CLASSIFICATION OF SPHERICAL FUSION CATEGORIES OF FROBENIUS-SCHUR EXPONENT 2

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Abstract. In this paper, we propose a new approach towards the classification of spherical fusion categories by their Frobenius-Schur exponents. We classify spherical fusion categories of Frobenius-Schur exponent 2 up to monoidal equivalence. We also classify modular categories of Frobenius-Schur exponent 2 up to braided monoidal equivalence. It turns out that the Gauss sum is a complete invariant for modular categories of Frobenius-Schur exponent 2. This result can be viewed as a categorical analog of Arf's theorem on the classification of non-degenerate quadratic forms over fields of characteristic 2.

1. Introduction

Let $\mathcal C$ be a spherical fusion category over $\mathbb C$. The higher Frobenius-Schur indicators $\nu_n(V)$ of $V \in Ob(\mathcal{C})$ and $n \in \mathbb{Z}$ are generalizations of the classical Frobenius-Schur indicator for irreducible finite group representations (see [\[15\]](#page-9-0) and the references therein). The Frobenius-Schur indicators are important invariants of a spherical fusion category, especially when the category is in addition non-degenerately braided (in other words, modular). For example, the congruence subgroup conjecture on the $SL(2,\mathbb{Z})$ representations arising from modular categories can be resolved using generalized Frobenius-Schur indicators [\[17\]](#page-9-1).

The Frobenius-Schur exponent of a spherical fusion category \mathcal{C} , denoted by $\text{FSexp}(\mathcal{C})$, is the smallest positive integer n such that $\nu_n(V) = \dim_{\mathcal{C}}(V)$ for any object $V \in Ob(\mathcal{C}),$ where $\dim_{\mathcal{C}}(V)$ is the categorical dimension of V in C. It is shown in [\[15\]](#page-9-0) that $F\text{Sexp}(\mathcal{C})$ is equal to the order of the T-matrix of $Z(\mathcal{C})$, the Drinfeld center of \mathcal{C} . Moreover, the Cauchy theorem for spherical fusion categories asserts that the prime ideals dividing $FS\exp(\mathcal{C})$ and those dividing the global dimension $\dim(\mathcal{C})$ are the same in the ring of algebraic integers [\[4\]](#page-9-2). It is then reasonable to pursue a classification of spherical fusion categories by their Frobenius-Schur exponents, as opposed to the usual method of classification by rank [\[18,](#page-9-3) [19,](#page-9-4) [3\]](#page-9-5).

In this paper, we give a full classification of spherical fusion categories of Frobenius-Schur exponent 2. We show that such a spherical fusion category $\mathcal C$ is equivalent, as a fusion category, to $\text{Rep}(\mathbb{Z}_2^n)$ for some positive integer n. In particular, the associativity constraints of C are all identities. We then show that if C is in addition modular, then C can be decomposed into a Deligne tensor product of two types of modular categories called $\mathcal{C}(\mathbb{Z}_2^2,q_1)$ and $\mathcal{C}(\mathbb{Z}_2^2,q_2)$. It is worth mentioning that in [\[5,](#page-9-6) Theorem 3.2], the authors showed that any modular category of Frobenius-Schur exponent 2 is a braided fusion subcategory of Rep($D^{\omega}(\mathbb{Z}_2^{2n})$) for some positive integer n. In this paper, we completely classify these modular categories by a categorical analog of Arf's theorem on the classification of nondegenerate quadratic forms over fields of characteristic 2. It turns out, in this case, the positive Gauss sum is a complete invariant.

The paper is structured as follows. In Section 2, we give a quick review of basic concepts and set up notations for future use. We also discuss the braided monoidal structure on the category of G -graded vector spaces for a finite abelian group G . In Section 3, we classify spherical fusion categories of Frobenius-Schur exponent 2. Finally, in Section 4, we classify modular categories of Frobenius-Schur exponent 2.

2. Preliminaries

2.1. Basic concepts and notations.

Now let C be a *fusion category* over C over \mathbb{C} [\[8\]](#page-9-7). In particular, C is rigid monoidal, \mathbb{C} linear, semisimple with finitely many isomorphism classes of simple objects such that the tensor unit $1 \in Ob(\mathcal{C})$ is simple. We fix a choice of representatives from the isomorphism class of simple objects and denote the set of all such representatives by Π_{*C*}. The *Frobenius-Perron dimension* of $V \in Ob(\mathcal{C})$, denoted by FP dim_C(V), is the largest non-negative eigenvalue of the fusion matrix of V . We define the Frobenius-Perron dimension of C by FP dim $(C) := \sum_{V \in \Pi_C} FP$ dim $_C(V)^2$.

A fusion category C is called *spherical* if it has a pivotal structure such that the left and right pivotal traces coincide on all endomorphisms. In this case, the left (or right) pivotal trace of id_V, the identity of $V \in Ob(\mathcal{C})$, is called the *categorical dimension* of V. We denote the categorical dimension of $V \in Ob(\mathcal{C})$ by $\dim_{\mathcal{C}}(V)$, and we define the *global* dimension C by dim(C) := $\sum_{V \in \Pi_{\mathcal{C}}} \dim_{\mathcal{C}}(V)^2$.

