Existence and multiplicity results for some generalized Hammerstein equations with a parameter *

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Keywords: Hammerstein equations, nonlinear boundary value problems, parameter dependence, degree theory, fixed points in cones. **MSC:** 34B08, 34B10, 34B15, 34B18, 34B27.

Abstract

This paper considers the existence and multiplicity of fixed points for the integral operator

$$\mathcal{T}u(t) = \lambda \int_0^T k(t,s)\, f(s, u(s), u'(s), \dots, u^{(m)}(s))\, \mathrm{d}\, s, \quad t \in [0,T] \equiv I,$$

where $\lambda > 0$ is a positive parameter, $k : I \times I \to \mathbb{R}$ is a kernel function such that $k \in W^{m,1}(I \times I)$, m is a positive integer with $m \geq 1$, and $f : I \times \mathbb{R}^{m+1} \to [0, +\infty[$ is a L^1-Carathéodory function.

The existence of solutions for these Hammerstein equations is obtained by fixed point index theory on new type of cones. Therefore some assumptions must hold only for, at least, one of the derivatives of the kernel or, even, for the kernel, on a subset of the domain. Assuming some asymptotic conditions on the nonlinearity f, we get sufficient conditions for multiplicity of solutions.

Two examples will illustrate the potentialities of the main results, namely the fact that the kernel function and/or some derivatives may only be positive on some subintervals, which can degenerate to a point. Moreover, an application of our method to general Lidstone problems improves the existent results on the literature in this field.

*First author was partially supported by Xunta de Galicia (Spain), project EM2014/032, AIE Spain and FEDER, grants MTM2013-43014-P, MTM2016-75140-P, and FPU scholarship, Ministerio de Educación, Cultura y Deporte, Spain.

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1 Introduction

In this work we will study the existence and multiplicity of fixed points of the integral operator

$$\mathcal{T}u(t) = \lambda \int_0^T k(t,s)\, f(s,u(s),u'(s),\dots,u^{(m)}(s))\,\mathrm{d}\,s, \quad t\in[0,T]\equiv I, \tag{1}$$

where $\lambda > 0$ is a positive parameter, $k : I\times I \to \mathbb{R}$ is a kernel function such that $k \in W^{m,1}(I\times I)$, m is a positive integer with $m \geq 1$, and $f : I\times\mathbb{R}^{m+1} \to [0,+\infty[$ is a L^1-Carathéodory function.

The solvability of these type of integral equations, known as Hammerstein equations (see [13]), has been considered by many authors. In fact they have become a generalization of differential equations and boundary value problems and a main field for applications of methods and techniques of nonlinear analysis, as it can be seen, for instance, in [3, 8, 12, 15–18, 22].

In [5], the authors consider a third order three-point boundary value problem, whose solutions are the fixed points of the integral operator

$$Tu(t) = \lambda \int_0^1 G(t,s)\, f(s,u(s),u'(s),u''(s))\,\mathrm{d}\,s, \quad t\in[0,1],$$

where $G(t,s)$ is an explicit Green's function, verifying some adequate properties such that $G(t,s)$ and $\frac{\partial G}{\partial t}(t,s)$ are bounded and non-negative in the square $[0,1]\times[0,1]$, but $\frac{\partial^2 G}{\partial t^2}(t,s)$ could change sign, being non-negative in a subset of the square.

In [9], it is studied a generalized Hammerstein equation

$$u(t) = \int_0^1 k(t,s)\; g(s)\; f(s,u(s),u'(s),\dots,u^{(m)}(s))\,\mathrm{d}\,s, \tag{2}$$

with $k : [0,1]^2 \to \mathbb{R}$ a kernel function where, in short, $k \in W^{m,1}\left([0,1]^2\right)$, $m\geq 1$ is integer, $g : [0,1]\to[0,\infty)$ in almost everywhere $t\in[0,1]$ non-negative, and $f : [0,1]\times\mathbb{R}^{m+1}\to[0,\infty)$ is a $L^\infty-$Carathéodory function. Moreover, the kernel $k(t,s)$ and its derivatives $\frac{\partial^i k}{\partial t^i}(t,s)$, for $i = 1,\dots,m$, are bounded and non-negative on the square $[0,1]\times[0,1]$.

Our work generalize the existing results in the literature introducing a new type of cone,

$$K = \left\{ \begin{array}{r} u\in\mathcal{C}^m(I,\mathbb{R}) : u^{(i)}(t)\geq 0,\ t\in[m_i,n_i],\ i\in J; \\ \min\limits_{t\in[a_j,b_j]} u^{(j)}(t) \geq \xi_j\ \|u^{(j)}\|_{[c_j,d_j]}, j\in J_1 \end{array} \right\},$$

where the non-negativeness of the functions may happen only on a subinterval, possibly degenerate, that is reduced to a point, and, as $J_1\subset J$, $J_1\neq\varnothing$, the second property can hold, locally, only for a restrict number of derivatives, including the function itself. In this way, it is not required, as it was usual, that $k(t,s)$ and $\frac{\partial^i k}{\partial t^i}(t,s)$ have constant sign on the square. Another important novelty is given by (H_4), where the bounds must hold only for, at least, one of the derivatives of the kernel or, even, for the kernel, on a subset of the domain. Assuming some asymptotic conditions on the nonlinearity f and applying index theory, we obtain sufficient conditions for multiplicity of solutions, more precisely, for the existence of two or more solutions.

Moreover, the application in last section contains new sufficient conditions for the solvability of $2n$-th order Lidstone problems. In fact, our method allows that the nonlinearities may depend on derivatives of even and odd order, which is new in the literature on this type of problems, as it can be seen, for instance, in [6,19,25,28]. In this way, our results fill some gaps and improve the study of Lidstone and complementary Lidstone problems.

The paper is organized in the following way: Section 2 contains the main assumptions, the definition of the new cone and some properties on the integral operator. In section 3, the existence results are obtained with several asymptotic assumptions on f of the sublinear or superlinear type, near 0 or $+\infty$. Section 4 presents existence and multiplicity results applying fixed point index theory. Section 5 has two examples to illustrate our main results and, moreover, to emphasize the importance that (H_4) holds only for some derivatives and that the subsets could be reduced to a point. Last section contains an application to $2n$-th order Lidstone problems, which allows the dependence of the nonlineariry on odd and even derivatives.

2 Hypothesis and auxiliary results

Let's consider $E = \mathcal{C}^m(I, \mathbb{R})$ equipped with the norm

$$\|u\| = \max\{\|u^{(i)}\|_\infty, \; i \in J\},$$

where $\|v\|_\infty = \sup_{t\in I} |v(t)|$ and $J \equiv \{0, 1, \dots, m\}$. It is very well-known that $(E, \|\cdot\|)$ is a Banach space.

Throughout the paper we will make the following assumptions:

(H_1) The kernel function $k\colon I \times I \to \mathbb{R}$ is such that $k \in W^{m,1}(I \times I)$, with $m \geq 1$. Moreover, for $i = 0, \dots, m-1$, it holds that for every $\varepsilon > 0$ and every fixed $\tau \in I$, there exists some $\delta > 0$ such that $|t - \tau| < \delta$ implies that

$$\left| \frac{\partial^i k}{\partial t^i}(t, s) - \frac{\partial^i k}{\partial t^i}(\tau, s) \right| < \varepsilon \text{ for a. e. } s \in I.$$

Finally, for the m-th derivative of the kernel, it holds that for every $\varepsilon > 0$ and every fixed $\tau \in I$, there exist a set $Z_\tau \in I$ with measure equal to zero and some $\delta > 0$ such that $|t - \tau| < \delta$ implies that

$$\left| \frac{\partial^m k}{\partial t^m}(t, s) - \frac{\partial^m k}{\partial t^m}(\tau, s) \right| < \varepsilon, \; \forall s \in I \setminus Z_\tau \text{ such that } s < \min\{t, \tau\} \text{ or } s > \max\{t, \tau\}.$$

(H_2) For each $i \in J_0 \subset J$, $J_0 \neq \varnothing$, there exists a subinterval $[m_i, n_i]$ such that

$$\frac{\partial^i k}{\partial t^i}(t, s) \geq 0 \;\text{ for all } t \in [m_i, n_i], \; s \in I.$$

It is possible that the interval is degenerated, that is, $m_i = n_i$.

(H_3) For $i \in J$, there are positive functions $h_i \in L^1(I)$ such that

$$\left|\frac{\partial^i k}{\partial t^i}(t,s)\right| \le h_i(s) \quad \text{for } t \in I \text{ and a.e. } s \in I.$$

(H_4) For each $j \in J_1 \subset J_0$, $J_1 \neq \varnothing$, there exist subintervals $[a_j, b_j]$ and $[c_j, d_j]$, positive functions $\phi_j \colon I \to [0,\infty)$ and constants $\xi_j \in (0,1)$ such that

$$\left|\frac{\partial^j k}{\partial t^j}(t,s)\right| \le \phi_j(s) \quad \text{for } t \in [c_j, d_j] \text{ and a.e. } s \in I,$$

and

$$\frac{\partial^j k}{\partial t^j}(t,s) \ge \xi_j \, \phi_j(s) \quad \text{for } t \in [a_j, b_j] \text{ and a.e. } s \in I.$$

Moreover, $\phi_j \in L^1(I)$ satisfies

$$\int_{a_j}^{b_j} \phi_j(s) \, \mathrm{d}\, s > 0.$$

(H_5) There exists $i_0 \in J_0$ such that either $[c_{i_0}, d_{i_0}] \equiv I$ or $[m_{i_0}, n_{i_0}] \equiv I$ and, moreover, $\{0, 1, \dots, i_0\} \subset J_0$.

