SEQUENCES OF ZEROS OF ANALYTIC FUNCTION SPACES AND WEIGHTED SUPERPOSITION OPERATORS

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Abstract. We use properties of the sequences of zeros of certain spaces of analytic functions in the unit disc $\mathbb{D}$ to study the question of characterizing the weighted superposition operators which map one of these spaces into another. We also prove that for a large class of Banach spaces of analytic functions in $\mathbb{D}$, Y, we have that if the superposition operator S_φ associated to the entire function φ is a bounded operator from X, a certain Banach space of analytic functions in $\mathbb{D}$, into Y, then the superposition operator $S_{\varphi'}$ maps X into Y.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disc in the complex plane $\mathbb{C}$ and let $\mathcal{H}ol(\mathbb{D})$ be the space of all analytic functions in $\mathbb{D}$ endowed with the topology of uniform convergence in compact subsets.

Given an entire function φ, the superposition operator

$$S_\varphi : \mathcal{H}ol(\mathbb{D}) \longrightarrow \mathcal{H}ol(\mathbb{D})$$

is defined by $S_\varphi(f) = \varphi \circ f$.

More generally, if φ is an entire function and $w \in \mathcal{H}ol(\mathbb{D})$, the weighted superposition operator

$$S_{\varphi,w} : \mathcal{H}ol(\mathbb{D}) \longrightarrow \mathcal{H}ol(\mathbb{D})$$

is defined by

$$S_{\varphi,w}(f)(z) \,=\, w(z)\,\varphi\big(f(z)\big), \quad f \in \mathcal{H}ol(\mathbb{D}), \quad z \in \mathbb{D}.$$

In other words, $S_{\varphi,w} = M_w \circ S_\varphi$, where M_w is the multiplication operator defined by

$$M_w(f)(z) = w(z)f(z), \quad f \in \mathcal{H}ol(\mathbb{D}), \quad z \in \mathbb{D}.$$

Note that $S_\varphi = S_{\varphi,w}$ with $w(z) = 1$, for all $z \in \mathbb{D}$.

The natural questions in this context are: If X and Y are two normed (metric) subspaces of $\mathcal{H}ol(\mathbb{D})$, for which pair of functions (φ, w), with φ entire and $w \in \mathcal{H}ol(\mathbb{D})$, does the operator $S_{\varphi,w}$ map X into Y? When is $S_{\varphi,w}$ a bounded (or continuous) operator from X to Y?

Let us remark that we are dealing with non-linear operators. Consequently, boundedness and continuity are not equivalent a priori. However, Boyd and Rueda [6] have

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shown that for a large class of Banach spaces of analytic functions X and Y, bounded superposition operators from X to Y are continuous.

These questions have been studied for different pair of spaces (X, Y), specially in the case of superposition operators. Let us mention that Cámera [10] considered the case when X and Y are Hardy spaces. Cámera and Giménez [9] characterized the entire functions φ so that S_φ maps a Bergman space into another. The superposition operators acting between various spaces of Dirichlet type were studied in [7, 8]. Superposition operators between weighted Banach spaces of analytic functions were studied in [4, 5, 22]. Álvarez, Márquez, and Vukotić [1] studied the superposition operators between a Bergman space and the Bloch space in both directions.

If X is a subspace of $\mathcal{H}ol(\mathbb{D})$, a sequence of points $\{z_k\}$ in $\mathbb{D}$ is said to be an "X-zero-sequence" or a "sequence of zeros of X" if there exists a function $f \in X$ which vanishes precisely in that sequence (counting multiplicities).

It has been shown in [13, 14] that information on the zero-sequences of certain spaces X, Y of analytic functions in $\mathbb{D}$ can be an useful tool to characterize the superposition operators mapping X into Y. Our main objective in this paper is to find new applications of these ideas in the more general context of weighted superposition operators. We shall present these applications and some further results in sections 2 and 3.

2. Weighted superposition operators between the Bloch space and Bergman spces

For $0 < p < \infty$ and $\alpha > -1$ the weighted Bergman space A^p_α consists of those $f \in \mathcal{H}ol(\mathbb{D})$ such that

$$\|f\|_{A^p_\alpha} \stackrel{\text{def}}{=} \left((\alpha+1) \int_{\mathbb{D}} (1-|z|^2)^\alpha |f(z)|^p \, dA(z) \right)^{1/p} < \infty.$$

Here, dA stands for the area measure on $\mathbb{D}$, normalized so that the total area of $\mathbb{D}$ is 1. Thus $dA(z) = \frac{1}{\pi}\, dx\, dy = \frac{1}{\pi}\, r\, dr\, d\theta$. The unweighted Bergman space A^p_0 is simply denoted by A^p. We refer to [12, 17, 24] for the notation and results about Bergman spaces.

