CLASSIFICATION OF SPHERICAL FUSION CATEGORIES OF FROBENIUS-SCHUR EXPONENT 2

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ABSTRACT. In this paper, we propose a new approach towards the classification of spherical fusion categories by their Frobenius-Schur exponents. We classify spherical fusion categories of Frobenius-Schur exponent 2 up to monoidal equivalence. We also classify modular categories of Frobenius-Schur exponent 2 up to braided monoidal equivalence. It turns out that the Gauss sum is a complete invariant for modular categories of Frobenius-Schur exponent 2. This result can be viewed as a categorical analog of Arf's theorem on the classification of non-degenerate quadratic forms over fields of characteristic 2.

1. INTRODUCTION

Let \mathcal{C} be a spherical fusion category over \mathbb{C} . The higher Frobenius-Schur indicators $\nu_n(V)$ of $V \in \operatorname{Ob}(\mathcal{C})$ and $n \in \mathbb{Z}$ are generalizations of the classical Frobenius-Schur indicator for irreducible finite group representations (see [15] and the references therein). The Frobenius-Schur indicators are important invariants of a spherical fusion category, especially when the category is in addition non-degenerately braided (in other words, modular). For example, the congruence subgroup conjecture on the $\operatorname{SL}(2,\mathbb{Z})$ representations arising from modular categories can be resolved using generalized Frobenius-Schur indicators [17].

The Frobenius-Schur exponent of a spherical fusion category \mathcal{C} , denoted by $\operatorname{FSexp}(\mathcal{C})$, is the smallest positive integer n such that $\nu_n(V) = \dim_{\mathcal{C}}(V)$ for any object $V \in \operatorname{Ob}(\mathcal{C})$, where $\dim_{\mathcal{C}}(V)$ is the categorical dimension of V in \mathcal{C} . It is shown in [15] that $\operatorname{FSexp}(\mathcal{C})$ is equal to the order of the T-matrix of $Z(\mathcal{C})$, the Drinfeld center of \mathcal{C} . Moreover, the Cauchy theorem for spherical fusion categories asserts that the prime ideals dividing $\operatorname{FSexp}(\mathcal{C})$ and those dividing the global dimension $\dim(\mathcal{C})$ are the same in the ring of algebraic integers [4]. It is then reasonable to pursue a classification of spherical fusion categories by their Frobenius-Schur exponents, as opposed to the usual method of classification by rank [18, 19, 3].

In this paper, we give a full classification of spherical fusion categories of Frobenius-Schur exponent 2. We show that such a spherical fusion category C is equivalent, as a fusion category, to $\operatorname{Rep}(\mathbb{Z}_2^n)$ for some positive integer n. In particular, the associativity constraints of C are all identities. We then show that if C is in addition modular, then C can be decomposed into a Deligne tensor product of two types of modular categories called $C(\mathbb{Z}_2^2, q_1)$ and $C(\mathbb{Z}_2^2, q_2)$. It is worth mentioning that in [5, Theorem 3.2], the authors showed that any modular category of Frobenius-Schur exponent 2 is a braided fusion subcategory of $\operatorname{Rep}(D^{\omega}(\mathbb{Z}_2^{2n}))$ for some positive integer n. In this paper, we completely classify these modular categories by a categorical analog of Arf's theorem on the classification of non-degenerate quadratic forms over fields of characteristic 2. It turns out, in this case, the positive Gauss sum is a complete invariant.

The paper is structured as follows. In Section 2, we give a quick review of basic concepts and set up notations for future use. We also discuss the braided monoidal structure on the category of G-graded vector spaces for a finite abelian group G. In Section 3, we classify spherical fusion categories of Frobenius-Schur exponent 2. Finally, in Section 4, we classify modular categories of Frobenius-Schur exponent 2.

2. Preliminaries

2.1. Basic concepts and notations.

Now let \mathcal{C} be a fusion category over \mathcal{C} over \mathbb{C} [8]. In particular, \mathcal{C} is rigid monoidal, \mathbb{C} linear, semisimple with finitely many isomorphism classes of simple objects such that the tensor unit $\mathbb{1} \in \mathrm{Ob}(\mathcal{C})$ is simple. We fix a choice of representatives from the isomorphism class of simple objects and denote the set of all such representatives by $\Pi_{\mathcal{C}}$. The Frobenius-Perron dimension of $V \in \mathrm{Ob}(\mathcal{C})$, denoted by $\mathrm{FP}\dim_{\mathcal{C}}(V)$, is the largest non-negative eigenvalue of the fusion matrix of V. We define the Frobenius-Perron dimension of \mathcal{C} by $\mathrm{FP}\dim_{\mathcal{C}}(\mathcal{C}) := \sum_{V \in \Pi_{\mathcal{C}}} \mathrm{FP}\dim_{\mathcal{C}}(V)^2$.

