Chromatic Schultz Polynomial of Certain Graphs

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Abstract

A topological index of a graph G is a real number which is preserved under isomorphism. Extensive studies on certain polynomials related to these topological indices have also been done recently. In a similar way, chromatic versions of certain topological indices and the related polynomials have also been discussed in the recent literature. In this paper, the chromatic version of the Schultz polynomial is introduced and determined this polynomial for certain fundamental graph classes.

Keywords: Graph colouring, Schultz polynomial, χ^--chromatic Schultz polynomial, χ^+-chromatic Schultz polynomial, modified chromatic Schultz polynomial.

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1 Introduction

For all terms and definitions, not defined specifically in this paper, we refer to [1, 2, 7, 12]. Further, for graph colouring, see [3, 8, 9]. Unless mentioned otherwise, all graphs considered here are undirected, simple, finite and connected.

A *proper vertex colouring* of a graph G is an assignment $\varphi : V(G) \to \mathcal{C}$ of the vertices of G, where $\mathcal{C} = \{c_1, c_2, c_3, \ldots, c_\ell\}$ is a set of colours such that adjacent vertices of G have different colours. The cardinality of the minimum set of colours which allows a proper colouring of G is called the *chromatic number* of G and is denoted $\chi(G)$. The set of all vertices of G which have the colour c_i is called the *colour class* of that colour c_i in G. The cardinality of the colour class of a colour c_i is said to be the *strength* of that colour in G and is denoted by $\theta(c_i)$. We can also define a function $\zeta : V(G) \to \{1, 2, 3, \ldots, \ell\}$ such that $\zeta(v_i) = s$ if and only if $\varphi(v_i) = c_s, c_s \in \mathcal{C}$.

A vertex colouring consisting of the colours having minimum subscripts may be called a *minimum parameter colouring* (see [10]). If we colour the vertices of G in such a way that c_1 is assigned to maximum possible number of vertices, then

c_2 is assigned to maximum possible number of remaining uncoloured vertices and proceed in this manner until all vertices are coloured, then such a colouring is called a χ^--*colouring* of G. In a similar manner, if c_ℓ is assigned to maximum possible number of vertices, then $c_{\ell-1}$ is assigned to maximum possible number of remaining uncoloured vertices and proceed in this manner until all vertices are coloured, then such a colouring is called a χ^+-*colouring* of G.

A *topological index* of a graph G is a real number which is preserved under isomorphism. The chromatic versions of certain topological indices have been introduced in [10]. In this paper, we discuss the chromatic versions of certain polynomials related to te topological indices of a graph G.

2 Chromatic Schultz Polynomial of Graphs

Note that throughout this study, we use the chromatic colourings of the graphs under consideration. Motivated by the studies on Schultz polynomial of graphs (see [5, 6]), we can now introduce the chromatic version of the Schultz polynomial as follows:

Definition 2.1. Let G be a connected graph with chromatic number $\chi(G)$. Then, the *chromatic Schultz polynomial* of G, denoted by $\mathcal{S}_\chi(G, x)$, is defined as

$$\mathcal{S}_\chi(G, x) = \sum_{u,v \in V(G)} (\zeta(u) + \zeta(v)) x^{d(u,v)}.$$

Definition 2.2. Let G be a connected graph with chromatic number φ^- and $varphi^+$ be the minimal and maximal parameter colouring of G. Then,

(i) the χ^--*chromatic Schultz polynomial* of G, denoted by $\mathcal{S}_{\chi^-}(G, x)$, is defined as

$$\mathcal{S}_{\chi^-}(G, x) = \sum_{u,v \in V(G)} (\zeta_{\varphi^-}(u) + \zeta_{\varphi^-}(v)) x^{d(u,v)};$$

and

(ii) the χ^+-*chromatic Schultz polynomial* of G, denoted by $\mathcal{S}_{\chi^+}(G, x)$, is defined as

$$\mathcal{S}_{\chi^+}(G, x) = \sum_{u,v \in V(G)} (\zeta_{\varphi^+}(u) + \zeta_{\varphi^+}(v)) x^{d(u,v)}.$$

Now, we can determine the chromatic Schultz polynomials of certain fundamental graph classes.