A spherical fusion category admitting a braiding is called a braided spherical fusion category (or premodular category). A braided spherical fusion category is called *modular* if the braiding is non-degenerate, or equivalently, if its S-matrix is non-degenerate [\[14\]](#page-9-8). For example, $Z(\mathcal{C})$, the Drinfeld center of a spherical fusion category \mathcal{C} , is modular [\[14\]](#page-9-8). Objects of $Z(\mathcal{C})$ are pairs (X, σ_X) , where $X \in Ob(\mathcal{C})$ and $\sigma_X : X \otimes -\widetilde{\to} -\otimes X$ is a half braiding. Since the pivotal structure of $Z(\mathcal{C})$ is inherited from \mathcal{C} , we have

(2.1)
$$
\dim_{Z(\mathcal{C})}(V, \sigma_V) = \dim_{\mathcal{C}}(V)
$$

for any $V \in Ob(\mathcal{C})$.

Let C be a spherical fusion category. For any $n \in \mathbb{Z}$, and for any $V \in Ob(\mathcal{C})$, the *n*-th *Frobenius-Schur indicator* ν_n of V is defined to be the operator trace of a linear operator $E_V^{(n)}$ $V(V)$: Hom $c(1, V^{\otimes n}) \to \text{Hom}_{\mathcal{C}}(1, V^{\otimes n})$ satisfying $(E_V^{(n)})$ $(V^{(n)})^n = \text{id}$. Here, $V^{\otimes n}$ is understood as inductively defined by $V^{\otimes(m+1)} = V \otimes V^{\otimes m}$ for $1 \leq m < n$, and associativity constraints are included in the definition of $E_V^{(n)}$ $V^{(n)}$, see [\[16\]](#page-9-9). In particular, if V is simple, then

$$
\nu_1(V) = \delta_{1,V}.
$$

We also have

(2.3)
$$
\nu_2(V) = 0, \text{ if } V \not\cong V^*, \ \nu_2(V) = 1 \text{ or } -1, \text{ if } V \cong V^*
$$

for all $V \in \Pi_{\mathcal{C}}$.

The Frobenius-Schur exponent of an object V in a spherical fusion category C , denoted by FSexp(V), is defined to be the smallest positive integer n such that $\nu_n(V) = \dim_{\mathcal{C}}(V)$. The *Frobenius-Schur exponent of* C , denoted by $FSexp(C)$, is defined to be the smallest positive integer n such that $\nu_n(V) = \dim_{\mathcal{C}}(V)$ for all $V \in \mathcal{C}$ [\[15\]](#page-9-0). When \mathcal{C} is the category of finite dimensional H-modules for a semisimple Hopf algebra H over \mathbb{C} , F Sexp(V) is

equal to the exponent of V as a finite dimensional H-module. In other words, F Sexp(V) is equal to the exponent of the image of G in $GL(V, \mathbb{C})$ [\[11\]](#page-9-10).

It is immediate from the definition and Equation (2.2) that if C is a spherical fusion category such that $\text{FSexp}(\mathcal{C}) = 1$, then for any $V \in \Pi_{\mathcal{C}}$, $\dim_{\mathcal{C}}(V) = \delta_{1,V}$. According to [\[9,](#page-9-11) Theorem 2.3, $\dim_{\mathcal{C}}(V) \neq 0$ for all $V \in \Pi_{\mathcal{C}}$, hence \mathcal{C} has the tensor unit 1 as its only simple object. Therefore, C is monoidally equivalent to Vec_C, the category of finite dimensional vector spaces over C.

It is worth mentioning that by [\[15\]](#page-9-0), for any $V \in Ob(\mathcal{C})$, $FSexp(V)$ does not depend on the choice of pivotal structures. In addition, F Sexp(C) of a spherical fusion category C depends only on the equivalence class of the modular category $Z(\mathcal{C})$.

2.2. Braided monoidal structure on G-graded vector spaces.

Let G be a finite multiplicative abelian group. Recall that the category Vec_G^{ω} of finite-dimensional G-graded vector spaces has simple objects ${V_q|g \in G}$ where $(V_q)_h$ = $\delta_{g,h}\mathbb{C}, \forall h \in G.$ The tensor product is given by $V_g \otimes V_h = V_{gh}$, and the tensor unit is V_1 , where 1 is the identity of G. The associator is given by a normalized 3-cocycle $\omega \in Z^3(G, \mathbb C^\times)$

$$
\omega(x, y, z): V_x \otimes (V_y \otimes V_z) \longrightarrow (V_x \otimes V_y) \otimes V_z.
$$

Now we equip Vec_G^{ω} with a braiding given by a normalized 2-cochain $c \in C^2(G, \mathbb{C}^\times)$

$$
c(x,y): V_x \otimes V_y \longrightarrow V_y \otimes V_x
$$

satisfying the hexagon axioms

(2.4)
$$
\frac{c(xy,z)}{c(x,z)c(y,z)} \frac{\omega(x,y,z)\omega(z,x,y)}{\omega(x,z,y)} = 1 = \frac{c(x,yz)}{c(x,y)c(x,z)} \frac{\omega(y,x,z)}{\omega(x,y,z)\omega(y,z,x)}
$$

for all $x, y, z \in G$. In other words, the pair (ω, c) is an *Eilenberg-MacLane 3-cocycle* of G. Finally, we equip Vec_G^{ω} with the canonical (spherical) pivotal structure, which is simply given by identities on objects, so that the categorical dimensions are all positive. We denote this braided spherical fusion category by $Vec_G^{(\omega,c)}$.