A fusion category \mathcal{C} is called *spherical* if it has a pivotal structure such that the left and right pivotal traces coincide on all endomorphisms. In this case, the left (or right) pivotal trace of id_V , the identity of $V \in \mathrm{Ob}(\mathcal{C})$, is called the *categorical dimension* of V. We denote the categorical dimension of $V \in \mathrm{Ob}(\mathcal{C})$ by $\dim_{\mathcal{C}}(V)$, and we define the global dimension \mathcal{C} by $\dim(\mathcal{C}) := \sum_{V \in \Pi_{\mathcal{C}}} \dim_{\mathcal{C}}(V)^2$.

A spherical fusion category admitting a braiding is called a braided spherical fusion category (or premodular category). A braided spherical fusion category is called *modular* if the braiding is non-degenerate, or equivalently, if its S-matrix is non-degenerate [14]. For example, $Z(\mathcal{C})$, the Drinfeld center of a spherical fusion category \mathcal{C} , is modular [14]. Objects of $Z(\mathcal{C})$ are pairs (X, σ_X) , where $X \in Ob(\mathcal{C})$ and $\sigma_X : X \otimes - \xrightarrow{\sim} - \otimes X$ is a half braiding. Since the pivotal structure of $Z(\mathcal{C})$ is inherited from \mathcal{C} , we have

(2.1)
$$\dim_{Z(\mathcal{C})}(V,\sigma_V) = \dim_{\mathcal{C}}(V)$$

for any $V \in Ob(\mathcal{C})$.

Let \mathcal{C} be a spherical fusion category. For any $n \in \mathbb{Z}$, and for any $V \in \mathrm{Ob}(\mathcal{C})$, the *n*-th Frobenius-Schur indicator ν_n of V is defined to be the operator trace of a linear operator $E_V^{(n)} : \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, V^{\otimes n}) \to \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, V^{\otimes n})$ satisfying $(E_V^{(n)})^n = \mathrm{id}$. Here, $V^{\otimes n}$ is understood as inductively defined by $V^{\otimes (m+1)} = V \otimes V^{\otimes m}$ for $1 \leq m < n$, and associativity constraints are included in the definition of $E_V^{(n)}$, see [16]. In particular, if V is simple, then

(2.2)
$$\nu_1(V) = \delta_{\mathbb{1},V}$$

We also have

(2.3)
$$\nu_2(V) = 0$$
, if $V \not\cong V^*$, $\nu_2(V) = 1$ or -1 , if $V \cong V^*$

for all $V \in \Pi_{\mathcal{C}}$.

The Frobenius-Schur exponent of an object V in a spherical fusion category \mathcal{C} , denoted by $\operatorname{FSexp}(V)$, is defined to be the smallest positive integer n such that $\nu_n(V) = \dim_{\mathcal{C}}(V)$. The Frobenius-Schur exponent of \mathcal{C} , denoted by $\operatorname{FSexp}(\mathcal{C})$, is defined to be the smallest positive integer n such that $\nu_n(V) = \dim_{\mathcal{C}}(V)$ for all $V \in \mathcal{C}$ [15]. When \mathcal{C} is the category of finite dimensional H-modules for a semisimple Hopf algebra H over \mathbb{C} , $\operatorname{FSexp}(V)$ is equal to the exponent of V as a finite dimensional H-module. In other words, FSexp(V) is equal to the exponent of the image of G in $GL(V, \mathbb{C})$ [11].

It is immediate from the definition and Equation (2.2) that if \mathcal{C} is a spherical fusion category such that $\operatorname{FSexp}(\mathcal{C}) = 1$, then for any $V \in \Pi_{\mathcal{C}}$, $\dim_{\mathcal{C}}(V) = \delta_{\mathbb{1},V}$. According to [9, Theorem 2.3], $\dim_{\mathcal{C}}(V) \neq 0$ for all $V \in \Pi_{\mathcal{C}}$, hence \mathcal{C} has the tensor unit $\mathbb{1}$ as its only simple object. Therefore, \mathcal{C} is monoidally equivalent to $\operatorname{Vec}_{\mathbb{C}}$, the category of finite dimensional vector spaces over \mathbb{C} .

It is worth mentioning that by [15], for any $V \in Ob(\mathcal{C})$, FSexp(V) does not depend on the choice of pivotal structures. In addition, $FSexp(\mathcal{C})$ of a spherical fusion category \mathcal{C} depends only on the equivalence class of the modular category $Z(\mathcal{C})$.

2.2. Braided monoidal structure on G-graded vector spaces.

Let G be a finite multiplicative abelian group. Recall that the category $\operatorname{Vec}_G^{\omega}$ of finite-dimensional G-graded vector spaces has simple objects $\{V_g | g \in G\}$ where $(V_g)_h = \delta_{g,h}\mathbb{C}, \forall h \in G$. The tensor product is given by $V_g \otimes V_h = V_{gh}$, and the tensor unit is V_1 , where 1 is the identity of G. The associator is given by a normalized 3-cocycle $\omega \in Z^3(G, \mathbb{C}^{\times})$

$$\omega(x, y, z): V_x \otimes (V_y \otimes V_z) \longrightarrow (V_x \otimes V_y) \otimes V_z.$$

