LÊ NUMBERS AND NEWTON DIAGRAM

CHRISTOPHE EYRAL, GRZEGORZ OLEKSIK, ADAM RÓŻYCKI

ABSTRACT. We give an algorithm to compute the Lê numbers of (the germ of) a Newton non-degenerate complex analytic function $f: (\mathbb{C}^n, 0) \to$ $(\mathbb{C},0)$ in terms of certain invariants attached to the Newton diagram of the function $f + z_1^{\alpha_1} + \cdots + z_d^{\alpha_d}$, where d is the dimension of the critical locus of f and $\alpha_1, \ldots, \alpha_d$ are sufficiently large integers. This is a version for non-isolated singularities of a famous theorem of A. G. Kouchnirenko. As a corollary, we obtain that Newton non-degenerate functions with the same Newton diagram have the same Lê numbers.

1. Introduction

The most important numerical invariant attached to a complex analytic function $f: (\mathbb{C}^n,0) \to (\mathbb{C},0)$ with an isolated singularity at 0 is its Milnor number at 0 (denoted by $\mu_f(0)$). By a theorem of A. G. Kouchnirenko [\[7\]](#page-16-0), we know that if f is (Newton) non-degenerate and such that its Newton diagram meets each coordinate axis (so-called "convenient" function), then f has an isolated singularity at 0 and $\mu_f(0)$ coincides with the Newton number $\nu(f)$ of f — a numerical invariant attached to the Newton diagram of f . Actually, if f is a non-degenerate function with an isolated singularity at 0, then $\mu_f(0) = \nu(f)$ even if f is not convenient (see [\[1\]](#page-16-1)). This provides an elegant and easy way to compute the Milnor number of such functions.

For a function with a non-isolated singularity at 0, the Milnor number is no longer relevant. However, we can attach to such a function a series of polar invariants which plays a similar role to that of the Milnor number for an isolated singularity. These polar invariants are called $L\hat{e}$ numbers. They were introduced by D. B. Massey in the 1990s (see $[10-12]$). Then we may wonder whether like for the Milnor number, the Lê numbers of a non-degenerate function f with a non-isolated singularity at 0 can also be described with the help of invariants attached to a Newton diagram. In this paper, we positively answer this question. More precisely, we show that the Lê numbers of a non-degenerate function f can be expressed in terms of certain invariants (which we shall call modified Newton numbers) attached to the Newton diagram of the function $f + z_1^{\alpha_1} + \ldots + z_d^{\alpha_d}$, where d is the dimension at 0 of the critical locus of f and $\alpha_1, \ldots, \alpha_d$ are sufficiently large integers (see Theorem [4.1\)](#page-7-0).

As an important corollary, we obtain that non-degenerate functions with the same Newton diagram have the same Lê numbers (see Corollary [5.1\)](#page-7-1).

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In particular, any 1-parameter deformation family of non-degenerate functions with constant Newton diagram has constant Lê numbers. We recall that families with constant Lê numbers satisfy remarkable properties. For example, in [\[11\]](#page-16-4), Massey proved that under appropriate conditions the diffeomorphism type of the Milnor fibrations associated to the members of such a family is constant. In [\[5\]](#page-16-5), J. Fernández de Bobadilla showed that in the special case of families of 1-dimensional singularities, the constancy of Lê numbers implies the topological triviality of the family at least if $n \geq 5$.

The paper is organized as follows. In Section [2,](#page-1-0) we recall the definition of the Lˆe numbers. In Section [3,](#page-3-0) following Kouchnirenko's definition of the Newton number, we introduce our modified Newton numbers. Our main result — the formulas for the Lê numbers of a non-degenerate function f in terms of the modified Newton numbers of the function $f + z_1^{\alpha_1} + \cdots + z_d^{\alpha_d}$ — is given in Section [4.](#page-6-0) Corollaries of these formulas are given in Section [5.](#page-7-2) In Section [6,](#page-8-0) we discuss a complete example. Finally, in Sections [7](#page-11-0) and [8,](#page-14-0) we give the proofs of our main result and main corollary respectively.

2. LÊ NUMBERS

 $L\hat{e}$ numbers are intersection numbers of certain analytic cycles — so-called $L\hat{e}$ cycles — with certain affine subspaces. The Lê cycles are defined using the notion of gap sheaf. In this section, we briefly recall these definitions which are essential for the paper. We also recall the notion of "polar ratio" which is involved in so-called *Iomdine-Lê-Massey formula*. This formula plays a crucial role in the proof of our main theorem.

We follow the presentation given by Massey in [\[10–](#page-16-2)[12\]](#page-16-3).