An Eilenberg-MacLane 3-cocycle (ω, c) is called a *coboundary* if there exists a 2-cochain $h \in C^2(G, \mathbb{C}^\times)$ such that

(2.5)
$$
\omega = \delta h \text{ and } c(x, y) = \frac{h(x, y)}{h(y, x)}.
$$

The *Eilenberg-MacLane cohomology group* $H_{ab}^3(G, \mathbb{C}^\times)$ is then defined by

$$
H^3_{ab}(G,\mathbb{C}^\times) = Z^3_{ab}(G,\mathbb{C}^\times)/B^3_{ab}(G,\mathbb{C}^\times),
$$

where $Z_{ab}^3(G, \mathbb{C}^\times)$ and $B_{ab}^3(G, \mathbb{C}^\times)$ are respectively the abelian groups of Eilenberg-MacLane 3-cocycles and 3-coboundaries. To $(\omega, c) \in Z_{ab}^3(G, \mathbb{C}^\times)$, one can assign the function $q(x) := c(x, x)$, called its *trace*. It is easy to show that $q(x)$ is a *quadratic form* (or a quadratic function). In other words, we have

(1) $q(x^a) = q(x)^{a^2}$ for any $a \in \mathbb{Z}$, and (2) $b_q(x, y) := \frac{q(xy)}{q(x)q(y)}$ defines a bicharacter of G. We will use the pair (G, q) to denote a quadratic form q on the finite abelian group G. When the group G is clear from the context, we will sometimes simply write q . Note that given two quadratic forms (G, q) and (G', q') , we can define a quadratic form on $G \oplus G'$, denoted by $q \oplus q'$, via the following formula:

$$
(q \oplus q')(x, x') := q(x)q'(x')
$$

for all $(x, x') \in G \oplus G'$. The quadratic form $(G \oplus G', q \oplus q')$ is called the *direct sum* of (G, q) and (G', q') .

We recall a theorem of Eilenberg-MacLane ([\[6\]](#page-9-12) and [\[7\]](#page-9-13)).

Theorem (Eilenberg-MacLane). *The map assigning* (ω, c) to its trace induces an isomor*phism of groups*

$$
H^3_{ab}(G, \mathbb{C}^\times) \xrightarrow{\cong} Q(G, \mathbb{C}^\times)
$$

where $Q(G, \mathbb{C}^\times)$ *is the abelian group of quadratic forms from* G *to* \mathbb{C}^\times *.*

We introduce the following notations before proceeding. Given a group homomorphism $f: G \to G'$ and a positive integer n, we use the standard notation for the n-fold product of f:

$$
f^n: G^n \to (G')^n, f^n(g_1, ..., g_n) := (f(g_1), ..., f(g_n)).
$$

For any *n*-cochain $\mu \in C^n(G', \mathbb{C}^\times)$, we define $f^*(\mu) := \mu \circ f^n$.

Two quadratic forms $q: G \to \mathbb{C}^\times$ and $q': G' \to \mathbb{C}^\times$ are *equivalent* if there exists a group isomorphism $f: G \to G'$ such that $q = f^*(q')$.

Lemma 2.1. $\text{Vec}_G^{(\omega,c)}$ and $\text{Vec}_{G'}^{(\omega',c')}$ are equivalent braided monoidal categories if and only *if the traces of* (ω, c) *and* (ω', c') *are equivalent quadratic forms.*

Proof. If $F : \text{Vec}_G^{(\omega,c)} \to \text{Vec}_{G'}^{(\omega',c')}$ is a braided monoidal equivalence with the natural isomorphism $\mu(x, y) : F(V_x) \otimes F(V_y) \to F(V_x \otimes V_y)$, then F induces a group isomorphism $f: G \to G'$ on simple objects. Moreover, the following diagrams commute:

$$
(F(V_x) \otimes F(V_y)) \otimes F(V_z) \xrightarrow{\mu(x,y) \otimes id} F(V_x \otimes V_y) \otimes F(V_z) \xrightarrow{\mu(xy,z)} F((V_x \otimes V_y) \otimes V_z)
$$

$$
\omega'(f(x), f(y), f(z)) \uparrow \qquad \qquad \uparrow F(\omega(x,y,z))
$$

$$
F(V_x) \otimes (F(V_y) \otimes F(V_z)) \xrightarrow{id \otimes \mu(y,z)} F(V_x) \otimes F(V_y \otimes V_z) \xrightarrow{\mu(x,yz)} F(V_x \otimes (V_y \otimes V_z))
$$

$$
F(V_x) \otimes F(V_y) \xrightarrow{c'(f(x), f(y))} F(V_y) \otimes F(V_x)
$$

$$
\mu(x,y) \downarrow \qquad \qquad \downarrow \mu(y,x)
$$

$$
F(V_x \otimes V_y) \qquad \xrightarrow{F(c(x,y))} \qquad F(V_y \otimes V_x)
$$

