LOCALLY PLURIPOLAR SETS ARE PLURIPOLAR

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ABSTRACT. We prove that every locally pluripolar set on a compact complex manifold is pluripolar. This extends similar results in Kähler case.

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1. INTRODUCTION

Pluripotential theory has been a crucial tool in complex geometry, complex dynamics as well as other fields of Mathematics. We refer to [8, 12, 20, 23] for some expositions of this theory and its applications. Among other things, locally pluripolar sets are important objects in the pluripotential theory which play the role of negligible sets as a counterpart to algebraic subvarieties in algebraic geometry, see the next section for definitions. To illustrate this comparison, we recall that locally pluripolar sets are of Hausdorff codimension at least 2 (see [22, Th. 3.13]) and their intersections with totally real submanifolds of the ambient manifold are of Lebesgue measure zero (see [27, Cor. 1.2]). We refer to [19, 26, 10, 18] for more information.

Josefson's theorem [17], which is a key result in the pluripotential theory on $\mathbb{C}^k$, affirms that locally pluripolar sets on $\mathbb{C}^k$ are in fact (globally) pluripolar. Simplified proofs of this fact were given by Bedford-Taylor [3] and Alexander-Taylor [2]. This result was generalized to the pluripolar sets on projective manifolds, compact Kähler manifolds, respectively, by Dinh-Sibony [11], Guedj-Zeriahi [14], see also Berman-Boucksom-Witt Nyström [4] for the case of manifolds equipped a big line bundle. Our main result below extends this property to pluripolar sets on *every* compact complex manifolds.

Theorem 1.1. *Every locally pluripolar set on a compact complex manifold is pluripolar.*

By the above theorem, there exist abundantly non-continuous quasi-p.s.h. functions on X. This is a fact which probably cannot be seen directly because unlike projective manifolds, a general compact complex manifold might have very few hypersurfaces. The key ingredients of the proof of Theorem 1.1 are the comparison (2.9) between capacities generalizing similar comparison results in [2, 14] and recent developments of the pluripotential theory for non-Kähler manifolds by Kołodziej, Dinew and Nguyen [9, 21].

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2. PROOF OF THEOREM 1.1

First of all, we need to recall some basic notations from the pluripotential theory. Let X be a compact complex manifold of dimension k. A function from X to $[-\infty, \infty)$ is said to be *quasi-p.s.h.* if it can be written locally as the sum of a plurisubharmonic (p.s.h.) function and a smooth one. Put $\mathrm{d}^c := i/(2\pi)(\overline{\partial} - \partial)$. For a continuous real $(1,1)$-form η, a quasi-p.s.h. function φ is said to be *η-p.s.h.* if $\mathrm{dd}^c\varphi + \eta \geq 0$. We have the following characterization of quasi-p.s.h. functions in terms of submean-type inequalities.

Lemma 2.1. *Let U be an open subset of $\mathbb{C}^k$ and η a continuous real $(1,1)$-form on U. A function $\varphi : U \to [-\infty, \infty)$ is η-p.s.h. if and only if it is upper semi-continuous, not identically $-\infty$ and for every $x \in U$ and every complex line $L_v := \{x + tv : t \in \mathbb{C}\}$, for some $v \in \mathbb{C}^k$, passing through x, we have*

$$\varphi(x) \leq \frac{1}{2\pi}\int_0^{2\pi} \varphi(x + \epsilon e^{i\theta} v) d\theta + \int_0^{\epsilon} \frac{dt}{t} \int_{\{|s| \leq t\}} \eta_v, \tag{2.1}$$

for every constant $\epsilon > 0$ small enough, where $\eta_v(t)$ is the restriction of η to L_v which is identified with $\mathbb{C}$ via $t \longmapsto x + tv$.

