GLOBAL $W^{2,1+\varepsilon}$ ESTIMATES FOR MONGE-AMPÈRE EQUATION WITH NATURAL BOUNDARY CONDITION

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ABSTRACT. For the Monge-Ampère equation with a right-hand side bounded away from 0 and infinity, we show that the solution, subject to the natural boundary condition arising in optimal transport, is in $W^{2,1+\varepsilon}$ up to the boundary.

1. INTRODUCTION

Let Ω and Ω^* be two bounded convex domains in $\mathbb{R}^d$, and f be a function on Ω satisfying

$$(1.1) \qquad \frac{1}{\Lambda} \le f \le \Lambda$$

for some positive constant Λ . In this work, we study the regularity of convex Alexandrov solutions to the following problem

$$(1.2) \qquad \begin{cases} \det(D^2 u) = f & \text{in } \Omega, \\ \nabla u(\Omega) = \Omega^*. \end{cases}$$

For the definition of Alexandrov solutions to the Monge-Ampère equation, the reader can consult Figalli [F1] or Giutérrez [G]. Here we point out that this is the natural boundary value problem arising from the study of the theory of optimal transport.

To be precise, suppose ν and ν^* are two probability measures supported on Ω and Ω^* with density functions g and g^* respectively, then (1.2) is satisfied by the potential of the optimal transport that pushes-forward $\nu = g dx$ to $\nu^* = g^* dx$ [B, V]. In this case, the right-hand side is $f = \frac{g}{g^* \circ \nabla u}$.

When f is continuous, the regularity of solutions to (1.2) has been studied extensively. Caffarelli showed that u is locally in $W^{2,p}$ in the interior of Ω for all $p > 0$ [C1]. If f is further assumed to be Hölder continuous, Caffarelli showed that $D^2 u$ is Hölder continuous in the interior of Ω. When the domains are $C^{1,1}$, Chen-Liu-Wang [CLW] proved that these estimates hold up to the boundary of Ω, based on earlier results by Caffarelli [C3] and Urbas [U]. In two dimensions, we recently established the optimal global $W^{2,p}$ estimate without any regularity assumptions on the domains except their convexity [SY]. Still in two dimension, if the domains are assumed to be $C^{1,\alpha}$, $D^2 u$ is shown to be Hölder continuous by Chen-Liu-Wang [CLW2].

For several important applications, however, it is necessary to understand the regularity of u when f fails to be continuous. In the optimal transport problem described above, f does not enjoy any regularity if the density functions g and g^*

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are only assumed to be bounded away from 0 and infinity. This problem also has deep implications in the study of semi-geomstrophic equations [F2].

When f satisfies (1.1) but is allowed to be discontinuous, much less is known about the regularity of u. Caffarelli showed that u is C^{1,α_0} up to the boundary for some small dimensional α_0 [C2]. In terms of Sobolev regularity, Wang [W] showed that for any $p > 1$, one can find sufficiently large Λ such that u fails to be in $W^{2,p}$ even in the interior of the domain. Nevertheless, for fixed Λ, De Philippis-Figalli [DF] was able to show that u is in $W^{2,1}$ in the interior of Ω. This was later improved to an interior $W^{2,1+\varepsilon}$-estimate independently by De Philippis-Figalli-Savin [DFS] and Schmidt [Sch].

In this work, we extend this interior $W^{2,1+\varepsilon}$-estimate up to the boundary. To be precise, our main result is

Theorem 1.1. *Suppose Ω and Ω^* are bounded convex domains in $\mathbb{R}^d$. Let u be an Alexandrov solution to (1.2) with $\frac{1}{\Lambda} \le f \le \Lambda$ for some positive constant Λ.*

Then there are positive constants ε, depending only on d and Λ, and C, further depending on the inner and outer radii of Ω and Ω^*, such that

$$\int_\Omega |D^2 u|^{1+\varepsilon} \le C.$$

The exponent $1 + \varepsilon$ is optimal due to the examples of Wang [W]. Also, the result is sharp in the sense that the estimate has to depend on d, Λ and the inner and outer radii of Ω and Ω^*.

We'd like to point out that no regularity of Ω and Ω^* is assumed. In this case, it remains an interesting problem whether a global $W^{2,p}$-estimate can be established in the spirit of [SY].

This paper is structured as follows: in Section 2, we introduce some notations and collect some useful preliminary results. In Section 3, we give estimates in the normalized picture. The scaled versions of these estimates are applied to our solution u in Section 4. In the last section we give the proof Theorem 1.1.

