K-WEIGHT BOUNDS FOR *γ*-HYPERELLIPTIC SEMIGROUPS

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ABSTRACT. In this note, we show that γ -hyperelliptic numerical semigroups of genus $g \gg \gamma$ satisfy a refinement of a well-known characteristic weight inequality due to Torres. The refinement arises from substituting the usual notion of weight by an alternative version, the K-weight, which we previously introduced in the course of our study of unibranch curve singularities.

1. K-WEIGHTS OF NUMERICAL SEMIGROUPS

Let $S \subset \mathbb{N}$ denote a numerical semigroup of genus g. Recall that this means that the complement $G_S = \mathbb{N} \setminus S$ is of cardinality g; say

$$G_{\rm S} = \{\ell_1, \ldots, \ell_g\}$$

where $\ell_i < \ell_j$ whenever $1 \le i < j \le g$. Following [3], we define the K-weight of S to be the quantity

$$W_{\rm K} := \sum_{i=1}^{g-1} (\ell_i - i) + g - 1.$$

The definition of the K-weight was motivated by the study of unibranch complex curve singularities. It is also closely related to a more familiar notion of weight, which we will call the S-weight, namely:

$$W_{\rm S} := \sum_{i=1}^{g} (\ell_i - i).$$

The S-weight emerges naturally in the study of Weierstrass semigroups of (points of) *smooth* complex curves. It is not hard to show that $W_{\rm K} = W_{\rm S}$ whenever S is the value semigroup of a Gorenstein singularity. In purely combinatorial terms, the K- and S-weights agree if and only if S is a *symmetric* semigroup.

2. K-weights of γ -hyperelliptic semigroups

Now let $\gamma \geq 0$ be an integer. Recall from [6] that S is γ -hyperelliptic if it satisfies the following conditions:

- (1) S has γ even elements in [2, 4γ]; and
- (2) The $(\gamma + 1)$ st positive element of S is $4\gamma + 2$.

In [3], we conjectured that when $g \gg \gamma$, any γ -hyperelliptic numerical semigroup S satisfies

(1)
$$\binom{g-2\gamma}{2} + 2\gamma \le W_{\rm K} \le \binom{g-2\gamma}{2} + 2\gamma^2.$$

The inequalities (1) are not far-removed from the inequalities

(2)
$$\binom{g-2\gamma}{2} \le W_{\rm S} \le \binom{g-2\gamma}{2} + 2\gamma^2$$

proved by Torres in [5, 6]. As the disparity between upper and lower bounds in (1) is in general smaller than that of (2), we regard the K-weight inequalities as a refinement of the S-weight inequalities.

In this note, we will prove the K-weight inequalities (1) are satisfied by any γ -hyperelliptic semigroup of sufficiently large genus. In doing so, we adapt the geometric interpretation of the S-weight given in [2]. Namely, each numerical semigroup may be represented as a Dyck path $\tau = \tau(S)$ on a $g \times g$ square grid Γ with axes labeled by $0, 1, \ldots, g$. Each path starts at (0, 0), ends at (g, g), and has unit steps upward or to the right. The *i*th step of τ is up if $i \notin S$, and is to the right otherwise. The weight W_S of S is then equal to the total number of boxes in the Young tableau T_S traced by the upper and left-hand borders of the grid and the Dyck path τ . The contribution of each gap ℓ of S to W_S is then computed by the number of boxes inside the grid and to the left of the corresponding path edge.

Theorem 2.1. Fix a choice of non-negative integer γ , and let S denote a γ -hyperelliptic semigroup of genus $g \geq 2\gamma + 1$. The K-weight of S then satisfies the inequalities (1).

Proof. Suppose S is a γ -hyperelliptic semigroup of genus g. For the calculation of the K-weight of S, the largest gap $\ell_g \in \mathbb{N} \setminus S$ is irrelevant, so we focus on the subdiagram of Γ given by omitting the uppermost row of boxes in the grid, and on the corresponding subtableau T_S , which we will denote by T_K .