Proof. Consider an η-p.s.h. function φ. We need to verify (2.1). For every positive constant r, let χ_r be a smooth multi-radial nonnegative function compactly supported on the polydisk of radius r in $\mathbb{C}^k$ with $\int_{\mathbb{C}^k} \chi_r(x) \operatorname{vol}(x) = 1$, where vol is the canonical volume form on $\mathbb{C}^k$. Since φ is locally integrable, we can define the convolution

$$\varphi^r(x) := \int_{\mathbb{C}^k} \varphi(x - y)\chi_r(y) \operatorname{vol}(y)$$

which is smooth. We have $\varphi^r \to \varphi$ pointwise as $r \to 0$ because φ can be written as the sum of a p.s.h. function and a smooth one. Denote by

$$\eta^r(x) := \int_{\mathbb{C}^k} \eta(x - y)\chi_r(y) \operatorname{vol}(y)$$

which converges uniformly to η as $r \to 0$ because η is continuous. Hence, $\mathrm{dd}^c\varphi + \eta \geq 0$ if and only if $\mathrm{dd}^c\varphi^r + \eta^r \geq 0$ for every r small. Similarly, (2.1) holds if it holds for (φ^r, η^r) in place of (φ, η) for every small r. It follows that it suffices to prove (2.1) for smooth φ and smooth η.

Hence we can assume φ, η are smooth and follow standard arguments in [16]. Let $v \in \mathbb{C}^k$ and $x \in U$. Put $\varphi_v(t) := \varphi(x + tv)$. We get $\mathrm{dd}^c\varphi_v + \eta_v \geq 0$. The Lelong-Jensen formula for $\varphi_v(t)$ gives

$$M_{\epsilon,v} - M_{\epsilon',v} = \int_{\epsilon'}^{\epsilon} \frac{dt}{t} \int_{\{|s| \leq t\}} \mathrm{dd}^c\varphi_v,$$

where $\epsilon > \epsilon'$ are positive constants and

$$M_{s,v} := \frac{1}{2\pi}\int_0^{2\pi} \varphi_v(\epsilon e^{i\theta}) d\theta$$

for every constant $s > 0$. It follows that

$$M_{\epsilon',v} \leq M_{\epsilon,v} + \int_{\epsilon'}^{\epsilon} \frac{dt}{t} \int_{\{|s| \leq t\}} \eta_v.$$

Letting $\epsilon' \to 0$ in the last inequality gives (2.1) because φ_v is continuous at 0.

Assume now (2.1). This combined with the hypothesis that $\varphi \not\equiv -\infty$ implies $\varphi \in L^1_{loc}$. Moreover, as in the case of p.s.h. functions, since φ is upper semi-continuous, (2.1) also tells us that φ is strongly semi-continuous in the sense that for every Borel subset A of U whose complement in U is of zero Lebesgue measure, we have

$$\limsup_{y \in A \to x} \varphi(y) = \varphi(x). \tag{2.2}$$

Consider first the case where $\varphi \in \mathscr{C}^2$. Direct computations show

$$\epsilon^{-2}\big(M_{\epsilon,v} - \varphi_v(0)\big) \to \pi \mathrm{dd}^c \varphi_v(0)/2$$

as $\epsilon \to 0$. Applying this to (2.1) gives $\mathrm{dd}^c\varphi_v(0) + \eta_v(0) \geq 0$. In other words, we get $\mathrm{dd}^c\varphi + \eta \geq 0$.

In general, let φ^r, η^r be as above. Since $\varphi \in L^1_{loc}$, $\varphi^r \to \varphi$ in L^1_{loc}. We see easily that (2.1) also holds for (φ^r, η^r) in place of (φ, η). By the above arguments, $\mathrm{dd}^c\varphi^r + \eta^r \geq 0$. Letting $r \to 0$ gives $\mathrm{dd}^c\varphi + \eta \geq 0$.

It remains to check that φ is the sum of a p.s.h. function and a smooth one. To this end, we only need to work locally. Thus, we can assume there is a smooth function ψ on U with $\mathrm{dd}^c\psi \geq \eta$. We deduce $\mathrm{dd}^c\varphi_1 \geq 0$ for $\varphi_1 := \varphi + \psi$ which is also strongly semi-upper continuous in the above sense. Let φ_1^r be the regularisation of φ_1 defined in the same way as φ^r. Notice that $\varphi_1^r \to \varphi_1$ in L^1_{loc} and φ_1^r is p.s.h. and decreasing to some p.s.h. function φ_1'. Hence, $\varphi_1 = \varphi_1'$ almost everywhere. Using this and (2.2) yield that $\varphi_1 = \varphi_1'$ everywhere. In other words, φ is quasi-p.s.h.. This ends the proof. □

The following extension result generalizes the similar property for p.s.h. functions.