2. Preliminaries

2.1. **Extension of the solution to $\mathbb{R}^d$.** Let u be an Alexandrov solution to (1.2), we can extend it to the entire $\mathbb{R}^d$ by

$$x \in \mathbb{R}^d \mapsto \sup_{y \in \Omega}(u(y) + \nabla u(y) \cdot (x - y)).$$

The resulting function, still denoted by u, is a convex function solving the following equation in the Alexandrov sense [C2]:

(2.1)
$$\begin{cases} \det(D^2 u) = f\chi_\Omega & \text{in } \mathbb{R}^d, \\ \nabla u(\Omega) = \Omega^*. \end{cases}$$

For a set S, χ_S denotes its characteristic function.

We assume $u \in C^2(\overline{\Omega})$ in the rest of the paper, and prove Theorem 1.1 for such solutions. This implies the estimate for general solutions via a standard approximation procedure.

2.2. **Sections and their properties.** Sections are a fundamental tool in the study the Monge-Ampère equation. Among several related notions of sections, the centered section introduced in [C1] is the most convenient for our purpose. We give its definition here.

Definition 2.1. Let $x_0 \in \overline{\Omega}$, $h > 0$, *the centered section* of u of height h at x_0 is defined by

$$S_h[u](x_0) = \{y \in \mathbb{R}^d | y < u(x_0) + p \cdot (y - x_0) + h\}.$$

Here $p \in \mathbb{R}^d$ is chosen such that the center of mass of $S_h[u](x_0)$ stays at x_0, that is,

$$\frac{1}{|S_h[u](x_0)|} \int_{S_h[u](x_0)} y dy = x_0.$$

For the existence of such p, see [C2].

By the convexity of u, these sections are bounded convex subsets of $\mathbb{R}^d$. In order to describe their shapes, we need the following lemma due to Fritz John [J]:

Lemma 2.1 (John's lemma). *For any bounded convex subset S of $\mathbb{R}^d$, there is an ellipsoid E with the same center of mass as S such that*

$$E \subset S \subset \alpha_d E.$$

This factor α_d depends only on the dimension d.

Such ellipsoid E is called the John ellipsoid of S.

For a set S and a positive constant c, cS denotes the dilation of S by a factor of c with respect to the center of mass of S.

When E is an ellipsoid, we write $E = x + \sum \lambda_j \omega_j$ when x is the center of E, ω_j's are the directions of the principal axises of E, and λ_j's are the length of the axis in the direction of ω_j.

To each such ellipsoid $E = x + \sum \lambda_j \omega_j$, we associate the matrix

$$(2.2) \qquad M_E = \sum \frac{1}{\lambda_j} \omega_j \otimes \omega_j,$$

where $\otimes$ denotes the tensor product. This is the matrix that maps E to a unit ball.

Sections share many properties with Euclidean balls. In particular, one has a Besicovitch-type covering lemma with sections. The following is based on Caffarelli-Giutiérrez [CG]:

Lemma 2.2. *Let A be a subset of Ω. Suppose for each $x \in A$, a section $S_{h_x}[u](x)$ is chosen such that the heights h_x are uniformly bounded. Let $\mathcal{F}$ denote this family of sections.*

There are constants $\eta_0 \in (0,1)$ and K, depending only on d and Λ, such that there is a countable subfamily $\{S_{h_j}[u](x_j)\}$ of $\mathcal{F}$ satisfying the following:

(1) $A \subset \cup S_{h_j}[u](x_j)$;
(2) $\sum \chi_{S_{\eta_0 h_j}[u](x_j)} \leq K$.

3. ESTIMATES FOR NORMALIZED SOLUTIONS

In this section we establish several key estimates in the normalized picture. Later these are applied to our solution after rescaling. The methods are motivated by De Philippis-Figalli [DF] and De Philippis-Figali-Savin [DFS]. However, since we are

dealing with global regularity estimates, we need more detailed analysis concerning the interaction between the sections and the boundary of the domain.

The following **assumptions** are in effect throughout this section:

(1) U is a convex domain in $\mathbb{R}^d$ containing a point x_0;

(2) v is a C^2 convex function in $\overline{U}$ extended to $\mathbb{R}^d$ by
$$x \mapsto \sup_{y \in U}(v(y) + \nabla v(y) \cdot (x - y));$$

(3) $Z := \{v < 0\}$ is centered at x_0 and normalized in the sense that $B_1(x_0) \subset Z \subset B_{\alpha_d}(x_0)$, where α_d is the constant in Lemma 2.1;

(4) $|\nabla v| \leq L_0$ in Z;

(5) $\det(D^2 v) = g\chi_U$ in Z with $\frac{1/\Lambda}{|Z \cap U|} \leq g \leq \frac{\Lambda}{|Z \cap U|}$;

(6) $E = \sum \lambda_j e_j$ is the John ellipsoid for $Z \cap U$. Here $\{e_j\}$ is the standard basis of $\mathbb{R}^d$. In particular, $Z \cap U$ is centered at 0.