A function $f \in \mathcal{H}ol(\mathbb{D})$ is said to be a Bloch function if

$$\|f\|_{\mathcal{B}} \stackrel{\text{def}}{=} |f(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)|f'(z)| < \infty.$$

The space of all Bloch functions is called the Bloch space and it will be denoted by $\mathcal{B}$. We mention [2] as a classical reference for Bloch functions.

One of the main results in [1] was showing that if $0 < p < \infty$ and φ is an entire function, then the superposition operator S_φ maps the Bergman space A^p into the Bloch space $\mathcal{B}$ if and only if φ is a constant function. The proof of this result given in [1] uses in a very precise way properties of the conformal mapping of the disc onto a sector. We shall use completely different ideas to prove the following more general result.

Theorem 1. *Suppose that $0 < p < \infty$ and $\alpha > -1$. Let w be a non-identically zero analytic function in $\mathbb{D}$ and let φ be an entire function with $\varphi \not\equiv 0$. Then the weighted* *superposition operator $S_{\varphi,w}$ maps A^p_α into the Bloch space $\mathcal{B}$ if and only if $w \in \mathcal{B}$ and φ is constant.*

When $\alpha = 0$ and $w \equiv 1$, Theorem 1 reduces to the just mentioned result of [1]. Before embarking in the proof, we shall recall some results about the sequences of zeros of the spaces A^p_α and $\mathcal{B}$. Horowitz proved in [19] the following result.

Theorem A. *Suppose that $0 < p < \infty$ and $\alpha > -1$.*

(i) *If $g \in A^p_\alpha$, $g(0) \neq 0$, and $\{z_k\}$ is its sequence of zeros then*

$$\prod_{k=1}^{n} \frac{1}{|z_k|} = \mathrm{O}\left(n^{(1+\alpha)/p}\right). \tag{2.1}$$

(ii) *The statement (i) is sharp in the following sense: For any given $\varepsilon > 0$, there exists a function $g \in A^p_\alpha$ with $f(0) \neq 0$ whose sequence of zero zeros $\{z_k\}$ satisfies*

$$\prod_{k=1}^{n} \frac{1}{|z_k|} \neq \mathrm{O}\left(n^{(1+\alpha)/(p(1+\varepsilon))}\right). \tag{2.2}$$

Actually, Horowitz proved these results for $\alpha > 0$ and Sedletskii [23] proved that they remain true for all $\alpha > -1$. Arguing as in the proof of [15, Theorem 1], we obtain that $\mathrm{O}\left(n^{(1+\alpha)/p}\right)$ can be replaced by $\mathrm{o}\left(n^{(1+\alpha)/p}\right)$ in (2.1).

Girela, Nowak and Waniurski proved in [15] that if f is a Bloch function with $f(0) \neq 0$ and $\{\xi_k\}$ is its sequence of zeros then

$$\prod_{k=1}^{n} \frac{1}{|\xi_k|} = \mathrm{O}\left((\log n)^{1/2}\right). \tag{2.3}$$

Later, Nowak [21] proved that $\mathrm{O}\left((\log n)^{1/2}\right)$ cannot be replaced by $\mathrm{o}\left((\log n)^{1/2}\right)$.

Now we are in disposition to prove Theorem 1.

Proof of Theorem 1. It is trivial that if φ is constant and $w \in \mathcal{B}$ then $S_{\varphi,w}(A^p_\alpha) \subset \mathcal{B}$.

If φ is constant, not identically 0, and $S_{\varphi,w}$ maps A^p_α into $\mathcal{B}$ then it is clear that $w \in \mathcal{B}$.

Suppose now that $w \not\equiv 0$, φ is not constant, and $S_{\varphi,w}(A^p_\alpha) \subset \mathcal{B}$. Take $a \in \mathbb{C}$ such that $\varphi(a) \neq 0$ and let f be the constant function defined by $f(z) = a$, for all $z \in \mathbb{D}$. Since $f \in A^p_\alpha$, it follows that

$$S_{\varphi,w}(f) = \varphi(a) \cdot w \in \mathcal{B}.$$

This implies that $w \in \mathcal{B}$.