Lemma 2.2. *Let U be an open subset in a complex manifold Y. Let η be a continuous real $(1,1)$-form on Y. Let ψ_1 be an η-p.s.h. function on U and ψ_2 an η-p.s.h function on Y such that $\limsup_{y\to x} \psi_1(y) \leq \psi_2(x)$ for every $x \in \partial U$. Define $\psi := \max\{\psi_1, \psi_2\}$ on U and $\psi := \psi_2$ on $Y \backslash U$. Then ψ is an η-p.s.h. function.*

Proof. This is a direct consequence of Lemma 2.1. □

A subset A of X is *locally pluripolar* if every point x in A there is an open neighborhood U_x of x in X and a p.s.h. function φ on U_x for which $A \cap U_x \subset \{\varphi = -\infty\}$. A subset A of X is *pluripolar* if $A \subset \{\varphi = -\infty\}$ for some quasi-p.s.h. function φ in X.

For every Borel set A' in an open subset U of $\mathbb{C}^k$, Bedford-Taylor [3] introduced the following notion of *capacity* of A' in U :

$$\mathrm{cap_{BT}(A', U)} := \sup\Big\{\int_{\mathrm{A}} (\mathrm{dd^c}\varphi)^{\mathrm{k}} : \varphi \text{ p.s.h. on } U\,, 0 \leq \varphi \leq 1 \text{ on U}\Big\}.$$

Fix, from now on, a Hermitian metric ω on X. For every Borel set $A \subset X$, define

$$\mathrm{cap_{BTK}(A)} := \sup\Big\{\int_{\mathrm{A}} (\mathrm{dd^c}\varphi + \omega)^{\mathrm{k}} : \varphi \ \omega\text{-p.s.h.}, 0 \leq \varphi \leq 1 \text{ on X}\Big\}.$$

The last capacity was introduced by Kołodziej [20] as an analogue to the local capacity $\mathrm{cap_{BT}}$ and is used to study complex Monge-Ampère equations on Hermitian manifolds, see for example [20, 9, 21, 24]. By Lemma 2.4 below, $\mathrm{cap_{BTK}(A)}$ is always finite. It is also clear that if we use another Hermitian metric to define $\mathrm{cap_{BTK}}$, then the resulted capacity is equivalent to that associated to ω.

We will need the following modified version of the classical Bedford-Taylor comparison principle due to Kołodziej and Nguyen, see [9] for a related result.

Proposition 2.3. [21, Th. 0.2] *Let φ, ψ be bounded ω-p.s.h functions on X. Let $0<\epsilon<1$ and $m_\epsilon := \inf_X(\varphi-(1-\epsilon)\psi)$. Then there exists a big constant $B>0$ depending only on ω, k such that for every constant $0<s<\epsilon^3/(16B)$ we have*

$$\int_{\{\varphi<(1-\epsilon)\psi+m_\epsilon+s\}} \big((1-\epsilon)\mathrm{dd}^c\psi+\omega\big)^k \leq (1+C\epsilon^{-k}s)\int_{\{\varphi<(1-\epsilon)\psi+m_\epsilon+s\}} (\mathrm{dd}^c\varphi+\omega)^k,$$

where C is a constant depending only on k, B.

A consequence of the last result is the following.

Lemma 2.4. *([9, 21]) Let M be a positive number. Then there exists a constant $c_M>0$ such that for every ω-p.s.h. function φ bounded by M, we have*

$$0<\int_X (\mathrm{dd}^c\varphi+\omega)^k \leq c_M. \tag{2.3}$$

However, we don't know whether

$$\inf_{\{\varphi\colon |\varphi|\leq M\}}\int_X (\mathrm{dd}^c\varphi+\omega)^k>0?$$