Throughout this section, constants depending only on d, Λ and L_0 are called *universal constants*.

Denote
$$h_0 = |\inf_Z v|,$$
assumptions (3) and (5) imply that $0 < c \leq h_0 \leq C$ for some universal c and C.

Inside $Z \cap U$, we expect v to behave like the parabola $p(x) = (M_E x) \cdot x$, where M_E is the matrix defined in (2.2). An application of the ABP estimate [G] shows that this is indeed true in a large portion of Z:

Lemma 3.1. *Let* $Z_{\eta_0} = S_{\eta_0 h_0}[v](x_0)$, *where* η_0 *is the constant in Lemma 2.2. Then there are universal constants* C_0 *and* δ_0 *such that*

(3.1)
$$\frac{|Z_{\eta_0} \cap U \cap \{C_0^{-1} M_E \leq D^2 v \leq C_0 M_E\}|}{|Z \cap U|} \geq \delta_0.$$

Proof. Step 1: Construction of comparison functions. By the engulfing property [G], there is a universal constant $0 < c_0 < 1$ such that $c_0 Z \subset Z_{\eta_0} \subset Z$. Consequently, the ellipsoid $\tilde{E} = c_0 \alpha_d \sum \lambda_j e_j$ satisfies $\frac{1}{\alpha_d}\tilde{E} \subset Z_{\eta_0} \cap U \subset \tilde{E}$.

Define a quadratic polynomial $\tilde{p} : \tilde{E} \to \mathbb{R}$ by $\tilde{p}(x) = \sum \frac{1}{c_0 \alpha_d \lambda_j} x_j^2$, and extend $\tilde{p}$ to the entire $\mathbb{R}^d$ by $\tilde{p}(x) = \sup_{y \in \tilde{E}}(\tilde{p}(y) + \nabla \tilde{p}(y) \cdot (x - y))$.

Then one has
$$\sup_{y \in \mathbb{R}^d} |\nabla \tilde{p}(y)| \leq \sup_{y \in \tilde{E}} |\nabla \tilde{p}(y)| \leq 1,$$
and

(3.2)
$$0 \leq \tilde{p} \leq C \text{ in } Z_{\eta_0}$$

for some universal C.

Up to subtracting an affine function, we have $v = \eta_0 h_0$ along ∂Z_{η_0}, and $v(x_0) = 0$.

In particular, if we define $p = \frac{1}{2C}\eta_0 h_0 \tilde{p} + \frac{1}{2}\eta_0 h_0$, where C is the constant in (3.2), then
$$p \leq v \text{ on } \partial Z_{\eta_0}$$
and
$$\frac{1}{2}\eta_0 h_0 \leq p \leq \eta_0 h_0 \text{ in } Z_{\eta_0}.$$

Let $w = v - p$.

Then
$$w \geq 0 \text{ on } \partial Z_{\eta_0},$$
and
$$|\inf_{Z_{\eta_0}} w| \geq \frac{1}{2}\eta_0 h_0.$$

Step 2: The ABP estimate. If Γ_w is the convex envelop of w in Z_{η_0}, then the ABP estimate [G] implies

(3.3) $$ch_0^d \leq |\nabla\Gamma_w(Z_{\eta_0} \cap \{\Gamma_w = w\})|$$

for some dimensional c.

For $\bar{x} \in Z_{\eta_0} \cap \{\Gamma_w = w\}$, there is an affine function ℓ such that
$$\ell + p \leq v \text{ in } Z_{\eta_0}$$
and
$$\ell(\bar{x}) + p(\bar{x}) = v(\bar{x}).$$

In particular, one has
$$\nabla\ell(\bar{x}) + \nabla p(\bar{x}) = \nabla v(\bar{x}).$$

Step 3: Localizing to U. By assumption (2) at the beginning of this section, either $\bar{x} \in U$, or there is a point $\bar{y} \in Z_{\eta_0} \cap U$ such that $\nabla v(\bar{y}) = \nabla v(\bar{x})$, and that v is affine along the line segment between $\bar{x}$ and $\bar{y}$.