Hence $f^*(\omega') = \omega \cdot \delta \mu$ and $f^*(c')(x, y) = c(x, y) \frac{\mu(x, y)}{\mu(y, x)}$ $\frac{\mu(x,y)}{\mu(y,x)}$. Therefore, (ω, c) and $(f^*(\omega'), f^*(c'))$ differ by an Eilenberg-MacLane 3-coboundary. By the theorem of Eilenberg-MacLane, $q = f^{*}(q')$.

Conversely, assume there exists a group isomorphism $f: G \to G'$ such that $q = f^*(q')$. By the theorem of Eilenberg-MacLane, (ω, c) and $(f^*(\omega'), f^*(c'))$ differ by an Eilenberg-MacLane 3-coboundary. In other words, there exists a 2-cochain μ of G such that $f^*(\omega') =$ $\omega \cdot \delta \mu$ and $f^*(c')(x, y) = c(x, y) \frac{\mu(x, y)}{\mu(y, x)}$ $\frac{\mu(x,y)}{\mu(y,x)}$. Define $F(V_x) := V_{f(x)}$ and $\mu(x,y) : F(V_x) \otimes F(V_y) \rightarrow$ $F(V_x \otimes V_y)$, then F together with μ extends to a braided monoidal equivalence between $\text{Vec}_G^{(\omega,c)}$ and $\text{Vec}_{G'}^{(\omega',c')}$ $\left(\begin{matrix} \omega',c' \end{matrix}\right)$.

Remark 2.2. *In the light of the Eilenberg-MacLane Theorem, we will denote any representative in the braided monoidal equivalence class of some* $Vec_G^{(\omega,c)}$ *by* $C(G,q)$ *where* q *is the trace of* (ω, c) *. Then Lemma [2.1](#page-3-0) can be rewritten as follows:* $\mathcal{C}(G, q) \cong \mathcal{C}(G', q')$ *as braided monoidal categories if and only if* q *and* q ′ *are equivalent quadratic forms.*

3. Classification of spherical fusion categories of Frobenius-Schur exponent 2

In this section, we classify spherical fusion categories of Frobenius-Schur exponent 2 up to monoidal equivalence. Let C be such a category. The Frobenius-Schur exponent of $Z(\mathcal{C})$ is then also 2 by Corollary 7.8 of [\[15\]](#page-9-0). Consequently, for any $V \in Ob(\mathcal{C}), \nu_2(V) = \dim_{\mathcal{C}}(V)$. In addition, if V is simple, then $\nu_2(V) = 0, \pm 1$ (cf. Equation [\(2.3\)](#page-1-1)). By [\[9,](#page-9-11) Theorem 2.3], $\dim_{\mathcal{C}}(V) \neq 0$. Hence, we have

(3.1)
$$
\dim_{\mathcal{C}}(V) = \nu_2(V) = \pm 1
$$

for any $V \in \Pi_{\mathcal{C}}$. By Proposition 8.22 of [\[9\]](#page-9-11),

$$
(\text{FP dim}(\mathcal{C}))^2 = \frac{(\dim(\mathcal{C}))^2}{\dim_{\mathcal{Z}(\mathcal{C})}((V, \sigma_V))^2}
$$

for some $(V, \sigma_V) \in \Pi_{Z(C)}$. Since $(V, \sigma_V) \in \Pi_{Z(C)}$ implies that $V \in \Pi_{C}$ [\[14\]](#page-9-8), by Equations (2.1) and (3.1) , we have $(FP \dim(\mathcal{C}))^2 = (\dim(\mathcal{C}))^2$. As both FP dim (\mathcal{C}) and dim (\mathcal{C}) are positive [\[9,](#page-9-11) Theorem 2.3], we have FP dim(\mathcal{C}) = dim(\mathcal{C}). Hence, \mathcal{C} is pseudo-unitary [\[9\]](#page-9-11). By Proposition 8.23 of $[9]$, there exists a unique spherical pivotal structure on C such that $\dim_{\mathcal{C}}(V) = FP \dim_{\mathcal{C}}(V) > 0$ for all $V \in \Pi_{\mathcal{C}}$. Since our classification is up to monoidal equivalence, we can assume without loss of generality that $\mathcal C$ is equipped with its unique spherical pivotal structure described above.