(H_6) The nonlinearity $f \colon I \times \mathbb{R}^{m+1} \to [0,\infty)$ satisfies L^1-Carathéodory conditions, that is,

- $f(\cdot, x_0, \dots, x_m)$ is measurable for each $(x_0, \dots, x_m)$ fixed.
- $f(t, \cdot, \cdots, \cdot)$ is continuous for a.e. $t \in I$.
- For each $r > 0$ there exists $\varphi_r \in L^1(I)$ such that

$$f(t, x_0, \dots, x_m) \le \varphi_r(t), \quad \forall\, (x_0, \dots, x_m) \in (-r, r)^{m+1}, \quad \text{a.e. } t \in I.$$

(H_7) Functions h_i defined in (H_3) and φ_r defined in (H_6) are such that $h_i \, \varphi_r \in L^1(I)$ for every $i \in J$ and $r > 0$.

We will look for fixed points of operator $\mathcal{T}$ on a suitable cone on the Banach space E. We recall that a cone K is a closed and convex subset of E satisfying the two following properties:

- If $x \in K$, then $\lambda\, x \in K$ for all $\lambda \ge 0$.
- $K \cap (-K) = \{0\}$.

Now, taking into account the properties satisfied by the kernel k, we define

$$K = \left\{ \begin{array}{l} u \in \mathcal{C}^m(I, \mathbb{R}) : u^{(i)}(t) \ge 0, \ t \in [m_i, n_i], \ i \in J_0; \\ \qquad\qquad\qquad\qquad \displaystyle\min_{t \in [a_j, b_j]} u^{(j)}(t) \ge \xi_j \, \|u^{(j)}\|_{[c_j, d_j]}, \ j \in J_1 \end{array} \right\},$$

where

$$\left\|u^{(j)}\right\|_{[c_j, d_j]} := \max_{t \in [c_j, d_j]} \left|u^{(j)}(t)\right|.$$

Lemma 1. *Hypothesis* (H_5) *warrants that* K *is a cone in* E.

Proof. We need to verify that K satisfies the two properties which characterize cones in a Banach space. First of all, from the definition of K, it is trivial to check that if $x \in K$, then $\lambda\, x \in K$ for all $\lambda \geq 0$.

Now, to prove that $K \cap (-K) = \{0\}$, we will distinguish between two different cases:

(I) There exists $i_0 \in J_0$ such that $[m_{i_0}, n_{i_0}] \equiv I$.

Suppose that u, $-u \in K$. Then $u^{(i_0)}(t) \geq 0$ and $-u^{(i_0)}(t) \geq 0$ for all $t \in I$, which implies that $u^{(i_0)} \equiv 0$ on I. If $i_0 \geq 1$, $u^{(i_0-1)}$ is constant on I.

Now, we have that $u^{(i_0-1)}(t) \geq 0$ and $-u^{(i_0-1)}(t) \geq 0$ for all $t \in [c_{i_0-1}, d_{i_0-1}]$, that is $u^{(i_0-1)} \equiv 0$ on $[c_{i_0-1}, d_{i_0-1}]$. Then, since $u^{(i_0-1)}$ is constant on I, we deduce that $u^{(i_0-1)} \equiv 0$ on I.

Using the same argument repeatedly we conclude that $u \equiv 0$ on I. In this way, we have proved that $K \cap (-K) = \{0\}$.

(II) There exists $i_0 \in J_0$ such that $[c_{i_0}, d_{i_0}] \equiv I$.

Suppose again that u, $-u \in K$. Then, from the fact that

$$\min_{t\in\left[a_{i_0}, b_{i_0}\right]} u^{(i_0)}(t) \geq \xi_{i_0}\, \|u^{(i_0)}\|_I \quad \text{and} \quad \min_{t\in\left[a_{i_0}, b_{i_0}\right]} \left(-u^{(i_0)}(t)\right) \geq \xi_{i_0}\, \|u^{(i_0)}\|_I,$$

it is deduced that $\|u^{(i_0)}\|_I = 0$, which implies that $u^{(i_0)} \equiv 0$ on I. Now, following the same arguments than in Case (I), we deduce the result.

□

In next section, considering some additional properties on the function f, we will ensure the existence of fixed points of operator $\mathcal{T}$. However, before doing that, we obtain some previous technical results.

Lemma 2. *If* $(H_1) - (H_7)$ *hold, then* $\mathcal{T}\colon K \to K$ *is a completely continuous operator.*

Proof. We divide the proof into several steps.

Step 1. $\mathcal{T}$ is well defined in K.

Let $u \in K$. The proof that $\mathcal{T}u \in \mathcal{C}^m(I, \mathbb{R})$ follow standard techniques and so we omit it. We will prove that $\mathcal{T}u \in K$.

It is obvious that, for $i \in J_0$, $(\mathcal{T}u)^{(i)}(t) \geq 0$ for all $t \in [m_i, n_i]$.

Moreover, for $j \in J_1$ and $t \in [c_j, d_j]$, we have that

$$\begin{aligned} \left|(\mathcal{T}u)^{(j)}(t)\right| &\leq \lambda \int_0^T \left|\frac{\partial^j k}{\partial t^j}(t,s)\right| f(s, u(s), \dots, u^{(m)}(s))\, \mathrm{d}\, s \\ &\leq \lambda \int_0^T \phi_j(s)\, f(s, u(s), \dots, u^{(m)}(s))\, \mathrm{d}\, s, \end{aligned}$$

and, taking the supremum for $t \in [c_j, d_j]$, we deduce that

$$\left\|(\mathcal{T}u)^{(j)}\right\|_{[c_j,d_j]} \leq \lambda \int_0^T \phi_j(s)\, f(s, u(s), \ldots, u^{(m)}(s))\, \mathrm{d}\, s.$$

On the other hand, for $t \in [a_j, b_j]$, we have

$$\begin{aligned}(\mathcal{T}u)^{(j)}(t) &= \lambda \int_0^T \frac{\partial^j k}{\partial t^j}(t,s)\, f(s, u(s), \ldots, u^{(m)}(s))\, \mathrm{d}\, s \\ &\geq \lambda \int_0^T \xi_j\, \phi_j(s)\, f(s, u(s), \ldots, u^{(m)}(s))\, \mathrm{d}\, s \geq \xi_j \left\|(\mathcal{T}u)^{(j)}\right\|_{[c_j,d_j]}\end{aligned}$$

and we deduce that

$$\min_{t\in[a_j,b_j]} (\mathcal{T}u)^{(j)}(t) \geq \xi_j \left\|(\mathcal{T}u)^{(j)}\right\|_{[c_j,d_j]}$$

for $j \in J_1$.

Therefore, we can conclude that $\mathcal{T}u \in K$.

Step 2. $\mathcal{T}$ is continuous in $\mathcal{C}^m(I, \mathbb{R})$.

This part also follows standard techniques.

Step 3. $\mathcal{T}$ is a compact operator.

Let's consider

$$B = \{u \in E;\ \|u\| \leq r\}.$$

First, we will prove that $T(B)$ is uniformly bounded in $\mathcal{C}^m(I)$.

We find the following bounds for $u \in B$ and $i \in J$:

$$\begin{aligned}\left\|(\mathcal{T}u)^{(i)}\right\|_{\infty} &= \sup_{t\in I} \left|\lambda \int_0^T \frac{\partial^i k}{\partial t^i}(t,s)\, f(s, u(s), \ldots, u^{(m)}(s))\, \mathrm{d}\, s\right| \\ &\leq \lambda \int_0^T h_i(s)\, f(s, u(s), \ldots, u^{(m)}(s))\, \mathrm{d}\, s \leq \lambda \int_0^T h_i(s)\, \varphi_r(s)\, \mathrm{d}\, s := M_i,\end{aligned}$$

with $M_i > 0$. Therefore,

$$\|\mathcal{T}u\| \leq \max\{M_i :\ i \in J\}, \quad \forall\, u \in B.$$

Now, we will prove that $\mathcal{T}(B)$ is equicontinuous in $\mathcal{C}^m(I)$. Let $t_2 \in I$ be fixed. Then, for every $\varepsilon > 0$, take $\delta > 0$ given in (H_1) and for $i = 0, \ldots, m-1$, it holds that $|t_1 - t_2| < \delta$ implies that

$$\begin{aligned}\left|(\mathcal{T}u)^{(i)}(t_1) - (\mathcal{T}u)^{(i)}(t_2)\right| &\leq \lambda \int_0^T \left|\frac{\partial^i k}{\partial t^i}(t_1,s) - \frac{\partial^i k}{\partial t^i}(t_2,s)\right| f(s, u(s), \ldots, u^{(m)}(s))\, \mathrm{d}\, s \\ &\leq \lambda \int_0^T \left|\frac{\partial^i k}{\partial t^i}(t_1,s) - \frac{\partial^i k}{\partial t^i}(t_2,s)\right| \varphi_r(s)\, \mathrm{d}\, s \leq \varepsilon\, \lambda \int_0^T \varphi_r(s)\, \mathrm{d}\, s,\end{aligned}$$

and, since $\varphi_r \in L^1(I)$, it is clear that there exists a positive constant κ_1 such that

$$\left|(\mathcal{T}u)^{(i)}(t_1) - (\mathcal{T}u)^{(i)}(t_2)\right| < \kappa_1\, \varepsilon$$

for all $u \in B$.