By convexity, one has
$$\begin{aligned}
\ell(\bar{y}) + p(\bar{y}) &\geq (\ell+p)(\bar{x}) + \nabla(\ell+p)(\bar{x}) \cdot (\bar{y}-\bar{x}) \\
&= v(\bar{x}) + \nabla v(\bar{x}) \cdot (\bar{y}-\bar{x}) \\
&= v(\bar{y}).
\end{aligned}$$

Together with $\ell + p \leq v$ in Z_{η_0}, this implies $\bar{y} \in Z_{\eta_0} \cap \{w = \Gamma_w\}$.

In particular,
$$\Gamma_w(\bar{x}) = \nabla\ell(\bar{x}) = \nabla\ell(\bar{y}) \in \Gamma_w(Z_{\eta_0} \cap U \cap \{w = \Gamma_w\}).$$

Since this is true for all $\bar{x} \in Z_{\eta_0} \cap \{\Gamma_w = w\}$, we conclude
$$\nabla\Gamma_w(Z_{\eta_0} \cap \{\Gamma_w = w\}) \subset \nabla\Gamma_w(Z_{\eta_0} \cap U \cap \{\Gamma_w = w\}).$$

Step 4: Proof of (3.1). Using this inclusion in (3.3) and note that $D^2\Gamma_w \leq D^2v$ in $\{\Gamma_w = w\}$, we have
$$\begin{aligned}
ch_0^d &\leq |\nabla\Gamma_w(Z_{\eta_0} \cap U \cap \{\Gamma_w = w\})| \\
&\leq \int_{Z_{\eta_0} \cap U \cap \{\Gamma_w = w\}} \det(D^2\Gamma_w) \\
&\leq \int_{Z_{\eta_0} \cap U \cap \{\Gamma_w = w\}} \det(D^2 v) \\
&\leq \frac{\Lambda}{|Z \cap U|}|Z_{\eta_0} \cap U \cap \{\Gamma_w = w\}|.
\end{aligned}$$

For the last inequality, we used assumption (5) at the beginning of this section.

Since h_0 is universal, the estimate above implies
$$|Z_{\eta_0} \cap U \cap \{\Gamma_w = w\}| \geq \delta_0|Z \cap U|$$

for some universal $\delta_0 > 0$.

To get (3.1), it suffices to note that in $Z_{\eta_0} \cap U \cap \{\Gamma_w = w\}$, one has

$$D^2 v \geq D^2 p = c M_E$$

for some universal c.   □

We now use the previous lemma to estimate the integral of pure second derivatives of v in $Z \cap U$ in terms of the integral over 'the good set'. We first estimate second order derivatives in the directions along the axises of $\mathbb{R}^d$:

Lemma 3.2. *Under the same assumptions as in Lemma 3.1, there is a universal constant C such that*

$$(3.4) \qquad \int_{Z \cap U} v_{jj} \leq C \int_{Z_{\eta_0} \cap U \cap \{C_0^{-1} M_E \leq D^2 v \leq C_0 M_E\}} v_{jj}$$

for each $j = 1, 2, \ldots, d$.

Proof. To simply our notations, let's denote 'the good set' by

$$G = Z_{\eta_0} \cap U \cap \{C_0^{-1} M_E \leq D^2 v \leq C_0 M_E\}.$$

With Lemma 3.1, the right-hand side of (3.4) can be bounded from below by

$$\int_G v_{jj} \geq C_0^{-1} \frac{1}{\lambda_j} |G|$$

$$\geq C_0^{-1} \frac{1}{\lambda_j} \delta_0 |Z \cap U|.$$

Since $E = \sum \lambda_j e_j$ is the John ellipsoid for $Z \cap U$, we have

$$|Z \cap U| \geq c \lambda_1 \lambda_2 \ldots \lambda_d$$

for some dimensional c. As a result,

$$(3.5) \qquad \int_G v_{jj} \geq c \lambda_1 \lambda_2 \ldots \lambda_d / \lambda_j$$

for some universal c.

Now we estimate the left-hand side of (3.4).

Define $v_{M_E}(x) = v(M_E^{-1} x) = v(\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_d x_d)$, then

$$\frac{\partial}{\partial x_j} v_{M_E}(x) = \lambda_j \frac{\partial}{\partial x_j} v(M^{-1} x).$$

By assumption (4) at the beginning of this section, we have $|\frac{\partial}{\partial x_j} v_{M_E}| \leq L_0 \lambda_j$ in $M_E(Z)$.