Next we take $\varepsilon > 0$ and use Theorem 2.1 (ii) to pick a function $g \in A^p_\alpha$ with $g(0) \neq 0$ such that its sequence of zeros $\{z_k\}$ satisfies (2.2). It is clear that $\varphi \circ g$ is not constant. Set $F = S_{\varphi,w}(g) - \varphi(0) \cdot w$. Since w and $S_{\varphi,w}(g)$ are Bloch functions, it follows that

$$F = S_{\varphi,w}(g) - \varphi(0) \cdot w = w\,[\varphi \circ g - \varphi(0)] \in \mathcal{B}.$$

Now, it is clear that $F \not\equiv 0$ and that all the zeros of g are zeros of F. In other words, the sequence $\{z_k\}$ is contained in the sequence $\{\xi_k\}$ of zeros of F. Since $\{z_k\}$ satisfies

(2.2), it follows that that $\{\xi_k\}$ does not satisfy (2.3). This is in contradiction with the fact that $F \in \mathcal{B}$. $\square$

Let us turn now our attention to the weighted superposition operators which map a weighted Bergman space into the Bloch space. We shall prove the following result which is related to [1, Theorem 3].

Theorem 2. *Suppose that:*

(i) $0 < p < \infty$ *and* $\alpha > -1$.
(ii) $w \in A^p_\beta$ *for some* β *with* $-1 < \beta < \alpha$.
(iii) φ *is an entire function of order less than one, or of order one and type zero.*

Then the weighted superposition operator $S_{\varphi,w}$ *is a bounded operator from* $\mathcal{B}$ *into* A^p_α.

Proof. Set $M(r) = \max_{|z|\le r} |f(z)|$ $(0 < r < \infty)$. Condition (iii) implies that

$$\frac{\log M(r)}{r} \to 0, \quad \text{as } r \to \infty \tag{2.4}$$

(see [18, Chapter 14]). Take $K > 0$ and let f be a Bloch function with norm at most K. It is easy to see (see [2], for example) that there exists an absolute constant $C > 0$ such that

$$|f(z)| \le \|f\|_{\mathcal{B}} \log \frac{C}{1-|z|}. \tag{2.5}$$

By (2.4), there exists $r_0 > 0$ (which depend only on φ and K) such that

$$\frac{\log M(r)}{r} \le \frac{\alpha-\beta}{Kp}, \quad r \ge r_0.$$

Thus, using (2.5), we see that whenever $|f(z)| \ge r_0$ we have

$$\begin{aligned} |S_\varphi(f)(z)| = |\varphi(f(z))| &\le \exp\left(\frac{\alpha-\beta}{Kp}|f(z)|\right) \\ &\le \exp\left(\frac{\alpha-\beta}{p}\log\frac{C}{1-|z|}\right) \\ &= D(1-|z|)^{(\beta-\alpha)/p}, \end{aligned}$$

with $D = C^{(\alpha-\beta)/p}$. Then it follows that

$$\int_{|f(z)|\ge r_0} (1-|z|)^\alpha |w(z)|^p |S_\varphi(f)(z)|^p \, dA(z) \le D^p \int_{\mathbb{D}} (1-|z|)^\beta |w(z)|^p \, dA(z). \tag{2.6}$$

If $|f(z)| \le r_0$ then, by the maximum principle, $|S_\varphi(f)(z)| \le M(r_0)$. Combining this with (2.6), we obtain

$$\begin{aligned} &\int_{\mathbb{D}} (1-|z|)^\alpha |w(z)|^p |S_\varphi(f)(z)|^p \, dA(z) \\ &\le D^p \int_{\mathbb{D}} (1-|z|)^\beta |w(z)|^p \, dA(z) + M(r_0)^p \int_{\mathbb{D}} (1-|z|)^\alpha |w(z)|^p \, dA(z). \end{aligned}$$

This is a positive number which depend on φ, w, p, α, β, and K, but not on f. Thus we have shown that $S_{\varphi,w}$ is a bounded operator from $\mathcal{B}$ into A^α_p. $\square$

3. Some further results

The arguments used to prove Theorem 1 actually lead to the following result.

Theorem 3. *Let X and Y be two spaces of analytic functions in $\mathbb{D}$ satisfying the following conditions.*

(i) *X contains the constant functions.*
(ii) *There exists a function $f \in X$ with $f(0) \neq 0$ whose sequence of zeros $\{z_k\}$ is not a sequence of zeros of Y.*

Let w be a non-identically zero analytic function in $\mathbb{D}$ and let φ be an entire function with $\varphi \not\equiv 0$. Then the weighted superposition operator $S_{\varphi,w}$ maps X into Y if and only if $w \in Y$ and φ is constant.