On the other hand, for the m-th derivative, for every $\varepsilon > 0$, take $\delta > 0$ given in (H_1) and $|t_1 - t_2| < \delta$, $t_1 < t_2$, implies that

$$\begin{aligned}
\left|(\mathcal{T}u)^{(m)}(t_1) - (\mathcal{T}u)^{(m)}(t_2)\right| \leq& \lambda \int_0^T \left|\frac{\partial^m k}{\partial t^m}(t_1,s) - \frac{\partial^m k}{\partial t^m}(t_2,s)\right| f(s,u(s),\dots,u^{(m)}(s))\, \mathrm{d}\, s \\
\leq& \lambda \int_0^T \left|\frac{\partial^m k}{\partial t^m}(t_1,s) - \frac{\partial^m k}{\partial t^m}(t_2,s)\right| \varphi_r(s)\, \mathrm{d}\, s \\
=& \lambda \int_0^{t_1} \left|\frac{\partial^m k}{\partial t^m}(t_1,s) - \frac{\partial^m k}{\partial t^m}(t_2,s)\right| \varphi_r(s)\, \mathrm{d}\, s \\
&+ \lambda \int_{t_1}^{t_2} \left|\frac{\partial^m k}{\partial t^m}(t_1,s) - \frac{\partial^m k}{\partial t^m}(t_2,s)\right| \varphi_r(s)\, \mathrm{d}\, s \\
&+ \lambda \int_{t_2}^{T} \left|\frac{\partial^m k}{\partial t^m}(t_1,s) - \frac{\partial^m k}{\partial t^m}(t_2,s)\right| \varphi_r(s)\, \mathrm{d}\, s.
\end{aligned}$$

From (H_1), it is clear that first and third integrals in last term of previous expression can be arbitrarily small when $|t_1 - t_2| < \delta$. Moreover, $\left|\frac{\partial^m k}{\partial t^m}(t_1,\cdot) - \frac{\partial^m k}{\partial t^m}(t_2,\cdot)\right| \varphi_r(\cdot) \in L^1[t_1,t_2]$ and so there exists some $\delta' > 0$ such that

$$\lambda \int_{t_1}^{t_2} \left|\frac{\partial^m k}{\partial t^m}(t_1,s) - \frac{\partial^m k}{\partial t^m}(t_2,s)\right| \varphi_r(s)\, \mathrm{d}\, s < \varepsilon$$

when $|t_1 - t_2| < \delta'$.

Therefore it is clear that, for $|t_1 - t_2| < \min\{\delta, \delta'\}$, $t_1 < t_2$, there exists a positive constant κ_2 such that

$$\left|(\mathcal{T}u)^{(m)}(t_1) - (\mathcal{T}u)^{(m)}(t_2)\right| < \kappa_2\, \varepsilon$$

for all $u \in B$.

Analogously, when $|t_1 - t_2| < \delta$, $t_1 > t_2$, there exists some some positive constant κ_3 such that

$$\left|(\mathcal{T}u)^{(m)}(t_1) - (\mathcal{T}u)^{(m)}(t_2)\right| < \kappa_3\, \varepsilon$$

for all $u \in B$.

We have proved the pointwise equicontinuity on I. Moreover, since I is compact, pointwise equicontinuity is equivalent to uniform equicontinuity.

This way, we conclude that $\mathcal{T}(B)$ is equicontinuous in $\mathcal{C}^m(I)$.

As a consequence, by Ascoli-Arzelà Theorem, we can affirm that $\mathcal{T}(B)$ is relatively compact in $\mathcal{C}^m(I)$ and so $\mathcal{T}$ is a completely continuous operator. □

3 Main results

We introduce now the following notation

$$\Lambda^i := \int_0^T h_i(s)\,\mathrm{d}\,s, \quad \Lambda_i := \int_{a_i}^{b_i} \xi_i\,\phi_i(s)\,\mathrm{d}\,s$$

and define

$$\bar{\Lambda} := (m+1)\,\max\{\Lambda^i:\ i \in J\} \quad \text{and} \quad \underline{\Lambda}: = \max\{\xi_i\,\Lambda_i:\ i \in J_1\}.$$

On the other hand, we denote

$$f_0 := \liminf_{|x_0|,\ldots,|x_m| \to 0}\ \min_{t \in I} \frac{f(t, x_0, \ldots, x_m)}{|x_0| + \cdots + |x_m|}$$

and

$$f^\infty := \limsup_{|x_0|,\ldots,|x_m| \to \infty}\ \max_{t \in I} \frac{f(t, x_0, \ldots, x_m)}{|x_0| + \cdots + |x_m|}.$$

We will give now our existence result.

Theorem 3. *Assume that hypotheses $(H_1) - (H_7)$ hold. If $\bar{\Lambda}\,f^\infty < \underline{\Lambda}\,f_0$, then for all*

$$\lambda \in \left(\frac{1}{\underline{\Lambda}\,f_0}, \frac{1}{\bar{\Lambda}\,f^\infty}\right)$$

operator $\mathcal{T}$ has a fixed point in the cone K, which is not a trivial solution.

Proof. Let $\lambda \in \left(\frac{1}{\underline{\Lambda}\,f_0}, \frac{1}{\bar{\Lambda}\,f^\infty}\right)$ and choose $\varepsilon \in (0, f_0)$ such that

$$\frac{1}{\underline{\Lambda}\,(f_0 - \varepsilon)} \leq \lambda \leq \frac{1}{\bar{\Lambda}\,(f^\infty + \varepsilon)}.$$

Taking into account the definition of f_0, we know that there exists $\delta_1 > 0$ such that when $\|u\| \leq \delta_1$,

$$f(t, u(t), \ldots, u^{(m)}(t)) >\ (f_0 - \varepsilon)\ \left(|u(t)| + \cdots + |u^{(m)}(t)|\right), \quad \forall\, t \in I.$$

Let

$$\Omega_{\delta_1} = \{u \in K;\ \|u\| < \delta_1\}$$

and choose $u \in \partial\,\Omega_{\delta_1}$. We will prove that $\mathcal{T}u \not\preceq u$. We have that for $t \in [a_i, b_i]$ and $j \in J_1$,

$$
\begin{aligned}
(\mathcal{T}u)^{(j)}(t) &= \lambda \int_0^T \frac{\partial^j k}{\partial t^j}(t,s)\, f(s,u(s),\dots,u^{(m)}(s))\,\mathrm{d}\,s \geq \lambda \int_{a_j}^{b_j} \frac{\partial^j k}{\partial t^j}(t,s)\, f(s,u(s),\dots,u^{(m)}(s))\,\mathrm{d}\,s \\
&\geq \lambda \int_{a_j}^{b_j} \xi_j\,\phi_j(s)\, f(s,u(s),\dots,u^{(m)}(s))\,\mathrm{d}\,s \\
&> \lambda \int_{a_j}^{b_j} \xi_j\,\phi_j(s)\,(f_0-\varepsilon)\,\left(|u(s)|+\dots+|u^{(m)}(s)|\right)\mathrm{d}\,s \\
&\geq \lambda\,(f_0-\varepsilon)\,\xi_j\,\|u^{(j)}\|_{[a_j,b_j]} \int_{a_j}^{b_j} \xi_j\,\phi_j(s)\,\mathrm{d}\,s \\
&= \lambda\,(f_0-\varepsilon)\,\xi_j\,\|u^{(j)}\|_{[a_j,b_j]}\,\Lambda_j \geq\ \lambda\,(f_0-\varepsilon)\,\Lambda_j\,\xi_j\,u^{(j)}(t).
\end{aligned}
$$

As a consequence we have that for some $j \in J_1$, $(\mathcal{T}u)^{(j)}(t) > u^{(j)}(t)$ for all $t \in [a_j, b_j]$ and so it is proved that $\mathcal{T}u \not\preceq u$. We deduce (see [11, Theorem 2.3.3]) that

$$i_K(\mathcal{T},\,\Omega_{\delta_1}) = 0.$$

On the other hand, due to the definition of f^∞, we know that there exists $\tilde{C} > 0$ such that when $\min\left\{|u^{(i)}(t)| : i \in J\right\} \geq \tilde{C}$,

$$f(t,u(t),\dots,u^{(m)}(t)) \leq (f^\infty+\varepsilon)\,\left(|u(t)|+\dots+|u^{(m)}(t)|\right) \leq (m+1)\,(f^\infty+\varepsilon)\,\|u\|, \quad \forall\, t \in I.$$

Let $C > \{\delta_1,\,\tilde{C}\}$ and define

$$\Omega_C = \bigcup_{i=0}^{m} \left\{ u \in K : \min_{t\in I} |u^{(i)}(t)| < C \right\}.$$

We note that Ω_C is an unbounded subset of the cone K. Because of this, the fixed point index of operator $\mathcal{T}$ with respect to Ω_C, $i_K(\mathcal{T},\,\Omega_C)$, is only defined in the case that the set of fixed points of operator $\mathcal{T}$ in Ω_C, that is, $(I-\mathcal{T})^{-1}(\{0\}) \cap \Omega_C$, is compact (see [10] for the details). We will see that $i_K(\mathcal{T},\,\Omega_C)$ can be defined in this case.