2.1. Gap sheaves. Let (X, \mathscr{O}_X) be a complex analytic space, $W \subseteq X$ be an analytic subset of X, and $\mathscr I$ be a coherent sheaf of ideals in $\mathscr O_X$. As usual, we denote by $V(\mathscr{I})$ the analytic space defined by the vanishing of \mathcal{I} . At each point $x \in V(\mathcal{I})$, we want to consider scheme-theoretically those components of $V(\mathscr{I})$ which are not contained in W. For this purpose, we look at a minimal primary decomposition of the stalk \mathscr{I}_x of \mathscr{I} in the local ring $\mathscr{O}_{X,x}$, and we consider the ideal $\mathscr{I}_x\neg W$ in $\mathscr{O}_{X,x}$ consisting of the intersection of those (possibly embedded) primary components Q of \mathscr{I}_x such that $V(Q) \nsubseteq W$. This definition does not depend on the choice of the minimal primary decomposition of \mathscr{I}_x . Now, if we perform the operation described above at the point x simultaneously at all points of $V(\mathscr{I})$, then we obtain a coherent sheaf of ideals called a *gap sheaf* and denoted by $\mathscr{I}\neg W$. Hereafter, we shall denote by $V(\mathscr{I})\neg W$ the scheme (i.e., the complex analytic space) $V(\mathscr{I}\neg W)$ defined by the vanishing of the gap sheaf $\mathscr{I}\neg W$.

2.2. Lê cycles and Lê numbers. Consider an analytic function $f: (U, 0) \rightarrow$ $(\mathbb{C},0)$, where U is an open neighbourhood of 0 in \mathbb{C}^n , and fix a system of linear coordinates $z = (z_1, \ldots, z_n)$ for \mathbb{C}^n . Let Σf be the critical locus of f. For $0 \leq k \leq n-1$, the kth (relative) polar variety of f with respect to the coordinates z is the scheme

$$
\Gamma^k_{f,z} := V\left(\frac{\partial f}{\partial z_{k+1}},\ldots,\frac{\partial f}{\partial z_n}\right) \neg \Sigma f.
$$

The analytic cycle

$$
[\Lambda_{f,z}^k]:=\left[\Gamma_{f,z}^{k+1}\cap V\bigg(\frac{\partial f}{\partial z_{k+1}}\bigg)\right]-\left[\Gamma_{f,z}^k\right]
$$

is called the kth $L\hat{e}$ cycle of f with respect to the coordinates z. (We always use brackets [·] to denote analytic cycles.) The kth Lê number $\lambda_{f,z}^k(0)$ of f at $0 \in \mathbb{C}^n$ with respect to the coordinates z is defined to be the intersection number

(2.1)
$$
\lambda_{f,z}^k(0) := ([\Lambda_{f,z}^k] \cdot [V(z_1, ..., z_k)])_0
$$

provided that this intersection is 0-dimensional or empty at 0; otherwise, we say that $\lambda_{f,z}^k(0)$ is undefined.^{[1](#page-2-0)} For $k=0$, the relation [\(2.1\)](#page-2-1) means

$$
\lambda_{f,z}^0(0) = \left([\Lambda_{f,z}^0] \cdot U \right)_0 = \left[\Gamma_{f,z}^1 \cap V \left(\frac{\partial f}{\partial z_1} \right) \right]_0.
$$

For any $\dim_0 \Sigma f \leq k \leq n-1$, the Lê number $\lambda_{f,z}^k(0)$ is always defined and equal to zero. For this reason, we usually only consider the Lê numbers

$$
\lambda_{f,z}^0(0),\ldots,\lambda_{f,z}^{\dim_0 \Sigma f}(0).
$$

Note that if 0 is an *isolated* singularity of f, then $\lambda_{f,z}^0(0)$ (which is the only possible non-zero Lê number) is equal to the Milnor number $\mu_f(0)$ of f at 0.

2.3. Polar ratios. As already mentioned above, a key ingredient in the proof of our main result is the Iomdine-Lê-Massey formula (see [\[12,](#page-16-3) Theorem 4.5]). Roughly, this formula says that if the Lê numbers of f at 0 with respect to z exist and if $d := \dim_0 \Sigma f \geq 1$, then for any integer α_1 large enough, $\dim_0 \Sigma(f + z_1^{\alpha_1}) = d - 1$ and the Lê numbers of $f + z_1^{\overline{\alpha}_1}$ at 0 with respect to the rotated coordinates $z^{(1)} := (z_2, \ldots, z_n, z_1)$ exist and they can be described in terms of the Lê numbers $\lambda_{f,z}^k(0)$ of the original function f. Moreover, the formula says that any $\alpha_1 > \rho_{f,z}(0)$ is suitable, where $\rho_{f,z}(0)$ is the maximum "polar ratio" of f at 0 with respect to z . In this section, we recall the definition of polar ratios (see [\[12,](#page-16-3) Definition 4.1]).