It is well-known (and follows easily from the semigroup structure in any case) that every even number greater than or equal to 4γ belongs to S. So the weight contributed by elements $m \ge 4\gamma$ of S will be minimized when there are no such *odd* elements m strictly less than ℓ_g . Geometrically, this means that T_K stabilizes to a *staircase*, i.e a path in which up- and rightward steps alternate. It follows easily T_K will be of minimal weight precisely when it *is* a staircase for which the γ even numbers $P_i \in [2, 4\gamma], 1 \le i \le \gamma$ that belong to S are maximal, namely when

$$P_i = 2\gamma + 2j, 1 \le j \le \gamma.$$

Since the first column of $T_{\rm K}$ has precisely $(g-1) - (2\gamma + 2 - 1) = g - 2\gamma - 2$ boxes, we deduce that its total weight is

$$W(T_{\rm K}) = \begin{pmatrix} g - 2\gamma - 1 \\ 2 \end{pmatrix}$$

and it follows that

$$W_{\mathrm{K}} = W(T_{\mathrm{K}}) + g - 1 = \begin{pmatrix} g - 2\gamma \\ 2 \end{pmatrix} + 2\gamma.$$

It is clear, moreover, that there are γ -hyperelliptic semigroups of genus g whose K-weights realize the minimum value of $W_{\rm K}$.

Remark 2.2. Strictly speaking, the preceding geometric argument requires $g \ge 2\gamma + 2$. However, it is easy to check that when $g = 2\gamma + 1$, the minimal K-weight is realized by a γ -hyperelliptic semigroup with an empty K-tableau $T_{\rm K}$. The K-weight of the corresponding semigroup is then $W_{\rm K} = g - 1 = 2\gamma$, as desired. The same



FIGURE 1. Tableau $T_{\rm K}$ associated with the weight-maximizing γ -hyperelliptic semigroup $S_0 = \langle 4, 4\gamma + 2, 2g - 4\gamma + 1 \rangle$ when $\gamma = 3$ and g = 20. The (irrelevant) uppermost line is left empty, while the disparity in weights between the maximizing and minimizing semigroups is in red.

argument shows that when $g = 2\gamma$, the minimal K-weight is $g - 1 = 2\gamma - 1$. So our lower bound on g is sharp.

We will now argue that the maximum possible $\gamma\text{-hyperelliptic K-weight is achieved precisely when$

$$\mathbf{S} = \mathbf{S}_0 := \langle 4, 4\gamma + 2, 2g - 4\gamma + 1 \rangle$$

just as is the case for S-weights [6]. See Figure 1 for the K-tableau associated with S_0 when $\gamma = 3$ and g = 20.

Since K-weight and S-weight agree for symmetric semigroups, we may (and shall) assume that S is nonsymmetric. We will exploit the dual relationship between ramification and weight already used in [6] to prove that S_0 is of maximal S-weight among γ -hyperelliptic semigroups of genus g. (Indeed, our argument is a modification of that used in [6].)

Namely, let R = R(S) denote the *total ramification* of S, given by

(3)
$$R := \sum_{i=1}^{g} (m_i - i) = \sum_{i=1}^{g} m_i - \binom{g+1}{2}$$

where $m_1 < \cdots < m_g$ are the g smallest nonzero elements of S. Using the structure theory for γ -hyperelliptic semigroups presented in [5, 6], we may rewrite the total ramification (3) of a γ -hyperelliptic semigroup S in the following form:

$$R = \sum_{i=1}^{\gamma} (n_i + u_i) + \sum_{i=1}^{g-2\gamma} (4\gamma + 2i) - {g+1 \choose 2}$$

where $n_1 < \cdots < n_{\gamma}$ are the smallest nonzero even elements and $u_1 > \cdots > u_{\gamma}$ are the odd nonzero elements of S less than 2g, respectively.