First of all, since $(I-\mathcal{T})$ is a continuous operator, it is obvious that $(I-\mathcal{T})^{-1}(\{0\}) \cap \Omega_C$ is closed.

Moreover, we can assume that $(I-\mathcal{T})^{-1}(\{0\}) \cap \Omega_C$ is bounded. Indeed, on the contrary, we would have infinite fixed points of operator $\mathcal{T}$ on Ω_C and it would be immediately deduced that $\mathcal{T}$ has an infinite number of fixed points in the cone K. Therefore we may assume that there exists a constant $M > 0$ such that $\|u\| < M$ for all $u \in (I-\mathcal{T})^{-1}(\{0\}) \cap \Omega_C$.

Finally, it is left to see that $(I-\mathcal{T})^{-1}(\{0\}) \cap \Omega_C$ is equicontinuous. This property follows from the fact that $(I-\mathcal{T})^{-1}(\{0\}) \cap \Omega_C$ is bounded. The proof is totally analogous to Step 3 in the proof of Lemma 2.

Now, we will calculate $i_K(\mathcal{T}, \Omega_C)$. In particular, we will prove that $\|\mathcal{T}u\| \leq \|u\|$ for all $u \in \partial\,\Omega_C$. Let $u \in \partial\,\Omega_C$, that is, $u \in K$ such that

$$\min\left\{\min_{t\in I}|u^{(i)}(t)|:\ i \in J\right\} = C.$$

Then, for $i \in J$,

$$\begin{aligned}|(\mathcal{T}u)^{(i)}(t)| &\leq \lambda\int_0^T\left|\frac{\partial^i k}{\partial t^i}(t,s)\right| f(s,u(s),\dots,u^{(m)}(s))\,\mathrm{d}\,s \leq \lambda\int_0^T h_i(s)\,f(s,u(s),\dots,u^{(m)}(s))\,\mathrm{d}\,s\\ &\leq (m+1)\,\lambda\int_0^T h_i(s)\,(f^\infty+\varepsilon)\,\|u\|\,\mathrm{d}\,s = (m+1)\,\lambda\,(f^\infty+\varepsilon)\,\|u\|\,\Lambda^i\\ &\leq \lambda\,(f^\infty+\varepsilon)\,\|u\|\,\bar{\Lambda} \leq \|u\|.\end{aligned}$$

We deduce that

$$\|\mathcal{T}u\| \leq \|u\|$$

and as a consequence ([10, Corollary 7.4]) we have that

$$i_K(\mathcal{T}, \Omega_C) = 1.$$

Therefore, we conclude that $\mathcal{T}$ has a fixed point in $\bar{\Omega}_C \setminus \Omega_{\delta_1}$. □

Consequently, we obtain the following

Corollary 4. *Assume that hypotheses* $(H_1)-(H_7)$ *hold. Then,*

(i) If $f_0 = \infty$ *and* $f^\infty = 0$*, then for all* $\lambda \in (0,\infty)$*,* $\mathcal{T}$ *has a fixed point in the cone.*

(ii) If $f_0 = \infty$ *and* $0 < f^\infty < \infty$*, then for all* $\lambda \in \left(0, \frac{1}{\bar{\Lambda}\,f^\infty}\right)$*,* $\mathcal{T}$ *has a fixed point in the cone.*

(iii) If $0 < f_0 < \infty$ *and* $f^\infty = 0$*, then for all* $\lambda \in \left(\frac{1}{\underline{\Lambda}\,f_0}, \infty\right)$*,* $\mathcal{T}$ *has a fixed point in the cone.*

4 Existence and multiplicity of solutions

In this section we will use the fixed point index theory to study the existence of multiple fixed points of operator $\mathcal{T}$. In [4] the authors apply similar arguments to functional equations that only depend on the values of u. First of all, we compile some classical results regarding to fixed point index (see [2, 11] for the details).

Lemma 5. *Let* D *be an open bounded set with* $D_K = D \cap K \neq \emptyset$ *and* $\bar{D}_K \neq K$*. Assume that* $F\colon \bar{D}_K \to K$ *is a compact map such that* $x \neq F\,x$ *for* $x \in \partial D_K$*. Then the fixed point index* $i_K(F, D_K)$ *has the following properties:*

(1) If there exists $e \in K \setminus \{0\}$ *such that* $x \neq F\,x + \alpha\,e$ *for all* $x \in \partial D_K$ *and all* $\alpha > 0$*, then* $i_K(F, D_K) = 0$*.*

(2) If $\mu x \neq F x$ for all $x \in \partial D_K$ and for every $\mu \geq 1$, then $i_K(F, D_K) = 1$.

(3) Let D^1 be open in X with $\bar{D}^1 \subset D_K$. If $i_K(F, D_K) = 1$ and $i_K(F, D_K^1) = 0$, then F has a fixed point in $D_K \setminus \bar{D}_K^1$. The same result holds if $i_K(F, D_K) = 0$ and $i_K(F, D_K^1) = 1$.

We will define the following sets:

$$K_\rho = \{u \in K;\ \|u\| < \rho\},$$

$$V_\rho = \left\{ u \in K :\ \min_{t \in [a_i, b_i]} u^{(i)}(t) < \rho,\ i \in J_2, \quad \|u^{(i)}\|_\infty < \rho,\ i \in J \setminus J_2 \right\},$$

where $J = \{0, \ldots, m\}$ and

$$J_2 = \{i \in J :\ [c_i, d_i] \equiv I\}.$$

To ensure that the sets K_ρ and V_ρ are not the same, we need to change condition (H_5) into

$(\widetilde{H}_5)$ There exists $i_0 \in \{0, \ldots, m\}$ such that $[c_{i_0}, d_{i_0}] \equiv I$ and, moreover, $\{0, 1, \ldots, i_0\} \subset J_0$.

In this situation, it is clear that $J_2 \neq \emptyset$ and therefore

$$K_\rho \subsetneq V_\rho \subsetneq K_{\frac{\rho}{c}}$$

where

$$c = \min\{\xi_i :\ i \in J_2\}. \tag{3}$$

Now we will give sufficient conditions under which the index of the previous sets is either 1 or 0.

Lemma 6. *Let*

$$\frac{1}{N} = \max\left\{ \sup_{t \in I} \int_0^T \left| \frac{\partial^i k}{\partial t^i}(t, s) \right| \mathrm{d}\, s :\ i \in J \right\}$$

and

$$f^\rho = \sup\left\{ \frac{f(t, x_0, \ldots, x_m)}{\rho};\ t \in I,\ x_i \in [-\rho, \rho],\ i \in J \right\}.$$

If there exists $\rho > 0$ such that

$$\lambda \frac{f^\rho}{N} < 1, \tag{I_ρ^1}$$

then $i_K(\mathcal{T}, K_\rho) = 1$.

Proof. We will prove that $\mathcal{T} u \neq \mu u$ for all $u \in \partial K_\rho$ and for every $\mu \geq 1$.

Suppose, on the contrary, that there exist some $u \in \partial K_\rho$ and $\mu \geq 1$ such that

$$\mu\, u^{(i)}(t) = \lambda \int_0^T \frac{\partial^i k}{\partial t^i}(t, s)\, f(s, u(s), \ldots, u^{(m)}(s))\, \mathrm{d}\, s.$$

Taking the supremum for $t \in I$, we obtain that

$$\mu \|u^{(i)}\|_\infty \leq \lambda \sup_{t\in I} \int_0^T \left|\frac{\partial^i k}{\partial t^i}(t,s)\right| f(s,u(s),\dots,u^{(m)}(s)) \,\mathrm{d}\, s \leq \lambda \,\rho\, f^\rho \sup_{t\in I} \int_0^T \left|\frac{\partial^i k}{\partial t^i}(t,s)\right| \mathrm{d}\, s$$
$$\leq \lambda\,\rho\,\frac{f^\rho}{N} < \rho.$$

Consequently, we deduce that

$$\mu\,\rho = \mu\,\max\{\|u^{(i)}\|_\infty : \, i \in J\} < \rho,$$

which contradicts the assumption that $\mu \geq 1$. Therefore, $i_K(\mathcal{T}, K_\rho) = 1$. □

Lemma 7. *For $i \in J_1$, let*

$$\frac{1}{M_i} = \inf_{t\in[a_i,b_i]} \int_{a_i}^{b_i} \frac{\partial^i k}{\partial t^i}(t,s)\,\mathrm{d}\, s,$$

and

$$f_\rho^i = \inf\left\{ \frac{f(t,x_0,\dots,x_m)}{\rho} : \; t\in[a_i,b_i], \; x_j \in \left[0, \frac{\rho}{\xi_j}\right], \, j\in J_2, \; x_k \in [0,\rho]\,, \, k \in J\setminus J_2 \right\}.$$

If there exists $\rho > 0$ and $i_0 \in J_1$ such that

$$\lambda\,\frac{f_\rho^{i_0}}{M_{i_0}} > 1, \qquad (I_\rho^0)$$

then $i_K(\mathcal{T}, V_\rho) = 0$.