Now we equip $\operatorname{Vec}_G^{\omega}$ with a braiding given by a normalized 2-cochain $c \in C^2(G, \mathbb{C}^{\times})$

$$c(x,y): V_x \otimes V_y \longrightarrow V_y \otimes V_x$$

satisfying the hexagon axioms

(2.4)
$$\frac{c(xy,z)}{c(x,z)c(y,z)}\frac{\omega(x,y,z)\omega(z,x,y)}{\omega(x,z,y)} = 1 = \frac{c(x,yz)}{c(x,y)c(x,z)}\frac{\omega(y,x,z)}{\omega(x,y,z)\omega(y,z,x)}$$

for all $x, y, z \in G$. In other words, the pair (ω, c) is an *Eilenberg-MacLane 3-cocycle* of G. Finally, we equip $\operatorname{Vec}_G^{\omega}$ with the canonical (spherical) pivotal structure, which is simply given by identities on objects, so that the categorical dimensions are all positive. We denote this braided spherical fusion category by $\operatorname{Vec}_G^{(\omega,c)}$.

An Eilenberg-MacLane 3-cocycle (ω, c) is called a *coboundary* if there exists a 2-cochain $h \in C^2(G, \mathbb{C}^{\times})$ such that

(2.5)
$$\omega = \delta h \text{ and } c(x,y) = \frac{h(x,y)}{h(y,x)}$$

The Eilenberg-MacLane cohomology group $H^3_{ab}(G, \mathbb{C}^{\times})$ is then defined by

$$H^3_{ab}(G,\mathbb{C}^\times)=Z^3_{ab}(G,\mathbb{C}^\times)/B^3_{ab}(G,\mathbb{C}^\times),$$

where $Z^3_{ab}(G, \mathbb{C}^{\times})$ and $B^3_{ab}(G, \mathbb{C}^{\times})$ are respectively the abelian groups of Eilenberg-MacLane 3-cocycles and 3-coboundaries. To $(\omega, c) \in Z^3_{ab}(G, \mathbb{C}^{\times})$, one can assign the function q(x) := c(x, x), called its *trace*. It is easy to show that q(x) is a *quadratic form* (or a quadratic function). In other words, we have

(1) $q(x^a) = q(x)^{a^2}$ for any $a \in \mathbb{Z}$, and (2) $b_q(x,y) := \frac{q(xy)}{q(x)q(y)}$ defines a bicharacter of G. We will use the pair (G, q) to denote a quadratic form q on the finite abelian group G. When the group G is clear from the context, we will sometimes simply write q. Note that given two quadratic forms (G, q) and (G', q'), we can define a quadratic form on $G \oplus G'$, denoted by $q \oplus q'$, via the following formula:

$$(q \oplus q')(x, x') := q(x)q'(x')$$

for all $(x, x') \in G \oplus G'$. The quadratic form $(G \oplus G', q \oplus q')$ is called the *direct sum* of (G, q) and (G', q').

We recall a theorem of Eilenberg-MacLane ([6] and [7]).

Theorem (Eilenberg-MacLane). The map assigning (ω, c) to its trace induces an isomorphism of groups

$$H^3_{ab}(G, \mathbb{C}^{\times}) \xrightarrow{\cong} Q(G, \mathbb{C}^{\times})$$

where $Q(G, \mathbb{C}^{\times})$ is the abelian group of quadratic forms from G to \mathbb{C}^{\times} .

We introduce the following notations before proceeding. Given a group homomorphism $f: G \to G'$ and a positive integer n, we use the standard notation for the *n*-fold product of f:

$$f^n: G^n \to (G')^n, \ f^n(g_1, ..., g_n) := (f(g_1), ..., f(g_n)).$$

For any *n*-cochain $\mu \in C^n(G', \mathbb{C}^{\times})$, we define $f^*(\mu) := \mu \circ f^n$.

Two quadratic forms $q: G \to \mathbb{C}^{\times}$ and $q': G' \to \mathbb{C}^{\times}$ are *equivalent* if there exists a group isomorphism $f: G \to G'$ such that $q = f^*(q')$.

Lemma 2.1. $\operatorname{Vec}_{G}^{(\omega,c)}$ and $\operatorname{Vec}_{G'}^{(\omega',c')}$ are equivalent braided monoidal categories if and only if the traces of (ω,c) and (ω',c') are equivalent quadratic forms.