There are a lot of possible choices of the spaces X and Y in Theorem 3 and each one of these choices leads us to a concrete result. Let us mention just a couple of them.

Just as in [4], in this paper a weight v on $\mathbb{D}$ will be a positive and continuous function defined on $\mathbb{D}$ which is radial, i. e. $v(z) = v(|z|)$, for all $z \in \mathbb{D}$, and satisfying that $v(r)$ is strictly decreasing in $[0, 1)$ and that $\lim_{r\to 1} v(r) = 0$. For such a weight, the weighted Banach space H_v^∞ is defined by

$$H_v^\infty = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : \|f\|_v \stackrel{\text{def}}{=} \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty \right\}.$$

We refer to [4] for the origin and relevance of these spaces and for examples of weights on $\mathbb{D}$.

We recall that a sequence $\{z_k\} \subset \mathbb{D}$ is said to be a Blaschke sequence if

$$\sum (1 - |z_k|) < \infty.$$

Blaschke sequences are the zero sequences of any of the Hardy spaces H^p $(0 < p \leq \infty)$ and also of the Nevannlinna class N (see [11]). Actually, using Jensen's formula it follows that if $f \in \mathcal{H}ol(\mathbb{D})$ and $f \not\equiv 0$ then its sequence of zeros $\{z_k\}$ satisfies the Blaschke condition if and only if

$$\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{it})|\, dt < \infty.$$

We can state the following result.

Corollary 4. *Let Y be a space of analytic functions in $\mathbb{D}$ such that all the sequences of zeros of Y are Blaschke sequences and let v be a weight on $\mathbb{D}$. Let w be a non-identically zero analytic function in $\mathbb{D}$ and let φ be an entire function with $\varphi \not\equiv 0$. Then the weighted superposition operator $S_{\varphi,w}$ maps H_v^∞ into Y if and only if $w \in Y$ and φ is constant.*

Proof. By Theorem 3, it suffices to show that there exists a function $f \in H_v^\infty$, $f \not\equiv 0$, whose sequence of zeros $\{z_k\}$ is not a Blaschke sequence.

Set $\psi(r) = 1/v(r)$ $(r \in [0, 1))$. Then ψ is a positive, continuous, and increasing function in $[0, 1)$, and $\lim_{r\to 1} \psi(r) \to \infty$. Using Lemma 2 in p. 80 of [12] we can choose an increasing sequence of positive integers $\{n_k\}_{k=1}^{\infty}$ such that

$$\sum_{k=1}^{\infty} r^{n_k} \leq \frac{1}{2}\log\phi(r), \quad 0 \leq r < 1.$$

Then the function f constructed in p. 94 of [12] with this sequence $\{n_k\}$ belongs to H_v^∞ and its sequence of zeros is not a Blaschke sequence. □

When v is the weight defined by $v(z) = \left(\log\frac{e}{1-|z|}\right)^{-1}$ $(z \in \mathbb{D})$, the space H_v^∞ is the space of those functions $f \in \mathcal{H}ol(\mathbb{D})$ such that

$$|f(z)| = \mathrm{O}\left(\log\frac{1}{1-|z|}\right), \quad \text{as } |z| \to 1. \tag{3.1}$$

This is the space which was called A^0 in [15] and $H_{\log}^\infty$ in [16] and in [3]. The space $H_{\log}^\infty$ and the Bloch space are closely related. It is well known that any Bloch function f satisfies (3.1). Hence, $\mathcal{B} \subset H_{\log}^\infty$. Since the function $f(z) = \log\frac{1}{1-z}$ is a Bloch function, the growth condition (3.1) is the best one possible for the Bloch space. Bao and Ye [3] have identified the space $H_{\log}^\infty$ as the dual of a certain Luecking-type subspace of A^1 and they have shown also that the Bloch space is not dense in $H_{\log}^\infty$. Using Theorem 3 and [15, Theorem 2] we obtain the following result.

Corollary 5. *Let w be a non-identically zero analytic function in $\mathbb{D}$ and let φ be an entire function with $\varphi \not\equiv 0$. Then the weighted superposition operator $S_{\varphi,w}$ maps $H_{\log}^\infty$ into the Bloch space if and only if $w \in \mathcal{B}$ and φ is constant.*

Let us recall that the order and type of an entire function φ are the same as those of its derivative (see, e. g., [18, Chapter 14]). Bearing this in mind and looking at the results in [1, 4, 5, 6, 7, 8, 9, 10, 13, 14], we see that for a good number of pairs of spaces of analytic functions in $\mathbb{D}$, (X, Y), we have that if the superposition operator S_φ maps X into Y then so does the operator $S_{\varphi'}$. Then it is natural to look for general conditions on the pair (X, Y) which imply this fact. We shall prove a result of this kind when Y is one of the weighted Banach spaces H_v^∞.