The left-hand side of (3.4) can be computed as

$$\int_{Z \cap U} v_{jj} = \int_{M_E(Z \cap U)} v_{jj}(M_E^{-1} x) \det(M_E^{-1}) dx$$

$$= \det(M_E^{-1}) \int_{M_E(Z \cap U)} \frac{1}{\lambda_j^2} \frac{\partial^2}{\partial x_j^2} v_{M_E}(x) dx$$

$$= \det(M_E^{-1}) \frac{1}{\lambda_j^2} \int_{\partial M_E(Z \cap U)} \frac{\partial}{\partial x_j} v_{M_E} \nu \cdot e_j,$$

where ν is the outward unit normal to $M_E(Z \cap U)$.

Consequently,

$$\int_{Z \cap U} v_{jj} \le \det(M_E^{-1}) \frac{1}{\lambda_j^2} C \lambda_j \mathcal{H}^{d-1}(\partial M_E(Z \cap U))$$

$$\le C \lambda_1 \lambda_2 \dots \lambda_d / \lambda_j$$

for some universal C. Here we used $B_1 \subset M_E(Z \cap U) \subset B_{\alpha_d}$ to control the $(d-1)$-dimensional Hausdorff measure of $\partial M_E(Z \cap U)$.

Combining this with (3.5), we get the desired estimate. $\qquad \square$

For a general vector $\xi = \sum \xi_j e_j$ in $\mathbb{R}^d$, define $v_{\xi\xi} := (D^2 v \xi) \cdot \xi$. A similar estimate as the one in Lemma 3.2 holds for these second order derivatives.

Lemma 3.3. *Under the same assumptions as in Lemma 3.1, there is a universal constant C such that*

$$(3.6) \qquad \int_{Z \cap U} v_{\xi\xi} \le C \int_{Z_{\eta_0} \cap U \cap \{C_0^{-1} M_E \le D^2 v \le C_0 M_E\}} v_{\xi\xi}.$$

Proof. Again we write

$$G = Z_{\eta_0} \cap U \cap \{C_0^{-1} M_E \le D^2 v \le C_0 M_E\}.$$

By convexity of v, $D^2 v \ge 0$. Thus for $1 \le i, j \le d$ one has

$$v_{ii} v_{jj} \ge v_{ij}^2.$$

Therefore,

$$v_{\xi\xi} = \sum_{j=1}^{d} v_{jj} \xi_j^2 + \sum_{i \ne j} v_{ij} \xi_i \xi_j$$

$$\le \sum_{j=1}^{d} v_{jj} \xi_j^2 + \frac{1}{2} \sum_{i \ne j} (v_{ii} \xi_i^2 + v_{jj} \xi_j^2)$$

$$\le d \sum v_{jj} \xi_j^2.$$

Combining this with Lemma 3.2, we can estimate the left-hand side of (3.6) as

$$\int_{Z \cap U} v_{\xi\xi} \le d \sum \xi_j^2 \int_{Z \cap U} v_{jj}$$

$$\le Cd \sum \xi_j^2 \int_G v_{jj}.$$

Now note that on G, $C_0^{-1} M \le D^2 v \le C_0 M$. Thus

$$\sum v_{jj} \xi_j^2 \le C_0 \sum \frac{1}{\lambda_j} \xi_j^2$$

and

$$(D^2 v \xi) \cdot \xi \ge C_0^{-1} \sum \frac{1}{\lambda_j} \xi_j^2.$$

Therefore we can continue the previous estimate by

$$\int_{Z \cap U} v_{\xi\xi} \le Cd \sum \xi_j^2 \int_G v_{jj}$$

$$\le CdC_0^2 \int_G v_{\xi\xi}.$$

This is the desired estimate. $\square$

4. Estimates in sections of our solution

In this section we rescale the estimates from the previous one, so that they can be applied to our solution u. These computations are more or less standard. Nevertheless, we include them here for completeness.

Lemma 4.1. *Suppose u is a solution to (2.1). For a point $x_0 \in \Omega$ and $h > 0$, let A be the symmetric matrix such that $B_1(x_0) \subset A(S_h[u](x_0)) \subset B_{\alpha_d}(x_0)$.*

Up to rotation and translation, suppose $E = \sum \lambda_j e_j$ is the John ellipsoid for $S_h[u](x_0) \cap \Omega$.

Then there are constants C and C_0, depending only on d and Λ, such that for all $\xi \in \mathbb{R}^d$

$$\int_{S_h[u](x_0) \cap \Omega} u_{\xi\xi} \leq C \int_G u_{\xi\xi},$$

where $G = S_{\eta_0 h[u](x_0)} \cap \Omega \cap \{C_0^{-1} hAM_E \leq D^2 u \leq C_0 hAM_E\}$.

The existence of this normalizing A is a consequence of Lemma 2.1.

Here η_0 is the constant in Lemma 2.2. M_E is the matrix defined in (2.2).