Proof. We will prove that there exists $e \in K\setminus\{0\}$ such that $u \neq \mathcal{T}u + \alpha\, e$ for all $u \in \partial V_\rho$ and all $\alpha > 0$.

Let us take $e(t) = 1$ and suppose that there exists some $u \in \partial V_\rho$ and $\alpha > 0$ such that $u = \mathcal{T}u + \alpha$. Then, for $t \in [a_{i_0}, b_{i_0}]$,

$$u^{(i_0)}(t) \geq \lambda \int_0^T \frac{\partial^{i_0} k}{\partial t^{i_0}}(t,s)\, f(s,u(s),\dots,u^{(m)}(s))\,\mathrm{d}\, s \geq \lambda \int_{a_{i_0}}^{b_{i_0}} \frac{\partial^{i_0} k}{\partial t^{i_0}}(t,s)\, f(s,u(s),\dots,u^{(m)}(s))\,\mathrm{d}\, s$$
$$\geq \lambda\,\rho\, f_\rho^{i_0} \int_{a_{i_0}}^{b_{i_0}} \frac{\partial^{i_0} k}{\partial t^{i_0}}(t,s)\,\mathrm{d}\, s > \rho.$$

Consequently, $u^{(i_0)}(t) > \rho$ for $t \in [a_{i_0}, b_{i_0}]$, which is a contradiction. Thus, $i_K(\mathcal{T}, V_\rho) = 0$. □

Combining the previous lemmas, it is possible to obtain some conditions under which operator $\mathcal{T}$ has multiple fixed points.

Theorem 8. *Assume that conditions $(H_1)-(H_4)$, $(\widetilde{H}_5)$ and $(H_6)-(H_7)$ hold and let c be defined in (3). The integral equation (1) has at least one non trivial solution in K if one of the following conditions hold*

(C1) There exist $\rho_1,\, \rho_2 \in (0,\infty)$, $\frac{\rho_1}{c} < \rho_2$, *such that* $(I^0_{\rho_1})$ *and* $(I^1_{\rho_2})$ *are verified.*

(C2) There exist $\rho_1,\, \rho_2 \in (0,\infty)$, $\rho_1 < \rho_2$, *such that* $(I^1_{\rho_1})$ *and* $(I^0_{\rho_2})$ *are verified.*

The integral equation (1) *has at least two non trivial solutions in* K *if one of the following conditions hold*

(C3) There exist $\rho_1,\, \rho_2,\, \rho_3 \in (0,\infty)$, $\frac{\rho_1}{c} < \rho_2 < \rho_3$, *such that* $(I^0_{\rho_1})$, $(I^1_{\rho_2})$ *and* $(I^0_{\rho_3})$ *are verified.*

(C4) There exist $\rho_1,\, \rho_2,\, \rho_3 \in (0,\infty)$, *with* $\rho_1 < \rho_2$ *and* $\frac{\rho_2}{c} < \rho_3$, *such that* $(I^1_{\rho_1})$, $(I^0_{\rho_2})$ *and* $(I^1_{\rho_3})$ *are verified.*

The integral equation (1) *has at least three non trivial solutions in* K *if one of the following conditions hold*

(C5) There exist $\rho_1,\, \rho_2,\, \rho_3,\, \rho_4 \in (0,\infty)$, *with* $\frac{\rho_1}{c} < \rho_2 < \rho_3$ *and* $\frac{\rho_3}{c} < \rho_4$, *such that* $(I^0_{\rho_1})$, $(I^1_{\rho_2})$, $(I^0_{\rho_3})$ *and* $(I^1_{\rho_4})$ *are verified.*

(C6) There exist $\rho_1,\, \rho_2,\, \rho_3,\, \rho_4 \in (0,\infty)$, *with* $\rho_1 < \rho_2$ *and* $\frac{\rho_2}{c} < \rho_3 < \rho_4$, *such that* $(I^1_{\rho_1})$, $(I^0_{\rho_2})$, $(I^1_{\rho_3})$ *and* $(I^0_{\rho_4})$ *are verified.*

The proof of the previous result is an immediate consequence of the properties of the fixed point index. Moreover, it must be point out that, despite of the fact that the previous theorem studies the existence of one, two or three solutions, similar results can be formulated to ensure the existence of four or more solutions.

5 Examples

Example 9. Consider the following boundary value problem:

$$\begin{cases} u^{(3)}(t) = \lambda\, \dfrac{e^t\,(|u(t)| + |u'(t)| + |u''(t)|)}{1 + (u(t))^2}, \quad t \in [0,1], \\[2ex] u(0) = -u(1),\ u'(0) = \dfrac{1}{2}\, u'(1),\ u''(0) = 0. \end{cases} \tag{4}$$

The Green's function related to the homogeneous problem

$$\begin{cases} u^{(3)}(t) = 0, \quad t \in [0,1], \\[1ex] u(0) = -u(1),\ u'(0) = \dfrac{1}{2}\, u'(1),\ u''(0) = 0, \end{cases}$$

is the following one

$$G(t,s) = \begin{cases} \frac{1}{4}\,(1-s)\,(-3+s+4\,t), & t \le s, \\ \frac{1}{4}\,(-3+s\,(s+4)+2\,t\,(t+2)-8\,s\,t), & s < t. \end{cases}$$

Therefore, solutions of the boundary value problem (4) correspond with the fixed points of the following operator:

$$\mathcal{T}u(t) = \lambda \int_0^1 G(t,s)\, f(s,u(s),u'(s),u''(s))\, \mathrm{d}\, s, \quad t \in [0,1],$$

which is a particular case of the operator defined in (1) for $T = 1$, $m = 2$, $k \equiv G$ and $f(t,x,y,z) = \frac{e^t(|x|+|y|+|z|)}{1+x^2}$. We will check now that the kernel G satisfies conditions $(H_1)-(H_5)$. To do that, we need to calculate the explicit expression of the first and second partial derivatives of the Green's function, that is,

$$\frac{\partial G}{\partial t}(t,s) = \begin{cases} 1 - s, & t \le s, \\ 1 - 2\,s + t, & s < t, \end{cases}$$

and

$$\frac{\partial^2 G}{\partial t^2}(t,s) = \begin{cases} 0, & t < s, \\ 1, & s < t. \end{cases}$$

Using this expressions, we are able to check that the required conditions hold:

(H_1) Let $\tau \in I$ be fixed. Both G and $\frac{\partial G}{\partial t}$ are uniformly continuous so the hypothesis is immediate for $i = 0,1$. Moreover, for the second derivative $\frac{\partial^2 G}{\partial t^2}$ (that is, for the case $i = m = 2$), we can take $Z_\tau = \{\tau\}$ and we have that

$$\left| \frac{\partial^2 G}{\partial t^2}(t,s) - \frac{\partial^2 G}{\partial t^2}(\tau,s) \right| = |1-1| = 0, \ \forall\, s < \min\{t,\tau\}$$

and

$$\left| \frac{\partial^2 G}{\partial t^2}(t,s) - \frac{\partial^2 G}{\partial t^2}(\tau,s) \right| = |0-0| = 0, \ \forall\, s > \max\{t,\tau\},$$

so the hypothesis hold.

(H_2) Numerically, it can be seen that

$$G(t,s) \ge 0, \quad \text{for all } t \in [t_0,1], \ s \in [0,1],$$

with $t_0 \approx 0.6133$. Therefore, in this case $[m_0,n_0] = [t_0,1]$.

Moreover, both $\frac{\partial G}{\partial t}$ and $\frac{\partial^2 G}{\partial t^2}$ are nonnegative on the square $[0,1] \times [0,1]$, which means that $[m_1,n_1] = [m_2,n_2] = [0,1]$.

(H_3) It can be checked that

$$|G(t,s)| \le \frac{1}{4}\,(3 - 4\,s + s^2), \quad \text{for all } t \in [0,1], \ s \in [0,1],$$

and the equality holds for $t = 0$ and $t = 1$ so the choice $h_0(s) = \frac{1}{4}(3 - 4s + s^2)$ is optimal. This inequality can be easily proved by taking into account that, since $\frac{\partial G}{\partial t}$ is nonnegative, then $G(\cdot, s)$ is nondecreasing for every $s \in [0, 1]$ and, therefore,

$$|G(t,s)| \le \max\{|G(0,s)|,\ |G(1,s)|\} = \frac{1}{4}(3 - 4s + s^2).$$

For the first derivative, it holds that

$$\left|\frac{\partial G}{\partial t}(t,s)\right| \le 2(1-s), \quad \text{for all } t \in [0,1],\ s \in [0,1],$$

and the equality holds for $t = 1$, so $h_1(s) = 2(1 - s)$ is also optimal.

Finally,

$$\left|\frac{\partial^2 G}{\partial t^2}(t,s)\right| \le 1, \quad \text{for } t \in [0,1] \text{ and a.e. } s \in [0,1],$$

and $h_2(s) = 1$ is trivially optimal.