The notation is as in Section [2.2.](#page-1-1) Suppose that $\dim_0 \Gamma^1_{f,z} = 1$. Let η be an irreducible component of $\Gamma^1_{f,z}$ (with its reduced structure) such that $\dim_0(\eta \cap V(z_1)) = 0$. The polar ratio of η at 0 is the number defined by

$$
\frac{\big([\eta]\cdot[V(f)]\big)_0}{\big([\eta]\cdot[V(z_1)]\big)_0} = \frac{\Big(\Big[\eta\Big]\cdot\Big[V\Big(\frac{\partial f}{\partial z_1}\Big)\Big]\Big)_0}{\big([\eta]\cdot[V(z_1)]\big)_0} + 1.
$$

If $\dim_0(\eta \cap V(z_1)) \neq 0$, then we say that the polar ratio of η at 0 is equal to 1. A polar ratio for f at 0 with respect to z is any one of the polar ratios at 0 of any component of $\Gamma^1_{f,z}$.

For example, if f is a homogeneous polynomial and if $\dim_0 \Gamma^1_{f,z} = 1$, then each component of $\Gamma^1_{f,z}$ is a line, and hence the polar ratios for f at 0 with respect to z are all equal to 1 or to the degree $\deg(f)$ of the polynomial f (see [\[12,](#page-16-3) Remark 4.2]).

¹As usual, $[V(z_1, \ldots, z_k)]$ denotes the analytic cycle associated to the analytic space defined by $z_1 = \cdots = z_k = 0$. The notation $\left([\Lambda_{f,z}^k] \cdot [V(z_1, \ldots, z_k)] \right)_0$ stands for the intersection number at 0 of the analytic cycles $[\Lambda_{f,z}^k]$ and $[V(z_1,\ldots,z_k)]$.

In [\[13,](#page-16-6) Section 3.2], M. Morgado and M. Saia gave an upper bound for the maximal polar ratio for a semi-weighted homogeneous arrangement.

3. Newton diagram and modified Newton numbers

Let $z := (z_1, \ldots, z_n)$ be a system of coordinates for \mathbb{C}^n , let U be an open neighbourhood of the origin in \mathbb{C}^n , and let

$$
f: (U,0) \to (\mathbb{C},0), \quad z \mapsto f(z) = \sum_{\alpha} c_{\alpha} z^{\alpha},
$$

be an analytic function, where $\alpha := (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$, $c_\alpha \in \mathbb{C}$, and z^α is a notation for the monomial $z_1^{\alpha_1} \cdots z_n^{\alpha_n}$.

3.1. Newton diagram. Here, the reference is Kouchnirenko [\[7\]](#page-16-0). The Newton polyhedron $\Gamma_{+}(f)$ of f (at the origin and with respect to the coordinates $z = (z_1, \ldots, z_n)$ is the convex hull in \mathbb{R}^n_+ of the set

$$
\bigcup_{c_{\alpha}\neq 0} (\alpha + \mathbb{R}^n_+).
$$

For any $v \in \mathbb{R}^n_+ \setminus \{0\}$, put

$$
\ell(v, \Gamma_+(f)) := \min\{\langle v, \alpha \rangle \, ; \, \alpha \in \Gamma_+(f)\},
$$

$$
\Delta(v, \Gamma_+(f)) := \{\alpha \in \Gamma_+(f) \, ; \, \langle v, \alpha \rangle = \ell(v, \Gamma_+(f))\},
$$

where $\langle \cdot , \cdot \rangle$ denotes the standard inner product in \mathbb{R}^n . A subset $\Delta \subseteq \Gamma_+(f)$ is called a *face* of $\Gamma_+(f)$ if there exists $v \in \mathbb{R}^n_+ \setminus \{0\}$ such that $\Delta = \Delta(v, \Gamma_+(f)).$ The dimension of a face Δ of $\Gamma_{+}(f)$ is the minimum of the dimensions of the affine subspaces of \mathbb{R}^n containing Δ . The *Newton diagram* (also called Newton boundary) of f is the union of the compact faces of $\Gamma_{+}(f)$. It is denoted by $\Gamma(f)$. We say that f is *convenient* if the intersection of $\Gamma(f)$ with each coordinate axis of \mathbb{R}^n_+ is non-empty (i.e., for any $1 \leq i \leq n$, the monomial $z_i^{\alpha_i}$, $\alpha_i \geq 1$, appears in the expression $\sum_{\alpha} c_{\alpha} z^{\alpha}$ with a non-zero coefficient).

For any face $\Delta \subseteq \Gamma(f)$, define the *face function* f_{Δ} by

$$
f_{\Delta}(z) := \sum_{\alpha \in \Delta} c_{\alpha} z^{\alpha}.
$$

We say that f is *Newton non-degenerate* (in short, non-degenerate) on the face Δ if the equations

$$
\frac{\partial f_{\Delta}}{\partial z_1}(z) = \dots = \frac{\partial f_{\Delta}}{\partial z_n}(z) = 0
$$

have no common solution on $(\mathbb{C} \setminus \{0\})^n$. We say that f is *(Newton)* nondegenerate if it is non-degenerate on every face Δ of $\Gamma(f)$.