Maximizing $W_{\rm S}$ is then equivalent to minimizing $R({\rm S})$. Similarly, maximizing $W_{\rm K}$ is equivalent to minimizing

(4)
$$R_{\rm K} := \sum_{i=1}^{\gamma} (n_i + u_i) + \sum_{i=1}^{g-2\gamma-1} (4\gamma + 2i) - {g \choose 2} - 2k$$

where k = k(S) is the number of odd elements of S greater than or equal to the conductor. Geometrically speaking, (4) computes the area of the complement of $T_{\rm K}$ in its minimal $(g-1) \times (g-1)$ bounding box inside of Γ .

On the other hand, when $S = S_0$ the sum $\sum_{i=1}^{\gamma} n_i$ is minimized, while $k(S_0) = 0$. So in light of (4), it suffices to check that

$$\sum_{i=1}^{k(S)} (u_i(S) - u_i(S_0)) - 2k(S)$$

is nonnegative for every nonsymmetric γ -hyperelliptic semigroup S, for which it suffices in turn to show that

(5)
$$u_i(\mathbf{S}) - u_i(\mathbf{S}_0) \ge 2$$

for all $1 \leq i \leq k$. The inequality (5) follows immediately, however, from the fact that S₀ is symmetric, while S is not.

Finally, just as in [4], it is natural to ask for refinements of Theorem 2.1. We have the following K-weight analogue of [4, Props. 2.10, 2.12(1)].

Proposition 2.3. Let $\gamma \geq 1$ be an integer, and let S denote a γ -hyperelliptic semigroup of genus $g \geq 3\gamma$. Assume that the multiplicity, i.e., the smallest nonzero element of S is 4. The K-weight of S then satisfies

$$W_{\rm K} \in \left\{ \begin{pmatrix} g - 2\gamma \\ 2 \end{pmatrix} + \gamma^2 + \gamma + k^2 - 3k + 2 : k = 1, \dots, \gamma + 1 \right\};$$

and $W_{\rm K} = {g-2\gamma \choose 2} + \gamma^2 + \gamma + k^2 - 3k + 2$ if and only if $S = \langle 4, 4\gamma + 2, 2g - 2\gamma - 2k + 3, 2g - 2\gamma + 2k + 1 \rangle$. In particular, every γ -hyperelliptic semigroup with multiplicity m = 4 of nonmaximal weight satisfies

$$\binom{g-2\gamma}{2} + \gamma^2 + \gamma \le W_{\mathrm{K}} \le \binom{g-2\gamma}{2} + 2(\gamma^2 - \gamma) + 2.$$

Proof. Our result is a consequence of [4, Prop. 2.10], which establishes that

$$W_{\rm S} \in \left\{ \begin{pmatrix} g-2\gamma\\ 2 \end{pmatrix} + \gamma^2 - \gamma + k^2 - k : k = 1, \dots, \gamma + 1 \right\};$$

and $W_{\rm S} = \binom{g-2\gamma}{2} + \gamma^2 - \gamma + k^2 - k$ if and only if ${\rm S} = \langle 4, 4\gamma + 2, 2g - 2\gamma - 2k + 3, 2g - 2\gamma + 2k + 1 \rangle$. Indeed, letting S₀ denote the latter semigroup, we have

$$S_0 = 2\langle 2, 2\gamma+1 \rangle \sqcup \{2g-2\gamma-2k+3, 2g-2\gamma-2k+7, \dots, 2g-2\gamma+2k-1, 2g-2\gamma+2k+1, \dots\}$$

in particular, the largest gap of S_0 is $\ell_q = 2g-2\gamma+2k-3$. We conclude immediately

using the general fact that $W_{\rm K} = W_{\rm S} + 2g - 1 - \ell_g$.

It would be interesting to extend the reach of Proposition 2.3 to higher multiplicities, and e.g., to establish an analogue of [4, Prop. 2.12(2)], which establishes an upper bound on the S-weight when $m \ge 6$. We leave this for future work.

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