(H_4) If we take $\phi_0(s) = h_0(s) = \frac{1}{4}(3 - 4s + s^2)$, $[c_0, d_0] = [0, 1]$, and $[a_0, b_0] = [t_1, 1]$ with $t_1 > t_0$ (t_0 given in (H_2)), it holds that there exists a constant $\xi_0(t_1) \in (0, 1)$ such that

$$G(t,s) \ge \xi_0(t_1)\,\phi_0(s), \quad \text{for all } t \in [t_1,1],\ s \in [0,1].$$

We note that the bigger t_1 is, the bigger the constant $\xi_0(t_1)$ is. For instance, if we take $t_1 = 0.62$, we can choose $\xi_0 = \frac{1}{75}$.

With regard to the first derivative of G, it satisfies that

$$\frac{\partial G}{\partial t}(t,s) \le 2(1-s), \quad \text{for all } t \in [0,1],\ s \in [0,1],$$

that is, we could take $\phi_1(s) = h_1(s) = 2(1-s)$, $[c_1, d_1] = [0, 1]$, $\xi_1 = \frac{1}{2}$ and $[a_1, b_1] = [0, 1]$.

Finally, for the second derivative of G, it does not exist a suitable function ϕ_2 and a constant ξ_2 for which the inequalities in (H_4) hold.

As a consequence, we deduce that $J_1 = \{0,\ 1\}$.

Moreover, it is obvious that $\int_{a_i}^{b_i} \phi_i(s)\,\mathrm{d}s > 0$ for $i = 0, 1$.

(H_5) It is immediately deduced from the proofs of the previous conditions.

Moreover, the nonlinearity f satisfies condition (H_6).

We will work in the cone

$$K = \left\{ \begin{array}{l} u \in \mathcal{C}^2([0,1],\mathbb{R}) : u(t) \ge 0,\ t \in [t_0,1], \quad u'(t), u''(t) \ge 0,\ t \in [0,1]; \\ \qquad\qquad \displaystyle\min_{t\in[t_1,1]} u(t) \ge \xi_0(t_1)\,\|u\|_{[0,1]}, \quad \min_{t\in[0,1]} u'(t) \ge \frac{1}{2}\,\|u'\|_{[0,1]} \end{array} \right\}.$$

With the notation introduced in Section 3, we obtain the following values for the constants involved in Theorem 3:

$$\Lambda^0 = \frac{1}{3}, \quad \Lambda^1 = 1, \quad \Lambda^2 = 1,$$

and therefore

$$\bar{\Lambda} = 3 \max \left\{ \Lambda^0, \Lambda^1, \Lambda^2 \right\} = 3,$$

$$\Lambda_0 = \xi_0(t_1) \left(\frac{1}{3} - \frac{3}{4} t_1 + \frac{1}{2} t_1^2 - \frac{1}{12} t_1^3 \right), \quad \Lambda_1 = \frac{1}{2},$$

and so

$$\underline{\Lambda} = \max \left\{ \xi_0^2(t_1) \left(\frac{1}{3} - \frac{3}{4} t_1 + \frac{1}{2} t_1^2 - \frac{1}{12} t_1^3 \right), \frac{1}{4} \right\}.$$

We note that, since $\xi_0(t_1) \in (0,1)$,

$$\xi_0^2(t_1) \left(\frac{1}{3} - \frac{3}{4} t_1 + \frac{1}{2} t_1^2 - \frac{1}{12} t_1^3 \right) < \frac{1}{3} - \frac{3}{4} t_1 + \frac{1}{2} t_1^2 - \frac{1}{12} t_1^3$$

and it is easy to see that the right hand side of previous inequality decreases with t_1 and, in particular, it is always smaller than $\frac{1}{4}$. Thus,

$$\underline{\Lambda} = \frac{1}{4}$$

independently of the value of t_1.

On the other hand, we obtain the following values for the limits over the nonlinearity f:

$$f_0 = \liminf_{|x|,|y|,|z| \to 0} \min_{t \in [0,1]} \frac{e^t \left(|x| + |y| + |z| \right)}{(1 + x^2) \left(|x| + |y| + |z| \right)} = \lim_{|x|,|y|,|z| \to 0} \frac{1}{(1 + x^2)} = 1,$$

$$f^\infty = \limsup_{|x|,|y|,|z| \to \infty} \max_{t \in [0,1]} \frac{e^t \left(|x| + |y| + |z| \right)}{(1 + x^2) \left(|x| + |y| + |z| \right)} = \lim_{|x|,|y|,|z| \to \infty} \frac{e}{(1 + x^2)} = 0.$$

Therefore, from Corollary 4, we deduce that for all $\lambda \in (4, \infty)$, $\mathcal{T}$ has at least a fixed point in the cone K, with independence of the choice of t_1. This fixed point is a nontrivial solution of problem (4).

On the other hand, we will prove that it is not possible to apply Theorem 8 to this example. With the notation introduced in Lemma 7, we have that

$$f_\rho^0 = \inf \left\{ \frac{e^t(|x| + |y| + |z|)}{\rho\,(x^2 + 1)} \, : \, t \in [t_1, 1], \; x \in \left[0, \frac{\rho}{\xi_0(t_1)} \right], \; y \in [0, 2\rho], \; z \in [0, \rho] \right\} = 0$$

and

$$f_\rho^1 = \inf \left\{ \frac{e^t(|x| + |y| + |z|)}{\rho\,(x^2 + 1)} \, : \, t \in [0, 1], \; x \in \left[0, \frac{\rho}{\xi_0(t_1)} \right], \; y \in [0, 2\rho], \; z \in [0, \rho] \right\} = 0,$$

and therefore it does not exist any ρ such that condition (I_ρ^0) holds. Thus Theorem 8 is not applicable to this example.

Example 10. Consider now the following Lidstone fourth order problem:

$$\begin{cases} u^{(4)}(t) = \lambda\, t\, \left(e^{u(t)} + (u'(t))^2 + (u''(t))^2 + (u'''(t))^2\right), & t \in [0,1], \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \tag{5}$$

Notice that fourth order differential equations with this type of boundary conditions have been applied for the study of the bending of simply supported elastic beams (see [21, 24]) or suspension bridges (see [7, 20]).
The Green's function related to the homogeneous problem

$$\begin{cases} u^{(4)}(t) = 0, & t \in [0,1], \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases}$$

has the following expression:

$$G(t,s) = \frac{1}{6} \begin{cases} t\,(1-s)\,(2\,s - s^2 - t^2), & t \le s, \\ s\,(1-t)\,(2\,t - t^2 - s^2), & s < t, \end{cases}$$

which implies that the solutions of problem (5) coincide with the fixed points of

$$\mathcal{T}u(t) = \lambda \int_0^1 G(t,s)\, f(s, u(s), u'(s), u''(s), u'''(s))\, \mathrm{d}\, s, \quad t \in [0,1].$$

Previous operator is a particular case of (1) for $T = 1$, $m = 3$, $k \equiv G$ and $f(t,x,y,z,w) = t\,\left(e^x + y^2 + z^2 + w^2\right)$.

Next, we will give the explicit expressions of the first, second and third derivatives of the Green's function:

$$\frac{\partial\, G}{\partial\, t}(t,s) = \frac{1}{6} \begin{cases} -(1-s)\,(-2\,s + s^2 + 3\,t^2), & t \le s, \\ s\,(2 + s^2 + 3\,t^2 - 6\,t), & s < t, \end{cases}$$

$$\frac{\partial^2\, G}{\partial\, t^2}(t,s) = \begin{cases} -t\,(1-s), & t \le s, \\ -s\,(1-t), & s < t, \end{cases}$$

$$\frac{\partial^3\, G}{\partial\, t^3}(t,s) = \begin{cases} -(1-s), & t < s, \\ s, & s < t, \end{cases}$$

and now we will see that they satisfy the required hypotheses:

(H_1) As in previous example, it is easy to verify that this condition holds.

(H_2) The Green's function G is nonnegative on $[0,1] \times [0,1]$ (in fact it is positive on $(0,1) \times (0,1)$). Therefore $[m_0, n_0] = [0,1]$.

For first derivative it holds that

$$\frac{\partial G}{\partial t}(t,s) \geq 0 \quad \text{for all } t \in [0,t_2],\ s \in [0,1],$$

with $t_2 = 1 - \frac{\sqrt{3}}{3} \approx 0.42265$. Thus $[m_1, n_1] = [0, t_2]$.

With respect to the second derivative, it is immediate to see that it is nonpositive on its square of definition. However it is zero on the boundary of the square, so we could take $[m_2, n_2] = \{0\}$ (it would also be possible to choose $[m_2, n_2] = \{1\}$).

Finally, the third derivative is nonnegative on the triangle

$$\{(t,s) \in [0,1] \times [0,1] :\ s < t\},$$

that is, $[m_3, n_3] = \{1\}$.

(H_3) We have that

$$|G(t,s)| = G(t,s) \leq h_0(s) \quad \text{for all } t \in [0,1],\ s \in [0,1],$$

where

$$h_0(s) = \frac{1}{9\sqrt{3}} \begin{cases} s\,(1-s^2)^{\frac{3}{2}}, & 0 \leq s \leq \frac{1}{2}, \\ (1-s)\,(2\,s - s^2)^{\frac{3}{2}}, & \frac{1}{2} < s \leq 1. \end{cases}$$

Previous inequality has been proved in [26].

Previous inequality is optimal in the sense that for each $s \in [0,1]$ there exists at least one value of $t \in [0,1]$ for which the equality is satisfied.