3.2. A bound for non-degeneracy of certain functions. Another important ingredient in the proof of our main theorem is Lemma 3.7 of [\[1\]](#page-16-1). This lemma asserts that if f is a non-degenerate function with a singularity at 0, then there exists a constant $m(f)$ such that for any $\alpha_i > m(f)$, the function $f + z_i^{\alpha_i}$ is non-degenerate too. Such a (non unique) number $m(f)$ is defined as follows. For each face $\Delta \subseteq \Gamma(f)$ with maximal dimension (i.e.,

 Δ is not contained in any other face), choose a vector $v_{\Delta} \in \mathbb{R}^n_+ \setminus \{0\}$ such that $\Delta = {\alpha \in \Gamma_+(f)$; $\langle v_\Delta, \alpha \rangle = \ell(v_\Delta, \Gamma_+(f))}$, and define

$$
W := \bigcup_{\substack{\Delta \subseteq \Gamma(f) \\ \text{max dim}}} \{ \alpha \in \mathbb{R}^n_+ \, ; \, \langle v_\Delta, \alpha \rangle \le \ell(v_\Delta, \Gamma_+(f)) \},
$$

where the union is taken over all maximal dimensional faces $\Delta \subseteq \Gamma(f)$. Clearly, W is a compact set and it intersects each coordinate axis of \mathbb{R}^n_+ in a closed interval, say $[0, w_i]$ for some w_i . Then define

$$
m(f) := \max_{1 \le i \le n} w_i.
$$

Of course, $m(f)$ depends on the choice of the vectors v_{Δ} . It is possible to define a "smallest" number $m_0(f)$ that also guarantees the non-degeneracy of the functions $f + z_i^{\alpha_i}$ with $\alpha_i > m_0(f)$ (see [\[6,](#page-16-7) Section 2]). However, for our purpose, we shall not need it.

3.3. Newton number. Again, the reference for this section is [\[7\]](#page-16-0). Throughout the paper, for any subsets $I \subseteq \{1, \ldots, n\}$ and $X \subseteq \mathbb{R}^n_+$, we shall use the following notation:

$$
X^{I} := \{ (x_1, \ldots, x_n) \in X \, ; \, x_i = 0 \, \text{if} \, i \notin I \}.
$$

In particular, for any $i \in \{1, \ldots, n\}$, the set $X^{\{i\}}$ is nothing but the intersection of X with the *i*th coordinate axis of \mathbb{R}^n_+ .

Let $\Gamma_{-}(f)$ denote the cone over $\Gamma(f)$ with the origin as vertex. If f is convenient, then the *Newton number* $\nu(f)$ of f is defined by

(3.1)
$$
\nu(f) := \sum_{I \subseteq \{1, ..., n\}} (-1)^{n-|I|} |I|! \operatorname{Vol}_{|I|}(\Gamma_-(f)^I),
$$

where |I| is the cardinality of I and $Vol_{|I|}(\Gamma_{-}(f)^{I})$ is the |I|-dimensional Euclidean volume of $\Gamma_-(f)^I$. For $I = \emptyset$, the subset $\Gamma_-(f)^{\emptyset}$ reduces to $\{0\},$ and we set $\text{Vol}_0(\Gamma_-(f)^\emptyset) = 1$.

The Newton number can also be defined even if f is not convenient. More precisely, if I is the non-empty subset of $\{1, \ldots, n\}$ such that $\Gamma(f)$ meets the *i*th coordinate axis of \mathbb{R}^n_+ if and only if $i \notin I$, then the Newton number $\nu(f)$ of f is defined as

$$
\nu(f):=\sup_{m\in\mathbb{Z}_+}\nu\bigg(f+\sum_{i\in I}z_i^m\bigg),
$$

where of course the Newton number of the (convenient) function $f + \sum_{i \in I} z_i^m$ is given by (3.1) .

3.4. Modified Newton numbers. Following Kouchnirenko's definition of the Newton number, we now introduce our modified Newton numbers.

Let I be a non-empty subset of $\{1,\ldots,n\}$ such that $\Gamma(f)^I \neq \emptyset$. By [\[3,](#page-16-8) Theorem 1, choose a simplicial decomposition of $\Gamma(f)^{I}$ in which the vertices of a simplex are 0-dimensional faces of $\Gamma(f)^I$ (such a decomposition is not unique). The cones spanned by the origin $0 \in \mathbb{R}^n$ and such simplexes give a simplicial decomposition

$$
\Xi_I := \{S_{I,r}\}_{1 \leq r \leq r_I}
$$

of $\Gamma_-(f)^I$. Note that

(3.2)
$$
\text{Vol}_{|I|}(\Gamma_{-}(f)^{I}) = \sum_{S_{I,r} \in \Xi_{I}, \dim S_{I,r} = |I|} \text{Vol}_{|I|}(S_{I,r}).
$$

