^{-1}$ and the long axis of size $c\delta^{-2}$ for a sufficiently small $c > 0$. We denote by $\mathcal{T}_j + a_j$ the translation of \mathcal{T}_j by $a_j \in \mathbb{R}^d$.

The following lemma is the Kakeya set construction appeared in [7, Lemma 3].

Lemma 4.1. *Let $0 < \delta \ll 1$, and let U_j and \mathcal{T}_j are given as above. Then there exists $\{a_j\}_{1 \leq j \leq \delta^{-(d-1)}}$ satisfying*

$$|\bigcup_j (\mathcal{T}_j + a_j)| \lesssim \frac{\log \log 1/\delta}{\log 1/\delta} \sum_j |\mathcal{T}_j + a_j|.$$

To show the failure for $p > \frac{2d}{d-1}$, it suffices to show the case $p = \infty$ and the other case follows since $L^\infty(\mathbb{S}^{d-1}) \subset L^{p,1}(\mathbb{S}^{d-1})$ for any $p < \infty$. Let $q_* = \frac{2d}{d-1}$ and let us assume that $\|\widehat{f d\sigma}\|_{L^{q_*,\infty}} \lesssim \|f\|_{L^\infty}$. We show this lead to a contradiction.

As in the proof of Theorem 1.1, for each j , let $\epsilon_j = \pm 1$ be the random variables with equal probability. Let us set $f = \sum_j \epsilon_j f_j$ where $f_j(\xi) = \chi_{U_j}(\xi) e^{-ia_j \cdot \xi}$. Then by Khintchine's inequality we obtain

$$\left\| \sum_j \widehat{f_j d\sigma} \right\|_{L^{q_*/2,\infty}}^{1/2} = \left\| \left(\sum_j \widehat{f_j d\sigma}^2 \right)^{1/2} \right\|_{L^{q_*,\infty}} \sim \mathbb{E} \left(\left| \sum_j \epsilon_j \widehat{f_j d\sigma} \right| \right)_{L^{q_*,\infty}}.$$

By Minkowski's integral inequality, it follows that

$$(4.6) \quad \left\| \sum_j \widehat{f_j d\sigma} \right\|_{L^{q_*/2,\infty}}^{1/2} \lesssim \mathbb{E} \left(\left\| \sum_j \epsilon_j \widehat{f_j d\sigma} \right\|_{L^{q_*,\infty}} \right) \lesssim \|f\|_{L^\infty}.$$

For the second inequality we use the assumption $\|\widehat{f d\sigma}\|_{L^{q_*,\infty}} \lesssim \|f\|_{L^\infty}$. Since $\widehat{f_j d\sigma}$ is essentially constant on $\mathcal{T}_j + a_j$, we note that

$$|\widehat{f_j d\sigma}|^2 \gtrsim |U_j|^2 \chi_{\mathcal{T}_j + a_j} \sim \delta^{2(d-1)} \chi_{\mathcal{T}_j + a_j}.$$

Thus, it follows that

$$\begin{aligned} \sum_j |\mathcal{T}_j + a_j| &\leq \int \sum_j \chi_{\mathcal{T}_j + a_j}(y) dy \leq \left\| \sum_j \chi_{\mathcal{T}_j + a_j} \right\|_{L^{q_*/2}, \infty} \left| \bigcup_j \mathcal{T}_j + a_j \right|^{1-2/q_*} \\ &\lesssim \delta^{-2(d-1)} \left\| \sum_j |\widehat{f_j} d\sigma|^2 \right\|_{L^{q_*/2}, \infty} \left| \bigcup_j \mathcal{T}_j + a_j \right|^{1-2/q_*}. \end{aligned}$$

Combining this with Lemma 4.1 and (4.6), we obtain

$$\left(\sum_{j=0}^{\ell} |\mathcal{T}_j + a_j| \right)^{2/q_*} \lesssim \delta^{-2(d-1)} \left(\frac{\log \log 1/\delta}{\log 1/\delta} \right)^{1-2/q_*}.$$

Note that $(\sum_{j=0}^{\ell} |\mathcal{T}_j + a_j|)^{2/q_*} \gtrsim (\delta^{-(d-1)} \delta^{-(d+1)})^{(d-1)/d}$. Since $1 - 2/q_* = 1/d > 0$, we have a contradiction as $\delta \rightarrow 0$. This completes the proof.

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