2.1 Chromatic Schultz Polynomials of Paths

In this section, we discuss the two types of Schultz polynomials of paths.

Theorem 2.3. *Let P_n be a path on n vertices. Then, we have*

$$\mathcal{S}_{\chi^-}(P_n, x) = \begin{cases} \sum_{i=0}^{\frac{n-1}{2}} \left[(3n - 6i - 3)x + (3n - 6i - 1) \right] x^{2i}; & \text{if } n \text{ is odd;} \\ 3 \cdot \sum_{i=0}^{n} (n-i) x^i; & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let $V = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of P_n, where the vertices are labelled consecutively from one end vertex to the other. Note that $\chi(P_n) = 2$. Let c_1, c_2 be the two colours we use for colouring P_n. We also note that te diameter of P_n is $n - 1$. Hence, the power of the variable x varies from 0 to $n - 1$ in the Schultz polynomial of P_n. Here, we need to consider the following two cases:

Case-1: Let n be odd. Then, with respect to a χ^--colouring, the vertices $v_1, v_3, v_5, \ldots v_n$ get the colour c_1 and the vertices $v_2, v_4, v_6, \ldots v_{n-1}$ get the colour c_2. The possible colour pairs and their numbers in G in terms of the distances between them are listed in the following table.

Distance $d(u,v)$	Colour pairs	Number of pairs	Total number of pairs
0	(c_1, c_1)	$\frac{n+1}{2}$	n
	(c_2, c_2)	$\frac{n-1}{2}$	
1	(c_1, c_2)	$n - 1$	$n - 1$
2	(c_1, c_1)	$\frac{n-1}{2}$	$n - 2$
	(c_2, c_2)	$\frac{n-3}{2}$	
3	(c_1, c_2)	$n - 3$	$n - 3$
4	(c_1, c_1)	$\frac{n-3}{2}$	$n - 4$
	(c_2, c_2)	$\frac{n-5}{2}$	
5	(c_1, c_2)	$n - 5$	$n - 5$
6	(c_1, c_1)	$\frac{n-5}{2}$	$n - 6$
	(c_2, c_2)	$\frac{n-7}{2}$	
....	....	....	....
....	....	....	....
$n - 3$	(c_1, c_1)	2	3
	(c_2, c_2)	1	
$n - 2$	(c_1, c_2)	2	2
$n - 1$	(c_1, c_1)	1	1
	(c_2, c_2)	0	

Table 1

In the above table, the possible distances between different pairs of vertices are written in the first column, the different colour pairs with respect to each distance is written in the second column and the number of corresponding colour pairs with respect to each distance is written in the third column. The total number of vertex pairs corresponding to each distance is written in the fourth column.

From Table-1, we note that for $1 \le r \le n$, the number of vertex pairs which are at a distance r is $n - r$ and in this case all colour pairs contain two colours. But, when r is even, all colour pairs contain the same colour - either (c_1, c_1) or (c_2, c_2). In this case, note that the number of (c_1, c_1)-colour pairs is $\frac{n-r+1}{2}$ and the number of (c_2, c_2)-colour pairs is $\frac{n-r-1}{2}$ so that total number of colour pairs is $n - r$. Hence,

$$\mathcal{S}_{\chi^-}(P_n, x) = \sum_{r \text{ odd}} (1 + 2)(n - r)x^r + \sum_{r \text{ even}} \left[\frac{n - r + 1}{2} \cdot 2 + \frac{n - r - 1}{2} \cdot 4 \right] x^r$$

$$= \sum_{r \text{ odd}} (3n - 3r)x^r + \sum_{r \text{ even}} (3n - 3r - 1)x^r$$

$$= \sum_{i=0}^{\frac{n-1}{2}} (3n - 6i - 3)x^{2i+1} + \sum_{i=0}^{\frac{n-1}{2}} (3n - 6i - 1)x^{2i}$$

$$= \sum_{i=0}^{\frac{n-1}{2}} \left[(3n - 6i - 3)x + (3n - 6i - 1) \right] x^{2i}.$$

Case-2: Let n be even. Then, Then, with respect to a χ^--colouring, the vertices $v_1, v_3, v_5, \ldots v_{n-1}$ get the colour c_1 and the vertices $v_2, v_4, v_6, \ldots v_n$ get the colour c_2. The possible colour pairs and their numbers in G in terms of the distances between them are listed in the following table.