Proof. If $F : \operatorname{Vec}_{G}^{(\omega,c)} \to \operatorname{Vec}_{G'}^{(\omega',c')}$ is a braided monoidal equivalence with the natural isomorphism $\mu(x,y) : F(V_x) \otimes F(V_y) \to F(V_x \otimes V_y)$, then F induces a group isomorphism $f : G \to G'$ on simple objects. Moreover, the following diagrams commute:

$$\begin{array}{cccc} (F(V_x) \otimes F(V_y)) \otimes F(V_z) & \xrightarrow{\mu(x,y) \otimes \mathrm{Id}} & F(V_x \otimes V_y) \otimes F(V_z) & \xrightarrow{\mu(xy,z)} & F((V_x \otimes V_y) \otimes V_z) \\ & & & \downarrow^{\prime}(f(x), f(y), f(z)) \uparrow & & \uparrow^{F(\omega(x,y,z))} \\ & & & \uparrow^{F(\omega(x,y,z))} \\ & & & F(V_x) \otimes (F(V_y) \otimes F(V_z)) & \xrightarrow{\mathrm{id} \otimes \mu(y,z)} & F(V_x) \otimes F(V_y \otimes V_z) & \xrightarrow{\mu(x,yz)} & F(V_x \otimes (V_y \otimes V_z)) \\ & & & & F(V_x) \otimes F(V_y) & \xrightarrow{c'(f(x), f(y))} & F(V_y) \otimes F(V_x) \\ & & & & & \downarrow^{\mu(y,x)} \end{array}$$

$$F(V_x \otimes V_y) \xrightarrow{F(c(x,y))} F(V_y \otimes V_x)$$

Hence $f^*(\omega') = \omega \cdot \delta \mu$ and $f^*(c')(x, y) = c(x, y) \frac{\mu(x, y)}{\mu(y, x)}$. Therefore, (ω, c) and $(f^*(\omega'), f^*(c'))$ differ by an Eilenberg-MacLane 3-coboundary. By the theorem of Eilenberg-MacLane, $q = f^*(q')$.

Conversely, assume there exists a group isomorphism $f: G \to G'$ such that $q = f^*(q')$. By the theorem of Eilenberg-MacLane, (ω, c) and $(f^*(\omega'), f^*(c'))$ differ by an Eilenberg-MacLane 3-coboundary. In other words, there exists a 2-cochain μ of G such that $f^*(\omega') = \omega \cdot \delta \mu$ and $f^*(c')(x, y) = c(x, y) \frac{\mu(x, y)}{\mu(y, x)}$. Define $F(V_x) := V_{f(x)}$ and $\mu(x, y) : F(V_x) \otimes F(V_y) \to 0$ $F(V_x \otimes V_y)$, then F together with μ extends to a braided monoidal equivalence between $\operatorname{Vec}_G^{(\omega,c)}$ and $\operatorname{Vec}_{C'}^{(\omega',c')}$.

Remark 2.2. In the light of the Eilenberg-MacLane Theorem, we will denote any representative in the braided monoidal equivalence class of some $\operatorname{Vec}_{G}^{(\omega,c)}$ by $\mathcal{C}(G,q)$ where q is the trace of (ω,c) . Then Lemma 2.1 can be rewritten as follows: $\mathcal{C}(G,q) \cong \mathcal{C}(G',q')$ as braided monoidal categories if and only if q and q' are equivalent quadratic forms.

3. Classification of spherical fusion categories of Frobenius-Schur exponent 2

In this section, we classify spherical fusion categories of Frobenius-Schur exponent 2 up to monoidal equivalence. Let \mathcal{C} be such a category. The Frobenius-Schur exponent of $Z(\mathcal{C})$ is then also 2 by Corollary 7.8 of [15]. Consequently, for any $V \in Ob(\mathcal{C})$, $\nu_2(V) = \dim_{\mathcal{C}}(V)$. In addition, if V is simple, then $\nu_2(V) = 0, \pm 1$ (cf. Equation (2.3)). By [9, Theorem 2.3], $\dim_{\mathcal{C}}(V) \neq 0$. Hence, we have

(3.1)
$$\dim_{\mathcal{C}}(V) = \nu_2(V) = \pm 1$$

for any $V \in \Pi_{\mathcal{C}}$. By Proposition 8.22 of [9],

$$(\operatorname{FP}\dim(\mathcal{C}))^2 = \frac{(\dim(\mathcal{C}))^2}{\dim_{Z(\mathcal{C})}((V,\sigma_V))^2}$$

for some $(V, \sigma_V) \in \Pi_{Z(\mathcal{C})}$. Since $(V, \sigma_V) \in \Pi_{Z(\mathcal{C})}$ implies that $V \in \Pi_{\mathcal{C}}$ [14], by Equations (2.1) and (3.1), we have $(\operatorname{FP}\dim(\mathcal{C}))^2 = (\dim(\mathcal{C}))^2$. As both $\operatorname{FP}\dim(\mathcal{C})$ and $\dim(\mathcal{C})$ are positive [9, Theorem 2.3], we have $\operatorname{FP}\dim(\mathcal{C}) = \dim(\mathcal{C})$. Hence, \mathcal{C} is pseudo-unitary [9]. By Proposition 8.23 of [9], there exists a unique spherical pivotal structure on \mathcal{C} such that $\dim_{\mathcal{C}}(V) = \operatorname{FP}\dim_{\mathcal{C}}(V) > 0$ for all $V \in \Pi_{\mathcal{C}}$. Since our classification is up to monoidal equivalence, we can assume without loss of generality that \mathcal{C} is equipped with its unique spherical pivotal structure described above.