According to Equation [\(3.1\)](#page-4-0), for any $V \in \Pi_{\mathcal{C}}$, V is self-dual. As a result, we have

$$
\dim_{\mathcal{C}}(V \otimes V^*) = \dim_{\mathcal{C}}(V \otimes V) = \dim_{\mathcal{C}}(V)^2 = 1.
$$

By rigidity, pseudo-unitarity and the fact that categorical dimension is a character of the fusion ring, we have $V \otimes V \cong \mathbb{1}$. Therefore, $\Pi_{\mathcal{C}}$ is a group of exponent 2, or $\Pi_{\mathcal{C}} = \mathbb{Z}_2^n$ for some positive integer *n*. As a result, $C = \text{Vec}_{\mathbb{Z}_2^n}^{\omega}$ for some $\omega \in H^3(\mathbb{Z}_2^n, \mathbb{C}^\times)$. By Theorem 9.2 of $[15]$, for any finite group G, we have

$$
FS \exp(\text{Vec}_G^{\omega}) = \text{lcm}_{g \in G} \text{ord}(\omega|_{\langle g \rangle}) \text{ord}(g),
$$

where $\omega|_{\langle g \rangle}$ denotes the restriction of ω on the subgroup generated by g. Since FS exp(C) = 2, we have $\omega|_{\langle x \rangle}$ is trivial for all $x \in \mathbb{Z}_2^n$.

For any $n \in \mathbb{Z}$, consider the map

$$
b: H^3(\mathbb{Z}_2^n, \mathbb{C}^\times) \longrightarrow {\pm 1}^{2^n - 1}
$$

$$
\lambda \mapsto (\dots, \lambda_C, \dots)
$$

where C ranges over the subgroups of \mathbb{Z}_2^n of order 2, and

$$
\lambda_C = \begin{cases} 1 & \text{if the restriction of } \lambda \text{ on } C \text{ is trivial,} \\ -1 & \text{otherwise.} \end{cases}
$$

By [\[12,](#page-9-14) Proposition 2.2], b is injective. Therefore, $\omega|_{\langle x \rangle}$ being trivial for all $x \in \mathbb{Z}_2^n$ implies that ω itself is cohomologous to the trivial 3-cocycle. Let $[\omega]$ be the cohomology class of ω in $H^3(G, \mathbb{C}^\times)$, we have $[\omega] = 1$, and $\text{Vec}_{\mathbb{Z}_2^n}^{\omega}$ is monoidally equivalent to $\text{Vec}_{\mathbb{Z}_2^n}^1$ by a standard argument. Note that the more familiar category of finite dimensional representations of \mathbb{Z}_2^n , denoted by $\text{Rep}(\mathbb{Z}_2^n)$, is nothing but an incarnation of $\text{Vec}_{\mathbb{Z}_2^n}^1$ as a fusion category.

We summarize the above discussion in the following theorem.

Theorem 3.1. *If* C *is a spherical fusion category of Frobenius-Schur exponent 2, then* $\mathcal C$ is pseudo-unitary. Moreover, $\mathcal C$ is monoidally equivalent to $\operatorname{Rep}(\mathbb Z_2^n)$ for some positive *integer* n*.*

Remark 3.2. *We can also obtain this result by the explicit formula of the normalized 3-cocycle* [\[10\]](#page-9-15)*,*

$$
\omega(x, y, z) = \prod_{r=1}^{n} (-1)^{a_r i_r \left[\frac{j_r + k_r}{2}\right]} \prod_{1 \le r < s \le n} (-1)^{a_{rs} k_r \left[\frac{i_s + j_s}{2}\right]} \prod_{1 \le r < s < t \le n} (-1)^{a_{rs} t k_r j_s i_t}
$$

where $x = (i_1, \ldots, i_n)$, $y = (j_1, \ldots, j_n)$, $z = (k_1, \ldots, k_n)$, $i_r, j_r, k_r, a_r, a_{rs}, a_{rst} \in \{0, 1\}$.

$$
\omega(x, x, x) = \prod_{r=1}^{n} (-1)^{a_r i_r^2} \prod_{1 \le r < s \le n} (-1)^{a_{rs} i_r i_s} \prod_{1 \le r < s < t \le n} (-1)^{a_{rst} i_r i_s i_t} = 1.
$$

Take $x = (0, \ldots, 0, 1, 0, \ldots, 0)$ *where* 1 is at the r-th position, we get $a_r = 0$ for $1 \leq$ $r \leq n$. Take $x = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0)$ where the first 1 is at the r-th posi*tion, the second 1 is at the s-th position, we get* $a_{rs} = 0$ *for* $1 \leq r \leq s \leq n$ *. Take* $x = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0)$ where the first 1 is at the r-th position, the second 1 is at the s-th position, the third 1 is at the t-th position, we get $a_{rst} = 0$ for $1 \leq r < s < t \leq n$. Hence $[\omega] = 1$.

4. Classification of modular categories of Frobenius-Schur exponent 2

In this section, we use the result in the previous section to classify modular categories of Frobenius-Schur exponent 2 up to braided monoidal equivalence. Let $\mathcal C$ be such a modular category. By the same argument as in the previous section, $\mathcal C$ is pseudo-unitary, and we will equip C with its canonical spherical pivotal structure such that $\dim_{\mathcal{C}}(V) = \text{FPdim}_{\mathcal{C}}(V) > 0$ for all $V \in \Pi_{\mathcal{C}}$. According to Theorem [3.1,](#page-5-0) \mathcal{C} is equivalent to $\text{Rep}(\mathbb{Z}_2^n)$ as a fusion category for some n. Consequently, as a braided fusion category, $\mathcal{C} \cong \text{Vec}_{\mathbb{Z}_2^n}^{(\omega,c)}$ for some Eilenberg-MacLane 3-cocycle (ω, c) . By the same argument as in the previous section, $[\omega] = 1$.