Proof. The second desired inequality is proved in [9] by using integration by parts. The first one is observed in [21]. To see it, it is enough to notice that by choosing $\epsilon:=1/2$ and $s>0$ small enough in Proposition 2.3, for every ω-p.s.h. ψ with $0\leq\psi\leq s$ and φ as in the hypothesis, we have

$$\int_{\{\varphi<\inf_X\varphi+s\}}\big(\mathrm{dd}^c\psi+\omega\big)^k \lesssim \int_{\{\varphi<(1-\epsilon)\psi+m_\epsilon+2s\}}\big((1-\epsilon)\mathrm{dd}^c\psi+\omega\big)^k \lesssim_s \int_X (\mathrm{dd}^c\varphi+\omega)^k$$

because

$$\{\varphi<\inf_X\varphi+s\}\subset\{\varphi<(1-\epsilon)\psi+m_\epsilon+2s\}.$$

It follows that

$$\int_X (\mathrm{dd}^c\varphi+\omega)^k \gtrsim_s \mathrm{cap}_{\mathrm{BTK}}\big(\{\varphi<\inf_{\mathrm{X}}\varphi+\mathrm{s}\}\big) \tag{2.4}$$

which is strictly positive because it is the capacity of a non-empty open set. The proof is finished. ☐

Let $(U_j)_{1\leq j\leq N}$ and $(U'_j)_{1\leq j\leq N}$ be finite open coverings of X such that $\overline{U}_j$ is smooth and contained in some local chart of X biholomorphic to a polydiscs for every $1\leq j\leq N$, $U_j=\{\psi_j<0\}$ for some p.s.h. function ψ_j defined on an open neighborhood of $\overline{U}_j$ with $\partial U_j=\{\psi_j=0\}$ and $U'_j\Subset U_j$ for $1\leq j\leq N$. In practice, it suffices to take U_j, U'_j to be balls and ψ_j are the differences of radius functions and constants.

Lemma 2.5. *([20, 9]) There exists strictly positive constants c_1, c_2 such that for every $A\subset X$ we have*

$$c_1\sum_{j=1}^N \mathrm{cap}_{\mathrm{BT}}\big(\mathrm{A}\cap\mathrm{U}'_\mathrm{j},\mathrm{U}_\mathrm{j}\big)\leq \mathrm{cap}_{\mathrm{BTK}}(\mathrm{A})\leq \mathrm{c}_2\sum_{\mathrm{j}=1}^{\mathrm{N}} \mathrm{cap}_{\mathrm{BT}}\big(\mathrm{A}\cap\mathrm{U}'_\mathrm{j},\mathrm{U}_\mathrm{j}\big).$$

Proof. Put $A'_j := A \cap U'_j$ which is a relatively compact subset of U_j. We have $\cup_j A'_j = A$. The second desired inequality is obvious from the definitions of capacities. We prove now the first desired inequality.

Fix an index $1 \leq j \leq N$. By our choice of U_j, for every p.s.h. function $0 \leq u \leq 1$ on U_j, we can find another p.s.h. function $-1 \leq \tilde{u} \leq 0$ on U_j satisfying $\tilde{u} = u - 1$ on some open neighborhood of $\overline{U}'_j$ and $\tilde{u} = 0$ on ∂U_j. Such a $\tilde{u}$ can be chosen to be $\max\{u - 1, A\psi_j\}$ for some constant A big enough. Clearly,

$$\int_{A'_j} (\mathrm{dd}^c u)^k = \int_{A'_j} (\mathrm{dd}^c \tilde{u})^k.$$

Since $-1 \leq \tilde{u} \leq 0$ and $\tilde{u} = 0$ on ∂U_j, there is a quasi-p.s.h. function $\tilde{u}_1$ on X such that $\mathrm{dd}^c \tilde{u}_1 + C\omega \geq 0$ for some constant C independent of $\tilde{u}$ and $\tilde{u}_1 = \tilde{u}$ on some open neighborhood of $\overline{U}'_j$. We deduce that

$$\int_{A'_j} (\mathrm{dd}^c u)^k = \int_{A'_j} (\mathrm{dd}^c \tilde{u}_1)^k \leq \int_{A'_j} (\mathrm{dd}^c \tilde{u}_1 + C\omega)^k \leq C^k \mathrm{cap}_{\mathrm{BTK}}(\mathrm{A}'_\mathrm{j}).$$