Distance $d(u,v)$	Colour pairs	Number of pairs	Total number of pairs
0	(c_1, c_1)	$\frac{n}{2}$	n
	(c_2, c_2)	$\frac{n}{2}$	
1	(c_1, c_2)	$n - 1$	$n - 1$
2	(c_1, c_1)	$\frac{n-2}{2}$	$n - 2$
	(c_2, c_2)	$\frac{n-2}{2}$	
3	(c_1, c_2)	$n - 3$	$n - 3$
4	(c_1, c_1)	$\frac{n-4}{2}$	$n - 4$
	(c_2, c_2)	$\frac{n-4}{2}$	
5	(c_1, c_2)	$n - 5$	$n - 5$
6	(c_1, c_1)	$\frac{n-6}{2}$	$n - 6$
	(c_2, c_2)	$\frac{n-6}{2}$	
....	....	....	....
....	....	....	....
$n - 3$	(c_1, c_2)	3	3
$n - 2$	(c_1, c_1)	1	2
	(c_2, c_2)	1	
$n - 1$	(c_1, c_2)	1	1

Table 2

From Table-2, we have

$$\begin{aligned} \mathcal{S}_{\chi^-}(P_n, x) &= \sum_{r \text{ odd}} (1+2)(n-r)x^r + \sum_{r \text{ even}} \left[\frac{n-r}{2} \cdot 2 + \frac{n-r}{2} \cdot 4 \right] x^r \\ &= \sum_{r \text{ odd}} (3n - 3r)x^r + \sum_{r \text{ even}} (3n - 3r)x^r \\ &= 3 \cdot \sum_{i=0}^{n} (n-i)x^i. \end{aligned}$$

This completes the proof.   □

Note that the χ^+-colouring of P_n can be obtained by interchanging the colours c_1 and c_2 in the χ^--colouring. Hence, as explained in the proof of above theorem, we have

Theorem 2.4. *Let P_n be a path on n vertices. Then, we have*

$$
\mathcal{S}_{\chi^+}(P_n, x) = \begin{cases} \sum_{i=0}^{\frac{n-1}{2}} \left[(3n-6i-3)x + (3n-6i+1)\right] x^{2i}; & \text{if } n \text{ is odd;} \\ 3 \cdot \sum_{i=0}^{n} (n-i)x^i; & \text{if } n \text{ is even.} \end{cases}
$$

2.2   Chromatic Schultz Polynomial of Cycles

Theorem 2.5. *Let C_n be a path on n vertices. Then, we have*

$$
\mathcal{S}_{\chi^-}(C_n, x) = \begin{cases} \frac{3n(1-x^{\frac{n+2}{2}})}{1-x}; & \text{if } n \text{ is even;} \\ \frac{3(n+1)(1-x^{\frac{n+3}{2}})}{1-x}; & \text{if } n \text{ is odd.} \end{cases}
$$

Proof. Let $V = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of C_n, where the vertices are labelled consecutively from one end vertex to the other in a clockwise manner. Note that if n is odd, then the diameter of C_n is $\frac{n+1}{2}$ and if n is even, the diameter of C_n is $\frac{n}{2}$. Hence, we have to consider the following two cases:

Case-1: Let n be even. Then, C_n is 2-colourable and we can label the vertices $v_1, v_3, v_5 \ldots, v_{n-1}$ by colour c_1 and the vertices $v_2, v_4, v_6 \ldots, v_n$ by colour c_2. Then, for $0 \le i \le n$, the possible colour pairs and their numbers can be obtained from the following table.