According to Equation (3.1), for any $V \in \Pi_{\mathcal{C}}$, V is self-dual. As a result, we have

$$\dim_{\mathcal{C}}(V \otimes V^*) = \dim_{\mathcal{C}}(V \otimes V) = \dim_{\mathcal{C}}(V)^2 = 1.$$

By rigidity, pseudo-unitarity and the fact that categorical dimension is a character of the fusion ring, we have $V \otimes V \cong 1$. Therefore, $\Pi_{\mathcal{C}}$ is a group of exponent 2, or $\Pi_{\mathcal{C}} = \mathbb{Z}_2^n$ for some positive integer *n*. As a result, $\mathcal{C} = \operatorname{Vec}_{\mathbb{Z}_2^n}^{\omega}$ for some $\omega \in H^3(\mathbb{Z}_2^n, \mathbb{C}^{\times})$. By Theorem 9.2 of [15], for any finite group *G*, we have

$$FS \exp(\operatorname{Vec}_{G}^{\omega}) = \operatorname{lcm}_{q \in G} \operatorname{ord}(\omega|_{\langle q \rangle}) \operatorname{ord}(g),$$

where $\omega|_{\langle g \rangle}$ denotes the restriction of ω on the subgroup generated by g. Since FS exp(\mathcal{C}) = 2, we have $\omega|_{\langle x \rangle}$ is trivial for all $x \in \mathbb{Z}_2^n$.

For any $n \in \mathbb{Z}$, consider the map

$$b: H^{3}(\mathbb{Z}_{2}^{n}, \mathbb{C}^{\times}) \longrightarrow \{\pm 1\}^{2^{n}-1}$$
$$\lambda \mapsto (..., \lambda_{C}, ...)$$

where C ranges over the subgroups of \mathbb{Z}_2^n of order 2, and

$$\lambda_C = \begin{cases} 1 & \text{if the restriction of } \lambda \text{ on } C \text{ is trivial,} \\ -1 & \text{otherwise.} \end{cases}$$

By [12, Proposition 2.2], b is injective. Therefore, $\omega|_{\langle x \rangle}$ being trivial for all $x \in \mathbb{Z}_2^n$ implies that ω itself is cohomologous to the trivial 3-cocycle. Let $[\omega]$ be the cohomology class of ω in $H^3(G, \mathbb{C}^{\times})$, we have $[\omega] = 1$, and $\operatorname{Vec}_{\mathbb{Z}_2^n}^{\omega}$ is monoidally equivalent to $\operatorname{Vec}_{\mathbb{Z}_2^n}^1$ by a standard argument. Note that the more familiar category of finite dimensional representations of \mathbb{Z}_2^n , denoted by $\operatorname{Rep}(\mathbb{Z}_2^n)$, is nothing but an incarnation of $\operatorname{Vec}_{\mathbb{Z}_2^n}^1$ as a fusion category.

We summarize the above discussion in the following theorem.

Theorem 3.1. If C is a spherical fusion category of Frobenius-Schur exponent 2, then C is pseudo-unitary. Moreover, C is monoidally equivalent to $\operatorname{Rep}(\mathbb{Z}_2^n)$ for some positive integer n.

Remark 3.2. We can also obtain this result by the explicit formula of the normalized 3-cocycle [10],

$$\omega(x,y,z) = \prod_{r=1}^{n} (-1)^{a_r i_r [\frac{j_r + k_r}{2}]} \prod_{1 \le r < s \le n} (-1)^{a_{rs} k_r [\frac{i_s + j_s}{2}]} \prod_{1 \le r < s < t \le n} (-1)^{a_{rst} k_r j_s i_t}$$

where $x = (i_1, \dots, i_n), y = (j_1, \dots, j_n), z = (k_1, \dots, k_n), i_r, j_r, k_r, a_r, a_{rs}, a_{rst} \in \{0, 1\}.$

$$\omega(x,x,x) = \prod_{r=1}^{n} (-1)^{a_r i_r^2} \prod_{1 \le r < s \le n} (-1)^{a_{rs} i_r i_s} \prod_{1 \le r < s < t \le n} (-1)^{a_{rst} i_r i_s i_t} = 1.$$

Take $x = (0, \ldots, 0, 1, 0, \ldots, 0)$ where 1 is at the r-th position, we get $a_r = 0$ for $1 \leq r \leq n$. Take $x = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0)$ where the first 1 is at the r-th position, the second 1 is at the s-th position, we get $a_{rs} = 0$ for $1 \leq r < s \leq n$. Take $x = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0)$ where the first 1 is at the r-th position, the second 1 is at the s-th position, the third 1 is at the t-th position, we get $a_{rst} = 0$ for $1 \leq r < s \leq n$. Take $1 \leq r < s < t \leq n$. Hence $[\omega] = 1$.