Consequently, $\mathrm{cap}_{\mathrm{BT}}(\mathrm{A}'_\mathrm{j}, \mathrm{U}_\mathrm{j}) \leq \mathrm{C}^\mathrm{k} \mathrm{cap}_{\mathrm{BTK}}(\mathrm{A}'_\mathrm{j})$. Summing over $1 \leq j \leq N$ in the last inequality gives the first desired inequality. This finishes the proof. □

Since we already know that if A is locally pluripolar in U, then $\mathrm{cap}_{\mathrm{BT}}(\mathrm{A}, \mathrm{U}) = 0$ (see [19, Th. 4.6.4] or [3]), we get $\mathrm{cap}_{\mathrm{BTK}}(\mathrm{A}) = 0$ if A is locally pluripolar in X. Let (u_j) be a family of p.s.h. functions on an open subset U of $\mathbb{C}^k$ locally bounded from above. Define $u := \sup_j u_j$ and $u^* := \sup_j^* u_j$ the upper semi-continuous regularisation of u. The set $\{u < u^*\}$ is called *a negligible set* in U. By Bedford-Taylor [3], the negligible sets are locally pluripolar. The following notion of capacity, which is related to those of Alexander [1] and Sibony-Wong [25], is due to Dinh-Sibony [11]: for $A \subset X$,

$$\mathrm{cap}_{\mathrm{ADS}}(\mathrm{A}) := \inf\{\exp(\sup_{\mathrm{A}} \varphi) : \varphi \ \omega\text{-p.s.h. on } X, \ \sup_{\mathrm{X}} \varphi = 0\},$$

see [15] for some of its applications.

Lemma 2.6. $\mathrm{cap}_{\mathrm{ADS}}(\mathrm{A}) = 0$ *if and only if* A *is pluripolar on* X.

Proof. If $A \subset \{\varphi = -\infty\}$ for some quasi-p.s.h. φ, it is clear that $\mathrm{cap}_{\mathrm{ADS}}(\mathrm{A}) = 0$. Consider now

$$\mathrm{cap}_{\mathrm{ADS}}(\mathrm{A}) = 0. \tag{2.5}$$

Recall that there exists a constant c such that for every ω-p.s.h. function φ with the normalization condition $\sup_X \varphi = 0$, we have

$$\|\varphi\|_{L^1(X)} \leq c. \tag{2.6}$$

We refer to [16, 11, 9] for a proof. Using (2.5), there exists a sequence of ω-p.s.h. functions (φ_n) with $\sup_X \varphi_n = 0$ such that $\sup_A \varphi_n \leq -n^3$. Put

$$\varphi := \sum_{n=1}^{\infty} \frac{\varphi_n}{n^2}$$

which is a well-defined quasi-p.s.h. function because of (2.6). On the other hand,

$$\sup_A \varphi \leq \sum_{n=1}^{\infty} \frac{-n^3}{n^2} = -\infty.$$

It means that $A \subset \{\varphi = -\infty\}$. This finishes the proof. $\square$

Let $(\varphi_j)_{j\in J}$ be a family of ω-p.s.h. functions uniformly bounded from above. Define

$$\varphi_J := \sup_{j\in J} \varphi_j.$$

Observe that φ_J^* is an ω-p.s.h. function. This can be seen by using Lemma 2.1 or noticing that for every ω-p.s.h. functions $\varphi_j, \varphi_{j'}$ we have $\max\{\varphi_j, \varphi_{j'}\} = \lim_{n\to\infty} n^{-1}\log(e^{n\varphi_j} + e^{n\varphi_{j'}})$ whose dd^c is $\geq -\omega$ for every n. As in the local setting, $\{\varphi_J^* > \varphi_J\}$ is a locally pluripolar set. We will present below an important case of $(\varphi_j)_{j\in J}$ and its associated extremal function φ_J^*.

Let A be a *non-pluripolar* subset of X. As in the local setting or in the Kähler case, we introduce the following extremal ω-p.s.h. function:

$$T_A := \sup\big\{\varphi \ \ \omega\text{-p.s.h.} : \varphi \leq 0 \text{ on } A\big\}.$$

It is clear that $T_A \geq 0$. Let T_A^* be the upper semi-continuous regularisation of T_A. We can check that

$$\mathrm{cap_{ADS}(A) = \exp(-\sup_X T_A)}. \tag{2.7}$$

Thus T_A is bounded from above because A is non-pluripolar. We deduce that T_A^* is a bounded ω-p.s.h. function and $Q_A := \{T_A^* > T_A\}$ is a locally pluripolar set. This combined with the fact that $T_A = 0$ on A implies that $T_A^* = 0$ on $A\backslash Q_A$. The following generalized a well-known property of T_A^* in the Kähler case.