Clearly, each simplex $S_{I,r} \subseteq (\mathbb{R}^n_+)^I$ may be identified to a simplex (still denoted by $S_{I,r}$) of $\mathbb{R}^{|I|}$, and with such an identification, the volume $Vol_{|I|}(S_{I,r})$ of a simplex $S_{I,r}$ with maximal dimension (i.e., with dimension |I|) is given by

(3.3)
$$
\text{Vol}_{|I|}(S_{I,r}) = \pm \frac{1}{|I|!} \det \begin{pmatrix} 0 & S_{I,r;1} & \cdots & S_{I,r;|I|} \\ 1 & 1 & \cdots & 1 \end{pmatrix},
$$

where $0, S_{I,r;1}, \ldots, S_{I,r;|I|}$ are the column vectors representing the coordinates of the vertices of the simplex $S_{I,r} \subseteq \mathbb{R}^{|I|}$. Note that each such column vector has $|I|$ components, so that the matrix in (3.3) has dimension $(|I| + 1) \times (|I| + 1).$

Let J be another subset of $\{1, \ldots, n\}$. We suppose that for any $i \in J$ the Newton boundary $\Gamma(f)$ meets the *i*th coordinate axis of \mathbb{R}^n_+ . Then to each $i_0 \in \{1, \ldots, n\}$, we associate a subset Ξ_{I,J,i_0} of Ξ_I (depending on I, J and i_0) as follows. If $i_0 \in I \cap J$, then we define Ξ_{I,J,i_0} as the set of all simplexes $S_{I,r} \in \Xi_I$ (as simplexes in $(\mathbb{R}^n_+)^I$) with maximal dimension |I| such that for any $i \in J$ the following property holds true:

$$
S_{I,r}^{\{i\}} = S_{I,r} \cap \Gamma_-(f)^{\{i\}}
$$
 is an edge of $S_{I,r} \Leftrightarrow i = i_0$.

(As usual, by an "edge" of a simplex we mean a 1-dimensional face.) If $i_0 \notin J$ (in particular if $J = \emptyset$) or if $i_0 \notin I$, then we set $\Xi_{I,J,i_0} := \emptyset$.

By definition, if $S_{I,r}$ is a simplex of Ξ_{I,J,i_0} , then it has maximal dimension and possesses a vertex with coordinates of the form $(0, \ldots, 0, \alpha_{i_0}, 0, \ldots, 0) \in$ \mathbb{R}^n (for some α_{i_0} located at the i_0 th place). To each such a simplex $S_{I,r}$ \in Ξ_{I,J,i_0} , we associate a (unique) "reduced" simplex $S_{I,r}$ defined by the same vertices as those of $S_{I,r}$ with the exception of the vertex $(0,\ldots,0,\alpha_{i_0},0,\ldots,0)$ which we replace by $(0, \ldots, 0, 1, 0, \ldots, 0)$. We denote by Ξ_{I,J,i_0} the set of such reduced simplexes.

By convention, for the next definition and all the statements hereafter, we agree that if I is a non-empty subset of $\{1,\ldots,n\}$ such that $\Gamma(f)^{I}$ is empty, then the corresponding "simplicial decomposition" Ξ_I is the empty set.

Definition 3.1. For each J , i_0 , and each collection $\Xi := {\{\Xi_I\}}_{I \subseteq {\{1,\ldots,n\}}, I \neq \emptyset}$ as above, we define a *modified Newton number* $\widetilde{\nu}_{\Xi, J,i_0}(f)$ for the function f by

$$
\widetilde{\nu}_{\Xi,J,i_0}(f):=\sum_{I\subseteq \{1,\ldots,n\},\,I\ni i_0}\Bigg(\sum_{\widetilde{S}_{I,r}\in \widetilde{\Xi}_{I,J,i_0}}(-1)^{n-|I|}|I|!\operatorname{Vol}_{|I|}(\widetilde{S}_{I,r})\Bigg).
$$

(If $\Xi_I = \emptyset$ or if $i_0 \notin J$, then $\Xi_{I,J,i_0} = \widetilde{\Xi}_{I,J,i_0} = \emptyset$, and the corresponding term in the above sum is zero by convention.)

Similarly, we introduce the subset $\Xi_{I,J,0}$ of Ξ_I consisting of those simplexes $S_{I,r} \in \Xi_I$ with maximal dimension and such that for any $i \in J$ the intersection $S_{I,r}^{\{i\}} = S_{I,r} \cap \Gamma_-(f)^{\{i\}}$ is not an edge of $S_{I,r}$.

Definition 3.2. For each J and each $\Xi := {\{\Xi_I\}}_{I \subseteq {\{1,\ldots,n\}}, I \neq \emptyset}$ as above, we define a special modified Newton number $\nu_{\Xi,J,0}(f)$ for the function f by

$$
\nu_{\Xi,J,0}(f):=\sum_{I\subseteq \{1,...,n\}, \, I\neq \emptyset} \Bigg(\sum_{S_{I,r}\in \Xi_{I,J,0}} (-1)^{n-|I|} |I|!\operatorname{Vol}_{|I|}(S_{I,r})\Bigg).
$$

Let us emphasize the fact that the simplexes involved in the definition of the modified Newton number $\widetilde{\nu}_{\Xi,J,i_0}(f)$ are *reduced* simplexes, while those used to define the special modified Newton number $\nu_{\Xi,J,0}(f)$ are not reduced.