4. Classification of modular categories of Frobenius-Schur exponent 2

In this section, we use the result in the previous section to classify modular categories of Frobenius-Schur exponent 2 up to braided monoidal equivalence. Let \mathcal{C} be such a modular category. By the same argument as in the previous section, \mathcal{C} is pseudo-unitary, and we will equip \mathcal{C} with its canonical spherical pivotal structure such that $\dim_{\mathcal{C}}(V) = \operatorname{FPdim}_{\mathcal{C}}(V) > 0$ for all $V \in \Pi_{\mathcal{C}}$. According to Theorem 3.1, \mathcal{C} is equivalent to $\operatorname{Rep}(\mathbb{Z}_2^n)$ as a fusion category for some n. Consequently, as a braided fusion category, $\mathcal{C} \cong \operatorname{Vec}_{\mathbb{Z}_2^n}^{(\omega,c)}$ for some Eilenberg-MacLane 3-cocycle (ω, c) . By the same argument as in the previous section, $[\omega] = 1$.

Therefore, $C \cong \operatorname{Vec}_{\mathbb{Z}_2^n}^{(1,c)}$ with (1,c) an Eilenberg-MacLane 3-cocycle. By Equation (2.4), we have c(1,x) = c(x,1) = 1, and $q(x)^2 = c(x,x)^2 = 1$ for all $x \in \mathbb{Z}_2^n$, in particular, q takes value in $\{\pm 1\}$. Therefore, by definition (cf. Section 2.2), the bilinear form associated to q is given by

$$b_q: \mathbb{Z}_2^n \oplus \mathbb{Z}_2^n \to \{\pm 1\}, \ b_q(x,y) = \frac{q(xy)}{q(x)q(y)} = c(x,y)c(y,x)$$

for any $(x, y) \in \mathbb{Z}_2^n \oplus \mathbb{Z}_2^n$. Moreover, since $b_q(x, y)$ is the entry of the S-matrix of \mathcal{C} [8], the modularity of \mathcal{C} then implies that q is a non-degenerate quadratic form. Hence, b_q is a non-degenerate alternating form (in particular, $b_q(x, x) = 1$ for any $x \in \mathbb{Z}_2^n$). Therefore, n = 2m is even. Moreover, there exists a symplectic basis $\{e_1, ..., e_m, f_1, ..., f_m\}$ of \mathbb{Z}_2^{2m} , with respect to which $b_q(e_j, e_k) = b_q(f_j, f_k) = 1$, and $b_q(e_j, f_k) = (-1)^{\delta_{j,k}}$ for any j, k = 1, ..., m.

For any non-degenerate quadratic form $q: \mathbb{Z}_2^{2m} \to \{\pm 1\}$, we define its additive version $Q: \mathbb{Z}_2^{2m} \to \mathbb{Z}_2$ such that $(-1)^{Q(x)} = q(x)$ for any $x \in \mathbb{Z}_2^{2m}$. Then the Arf invariant of q, denoted by $\operatorname{Arf}(q)$, is given by the classical Arf invariant of Q. More precisely, we have

$$\operatorname{Arf}(q) := \operatorname{Arf}(Q) = \sum_{j=1}^{m} Q(e_j)Q(f_j),$$

where $\{e_1, ..., e_m, f_1, ..., f_m\}$ is the symplectic basis given above. Note that the Arf invariant takes value in \mathbb{Z}_2 , where we use the standard notation $\mathbb{Z}_2 = \{0, 1\}$. We also view \mathbb{Z}_2 as a field here.

Arf showed in [1] that the Arf invariant is independent of the choice of basis, and is additive with respect to the direct sum of quadratic forms. More importantly, Arf showed that the dimension 2m (of \mathbb{Z}_2^{2m} as a vector space over \mathbb{Z}_2) and the Arf invariant Arf(q) completely determine the equivalence class of a non-degenerate quadratic form (\mathbb{Z}_2^{2m}, q) over \mathbb{Z}_2 . The readers, especially those who are not fluent in German, are highly recommended to consult Appendix 1 of [13] for a beautiful exposition of Arf invariant.

As a consequence of Arf's theorems, for any positive integer m, there are only two equivalence classes of non-degenerate quadratic forms on \mathbb{Z}_2^{2m} , and they can be obtained as direct sums from two inequivalent quadratic forms on \mathbb{Z}_2^2 . We give explicit representatives for the two equivalence classes of non-degenerate quadratic forms on \mathbb{Z}_2^2 as follows:

(4.1)
$$q_1: \mathbb{Z}_2^2 \to \{\pm 1\}, \ q_1(x,y) = (-1)^{xy}$$

and

(4.2)
$$q_2: \mathbb{Z}_2^2 \to \{\pm 1\}, \ q_2(x,y) = (-1)^{x^2 + xy + y^2}$$

for any $x, y \in \mathbb{Z}_2$. In other words, we have $Q_1(x, y) = xy$ and $Q_2(x, y) = x^2 + xy + y^2$. Therefore, any quadratic form (\mathbb{Z}_2^{2m}, q) is equivalent to $q_1^a \oplus q_2^{m-a}$ for some $a \ge 0$. The presentation of q may not be unique, but they are all equivalent to the representatives given as follows.