Proposition 2.7. *Let A be a nonpluripolar compact subset of X. We have*

$$(\mathrm{dd}^c T_A^* + \omega)^k = 0 \tag{2.8}$$

on $X\backslash A$.

Proof. We follow the usual strategy. The key points are the existence of solutions of the Dirichlet problems proved in [21, 13, 7, 6] and Lemma 2.2 above.

By Choquet's lemma, there exists an increasing sequence of ω-p.s.h. function φ_n for which $T_A^* = (\lim_{n\to\infty}\varphi_n)^*$. For every ω-p.s.h function φ and every positive constant ϵ, using a regularisation of φ (see [5]), Hartog's lemma and the compactness of A, we deduce that there exists a smooth ω-p.s.h. function φ' such that $\varphi \leq \varphi'$ and $\varphi' \leq \sup_K \varphi + \epsilon$ on K. We construct a sequence (φ_n'') of smooth ω-p.s.h. functions from (φ_n) inductively as follows. Let φ_1' be a smooth ω-p.s.h. function such that $\varphi_1 \leq \varphi_1'$ and $\varphi_1' \leq 1$ on A. For $n \geq 2$, let φ_n' be a smooth ω-p.s.h. function such that

$$\max\{\varphi_n, \varphi_{n-1}' - (n-1)^{-2}\} \leq \varphi_n'$$

and $\varphi_n' \leq 1/n^2$ on A. Put

$$\varphi_n'' := \varphi_n' - \sum_{j=n}^{\infty} j^{-2}.$$

By our construction, (φ_n'') is increasing and $\varphi_n'' \leq 0$ on A and $\varphi_n'' \geq \varphi_n - (n-1)^{-1}$ for $n \geq 2$. We infer that

$$T_A^* = (\lim_{n\to\infty}\varphi_n'')^*.$$

Let $\mathbb{B}$ be an open ball in $X\backslash A$. By [21, Th. 4.2], there exists ω-p.s.h. functions u_n on $\mathbb{B}$ which is in $\mathscr{C}^0(\overline{\mathbb{B}})$ for which $(\mathrm{dd}^c u_n + \omega)^n = 0$ on $\mathbb{B}$ and $u_n = \varphi_n''$ on $\partial\mathbb{B}$. Define $\tilde{\varphi}_n'' := u_n$ on

$\overline{\mathbb{B}}$ and $\tilde{\varphi}''_n := \varphi''_n$ outside $\mathbb{B}$. By the domination principle [21, Cor. 3.4], we get $u_n \geq \varphi''_n$ on $\mathbb{B}$ and $u_{n+1} \geq u_n$ because $\varphi''_{n+1} \geq \varphi''_n$. By Lemma 2.2, $\tilde{\varphi}''_n$ is an ω-p.s.h. function. We have obtained a sequence $(\tilde{\varphi}''_n)$ of continuous ω-p.s.h functions increasing almost everywhere to T^*_A. Hence,

$$(dd^c\tilde{\varphi}''_n + \omega)^k \to (dd^c T^*_A + \omega)^k$$

as $n \to \infty$. We thus get $(dd^c T^*_A + \omega)^k = 0$ on $\mathbb{B}$ for every $\mathbb{B}$ in $X \backslash A$. The desired equality follows. This finishes the proof. □

Proposition 2.8. *Let A be a nonpluripolar compact subset of X. Then there exist strictly positive constants $c_1, c_2, \lambda_1, \lambda_2$ independent of A such that*

$$\exp\big(-\lambda_1 \mathrm{cap}_{\mathrm{BTK}}^{-1}(\mathrm{A})\big) \leq \mathrm{cap}_{\mathrm{ADS}}(\mathrm{A}) \leq \mathrm{c}_2 \exp\big(-\lambda_2 \mathrm{M}_{\mathrm{A}}^{1/\mathrm{k}} \mathrm{cap}_{\mathrm{BTK}}^{-1/\mathrm{k}}(\mathrm{A})\big). \tag{2.9}$$

where $M_A := \int_X (dd^c T^*_A + \omega)^k > 0$.