Therefore, $C \cong \text{Vec}_{\mathbb{Z}_2^n}^{(1,c)}$ with $(1,c)$ an Eilenberg-MacLane 3-cocycle. By Equation (2.4) , we have $c(1, x) = c(x, \tilde{1}) = 1$, and $q(x)^2 = c(x, x)^2 = 1$ for all $x \in \mathbb{Z}_2^n$, in particular, q takes value in $\{\pm 1\}$. Therefore, by definition (cf. Section [2.2\)](#page-2-1), the bilinear form associated to q is given by

$$
b_q: \mathbb{Z}_2^n \oplus \mathbb{Z}_2^n \to \{\pm 1\}, \ b_q(x, y) = \frac{q(xy)}{q(x)q(y)} = c(x, y)c(y, x)
$$

for any $(x, y) \in \mathbb{Z}_2^n \oplus \mathbb{Z}_2^n$. Moreover, since $b_q(x, y)$ is the entry of the S-matrix of C [\[8\]](#page-9-7), the modularity of C then implies that q is a non-degenerate quadratic form. Hence, $\overline{b_q}$ is a non-degenerate alternating form (in particular, $b_q(x,x) = 1$ for any $x \in \mathbb{Z}_2^n$). Therefore, $n = 2m$ is even. Moreover, there exists a symplectic basis $\{e_1, ..., e_m, f_1, ..., f_m\}$

of \mathbb{Z}_2^{2m} , with respect to which $b_q(e_j, e_k) = b_q(f_j, f_k) = 1$, and $b_q(e_j, f_k) = (-1)^{\delta_{j,k}}$ for any $j, k = 1, ..., m$.

For any non-degenerate quadratic form $q : \mathbb{Z}_2^{2m} \to {\pm 1}$, we define its additive version $Q: \mathbb{Z}_2^{2m} \to \mathbb{Z}_2$ such that $(-1)^{Q(x)} = q(x)$ for any $x \in \mathbb{Z}_2^{2m}$. Then the *Arf invariant* of q, denoted by $\text{Arf}(q)$, is given by the classical Arf invariant of Q. More precisely, we have

$$
Arf(q) := Arf(Q) = \sum_{j=1}^{m} Q(e_j)Q(f_j),
$$

where $\{e_1, ..., e_m, f_1, ..., f_m\}$ is the symplectic basis given above. Note that the Arf invariant takes value in \mathbb{Z}_2 , where we use the standard notation $\mathbb{Z}_2 = \{0, 1\}$. We also view \mathbb{Z}_2 as a field here.

Arf showed in [\[1\]](#page-9-16) that the Arf invariant is independent of the choice of basis, and is additive with respect to the direct sum of quadratic forms. More importantly, Arf showed that the dimension $2m$ (of \mathbb{Z}_2^{2m} as a vector space over \mathbb{Z}_2) and the Arf invariant $Arf(q)$ completely determine the equivalence class of a non-degenerate quadratic form (\mathbb{Z}_2^{2m}, q) over \mathbb{Z}_2 . The readers, especially those who are not fluent in German, are highly recommended to consult Appendix 1 of [\[13\]](#page-9-17) for a beautiful exposition of Arf invariant.

As a consequence of Art's theorems, for any positive integer m , there are only two equivalence classes of non-degenerate quadratic forms on \mathbb{Z}_2^{2m} , and they can be obtained as direct sums from two inequivalent quadratic forms on \mathbb{Z}_2^2 . We give explicit representatives for the two equivalence classes of non-degenerate quadratic forms on \mathbb{Z}_2^2 as follows:

(4.1)
$$
q_1: \mathbb{Z}_2^2 \to \{\pm 1\}, \ q_1(x, y) = (-1)^{xy}
$$

and

(4.2)
$$
q_2 : \mathbb{Z}_2^2 \to \{\pm 1\}, q_2(x, y) = (-1)^{x^2 + xy + y^2}
$$

for any $x, y \in \mathbb{Z}_2$. In other words, we have $Q_1(x, y) = xy$ and $Q_2(x, y) = x^2 + xy + y^2$. Therefore, any quadratic form (\mathbb{Z}_2^{2m}, q) is equivalent to $q_1^a \oplus q_2^{m-a}$ for some $a \geq 0$. The presentation of q may not be unique, but they are all equivalent to the representatives given as follows.