By direct computation, we have $\operatorname{Arf}(q_1) = 0$, $\operatorname{Arf}(q_2) = 1$. Therefore, $\operatorname{Arf}(q_1 \oplus q_1) = \operatorname{Arf}(q_2 \oplus q_2) = 0$. Since both $q_1 \oplus q_1$ and $q_2 \oplus q_2$ are quadratic forms on \mathbb{Z}_2^4 , by Arf's theorem, $q_1 \oplus q_1$ is equivalent to $q_2 \oplus q_2$. As a result, if a non-degenerate quadratic form (\mathbb{Z}_2^{2m}, q) is equivalent to $q_1^a \oplus q_2^{m-a}$ for some $a \ge 0$, then its Arf invariant is given by

$$\operatorname{Arf}(q) = \begin{cases} 0, & \text{if } m - a \text{ is even,} \\ 1, & \text{otherwise.} \end{cases}$$

by the additivity of the Arf invariant. Now that we can change any summand of the form $q_2 \oplus q_2$ into $q_1 \oplus q_1$ without changing the equivalence class of q, we have q is equivalent to q_1^m if $\operatorname{Arf}(q) = 0$, and q is equivalent to $q_1^{m-1} \oplus q_2$ if $\operatorname{Arf}(q) = 1$. We will assume for the rest of this article, that any non-degenerate quadratic form (\mathbb{Z}_2^{2m}, q) is represented in this way.

Next, we analyze the categorical interpretation of the direct sum of quadratic forms (c.f. Section 2.2). Let (G, q) and (G', q') be two non-degenerate quadratic forms. We consider the Deligne tensor product of the modular categories $\mathcal{C}(G, q)$ and $\mathcal{C}(G', q')$, denoted by $\mathcal{D} := \mathcal{C}(G, q) \boxtimes \mathcal{C}(G', q')$ [8]. By definition, \mathcal{D} is also a modular category, and its fusion rule is given by the multiplication of the abelian group $G \oplus G'$. Therefore, $\Pi_{\mathcal{D}} = G \oplus G'$, hence $\mathcal{D} \cong \operatorname{Vec}_{G \oplus G'}^{(\omega, c)}$ for some Eilenberg-MacLane 3-cocycle (ω, c) of $G \oplus G'$. Let p(x) = c(x, x) be the corresponding trace. In other words, $\mathcal{D} \cong \mathcal{C}(G \oplus G', p)$.

Let (ω_1, c_1) and (ω_2, c_2) be representatives of the Eilenberg-MacLane 3-cohomology classes corresponding to q and q' respectively. By the definition of the Deligne tensor product, the associativity constraints in \mathcal{D} is the tensor product of those in $\mathcal{C}(G,q)$ and $\mathcal{C}(G',q')$. In other words, for any $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in G \oplus G'$, we have

$$\omega((x_1, x_2), (y_1, y_2), (z_1, z_2)) = \omega_1(x_1, y_1, z_1)\omega_2(x_2, y_2, z_2).$$

Similarly, we have the following equality from the definition of the braiding on \mathcal{D}

$$c((x_1, x_2), (y_1, y_2)) = c_1(x_1, y_1)c_2(x_2, y_2).$$

In particular, for any $(x_1, x_2) \in G \oplus G'$, we have

$$p(x_1, x_2) = q(x_1)q'(x_2) = (q \oplus q')(x_1, x_2).$$

Therefore, by Lemma 2.1, we have $\mathcal{D} \cong \mathcal{C}(G \oplus G', p) \cong \mathcal{C}(G \oplus G', q \oplus q')$.

We summarize the above discussion in the following lemma.

Lemma 4.1. $\mathcal{C}(G \oplus G', q \oplus q') \cong \mathcal{C}(G, q) \boxtimes \mathcal{C}(G', q')$ as modular categories.

Combining the discussions in this section gives rise to the following classification result.

Theorem 4.2. If C is a modular category of Frobenius-Schur exponent 2, then C is pseudounitary, and C is braided monoidally equivalent to $C(\mathbb{Z}_2^{2m}, q)$ for a positive integer m and a non-degenerate quadratic form q. Moreover, we have the following Deligne tensor product decomposition

$$\mathcal{C} \cong \begin{cases} \mathcal{C}(\mathbb{Z}_2^2, q_1)^{\boxtimes m} & \text{if } \operatorname{Arf}(q) = 0, \\ \\ \mathcal{C}(\mathbb{Z}_2^2, q_1)^{\boxtimes (m-1)} \boxtimes \mathcal{C}(\mathbb{Z}_2^2, q_2) & \text{if } \operatorname{Arf}(q) = 1, \end{cases}$$

where q_1 and q_2 are given in Equations (4.1) and (4.2).