Distance $d(u,v)$	Colour pairs	Number of pairs	Total number of pairs
i, even	(c_1, c_1)	$\frac{n}{2}$	n
	(c_2, c_2)	$\frac{n}{2}$	
i, odd	(c_1, c_2)	n	n

Table 3

Then, from Table 3, we have

$$
\begin{aligned}
\mathcal{S}_{\chi^-}(C_n, x) &= \sum_{i \text{ odd}} 3n x^i + \sum_{i \text{ even}} \left[\frac{n}{2} \cdot 2 + \frac{n}{2} \cdot 4\right] x^i \\
&= \sum_{i \text{ odd}} 3n x^i + \sum_{i \text{ even}} (n + 2n) x^i \\
&= \sum_{i=0}^{\frac{n}{2}} 3n x^i \\
&= \frac{3n(1 - x^{\frac{n+2}{2}})}{1 - x}.
\end{aligned}
$$

Case-2: Let n be odd. Then, $\chi(C_n) = 3$ and the vertices $v_1, v_3, v_5 \ldots, v_{n-1}$ by colour c_1 and the vertices $v_2, v_4, v_6 \ldots, v_{n-2}$ by colour c_2 and the vertex v_n gets colour c_3. Then, for $0 \le i \le n$, the possible colour pairs and their numbers can be obtained from Table 4.

Distance $d(u,v)$	Colour pairs	Number of pairs	Total number of pairs
$i = 0$	(c_1, c_1)	$\frac{n-1}{2}$	n
	(c_2, c_2)	$\frac{n-1}{2}$	
	(c_3, c_3)	1	
$i > 0$ and even	(c_1, c_1)	$\frac{n-r-1}{2}$	n
	(c_1, c_2)	$r - 1$	
	(c_2, c_2)	$\frac{n-r-1}{2}$	
	(c_1, c_3)	1	
	(c_2, c_3)	1	
$i >$, odd	(c_1, c_1)	$\frac{r-1}{2}$	n
	(c_1, c_2)	$n - r - 1$	
	(c_2, c_2)	$\frac{r-1}{2}$	
	(c_1, c_3)	1	
	(c_2, c_3)	1	

Table 4

When $i = 0$, we have

$$\sum_{v \in V} (\zeta(v) + \zeta(v)) x^{d(v,v)} = \left[(2+4) \cdot \frac{n-1}{2} + 6 \cdot 1 \right] x^0$$
$$= 3(n+1)x^0$$

When $i > 0$ and is even, we have

$$\sum_{d(u,v)=i} (\zeta(u) + \zeta(v)) x^{d(u,v)} = \left[(2+4) \cdot \frac{n-r-1}{2} + 3(r-1) + (4+5) \cdot 1 \right] x^i$$
$$= 3(n+1)x^i$$

Similarly, when $i > 0$ and is odd, we have

$$\sum_{d(u,v)=i} (\zeta(u) + \zeta(v)) x^{d(u,v)} = \left[(2+4) \cdot \frac{r-1}{2} + 3(n-r-1) + (4+5) \cdot 1 \right] x^i$$
$$= 3(n+1)x^i$$

Therefore, $S_{\chi^-}(C_n, x) = \sum_{i=0}^{\frac{n+1}{2}} 3(n+1)x^i = \frac{3(n+1)(1-x^{\frac{n+3}{2}})}{1-x}$, completing the proof. $\square$

Note that in the χ^--colouring of an even cycle C_n if we the colours c_1 and c_2, we get its χ^+-colouring. It can be observed that this change makes no change in the corresponding Schultz polynomial. But, for an odd cycle C_n, we have to interchange the colours c_1 and c_3 in its χ^--colouring and keep c_2 as it is to get a χ^+-colouring.

In view of this fact, the χ^+-chromatic Schultz polynomial of C_n is obtained in the following theorem.

Theorem 2.6. *Let C_n be a path on n vertices. Then, we have*

$$\mathcal{S}_{\chi^+}(C_n, x) = \begin{cases} \frac{3n(1-x^{\frac{n+2}{2}})}{1-x}; & \text{if } n \text{ is even;} \\ \frac{(5n-3)(1-x^{\frac{n+3}{2}})}{1-x}; & \text{if } n \text{ is odd.} \end{cases}$$

2.3 Chromatic Schultz Polynomial of Complete Graphs

Next, we consider the complete graph K_n. In K_n, we have $d(u, v) = 1$ for any two $u, v \in V(G)$. Therefore, $\mathcal{S}_{\chi^-}(K_n, x)$ and $\mathcal{S}_{\chi^+}(K_n, x)$ are the same and are first degree polynomials. The following result provides the Schultz polynomial of a complete graph K_n.