4. Formulas for the Lˆe numbers of a non-degenerate function

Let $z := (z_1, \ldots, z_n)$ be a system of linear coordinates for \mathbb{C}^n , let U be an open neighbourhood of the origin in \mathbb{C}^n , and let $f: (U, 0) \to (\mathbb{C}, 0)$ be a nondegenerate analytic function. We denote by Σf the critical locus of f, and we suppose that $d := \dim_0 \Sigma f \geq 1$. We also assume that the Lê numbers

$$
\lambda_{f,z}^0(0),\ldots,\lambda_{f,z}^d(0)
$$

of f at 0 with respect to the coordinates $z = (z_1, \ldots, z_n)$ are defined. For example, if the coordinates are "prepolar" for f (see [\[12,](#page-16-3) Definition 1.26]), then the corresponding Lê numbers do exist. In particular, this is the case if f has an "aligned" singularity at 0 (e.g., a line singularity) and the coordinates are "aligning" for f at 0 (see [\[12,](#page-16-3) Definition 7.1]).

For any $1 \leq q \leq d$, we consider the function

(4.1)
$$
f_q(z) := f(z) + z_1^{\alpha_1} + \cdots + z_q^{\alpha_q},
$$

where $\alpha_1, \ldots, \alpha_q$ are integers such that, for any $1 \leq p \leq q$,

$$
\alpha_p > \max\{2, \rho_{f_{p-1}, z^{(p-1)}}(0), m(f_{p-1})\}.
$$

Here, $\rho_{f_{p-1},z^{(p-1)}}(0)$ is the maximum polar ratio for f_{p-1} at 0 with respect to the rotated coordinates

$$
z^{(p-1)} := (z_p, \ldots, z_n, z_1, \ldots, z_{p-1}),
$$

and $m(f_{p-1})$ is a bound which guarantees the non-degeneracy of the function f_p (see Sections [2.3](#page-2-2) and [3.2\)](#page-3-1). (By f_0 and $z^{(0)}$ we mean f and z respectively.) For example, if f is a homogeneous polynomial such that $d := \dim_0 \Sigma f = 1$, then we can take $f_d(z) = f_1(z) := \tilde{f}(z) + z_1^{\alpha_1}$, where $\alpha_1 > \max\{2, \deg(f)\}.$

Hereafter, we are mainly interested in the modified Newton numbers of the function f_d . For each non-empty subset $I \subseteq \{1, \ldots, n\}$, we choose a simplicial decomposition

$$
\Xi_I := \{S_{I,r}\}_{1 \leq r \leq r_I}
$$

of $\Gamma_-(f_d)^I$ as in Section [3.4](#page-4-1) (again, if $\Gamma(f_d)^I = \emptyset$, then $\Xi_I = \emptyset$), and we write $\Xi := {\{\Xi_I\}}_{I \subseteq \{1,\ldots,n\}, I \neq \emptyset}$. Since throughout this section we shall only consider modified Newton numbers of the form

$$
\nu_{\Xi,\{1,\dots,d\},0}(f_d) \quad \text{and} \quad \widetilde{\nu}_{\Xi,\{1,\dots,d\},k}(f_d)
$$

 $(1 \leq k \leq d)$ where d is the dimension at 0 of the critical locus Σf , we may simplify the notation as follows:

 $\nu_{\Xi,0}(f_d) := \nu_{\Xi,\{1,\dots,d\},0}(f_d) \text{ and } \tilde{\nu}_{\Xi,k}(f_d) := \tilde{\nu}_{\Xi,\{1,\dots,d\},k}(f_d).$

Here is our main result.

Theorem 4.1. Suppose that f is non-degenerate, $d := \dim_0 \Sigma f \geq 1$ and the Lê numbers $\lambda_{f,z}^k(0)$ of f at 0 with respect to the coordinates $z = (z_1, \ldots, z_n)$ are defined for any $0 \leq k \leq d$. Then the following two assertions hold true.