Note that $M_A > 0$ because of Lemma 2.4.

Proof. Since A non-pluripolar, T^*_A is a bounded ω-p.s.h. function. By (2.7), the desired inequalities are equivalent to the following:

$$\lambda_1 \mathrm{cap}_{\mathrm{BTK}}^{-1}(\mathrm{A}) \geq \sup_{\mathrm{X}} \mathrm{T}_{\mathrm{A}} \geq \mathrm{c}'_2 + \lambda_2 \mathrm{M}_{\mathrm{A}}^{1/\mathrm{k}} \mathrm{cap}_{\mathrm{BTK}}^{-1/\mathrm{k}}(\mathrm{A}), \tag{2.10}$$

where $c'_2 := -\log c_2$.

We prove now the first inequality of (2.10). We can assume $\sup_X T_A > 0$ because otherwise the desired inequality is trivial for any $\lambda_1 \geq 0$. Put $\varphi_A := T^*_A - \sup_X T^*_A$ which is an ω-p.s.h. function with $\sup_X \varphi_A = 0$. It follows that

$$\|\varphi_A\|_{L^p} \lesssim 1 \tag{2.11}$$

for every $p \geq 1$.

Let φ be an ω-p.s.h. function such that $0 \leq \varphi \leq 1$. Since $(\sup_X T_A)^{-1}\varphi_A = -1$ on $A \backslash Q_A$, and $\mathrm{cap}_{\mathrm{BTK}}(\mathrm{Q_A}) = 0$, we obtain

$$\int_A (dd^c\varphi + \omega)^k \leq (\sup_X T_A)^{-1} \int_X [-\varphi_A](dd^c\varphi + \omega)^k \lesssim (\sup_X T_A)^{-1} \|\varphi_A\|_{L^1} \tag{2.12}$$

for every φ with $0 \leq \varphi \leq 1$ by the Chern-Levine-Nirenberg inequality. Combining (2.12) with (2.11) gives the first inequality of (2.10). It remains to prove the second one.

Recall that $-1 \leq (\sup_X T_A)^{-1}\varphi_A \leq 0$ and $(\sup_X T_A)^{-1}\varphi_A$ is an $(\sup_X T_A)^{-1}\omega$-p.s.h. function. Hence $(\sup_X T_A)^{-1}\varphi_A$ is ω-p.s.h. if $(\sup_X T_A)^{-1} \leq 1$. Consider the case where $(\sup_X T_A)^{-1} \leq 1$. By definition of $\mathrm{cap}_{\mathrm{BTK}}$, we get

$$\mathrm{cap}_{\mathrm{BTK}}(\mathrm{A}) \geq (\sup_{\mathrm{X}} \mathrm{T_A})^{-\mathrm{k}} \int_{\mathrm{A}} (\mathrm{dd^c}\varphi_{\mathrm{A}} + \omega)^{\mathrm{k}} = (\sup_{\mathrm{X}} \mathrm{T_A})^{-\mathrm{k}} \int_{\mathrm{A}} (\mathrm{dd^c T^*_A} + \omega)^{\mathrm{k}} \tag{2.13}$$

By Proposition 2.7, we have

$$\int_A (dd^c T^*_A + \omega)^k = \int_X (dd^c T^*_A + \omega)^k.$$

Hence the second inequality of (2.10) follows if $(\sup_X T_A)^{-1} \leq 1$. When $(\sup_X T_A)^{-1} \geq 1$, then $T^*_A - 1 \leq 0$ on X and ≤ -1 on $A \backslash Q_A$. We imply that

$$\mathrm{cap}_{\mathrm{BTK}}(\mathrm{A}) = \mathrm{cap}_{\mathrm{BTK}}(\mathrm{A} \backslash \mathrm{Q_A}) \geq \int_{\mathrm{A}} (\mathrm{dd^c T^*_A} + \omega)^{\mathrm{k}} > 0$$

which combined with the fact that $\sup_X T_A \geq 0$ yields the second inequality of (2.10) in this case. The proof is finished. □