Proposition 2.7. *For $n \geq 2$, $\mathcal{S}_{\chi^-}(K_n, x) = \mathcal{S}_{\chi^+}(K_n, x) = (n^2 + n) + (2n^2 - n - 3)x$.*

Proof. In any proper colouring, distinct vertices in K_n get distinct colours. Now, $\sum_{v \in V} 2\zeta(v)x^0 = (2 + 4 + 6 + \ldots + 2n)x^0 = n(n + 1)$. Also, we have

$$\sum_{d(u,v)=1} (\zeta(u) + \zeta(v))x^1 = (3 + 4 + 5 + \ldots + (2n-1))x = \left(\frac{2n-3}{2}(2n+2)\right)x$$

$$= (2n-3)(n+1)x.$$

Therefore, $\mathcal{S}_{\chi^-}(K_n, x) = (n^2 + n) + (2n^2 - n - 3)x = \mathcal{S}_{\chi^+}(K_n, x)$. □

Next, let us consider the complete bipartite graphs $K_{a,b}$, where $a \geq b$.

Theorem 2.8. *For a complete bipartite $K_{a,b}, a \geq b, a+b = n$, we have $\mathcal{S}_{\chi^-}(K_n, x) = (2a + 4b) + 3abx + (a(a-1) + 2b(b-1))x^2$ and $\mathcal{S}_{\chi^+}(K_n, x) = (4a + 2b) + 3abx + (2a(a-1) + b(b-1))x^2$.*

Proof. Note that $K_{a,b}$ is 2-colourable and its diameter is 2. Since $a \geq b$, with respect to all a vertices in the first partition get the colour c_1 and all b vertices in the second partition get colour c_2. Then, we have the following table. Then,

Distance $d(u, v)$	Colour pairs	Number of pairs	Total number of pairs
$i = 0$	(c_1, c_1)	a	$a + b$
	(c_2, c_2)	b	
$i = 1$	(c_1, c_2)	ab	ab
$i = 2$	(c_1, c_1)	$\binom{a}{2}$	$\binom{a}{2} + \binom{b}{2}$
	(c_2, c_2)	$\binom{b}{2}$	

Table 5

$$\mathcal{S}_{\chi^-}(K_{m,n}, x) = (2a + 4b) + 3abx + (2 \cdot \binom{a}{2} + 4 \cdot \binom{b}{2})x^2$$

$$= (2a + 4b) + 3abx + (a(a-1) + 2b(b-1))x^2.$$

In a similar way, by interchanging c_1 and c_2, we can prove that $\mathcal{S}_{\chi^+}(K_{m,n}, x) = (4a + 2b) + 3abx + (2a(a-1) + b(b-1))x^2$. □

3   Modified Chromatic Schultz Polynomials

Definition 3.1. Let G be a connected graph with chromatic number $\chi(G)$. Then, the *modified chromatic Schultz polynomial* of G, denoted by $\mathcal{S}^*_\chi(G, x)$, is defined as

$$\mathcal{S}^*_\chi(G, x) = \sum_{u,v \in V(G)} (\zeta(u)\zeta(v))x^{d(u,v)}.$$

Definition 3.2. Let G be a connected graph with chromatic number φ^- and $varphi^+$ be the minimal and maximal parameter colouring of G. Then,

(i) the *modified χ^--chromatic Schultz polynomial* of G, denoted by $\mathcal{S}^*_{\chi^-}(G, x)$, is defined as

$$\mathcal{S}^*_{\chi^-}(G, x) = \sum_{u,v \in V(G)} (\zeta_{\varphi^-}(u) \cdot \zeta_{\varphi^-}(v))x^{d(u,v)};$$

and

(ii) the *χ^+-chromatic Schultz polynomial* of G, denoted by $\mathcal{S}^*_{\chi^+}(G, x)$, is defined as

$$\mathcal{S}^*_{\chi^+}(G, x) = \sum_{u,v \in V(G)} (\zeta_{\varphi^+}(u) \cdot \zeta_{\varphi^+}(v))x^{d(u,v)}.$$

The following theorems discuss the modified chromatic Schultz polynomials of paths.