Theorem 6. *Let v be weight on $\mathbb{D}$ and let $(X, \|\cdot\|)$ be a Banach space of analytic function in $\mathbb{D}$. Let φ be an entire function. If the superposition operator S_φ is a bounded operator from X into H_v^∞, then $S_{\varphi'}$ maps X into H_v^∞.*

Proof. Take $f \in X$ and $z \in \mathbb{D}$.

Suppose first that $|f(z)| \leq 1$. Set $A = \sup_{|\xi|\leq 1}|\varphi'(\xi)|$. Then we have

$$|S_{\varphi'}(f)(z)| = |\varphi'(f(z))| \leq A \leq \frac{Av(0)}{v(z)}.$$

Suppose now that $|f(z)| \geq 1$ and set $R = 2|f(z)|$. As usual, if F is an entire function and $0 \leq r < \infty$, we set

$$M(r, F) = \sup_{|z|=r}|F(z)|.$$

Using Cauchy's integral formula for the derivative and some simple estimates, we obtain

$$\begin{aligned}|S_{\varphi'}(f)(z)| &= |\varphi'(f(z))| = \left|\frac{1}{2\pi i}\int_{|\zeta|=R}\frac{\varphi(\zeta)}{(\zeta-f(z))^2}\,d\zeta\right| \\ &\leq \frac{1}{2\pi}\,2\pi R\,\frac{M(2|f(z)|,\varphi)}{R^2} = \frac{M(2|f(z)|,\varphi)}{2|f(z)|} \\ &\leq \frac{1}{2}\,M(2|f(z)|,\varphi).\end{aligned}$$

Take now $\theta \in \mathbb{R}$ such that

$$M(2|f(z)|,\varphi) = \left|\varphi\left(2e^{i\theta}f(z)\right)\right|$$

and set

$$g(\xi) = 2e^{i\theta}f(\xi), \quad \xi \in \mathbb{D}.$$

Then we have

$$|S_{\varphi'}(f)(z)| \leq \frac{1}{2}\,|\varphi(g(z))| = \frac{1}{2}\,|S_{\varphi}(g)(z)|\,. \tag{3.2}$$

Now, since $f \in X$ we also have that $g \in X$ and $\|g\| = 2\|f\|$. Since S_φ is a bounded operator from X into H_v^∞, there exists a positive constant L such that

$$|S_\varphi(g)(\xi)| \leq \frac{L\|f\|}{v(\xi)}, \quad \xi \in \mathbb{D}.$$

Using this in (3.2), we obtain

$$|S_{\varphi'}(f)(z)| \leq \frac{L\|f\|}{2v(z)}.$$

Putting both cases together, we obtain

$$|S_{\varphi'}(f)(z)| \leq \frac{C}{v(z)}, \quad \text{for all } z \in \mathbb{D},$$

with $C = \max\left(Av(0), \frac{L\|f\|}{2}\right)$. This gives that $S_{\varphi'}(f) \in H_v^\infty$. □

If v is a weight on $\mathbb{D}$, we define DH_v^∞ as follows

$$DH_v^\infty = \{f \in \mathcal{H}ol(\mathbb{D}) : f' \in H_v^\infty\}\,.$$

The space DH_v^∞ is a Banach space with the norm $\|\cdot\|_{D,v}$ defined by

$$\|f\|_{D,v} = |f(0)| + \|f'\|_v.$$

We have the following result.

Theorem 7. *Let v be a weight on $\mathbb{D}$ and let $(X, \|\cdot\|)$ be a Banach space of analytic function in $\mathbb{D}$. Let φ be an entire function. If the superposition operator S_φ is a bounded operator from X into DH_v^∞, then $S_{\varphi'}$ maps X into DH_v^∞.*

This result can be proved with arguments very similar to those used in the proof of Theorem 6, we omit the details. Notice that when $v(z) = (1-|z|)$ the space DH_v^∞ reduces to the Bloch space. Hence, as a particular case of Theorem 7 we obtain.

Corollary 8. *Let v be weight on $\mathbb{D}$ and let $(X, \|\cdot\|)$ be a Banach space of analytic function in $\mathbb{D}$. Let φ be an entire function. If the superposition operator S_φ is a bounded operator from X into the Bloch space $\mathcal{B}$, then $S_{\varphi'}$ maps X into $\mathcal{B}$.*

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