Analogously, it holds that

$$\left|\frac{\partial G}{\partial t}(t,s)\right| \leq h_1(s) \quad \text{for all } t \in [0,1],\ s \in [0,1],$$

for

$$h_1(s) = \frac{1}{6}\,s\,(1-s) \begin{cases} 2-s, & 0 \leq s \leq \frac{1}{2}, \\ 1+s, & \frac{1}{2} < s \leq 1, \end{cases}$$

and the equality holds for $0 \leq s \leq \frac{1}{2}$ at $t = 0$ and for $\frac{1}{2} < s \leq 1$ at $t = 1$, so this choice of h_1 is optimal.

For the second derivative, we have that

$$\left|\frac{\partial^2 G}{\partial t^2}(t,s)\right| \leq s(1-s) \equiv h_2(s) \quad \text{for all } t \in [0,1],\ s \in [0,1],$$

and the inequality is optimal in the same way than for the Green's function G.

With regard to the third derivative, it satisfies that

$$\left|\frac{\partial^3 G}{\partial t^3}(t,s)\right| \leq \max\{s, 1-s\} \equiv h_3(s) \quad \text{for } t \in [0,1] \text{ and a.e. } s \in [0,1],$$

and the inequality is also optimal.

(H_4) If we choose $\phi_0(s) = h_0(s)$, given in (H_3), and $[c_0, d_0] = [0, 1]$, then for any closed interval $[a_0, b_0] \subset (0, 1)$, it is possible to find a constant $\xi_0(a_0, b_0) \in (0, 1)$ such that

$$G(t, s) \geq \xi_0(a_0, b_0)\, \phi_0(s), \quad \text{for all } t \in [a_0, b_0],\ s \in [0, 1].$$

This has been proved in [26] with an explicit function. Of course, it is satisfied that the bigger the interval $[a_0, b_0]$ is, the smaller $\xi_0(a_0, b_0)$ needs to be.

Analogously, we can take $\phi_1(s) = h_1(s)$ and $[c_1, d_1] = [0, 1]$ and it holds that for any interval $[0, b_1]$, with $b_1 < 1 - \frac{\sqrt{3}}{3}$, there exists $\xi_1(b_1) \in (0, 1)$ such that

$$\frac{\partial G}{\partial t}(t, s) \geq \xi_1(b_1)\, \phi_1(s), \quad \text{for all } t \in [0, b_1],\ s \in [0, 1].$$

Finally, with respect to the second derivative of the Green's function G, it does not exist any pair of function ϕ_2 and constant ξ_2 such that the inequalities in (H_4) hold. The same occurs with the third derivative of G. Therefore $J_1 = \{0,\ 1\}$.

(H_5) It is a direct consequence of (H_2).

Clearly, f satisfies (H_6) and (H_7).

As a consequence of the properties of the Green's function that we have just seen, we will work in the cone

$$K = \left\{ \begin{array}{c} u \in \mathcal{C}^3([0, 1], \mathbb{R}) : u(t) \geq 0,\ t \in [0, 1], \quad u'(t) \geq 0,\ t \in [0, t_2], \\ u''(t) \geq 0,\ t \in \{0, 1\}, \quad u'''(1) \geq 0, \\ \min\limits_{t \in [a_0, b_0]} u(t) \geq \xi_0(a_0, b_0)\, \|u\|_{[0,1]}, \\ \min\limits_{t \in [0, b_1]} u'(t) \geq \xi_1(b_1)\, \|u'\|_{[0,1]} \end{array} \right\}.$$

Moreover, we will make all the calculations with the values $[a_0, b_0] = [0.1, 0.9]$, $\xi_0 = \frac{1}{4}$, $[0, b_1] = \left[0, \frac{1}{3}\right]$ and $\xi_1 = \frac{1}{6}$.

In this case, with the notation introduced in Lemma 6, we have that

$$\frac{1}{N} = \max\left\{ \frac{5}{384}, \frac{1}{24}, \frac{1}{8}, \frac{1}{2} \right\} = \frac{1}{2}$$

and

$$f^{\rho_2} = \sup\left\{ \frac{t\left(e^x + y^2 + z^2 + w^2\right)}{\rho_2} :\ t \in [0, 1],\ x,\ y,\ z,\ w \in [-\rho_2, \rho_2] \right\} = \frac{e^{\rho_2} + 3\,\rho_2^2}{\rho_2},$$

and so $\left(I^1_{\rho_2}\right)$ holds for any $\lambda < \frac{2\,\rho_2}{e^{\rho_2} + 3\,\rho_2^2}$.

Analogously, with the notation used in Lemma 7,

$$\frac{1}{M_0} = \frac{29}{7500}, \quad \frac{1}{M_1} = \frac{7}{1944},$$

$$f^0_{\rho_1} = \inf\left\{\frac{t\left(e^x + y^2 + z^2 + w^2\right)}{\rho_1} : \ t \in [0.1, 0.9],\ x \in [0, 4\,\rho_1]\,,\ y \in [0, 6\,\rho_1],\ z,\, w \in [0, \rho_1]\right\} = \frac{0.1}{\rho_1}$$

and

$$f^1_{\rho} = \inf\left\{\frac{t\left(e^x + y^2 + z^2 + w^2\right)}{\rho_1} : \ t \in \left[0, \frac{1}{3}\right],\ x \in [0, 4\,\rho_1]\,,\ y \in [0, 6\,\rho_1],\ z,\, w \in [0, \rho_1]\right\} = 0,$$

and thus $\left(I^0_{\rho_1}\right)$ holds for $\lambda > \frac{75000\,\rho_1}{29}$.

Therefore, as a consequence of (C_1) in Theorem 8, for any pair of values ρ_1, $\rho_2 > 0$ such that $\rho_1 < c\,\rho_2 = \frac{\rho_2}{6}$ and

$$\frac{75000\,\rho_1}{29} < \frac{2\,\rho_2}{e^{\rho_2} + 3\,\rho_2^2},$$

problem (5) has at least a nontrivial solution for all

$$\lambda \in \left(\frac{75000\,\rho_1}{29}, \frac{2\,\rho_2}{e^{\rho_2} + 3\,\rho_2^2}\right).$$

In particular, there exists at least a nontrivial solution of (5) for all

$$\lambda \in (0,\, 0.4171)\,.$$

On the other hand, we obtain that:

$$f_0 = \liminf_{|x|,|y|,|z|,|w| \to 0}\ \min_{t \in [0,1]} \frac{t\left(e^x + y^2 + z^2 + w^2\right)}{|x| + |y| + |z| + |w|} = 0,$$

and thus neither Theorem 3 nor Corollary 4 can be applied to this example.

6 Application to some $2n$-*th* order problems

In this section we contribute to fill some gaps on the study of general $2n$-*th* order Lidstone boundary value problems, for $n \geq 1$, usually the nonlinearities may depend only on the even derivatives (see, for example, [6, 19, 25, 28]), or general complementary Lidstone problems, (see [27] and the references therein). Therefore, we consider the following problem, with a full nonlinearity,

$$\begin{cases} u^{(2n)}(t) = f\left(t, u(t), \ldots, u^{(2n-1)}(t)\right), \quad t \in [0,1], \\ u^{(2k)}(0) = u^{(2k)}(1) = 0, \quad k = 0, \ldots, n-1. \end{cases} \tag{6}$$

Let $G(t,s)$ be the Green's function related to the homogeneous problem

$$\begin{cases} u^{(2n)}(t) = 0, \quad t \in [0,1], \\ u^{(2k)}(0) = u^{(2k)}(1) = 0, \quad k = 0, \dots, n-1. \end{cases}$$

It can be checked that, for $n \geq 2$, $g(t,s) = \frac{\partial^{2n-4} G}{\partial t^{2n-4}}(t,s)$ is the Green's function related to the problem

$$\begin{cases} u^{(4)}(t) = 0, \quad t \in [0,1], \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases}$$

whose explicit expression has been calculated in Example 10. As a consequence of the calculations made in that example we know that the following facts hold, for $n \geq 2$:

- $\frac{\partial^{2n-4} G}{\partial t^{2n-4}}(t,s) = g(t,s) \geq 0$ on $[0,1] \times [0,1]$ and $\frac{\partial^{2n-4} G}{\partial t^{2n-4}}(t,s) = 0$ on the boundary of the square.
- $\frac{\partial^{2n-3} G}{\partial t^{2n-3}}(t,s) = \frac{\partial g}{\partial t}(t,s) \geq 0$ on $[0,t_2] \times [0,1]$, with $t_2 = 1 - \frac{\sqrt{3}}{3}$.
- $\frac{\partial^{2n-2} G}{\partial t^{2n-2}}(t,s) = \frac{\partial^2 g}{\partial t^2}(t,s) \leq 0$ on $[0,1] \times [0,1]$, and $\frac{\partial^{2n-2} G}{\partial t^{2n-2}}(t,s) = 0$ on the boundary of the square.
- $\frac{\partial^{2n-1} G}{\partial t^{2n-1}}(t,s) = \frac{\partial^3 g}{\partial t^3}(1,s) \geq 0$ for $s \in [0,1]$.

With this information, we can obtain some results about the constant sign both of the derivatives of smaller order of G and of the Green's function itself.