Remark 4.3. A braiding of $\mathcal{C}(\mathbb{Z}_2^2, q_1)$ can be given by

$$c_1((x,y),(a,b)) = (-1)^{xb},$$

and a braiding of $\mathcal{C}(\mathbb{Z}_2^2, q_2)$ can be given by

$$c_2((x,y),(a,b)) = (-1)^{xa+yb+ay}.$$

We would like to interpret the Arf invariant in the modular category setting. Firstly, note that for any non-degenerate quadratic form (\mathbb{Z}_2^{2m}, q) , by direct computation, we have

$$(-1)^{\operatorname{Arf}(q)} = \frac{1}{\sqrt{|\mathbb{Z}_2^{2m}|}} \sum_{x \in \mathbb{Z}_2^{2m}} q(x) = \frac{1}{2^m} \sum_{x \in \mathbb{Z}_2^{2m}} q(x)$$

(by Arf's theorems, we only have to check this equality for (\mathbb{Z}_2^2, q_1) and (\mathbb{Z}_2^2, q_2) , which is immediate). In the literature, the above quantity is also referred to as the *Gaussian sum* for the quadratic form q on the finite abelian group \mathbb{Z}_2^{2m} (for example, see [20]).

On the category-theoretical side, recall (for example, [8]) that the *positive Gauss sum* of a modular category C is defined by

$$\tau_+ := \sum_{X \in \Pi_{\mathcal{C}}} \theta_X \dim_{\mathcal{C}} (X)^2,$$

where θ_X is the twist of the simple object X. It is standard [2] that in a modular category C, the global dimension is the square of the complex absolute value of τ_+ . In other words,

$$\dim(\mathcal{C}) = |\tau_+|^2.$$

The multiplicative central charge of \mathcal{C} is defined by

$$\xi := \frac{\tau_+(\mathcal{C})}{\sqrt{\dim(\mathcal{C})}} = \frac{\tau_+}{|\tau_+|}.$$

Note that $\xi(\mathcal{C})$ is well-defined as dim (\mathcal{C}) is a totally positive algebraic integer [8].

In particular, when $C = C(\mathbb{Z}_2^{2m}, q)$ for a non-degenerate quadratic form (\mathbb{Z}_2^{2m}, q) , we can compute the dimension m and the Arf invariant $\operatorname{Arf}(q)$ of (\mathbb{Z}_2^{2m}, q) by the positive Gauss sum τ_+ as follows. We have $\Pi_{\mathcal{C}} = \mathbb{Z}_2^{2m}$. We also have, for any $x \in \mathbb{Z}_2^{2m}$, that $\dim_{\mathcal{C}}(x) = 1$, hence

(4.3)
$$|\tau_+|^2 = \dim(\mathcal{C}) = \sum_{x \in \mathbb{Z}_2^{2m}} \dim_{\mathcal{C}}(x)^2 = |\mathbb{Z}_2^{2m}| = 2^{2m},$$

in particular, $|\tau_+| = 2^m$, or $m = \log_2(|\tau_+|)$. Moreover, since for any $x \in \mathbb{Z}_2^{2m}$, $\theta_x = q(x)$ [8], we have

(4.4)
$$\frac{\tau_+}{2^m} = \frac{\tau_+}{|\tau_+|} = \xi(\mathcal{C}(\mathbb{Z}^{2m}, q)) = \frac{1}{\sqrt{|\mathbb{Z}_2^{2m}|}} \sum_{x \in \mathbb{Z}_2^{2m}} q(x) = (-1)^{\operatorname{Arf}(q)}$$

Hence, $\operatorname{Arf}(q)$ is 0 or 1 depending on whether τ_+ is positive or negative, respectively.

Conversely, by Equations (4.3) and (4.4), we have $\tau_{+} = (-1)^{\operatorname{Arf}(q)} 2^{m}$.

The argument above shows that both the dimension and the Arf invariant of the quadratic form (\mathbb{Z}_2^{2m}, q) are completely determined by positive Gauss sum τ_+ of the modular category $\mathcal{C}(\mathbb{Z}_2^{2m}, q)$ and vice versa.

Recall that by Arf, a non-degenerate quadratic form is completely determined (up to equivalence) by its dimension and its Arf invariant. In the same vein, we restate Theorem 4.2 as a categorical analog of Arf's theorem.

Theorem 4.4. If C is a modular category of Frobenius-Schur exponent 2, then C is pseudounitary, and C is completely determined, up to braided monoidal equivalence, by its positive Gauss sum τ_+ . More precisely, we have

$$\mathcal{C} \cong \begin{cases} \mathcal{C}(\mathbb{Z}_2^2, q_1)^{\boxtimes \log_2(|\tau_+|)} & \text{if } \tau_+ > 0, \\ \\ \mathcal{C}(\mathbb{Z}_2^2, q_1)^{\boxtimes (\log_2(|\tau_+|) - 1)} \boxtimes \mathcal{C}(\mathbb{Z}_2^2, q_2) & \text{if } \tau_+ < 0. \end{cases}$$

Finally, we make a remark on the prime factorization of modular categories.

ZHEYAN WAN AND YILONG WANG

A modular category is *non-trivial* if its rank is larger than 1. A non-trivial modular category called a *prime modular category* if it is not braided monoidally equivalent to a Deligne tensor product of two non-trivial modular categories.

A direct consequence of Theorem 4.2 is that there are only two (pseudo-unitary) prime modular categories of Frobenius-Schur exponent 2. In view of [4, Lemma 2.4], there are finitely many prime modular categories of Frobenius-Schur exponent 2.

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10

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