Theorem 3.3. *Let P_n be a path on n vertices. Then, we have*

$$\mathcal{S}^*_{\chi^-}(P_n, x) = \begin{cases} \sum\limits_{i=0}^{\frac{n-1}{2}} \left[(2n - 4i - 2)x + \left(\frac{5n-10i-3}{2} \right) \right] x^{2i}; & \text{if } n \text{ is odd;} \\ \sum\limits_{i=0}^{\frac{n-1}{2}} \left[(2n - 4i - 2)x + \left(\frac{5n-10i}{2} \right) \right] x^{2i}; & \text{if } n \text{ is even.} \end{cases}$$

Proof. If n is odd, then from Table 1, we have

$$\begin{aligned} \mathcal{S}^*_{\chi^-}(P_n, x) &= \sum_{r \text{ odd}} 2(n - r)x^r + \sum_{r \text{ even}} \left[\frac{n - r + 1}{2} \cdot 1 + \frac{n - r - 1}{2} \cdot 4 \right] x^r \\ &= \sum_{r \text{ odd}} (2n - 2r)x^r + \sum_{r \text{ even}} \left(\frac{5n - 5r - 3}{2} \right) x^r \\ &= \sum_{i=0}^{\frac{n-1}{2}} (2n - 4i - 2)x^{2i+1} + \sum_{i=0}^{\frac{n-1}{2}} \left(\frac{5n - 10i - 3}{2} \right) x^{2i} \\ &= \sum_{i=0}^{\frac{n-1}{2}} \left[(2n - 4i - 2)x + \left(\frac{5n - 10i - 3}{2} \right) \right] x^{2i}. \end{aligned}$$

If n is even, then from Table 2, we have

$$\mathcal{S}^*_{\chi^-}(P_n, x) = \sum_{r \text{ odd}} 2(n - r)x^r + \sum_{r \text{ even}} \left[\frac{n - r}{2} \cdot 1 + \frac{n - r}{2} \cdot 4 \right] x^r$$

$$= \sum_{r \text{ odd}} (2n - 2r)x^r + \sum_{r \text{ even}} \left(\frac{5n - 5r}{2}\right) x^r$$

$$= \sum_{i=0}^{\frac{n-1}{2}} (2n - 4i - 2)x^{2i+1} + \sum_{i=0}^{\frac{n-1}{2}} \left(\frac{5n - 10i}{2}\right) x^{2i}$$

$$= \sum_{i=0}^{\frac{n-1}{2}} \left[(2n - 4i - 2)x + \left(\frac{5n - 10i}{2}\right)\right] x^{2i}.$$

This completes the proof. □

Similarly, by interchanging the colours c_1 and c_2, we have the following result.

Theorem 3.4. *Let P_n be a path on n vertices. Then,*

$$S^*_{\chi^+}(P_n, x) = \begin{cases} \sum_{i=0}^{\frac{n-1}{2}} \left[(2n - 4i - 2)x + \left(\frac{5n - 10i + 3}{2}\right)\right] x^{2i}; & \text{if } n \text{ is odd;} \\ \sum_{i=0}^{\frac{n-1}{2}} \left[(2n - 4i - 2)x + \left(\frac{5n - 10i}{2}\right)\right] x^{2i}; & \text{if } n \text{ is even.} \end{cases}$$

The following theorems discuss the modified chromatic Schultz polynomials of cycles.

Theorem 3.5. *Let C_n be a path on n vertices. Then, we have*

$$S^*_{\chi^-}(C_n, x) = \begin{cases} \sum_{i=0}^{\frac{n}{2}} \left(2nx + \frac{5n}{2}\right) x^{2i}; & \text{if } n \text{ is even;} \\ \frac{5n+17}{2} + \sum_{i=1}^{\frac{n-1}{2}} \left[(2n + 9i + 10)x + \frac{5n - 18i + 13}{2}\right] x^{2i}; & \text{if } n \text{ is odd.} \end{cases}$$