1. Since $\frac{\partial^{2n-4} G}{\partial t^{2n-4}}(t,s) \geq 0$, for $n \geq 3$, it holds that for each fixed $s \in [0,1]$, $\frac{\partial^{2n-5} G}{\partial t^{2n-5}}(\cdot,s)$ is increasing.

   Assume that it is nonnegative. Then it would occur that $\frac{\partial^{2n-6} G}{\partial t^{2n-6}}(\cdot,s)$ is also increasing and, since from the boundary value conditions it holds that $\frac{\partial^{2n-6} G}{\partial t^{2n-6}}(0,s) = \frac{\partial^{2n-6} G}{\partial t^{2n-6}}(1,s) = 0$, we would conclude that $\frac{\partial^{2n-6} G}{\partial t^{2n-6}}(t,s) = 0$ on $[0,1] \times [0,1]$, which is not possible.

   The same argument holds if we assume that $\frac{\partial^{2n-5} G}{\partial t^{2n-5}}(\cdot,s)$ is nonpositive.

   Therefore, necessarily $\frac{\partial^{2n-5} G}{\partial t^{2n-5}}(\cdot,s)$ is sign-changing and, since it is increasing, we know for sure that $\frac{\partial^{2n-5} G}{\partial t^{2n-5}}(0,s) < 0$ and $\frac{\partial^{2n-5} G}{\partial t^{2n-5}}(1,s) > 0$ for all $s \in [0,1]$.

2. Now, since $\frac{\partial^{2n-5} G}{\partial t^{2n-5}}(\cdot,s)$ is sign-changing and increasing, $\frac{\partial^{2n-6} G}{\partial t^{2n-6}}(\cdot,s)$ will be first decreasing and then increasing. This together with the boundary value conditions $\frac{\partial^{2n-6} G}{\partial t^{2n-6}}(0,s) = \frac{\partial^{2n-6} G}{\partial t^{2n-6}}(1,s) = 0$ implies that $\frac{\partial^{2n-6} G}{\partial t^{2n-6}}$ is nonpositive.

3. Since $\frac{\partial^{2n-6} G}{\partial t^{2n-6}}$ is nonpositive, we can follow an analogous argument to the one made in 1. to deduce that $\frac{\partial^{2n-7} G}{\partial t^{2n-7}}$ is sign-changing and decreasing. In particular this implies that $\frac{\partial^{2n-7} G}{\partial t^{2n-7}}(0,s) > 0$ and $\frac{\partial^{2n-7} G}{\partial t^{2n-7}}(1,s) < 0$ for all $s \in [0,1]$ and $n \geq 4$.

4. Finally, arguing analogously to 2., we can deduce that $\frac{\partial^{2n-8} G}{\partial t^{2n-8}}$ is nonnegative on $[0,1] \times [0,1]$, for $n \geq 4$.

We note that we could repeat all the previous arguments iteratively and this way we could deduce the following sign-criteria for the derivatives of G. So, for $n \geq \frac{k}{2}$:

- If $k \equiv 0 \pmod 4$, then $\frac{\partial^{2n-k} G}{\partial t^{2n-k}}(t,s) \geq 0$ on $[0,1] \times [0,1]$.
- If $k \equiv 1 \pmod 4$, then $\frac{\partial^{2n-k} G}{\partial t^{2n-k}}(\cdot,s)$ is sign-changing and increasing for every $s \in [0,1]$. In particular, $\frac{\partial^{2n-k} G}{\partial t^{2n-k}}(0,s) < 0$ and $\frac{\partial^{2n-k} G}{\partial t^{2n-k}}(1,s) > 0$ for every $s \in [0,1]$.
- If $k \equiv 2 \pmod 4$, then $\frac{\partial^{2n-k} G}{\partial t^{2n-k}}(t,s) \leq 0$ on $[0,1] \times [0,1]$.
- If $k \equiv 3 \pmod 4$, then $\frac{\partial^{2n-k} G}{\partial t^{2n-k}}(\cdot,s)$ is sign-changing and decreasing for every $s \in [0,1]$. In particular, $\frac{\partial^{2n-k} G}{\partial t^{2n-k}}(0,s) > 0$ and $\frac{\partial^{2n-k} G}{\partial t^{2n-k}}(1,s) < 0$ for every $s \in [0,1]$.

In particular, if n is even, we could deduce that $G(t,s) \geq 0$ on $[0,1] \times [0,1]$ and, for n odd, $G(t,s) \leq 0$ on $[0,1] \times [0,1]$.

Therefore, the Green's function and its derivatives satisfy the required hypotheses:

(H_1) As in Example 10, this condition holds as a direct consequence of the general properties of the Green's function.

(H_2) As we have just proved, we could take $[m_{2n-i}, n_{2n-i}] = [0,1]$ for $i \equiv 0 \pmod 4$, $[m_{2n-i}, n_{2n-i}] = \{1\}$ for $i \equiv 1 \pmod 4$, $[m_{2n-i}, n_{2n-i}] = \{0\}$ for $i \equiv 2 \pmod 4$ and $[m_{2n-i}, n_{2n-i}] = \{0\}$ for $i \equiv 3 \pmod 4$.

(H_3) It is enough to take $h_i(s) = \max\left\{ \left| \frac{\partial^i G}{\partial t^i}(t,s) \right| : \ t \in [0,1] \right\}$, for $i \in J$.

(H_4) For $n \geq 2$, we could take $J_1 = \{2n-4, 2n-3\}$. As a consequence of Example 10, we know that

$$\left| \frac{\partial^{2n-4} G}{\partial t^{2n-4}}(t,s) \right| = \frac{\partial^{2n-4} G}{\partial t^{2n-4}}(t,s) = g(t,s) \leq \phi_{2n-4}(s),$$

with

$$\phi_{2n-4}(s) = \frac{1}{9\sqrt{3}} \begin{cases} s\,(1-s^2)^{\frac{3}{2}}, & 0 \leq s \leq \frac{1}{2}, \\ (1-s)\,(2\,s-s^2)^{\frac{3}{2}}, & \frac{1}{2} < s \leq 1. \end{cases}$$

Moreover, it holds that for any closed interval $[a_{2n-4}, b_{2n-4}] \subset [0,1]$, there exists a constant $\xi_{2n-4}(a_{2n-4}, b_{2n-4}) \in (0,1)$ such that

$$\frac{\partial^{2n-4} G}{\partial t^{2n-4}}(t,s) \geq \xi_{2n-4}(a_{2n-4}, b_{2n-4})\, \phi_{2n-4}(s), \quad \text{for all } t \in [a_{2n-4}, b_{2n-4}], \ s \in [0,1].$$

Analogously, from Example 10 we know that

$$\left|\frac{\partial^{2n-3} G}{\partial t^{2n-3}}(t,s)\right| = \left|\frac{\partial g}{\partial t}(t,s)\right| \le \phi_{2n-3}(s) = \frac{1}{6}\, s\,(1-s) \begin{cases} 2-s, & 0 \le s \le \frac{1}{2}, \\ 1+s, & \frac{1}{2} < s \le 1, \end{cases}$$

for all $t \in [0,1]$, $s \in [0,1]$ and for any interval $[0, b_{2n-3}]$, with $b_{2n-3} < 1 - \frac{\sqrt{3}}{3}$ there exists $\xi_{2n-3}(b_{2n_3}) \in (0,1)$ such that

$$\frac{\partial^{2n-3} G}{\partial t^{2n-3}}(t,s) \ge \xi_{2n-3}(b_{2n_3})\, \phi_{2n-3}(s), \quad \text{for all } t \in [0, b_{2n-3}],\ s \in [0,1].$$

(H_5) As we have already seen, it holds that $[m_{2n-4}, n_{2n-4}] = [0,1]$.

Then, we could work in the cone, for $n \in \mathbb{N}$ such that $n \ge \max\left\{2, \frac{i}{2}\right\}$,

$$K = \left\{ \begin{array}{ll} u \in \mathcal{C}^{2n-1}([0,1],\mathbb{R}) : & u^{(2n-i)}(t) \ge 0,\ t \in [0,1],\ i \equiv 0 \mod 4, \\ & u^{(2n-i)}(1) \ge 0,\ i \equiv 1 \mod 4, \\ & u^{(2n-i)}(0) \ge 0,\ i \equiv 2 \mod 4, \\ & u^{(2n-i)}(0) \ge 0,\ i \equiv 3 \mod 4 \\ & \min\limits_{t \in [a_{2n-4}, b_{2n-4}]} u^{(2n-4)}(t) \ge \xi_{2n-4}(a_{2n-4}, b_{2n-4})\, \|u^{(2n-4)}\|_{[0,1]}, \\ & \min\limits_{t \in [0, b_{2n-3}]} u^{(2n-3)}(t) \ge \xi_{2n-3}(b_{2n-3})\, \|u^{(2n-3)}\|_{[0,1]} \end{array} \right\}.$$

Thus, for any nonlinearity f satisfying (H_6) and either conditions of Theorem 3 or of Theorem 8, it is possible to find nontrivial solution of problem (6).

Acknowledgement

This paper was mostly written during a stay of Lucía López-Somoza in Évora. Lucía López-Somoza would like to acknowledge her gratitude towards the Department of Mathematics of the University of Évora, and specially towards Professor Feliz Minhós for his kindness and hospitality.

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