By direct computation, we have $\text{Arf}(q_1) = 0$, $\text{Arf}(q_2) = 1$. Therefore, $\text{Arf}(q_1 \oplus q_1) =$ $\text{Arf}(q_2 \oplus q_2) = 0$. Since both $q_1 \oplus q_1$ and $q_2 \oplus q_2$ are quadratic forms on \mathbb{Z}_2^4 , by Arf's theorem, $q_1 \oplus q_1$ is equivalent to $q_2 \oplus q_2$. As a result, if a non-degenerate quadratic form (\mathbb{Z}_2^{2m}, q) is equivalent to $q_1^a \oplus q_2^{m-a}$ for some $a \geq 0$, then its Arf invariant is given by

$$
Arf(q) = \begin{cases} 0, & \text{if } m - a \text{ is even,} \\ 1, & \text{otherwise.} \end{cases}
$$

by the additivity of the Arf invariant. Now that we can change any summand of the form $q_2 \oplus q_2$ into $q_1 \oplus q_1$ without changing the equivalence class of q, we have q is equivalent to q_1^m if Arf $(q) = 0$, and q is equivalent to $q_1^{m-1} \oplus q_2$ if Arf $(q) = 1$. We will assume for the rest of this article, that any non-degenerate quadratic form (\mathbb{Z}_2^{2m}, q) is represented in this way.

Next, we analyze the categorical interpretation of the direct sum of quadratic forms (c.f. Section [2.2\)](#page-2-1). Let (G, q) and (G', q') be two non-degenerate quadratic forms. We consider the Deligne tensor product of the modular categories $\mathcal{C}(G,q)$ and $\mathcal{C}(G',q')$, denoted by $\mathcal{D} := \mathcal{C}(G, q) \boxtimes \mathcal{C}(G', q')$ [\[8\]](#page-9-7). By definition, \mathcal{D} is also a modular category, and its fusion rule is given by the multiplication of the abelian group $G \oplus G'$. Therefore, $\Pi_{\mathcal{D}} = G \oplus G'$, hence $\mathcal{D} \cong \text{Vec}_{G\oplus G'}^{(\omega,c)}$ for some Eilenberg-MacLane 3-cocycle (ω, c) of $G \oplus G'$. Let $p(x) = c(x, x)$ be the corresponding trace. In other words, $\mathcal{D} \cong \mathcal{C}(G \oplus G', p)$.

Let (ω_1, c_1) and (ω_2, c_2) be representatives of the Eilenberg-MacLane 3-cohomology classes corresponding to q and q' respectively. By the definition of the Deligne tensor product, the associativity constraints in $\mathcal D$ is the tensor product of those in $\mathcal C(G, q)$ and $\mathcal{C}(G',q')$. In other words, for any $(x_1,x_2), (y_1,y_2), (z_1,z_2) \in G \oplus G'$, we have

$$
\omega((x_1,x_2),(y_1,y_2),(z_1,z_2))=\omega_1(x_1,y_1,z_1)\omega_2(x_2,y_2,z_2).
$$

Similarly, we have the following equality from the definition of the braiding on $\mathcal D$

$$
c((x_1, x_2), (y_1, y_2)) = c_1(x_1, y_1)c_2(x_2, y_2).
$$

In particular, for any $(x_1, x_2) \in G \oplus G'$, we have

$$
p(x_1, x_2) = q(x_1)q'(x_2) = (q \oplus q')(x_1, x_2).
$$

Therefore, by Lemma [2.1,](#page-3-0) we have $\mathcal{D} \cong \mathcal{C}(G \oplus G', p) \cong \mathcal{C}(G \oplus G', q \oplus q').$

We summarize the above discussion in the following lemma.

Lemma 4.1. $\mathcal{C}(G \oplus G', q \oplus q') \cong \mathcal{C}(G, q) \boxtimes \mathcal{C}(G', q')$ *as modular categories.*

Combining the discussions in this section gives rise to the following classification result.

Theorem 4.2. *If* C *is a modular category of Frobenius-Schur exponent 2, then* C *is pseudounitary, and* C *is braided monoidally equivalent to* $C(\mathbb{Z}_2^{2m}, q)$ *for a positive integer* m *and a non-degenerate quadratic form* q*. Moreover, we have the following Deligne tensor product decomposition*

$$
\mathcal{C} \cong \begin{cases} \mathcal{C}(\mathbb{Z}_2^2, q_1)^{\boxtimes m} & \text{if } \operatorname{Arf}(q) = 0, \\ \\ \mathcal{C}(\mathbb{Z}_2^2, q_1)^{\boxtimes (m-1)} \boxtimes \mathcal{C}(\mathbb{Z}_2^2, q_2) & \text{if } \operatorname{Arf}(q) = 1, \end{cases}
$$

where q_1 *and* q_2 *are given in Equations* (4.1) *and* (4.2) *.*

Remark 4.3. *A braiding of* $C(\mathbb{Z}_2^2, q_1)$ *can be given by*

$$
c_1((x, y), (a, b)) = (-1)^{xb},
$$

and a braiding of $C(\mathbb{Z}_2^2, q_2)$ *can be given by*

$$
c_2((x, y), (a, b)) = (-1)^{xa + yb + ay}.
$$

We would like to interpret the Arf invariant in the modular category setting. Firstly, note that for any non-degenerate quadratic form (\mathbb{Z}_2^{2m}, q) , by direct computation, we have

$$
(-1)^{\text{Arf}(q)} = \frac{1}{\sqrt{|\mathbb{Z}_2^{2m}|}} \sum_{x \in \mathbb{Z}_2^{2m}} q(x) = \frac{1}{2^m} \sum_{x \in \mathbb{Z}_2^{2m}} q(x)
$$

(by Arf's theorems, we only have to check this equality for (\mathbb{Z}_2^2, q_1) and (\mathbb{Z}_2^2, q_2) , which is immediate). In the literature, the above quantity is also referred to as the *Gaussian sum* for the quadratic form q on the finite abelian group \mathbb{Z}_2^{2m} (for example, see [\[20\]](#page-9-18)).