Proof. If n is even and $r = d(u, v), u, v \in V(C_n)$, then from Table 3, we have

$$S^*_{\chi^-}(C_n, x) = \sum_{r \text{ odd}} 2nx^r + \sum_{r \text{ even}} \left(\frac{n}{2} \cdot 1 + \frac{n}{2} \cdot 4\right) x^r$$

$$= \sum_{r \text{ odd}} 2nx^r + \sum_{r \text{ even}} \frac{5n}{2} x^r$$

$$= \sum_{i=0}^{\frac{n}{2}} 2nx^{2i+1} + \sum_{i=0}^{\frac{n}{2}} \frac{5n}{2} x^{2i}$$

$$= \sum_{i=0}^{\frac{n}{2}} \left[2nx + \frac{5n}{2}\right] x^{2i}.$$

Let n be odd. Then, from Table 4,

$$S^*_{\chi^-}(C_n, x) = \sum_{r=0} \left(\frac{n-1}{2} \cdot 1 + \frac{n-1}{2} \cdot 4 + 9 \cdot 1\right) +$$

$$\sum_{r>0 \text{ and odd}} \left(\frac{r-1}{2}(1+4) + 2(n-r-1) + (3+6)\cdot 1 \right) x^r +$$

$$\sum_{r>0 \text{ and even}} \left((1+4)\frac{n-r-1}{2} + 2(r-1) + (3+6)\cdot 1 \right) x^r$$

$$= \frac{5n+17}{2} + \sum_{r \text{ odd}} \frac{4n+9r+11}{2}x^r + \sum_{r \text{ even}} \frac{5n-9r+13}{2}x^r$$

$$= \frac{5n+17}{2} + \sum_{i=1}^{\frac{n-1}{2}} \frac{4n+18i+20}{2}x^{2i+1} + \sum_{i=1}^{\frac{n-1}{2}} \frac{5n-18i+13}{2}x^{2i}$$

$$= \frac{5n+17}{2} + \sum_{i=1}^{\frac{n-1}{2}} \left[(2n+9i+10)x + \frac{5n-18i+13}{2} \right] x^{2i}.$$

This completes the proof. $\qquad\square$

Similarly, interchanging c_1 and c_2 in even cycles and interchanging c_1 and c_3 in even cycles, we get

Theorem 3.6. *Let C_n be a path on n vertices. Then, we have*

$$\mathcal{S}^*_{\chi^+}(C_n, x) = \begin{cases} \frac{13n-11}{2} + \sum_{i=1}^{\frac{n-1}{2}} \left[(6n+i-7)x + \frac{13n-2i-15}{2} \right] x^{2i}; & \text{if } n \text{ is odd}; \\ \sum_{i=0}^{\frac{n}{2}} \left(2nx + \frac{5n}{2} \right) x^{2i}; & \text{if } n \text{ is even}. \end{cases}$$

The following result provides the Schultz polynomial of a complete bipartite graph $K_{a,b}$.

Theorem 3.7. *For a complete bipartite $K_{a,b}, a \geq b, a+b = n$, we have $\mathcal{S}^*_{\chi^-}(K_n, x) = (a+4b) + 2abx + \left(\frac{a(a-1)}{2} + 2b(b-1) \right) x^2$ and $\mathcal{S}_{\chi^+}(K_n, x) = (4a+b) + 2abx + \left(2a(a-1) + \frac{b(b-1)}{2} \right) x^2$.*

Proof. The proof similar to that of Theorem 2.8. $\qquad\square$

4   Conclusion

In this article, we have introduced a particular type of polynomial, called chromatic Schultz polynomial of graphs, as an analogue of the Schultz polynomial of graphs and determined this polynomial for certain fundamental graphs.

The study seems to be promising for further studies as the polynomial can be computed for many graph classes and classes of derived graphs. The chromatic Schultz polynomial can be determined for graph operations, graph products and graph powers. The study on Schultz polynomials with respect to different types of graph colourings also seem to be much promising. The concept can be extended to edge colourings and map colourings also.

These polynomials have so many applications in various fields like Mathematical Chemistry, Distribution Theory, Optimisation Techniques etc. In Chemistry, some interesting studies using the above-mentioned concepts are possible if $c(v_i)$ (or $\zeta(v_i)$) assumes the values such as energy, valency, bond strength etc. Similar studies are possible in various other fields.

All these facts highlight the wide scope for further research in this area.

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