Proof. Let p be the vector such that $S_h[u](x_0) = \{y | u(y) < u(x_0) + p \cdot (y - x_0) + h\}$. Let ℓ be the affine function $x \mapsto p \cdot (x - x_0) + h$.

Define $v(x) = \frac{1}{h}(u - \ell)(A^{-1}x)$, $Z = A(S_h[u](x_0))$, $U = A(\Omega)$ and $g(x) = \frac{1}{|Z \cap U|} f(A^{-1}x)$. Then it is not difficult to see the assumptions (1)-(3) and (5) at the beginning of Section 3 are satisfied, up to a dimensional change of the value of Λ.

Moreover, by the doubling property, $Z' = \{v < 1\}$ is at a positive distance to Z, where the distance depending only on d and Λ. Therefore, the value L_0 as in assumption (4) in Section 3 depends only on d and Λ.

Let $\tilde{E} = A(E)$. Then $\tilde{E}$ is the John ellipsoid for $Z \cap U$ as in assumption (6) at the beginning of Section 3. Suppose $\tilde{E} = \sum \tilde{\lambda}_j \tilde{e}_j$. Denote by $\tilde{M}$ the matrix $\sum \frac{1}{\tilde{\lambda}_j} \tilde{e}_j \otimes \tilde{e}_j$, where $\otimes$ denotes the tensor product. Then Lemma 3.3, applied to the direction $\tilde{\xi} = A\xi$, gives

$$\int_{A(S_h[u](x_0) \cap \Omega)} v_{\tilde{\xi}\tilde{\xi}} \leq C \int_{A(S_{\eta_0 h}[u](x_0) \cap \Omega) \cap \{C_0^{-1} \tilde{M} \leq D^2 v \leq C_0 \tilde{M}\}} v_{\tilde{\xi}\tilde{\xi}}$$

for some C and C_0 depending only on d and Λ.

Back to the original variables, this means

$$\int_{S_h[u](x_0) \cap \Omega} (D^2 u \xi) \cdot \xi \leq C \int_{\tilde{G}} (D^2 u \xi) \cdot \xi,$$

where

$$\tilde{G} = S_{\eta_0 h}[u](x_0) \cap \Omega \cap \{C_0^{-1} hA\tilde{M}A \leq D^2 u \leq C_0 hA\tilde{M}A\}.$$

To conclude, it suffices to note that $\tilde{M}A = M_E$. $\square$

Up to a dimensional constant, this can be upgraded to an estimate for the integral of $|D^2 u|$:

Proposition 4.1. *Under the same assumptions as in Lemma 4.1, there are constants C and C_0, depending only on d and Λ, such that*

$$\int_{S_h[u](x_0) \cap \Omega} |D^2 u| \leq C \int_G |D^2 u|$$

for $G = S_{\eta_0 h[u](x_0)} \cap \Omega \cap \{C_0^{-1} h A M_E \leq D^2 u \leq C_0 h A M_E\}$.

Proof. By summing up the estimate in Lemma 4.1 in d orthogonal directions, we get a similar estimate where the integrand is Δu. From here it suffices to note that for convex functions $|D^2 u| \leq \Delta u \leq d|D^2 u|$. $\square$

Under the assumptions as in Lemma 4.1, the matrix that defines the 'good set' G associated with $S_h[u](x_0)$ is hAM_E. The next result says that this matrix has the correct behaviour when h is large and when $h \to 0$.

To simplify our notations, let's define the matrix A_h and M_h to be the matrices A and M_E as in Lemma 4.1 for the section $S_h[u](x_0)$.

Let $T_h = hA_h M_h$. Then one has

Proposition 4.2. *There is a constant C_1, depending only on d, Λ, and the inner and outer radii of Ω and Ω^*, such that*

$$\frac{1}{C_1} \leq T_1 \leq C_1.$$

There is a constant C_2, depending only on d, such that for $h > 0$ small,

$$\frac{1}{C_2} D^2 u(x_0) \leq T_h \leq C_2 D^2 u(x_0).$$

Proof. By Lipschitz estimate and uniform strict convexity of u [C2], we have

$$B_{r_1}(x_0) \subset S_1[u](x_0) \subset B_{R_1}(x_0)$$

for some r_1 and R_1 depending on d, Λ, and inner and outer radii of Ω and Ω^*.

Consequently, A_1 is bounded from both sides by constants depending only on d, Λ, and inner and outer radii of Ω and Ω^*.

Meanwhile, $S_1[u](x_0) \cap \Omega \subset B_{R_1}(x_0)$ and $|S_1[u](x_0) \cap \Omega| \geq |B_{r_1}(x_0) \cap \Omega| \geq c|B_{r_1}(x_0)|$ for some c depending only on the inner and outer radii of Ω.