(1) The modified Newton numbers $\nu_{\Xi,0}(f_d)$ and $\widetilde{\nu}_{\Xi,k}(f_d)$ of the function f_d do not depend on the choice of $\Xi := {\{\Xi_I\}}_{I \subseteq {\{1,\ldots,n\}},\,I \neq \emptyset}$. Therefore, we may further simplify the notation as follows:

$$
\nu_0(f_d) := \nu_{\Xi,0}(f_d) \quad \text{and} \quad \widetilde{\nu}_k(f_d) := \widetilde{\nu}_{\Xi,k}(f_d).
$$

(2) The Lê numbers $\lambda_{f,z}^0(0), \ldots, \lambda_{f,z}^d(0)$ are given by the following formulas:

$$
\lambda_{f,z}^0(0) = (-1)^n + \nu_0(f_d) + \widetilde{\nu}_1(f_d);
$$

\n
$$
\lambda_{f,z}^k(0) = (-1)^{k-1} (\widetilde{\nu}_k(f_d) - \widetilde{\nu}_{k+1}(f_d)) \text{ for } 1 \le k \le d-1 \text{ (if } d \ge 2);
$$

\n
$$
\lambda_{f,z}^d(0) = (-1)^{d-1} \widetilde{\nu}_d(f_d).
$$

Theorem [4.1](#page-7-0) is a version for non-isolated singularities of the Kouchnirenko theorem mentioned in the introduction. It will be proved in Section [7.](#page-11-0) The formulas given in item (2) reduce the calculation of the Lê numbers of a non-degenerate function to a simple computation of volumes of simplexes. Certainly, these formulas are well suited for computer algebra programs.

5. Corollaries

Let $z = (z_1, \ldots, z_n)$ be linear coordinates for \mathbb{C}^n . The first important corollary of Theorem [4.1](#page-7-0) is the invariance of the Lê numbers within the class of non-degenerate functions with fixed Newton diagram. More precisely we have the following statement.

Corollary 5.1. Let $f, g : (U, 0) \rightarrow (\mathbb{C}, 0)$ be two non-degenerate analytic functions, where U is an open neighbourhood of the origin of \mathbb{C}^n . Suppose that the dimensions at 0 of the critical loci Σf and Σg of f and g, respectively, are greater than or equal to 1. If furthermore $\Gamma(f) = \Gamma(g)$ and the Lê numbers of f and g at 0 with respect to the coordinates $z = (z_1, \ldots, z_n)$ exist, then $\dim_0 \Sigma f = \dim_0 \Sigma g$, and for any $0 \leq k \leq n-1$, we have

$$
\lambda_{f,z}^k(0) = \lambda_{g,z}^k(0).
$$

Corollary [5.1](#page-7-1) will be proved in Section [8.](#page-14-0) In particular, it implies that any 1-parameter deformation family of non-degenerate functions with constant Newton diagram has constant Lê numbers, provided that these numbers exist. Here is a more precise statement.

Corollary 5.2. Let $\{f_t\}$ be a 1-parameter deformation family of analytic functions f_t defined in an open neighbourhood of $0 \in \mathbb{C}^n$ and depending analytically on the parameter $t \in \mathbb{C}$. If for any sufficiently small t (say $|t| \leq \varepsilon$ for some $\varepsilon > 0$), the function f_t is non-degenerate, $\Gamma(f_t) = \Gamma(f_0)$ and all the Lê numbers $\lambda_{f_t,z}^k(0)$ are defined, then $\dim_0 \Sigma f_t = \dim_0 \Sigma f_0$ and $\lambda_{f_t,z}^k(0) = \lambda_{f_0,z}^k(0)$ for all small t.

By combining Corollary [5.2](#page-7-3) with [\[12,](#page-16-3) Theorem 9.4] and [\[5,](#page-16-5) Theorem 42], we obtain a new proof of the following result, which is a special case of a much more general theorem of J. Damon [\[2\]](#page-16-9).

Corollary 5.3 (Damon). Let $\{f_t\}$ be a family as in Corollary [5.2,](#page-7-3) that is, such that for any sufficiently small t, the function f_t is non-degenerate, $\Gamma(f_t) = \Gamma(f_0)$ and all the Lê numbers $\lambda_{f_t,z}^k(0)$ are defined. Under these assumptions, the following two assertions hold true.

- (1) If for all small t, the coordinates $z = (z_1, \ldots, z_n)$ are prepolar for f_t and dim₀ $\Sigma f_t \leq n-4$, then the diffeomorphism type of the Milnor fibration of f_t at 0 is independent of t for all small t.
- (2) If $n \geq 5$ and $\dim_0 \Sigma f_t = 1$ for all small t, then the family $\{f_t\}$ is topologically trivial.

Indeed, by Corollary [5.2,](#page-7-3) the family $\{f_t\}$ has constant Lê numbers with respect to the coordinates $z = (z_1, \ldots, z_n)$. Item (1) then follows from [\[12,](#page-16-3) Theorem 9.4] while item (2) is a consequence of [\[5,](#page-16-5) Theorem 42].

In fact, in [\[2\]](#page-16-9), Damon obtains the topological triviality without the restrictions $n \geq 5$ or dim₀ $\Sigma f_t = 1$. A third proof (based on so-called "uniform" stable radius") of item (2) for line singularities is also given in [\[4\]](#page-16-10).

Finally, combined with [\[12,](#page-16-3) Theorem 3.3], Theorem [4.1](#page-7-0) has the following corollary about the Euler characteristic of the Milnor fibre associated to a non-degenerate function.