On the category-theoretical side, recall (for example, [\[8\]](#page-9-7)) that the *positive Gauss sum* of a modular category $\mathcal C$ is defined by

$$
\tau_+ := \sum_{X \in \Pi_{\mathcal{C}}} \theta_X \dim_{\mathcal{C}} (X)^2,
$$

where θ_X is the twist of the simple object X. It is standard [\[2\]](#page-9-19) that in a modular category C, the global dimension is the square of the complex absolute value of τ_{+} . In other words,

$$
\dim(\mathcal{C}) = |\tau_+|^2.
$$

The *multiplicative central charge* of C is defined by

$$
\xi := \frac{\tau_+(\mathcal{C})}{\sqrt{\dim(\mathcal{C})}} = \frac{\tau_+}{|\tau_+|}.
$$

Note that $\xi(\mathcal{C})$ is well-defined as $\dim(\mathcal{C})$ is a totally positive algebraic integer [\[8\]](#page-9-7).

In particular, when $\mathcal{C} = \mathcal{C}(\mathbb{Z}_2^{2m}, q)$ for a non-degenerate quadratic form (\mathbb{Z}_2^{2m}, q) , we can compute the dimension m and the Arf invariant Arf (q) of (\mathbb{Z}_2^{2m}, q) by the positive Gauss sum τ_+ as follows. We have $\Pi_{\mathcal{C}} = \mathbb{Z}_2^{2m}$. We also have, for any $x \in \mathbb{Z}_2^{2m}$, that $\dim_{\mathcal{C}}(x) = 1$, hence

(4.3)
$$
|\tau_+|^2 = \dim(\mathcal{C}) = \sum_{x \in \mathbb{Z}_2^{2m}} \dim(\mathcal{C}) = |\mathbb{Z}_2^{2m}| = 2^{2m},
$$

in particular, $|\tau_+| = 2^m$, or $m = \log_2(|\tau_+|)$. Moreover, since for any $x \in \mathbb{Z}_2^{2m}$, $\theta_x = q(x)$ [\[8\]](#page-9-7), we have

(4.4)
$$
\frac{\tau_+}{2^m} = \frac{\tau_+}{|\tau_+|} = \xi(\mathcal{C}(\mathbb{Z}^{2m}, q)) = \frac{1}{\sqrt{|\mathbb{Z}_2^{2m}|}} \sum_{x \in \mathbb{Z}_2^{2m}} q(x) = (-1)^{\text{Arf}(q)}.
$$

Hence, Arf(q) is 0 or 1 depending on whether τ_{+} is positive or negative, respectively.

Conversely, by Equations [\(4.3\)](#page-8-0) and [\(4.4\)](#page-8-1), we have $\tau_{+} = (-1)^{\text{Arf}(q)} 2^m$.

The argument above shows that both the dimension and the Arf invariant of the quadratic form (\mathbb{Z}_2^{2m}, q) are completely determined by positive Gauss sum τ_+ of the modular category $\mathcal{C}(\mathbb{Z}_2^{2m}, q)$ and vice versa.

Recall that by Arf, a non-degenerate quadratic form is completely determined (up to equivalence) by its dimension and its Arf invariant. In the same vein, we restate Theorem [4.2](#page-7-0) as a categorical analog of Arf's theorem.

Theorem 4.4. *If* C *is a modular category of Frobenius-Schur exponent 2, then* C *is pseudounitary, and* C *is completely determined, up to braided monoidal equivalence, by its positive Gauss sum* τ+*. More precisely, we have*

$$
\mathcal{C} \cong \begin{cases} \mathcal{C}(\mathbb{Z}_2^2, q_1)^{\boxtimes \log_2(|\tau_+|)} & \text{if } \tau_+ > 0, \\ \\ \mathcal{C}(\mathbb{Z}_2^2, q_1)^{\boxtimes (\log_2(|\tau_+|)-1)} \boxtimes \mathcal{C}(\mathbb{Z}_2^2, q_2) & \text{if } \tau_+ < 0. \end{cases}
$$

 \Box

Finally, we make a remark on the prime factorization of modular categories.

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A modular category is *non-trivial* if its rank is larger than 1. A non-trivial modular category called a *prime modular category* if it is not braided monoidally equivalent to a Deligne tensor product of two non-trivial modular categories.

A direct consequence of Theorem [4.2](#page-7-0) is that there are only two (pseudo-unitary) prime modular categories of Frobenius-Schur exponent 2. In view of [\[4,](#page-9-2) Lemma 2.4], there are finitely many prime modular categories of Frobenius-Schur exponent 2.

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