As a result, the John ellipsoid for $S_1[u](x_0) \cap \Omega$ has a diameter that is bounded from above and a volume that is bounded from below. Thus M_h is also bounded from both sides by constants depending only on d, Λ, and inner and outer radii of Ω and Ω^*.

Therefore, $\frac{1}{C_1} \leq T_1 = A_1 M_1 \leq C_1$ for some C_1 depending only on d, Λ, and inner and outer radii of Ω and Ω^*.

To see the second statement in the proposition, we first note that when h is small, $S_h[u](x_0) \subset \Omega$ since $x_0 \in \Omega$.

Consequently $A_h = M_h$ for small h and $T_h = hA_h^2$.

By C^2 regularity of u inside Ω, up to subtracting an affine function,

$$u(x) = (D^2 u(x_0)(x - x_0)) \cdot (x - x_0) + o(|x - x_0|^2).$$

Up to a rotation,

$$D^2 u(x_0) = \begin{pmatrix} \eta_1 & 0 & 0 & \dots \\ 0 & \eta_2 & 0 & \dots \\ 0 & 0 & \eta_3 & \dots \\ \vdots & \ddots & & \\ 0 & 0 & \dots & \eta_d \end{pmatrix}.$$

Then $S_h[u](x_0)$ is comparable to $x_0 + \sum (\frac{h}{\eta_j})^{1/2} e_j$. Therefore, up to a dimensional constant, A_h is comparable to

$$\begin{pmatrix} (\frac{\eta_1}{h})^{1/2} & 0 & 0 & \dots \\ 0 & (\frac{\eta_2}{h})^{1/2} & 0 & \dots \\ 0 & 0 & (\frac{\eta_3}{h})^{1/2} & \dots \\ \vdots & \ddots & & \\ 0 & 0 & \dots & (\frac{\eta_d}{h})^{1/2} \end{pmatrix}.$$

As a result, $T_h = hA_h^2$ is comparable $D^2 u(x_0)$ up to a dimensional constant for small h. $\qquad\square$

5. PROOF OF THEOREM 1.1

In this final section of the paper, we give the proof of the main result.

To simplify our notations, define $\kappa = \max\{C_0 C_2, C_0^2\}$, where C_0 is the constant in Proposition 4.1 and C_2 is the constant in Proposition 4.2. In particular, κ depends only on d and Λ.

For each integer m, let's define

$$D_m = \{x \in \Omega | |D^2 u(x)| \geq \kappa^m\}.$$

The $W^{2,1+\varepsilon}$-estimate is a direct consequence of the following lemma concerning the decay of integrals over D_m:

Lemma 5.1. *Suppose u is a solution to (2.1). There is a constant $\tau \in (0,1)$, depending only on d and Λ, such that*

$$\int_{D_{m+1}} |D^2 u| \leq (1-\tau) \int_{D_m} |D^2 u|$$

for each $m \geq m_0$.

Here m_0 is an integer depending on d, Λ, and the inner and outer radii of Ω and Ω^*.

Proof. Step 1: Covering D_{m+1} by sections with the correct height. For $x \in D_{m+1}$, $|D^2 u(x)| \geq \kappa^{m+1}$. By Proposition 4.2, $|T_h|$ ranges from $1/C_1$ to κ^{m+1}/C_2 as h changes from 1 to 0. T_h is the matrix defined before Proposition 4.2.

By our choice of κ, $C_0 \kappa^m \leq \kappa^{m+1}/C_2$. We can also choose m_0, depending also on the inner and outer radii of Ω and Ω^*, such that $C_0^{-1} \kappa^{m+1} \geq C_0^{-1} \kappa^{m_0+1} \geq 1/C_1$.

Thus we can pick $h_x > 0$ such that

$$C_0 \kappa^m \leq |T_{h_x}| < C_0^{-1} \kappa^{m+1}.$$

Let $\mathcal{F}$ denote the family of sections corresponding to such choice of heights, namely, $\mathcal{F} = \{S_{h_x}[u](x)\}_{x \in D_{m+1}}$. Then Lemma 2.2 gives a countable subfamily

$\{S_{h_j}[u](x_j)\}$ such that

$$D_{m+1} \subset \cup S_{h_j}[u](x_j), \text{ and } \sum \chi_{S_{\eta_0 h_j}[u](x_j)} \leq K.$$

Step 2: Estimate in each section. Let $S_h[u](x)$ denote a generic section in this subfamily. Then Proposition 4.1 implies

$$\int_{S_h[u](x) \cap \Omega} |D^2 u| \leq C \int_G |D^2 u|$$

where $G = S_{\eta_0 h[u](x)} \cap \Omega \cap \{C_0^{-1} T_h \leq D^2 u \leq C_0 T_h\}$.