End of the proof of Theorem 1.1. First observe that a countable union of pluripolar sets is again a pluripolar set. Indeed, let $(V_n)_{n\in\mathbb{N}}$ be a countable family of pluripolar sets on X. Hence we have $V_n \subset \{\varphi_n = -\infty\}$ for some ω-p.s.h function φ_n with $\sup_X \varphi_n = 0$. Define

$$\varphi := \sum_{n=1}^{\infty} \varphi_n/n^2$$

which is of bounded L^1-norm because $\|\varphi_n\|_{L^1}$ is uniformly bounded in n. Hence φ is a quasi-p.s.h. function and $V_n \subset \{\varphi = -\infty\}$ for every n.

Let V be a locally pluripolar set. We need to prove V is pluripolar. If V is compact, the desired claim is a direct application of (2.9). For the general case, we need some more arguments.

By Lindelöf's property, we can cover V by at most countably many sets of form $\{\varphi_j = -\infty\}$ for some p.s.h functions φ_j on some open subset U_j of X. Hence in order to prove the desired assertion, we only need to consider $V = \{\varphi = -\infty\}$ for some p.s.h. function φ in an open subset U of X which is biholomorphic to a ball in $\mathbb{C}^k$.

Let U_1 be a relatively compact open subset of U. Suppose that $V \cap U_1$ is not pluripolar. Hence $T^*_{V\cap U_1}$ is a bounded ω-p.s.h function. Consider a decreasing sequence of smooth p.s.h. functions $(\varphi_n)_{n\in\mathbb{N}}$ defining on an open neighborhood of $\overline{U}_1$ converging pointwise to φ. For every positive integer N, put

$$V_{n,N} := \{\varphi_n \leq -N\} \cap \overline{U}_1$$

which is a compact subset increasing in n. Hence $(T^*_{V_{n,N}})_{n\in\mathbb{N}}$ is a decreasing sequence of ω-p.s.h. functions which converges pointwise to an ω-p.s.h. function T_N.

Since $\{\varphi_n < -N\}$ is open, $T^*_{V_{n,N}} = T_{V_{n,N}} = 0$ on $\{\varphi_n < -N\} \cap U_1$. Thus $T_N = 0$ on $\{\varphi < -N\} \cap U_1$ which contains $V \cap U_1$. We infer that

$$0 \leq T_N \leq T^*_{V\cap U_1}$$

for every N. This combined with the fact that $(T_N)_{N\in\mathbb{N}}$ is increasing gives

$$0 \leq T_\infty := (\lim_{N\to\infty} T_N)^* \leq T^*_{V\cap U_1} \tag{2.14}$$

and T_∞ is an ω-p.s.h. function. Applying (2.9) to $A := V_{n,N}$ we get

$$\sup_X T^*_{V_{n,N}} \geq c_2' + \lambda_2' M_{n,N}^{1/k} \mathrm{cap_{BTK}(V_{n,N})^{-1/k}} \tag{2.15}$$

where $M_{n,N} := \int_X (dd^c T^*_{V_{n,N}} + \omega)^k$. By the convergence of Monge-Ampère operators, we have

$$\lim_{n\to\infty} M_{n,N} = \int_X (dd^c T_N + \omega)^k =: M_N, \quad \lim_{N\to\infty} M_N = \int_X (dd^c T_\infty + \omega)^k =: M_\infty \tag{2.16}$$

Note that $M_\infty > 0$ by Lemma 2.4. On the other hand, we have

$$\mathrm{cap_{BTK}(V_{n,N}) \lesssim N^{-1}}$$

by the Chern-Levine-Nirenberg inequality. This together with (2.16) and (2.15) implies

$$\sup_X T_N \geq c_2' + \lambda_2' M_N^{1/k} N^{1/k}. \tag{2.17}$$

Letting $N \to \infty$ in the last inequality and using (2.16), (2.14), we get

$$\sup_X T^*_{V\cap U_1} \geq \sup_X T_\infty = \infty.$$

This is a contradiction. Hence $V \cap U_1$ is pluripolar for every relatively compact open subset U_1 of U. It follows that V is pluripolar. This finishes the proof. □

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