Now by our choice of h, $C_0 \kappa^m \leq |T_h| < C_0^{-1} \kappa^{m+1}$. In particular, we have

$$G \subset S_{\eta_0 h[u](x)} \cap \Omega \cap \{\kappa^m \leq |D^2 u| < \kappa^{m+1}\} \subset S_{\eta_0 h[u](x)} \cap (D_m \backslash D_{m+1}).$$

Hence the previous estimate leads to

$$\int_{S_h[u](x) \cap \Omega} |D^2 u| \leq C \int_{S_{\eta_0 h[u](x)} \cap (D_m \backslash D_{m+1})} |D^2 u|.$$

Step 3: The covering argument. With this estimate and the two properties at the end of Step 1, we have the following

$$\int_{D_{m+1}} |D^2 u| \leq \sum \int_{S_{h_j}[u](x_j) \cap \Omega} |D^2 u|$$

$$\leq \sum C \int_{S_{\eta_0 h_j}[u](x_j) \cap (D_m \backslash D_{m+1})} |D^2 u|$$

$$= C \int_{D_m \backslash D_{m+1}} |D^2 u| \sum \chi_{S_{\eta_0 h_j}[u](x_j)}$$

$$\leq CK \int_{D_m \backslash D_{m+1}} |D^2 u|$$

$$= CK \int_{D_m} |D^2 u| - CK \int_{D_{m+1}} |D^2 u|.$$

Consequently,

$$\int_{D_{m+1}} |D^2 u| \leq \frac{CK}{1 + CK} \int_{D_m} |D^2 u|,$$

where C and K are constants depending only on d and Λ. □

Now Theorem 1.1 follows from a standard iteration:

Proof of Theorem 1.1. For $t > \kappa^{m_0}$, we find an integer k such that

$$\kappa^{m_0+k} \leq t < \kappa^{m_0+k+1},$$

that is, $k \leq \log_\kappa t - m_0 < k + 1$.

An iteration of Lemma 5.1 gives

$$\int_{\{|D^2 u| \geq t\} \cap \Omega} |D^2 u| \leq \int_{D_{m_0+k}} |D^2 u|$$

$$\leq (1 - \tau)^k \int_{D_{m_0}} |D^2 u|$$

$$\leq (1 - \tau)^{-1-m_0} \cdot t^{\log_\kappa (1-\tau)} \cdot \int_{D_{m_0}} |D^2 u|.$$

Note that
$$\int_{D_{m_0}} |D^2 u| \le \int_\Omega \Delta u = \int_{\partial \Omega} \nabla u \cdot \nu \le C$$
for come C depending only on d and the inner and outer radii of Ω and Ω^*, the previous estimate gives
$$\int_{\{|D^2 u| \ge t\} \cap \Omega} |D^2 u| \le C t^{-\varepsilon_0},$$
where $\varepsilon_0 > 0$ depends only on d and Λ, and C depends further on the inner and outer radii of Ω and Ω^*.

By Markov's inequality, this gives $|\{|D^2 u| \ge t\} \cap \Omega| \le C t^{-1-\varepsilon_0}$ for $t \ge \kappa^{m_0}$.

Therefore, we can pick $\varepsilon \in (0, \varepsilon_0)$ depending only on d and Λ. Then it follows
$$\begin{aligned}
\int_\Omega |D^2 u|^{1+\varepsilon} &= \int_{D_{m_0}} |D^2 u|^{1+\varepsilon} + \int_{\{|D^2 u| \le \kappa^{m_0}\} \cap \Omega} |D^2 u|^{1+\varepsilon} \\
&\le C_\varepsilon \int_{\kappa^{m_0}}^\infty t^\varepsilon |\{|D^2 u| \ge t\} \cap \Omega| dt + \kappa^{m_0(1+\varepsilon)} |\Omega| \\
&\le C \int_{\kappa^{m_0}}^\infty t^{-1-\varepsilon_0+\varepsilon} dt + \kappa^{m_0(1+\varepsilon)} |\Omega| \\
&= C \kappa^{m_0(\varepsilon - \varepsilon_0)} + \kappa^{m_0(1+\varepsilon)} |\Omega|.
\end{aligned}$$
which is controlled by a constant depending on d, Λ and the inner and outer radii of Ω and Ω^*. $\qquad\square$

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