Corollary 5.4. Again, assume that f is non-degenerate, $d := \dim_0 \Sigma f \geq 1$ and the Lê numbers $\lambda_{f,z}^k(0)$ of f at 0 with respect to the coordinates $z =$ (z_1, \ldots, z_n) are defined for any $0 \leq k \leq d$. If furthermore the coordinates $z = (z_1, \ldots, z_n)$ are prepolar for f, then the reduced Euler characteristic $\widetilde{\chi}(F_{f,0})$ of the Milnor fibre $F_{f,0}$ of f at 0 is given by

$$
\widetilde{\chi}(F_{f,0}) = (-1)^{n-1} (\nu_0(f_d) + (-1)^n),
$$

where f_d is defined by (4.1) .

Indeed, by [\[12,](#page-16-3) Theorem 3.3], we have

$$
\widetilde{\chi}(F_{f,0}) = \sum_{k=0}^d (-1)^{n-1-k} \lambda_{f,z}^k(0).
$$

Thus, to get the formula in Corollary [5.4,](#page-8-1) it suffices to replace $\lambda_{f,z}^k(0)$ by its expression in terms of the modified Newton numbers given in Theorem [4.1.](#page-7-0)

6. Example

Consider the homogeneous polynomial function

$$
f(z_1, z_2, z_3) := z_1^2 z_2^2 + z_2^4 + z_3^4.
$$

The Newton diagram $\Gamma(f)$ of f is nothing but the triangle in \mathbb{R}^3_+ (with coordinates (x_1, x_2, x_3) defined by the vertices $A = (2, 2, 0), B = (0, 4, 0)$ and $C = (0, 0, 4)$ (see Figure [1\)](#page-9-0). We easily check that f is non-degenerate. The critical locus Σf of f is given by the z_1 -axis, and the restriction of f to the hyperplane $V(z_1)$ defined by $z_1 = 0$ has an isolated singularity at 0.

FIGURE 1. Newton diagrams of f and f_1

In other words, f has a *line singularity* at 0 in the sense of $[9, §4]$. Then, by [\[12,](#page-16-3) Remark 1.29], the partition of $V(f) := f^{-1}(0)$ given by

$$
\mathcal{S} := \{ V(f) \setminus \Sigma f, \Sigma f \setminus \{0\}, \{0\} \}
$$

is a "good stratification" for f in a neighbourhood of 0, and the hyperplane $V(z_1)$ is a "prepolar slice" for f at 0 with respect to \mathscr{S} (see [\[12,](#page-16-3) Definitions 1.24 and 1.26]). In other words, the coordinates $z = (z_1, z_2, z_3)$ are prepolar for f . In particular, combined with [\[12,](#page-16-3) Proposition 1.23], this implies that the Lê numbers $\lambda_{f,z}^0(0)$ and $\lambda_{f,z}^1(0)$ are defined. We can compute these numbers either using the definition or by applying Theorem [4.1.](#page-7-0)

6.1. Calculation using the definition. We need to compute the polar varieties $\Gamma_{f,z}^2$ and $\Gamma_{f,z}^1$ and the Lê cycles $[\Lambda_{f,z}^1]$ and $[\Lambda_{f,z}^0]$. By definition,

$$
\Gamma_{f,z}^2 = V\left(\frac{\partial f}{\partial z_3}\right) \neg V(z_2, z_3) = V(z_3^3) \neg V(z_2, z_3) = V(z_3^3),
$$

while

$$
\Gamma_{f,z}^1 = V\left(\frac{\partial f}{\partial z_2}, \frac{\partial f}{\partial z_3}\right) \neg V(z_2, z_3)
$$

= $V(z_2(2z_1^2 + 4z_2^2), z_3^3) \neg V(z_2, z_3) = V(2z_1^2 + 4z_2^2, z_3^3).$

It follows that the Lê cycles are given by

$$
\begin{aligned} [\Lambda_{f,z}^1] &= \left[\Gamma_{f,z}^2 \cap V\left(\frac{\partial f}{\partial z_2}\right) \right] - \left[\Gamma_{f,z}^1 \right] \\ &= [V(z_3^3) \cap V(z_2(2z_1^2 + 4z_2^2))] - [V(2z_1^2 + 4z_2^2, z_3^3)] \\ &= [V(z_2, z_3^3)] \end{aligned}
$$

and

$$
\begin{aligned} [\Lambda_{f,z}^0] &= \left[\Gamma_{f,z}^1 \cap V \left(\frac{\partial f}{\partial z_1} \right) \right] \\ &= [V(2z_1^2 + 4z_2^2, z_3^3) \cap V(z_1 z_2^2)] \\ &= [V(z_1, z_2^2, z_3^3)] + [V(z_1^2, z_2^2, z_3^3)]. \end{aligned}
$$

Finally the Lê numbers are given by

$$
\lambda_{f,z}^1 = ([\Lambda_{f,z}^1] \cdot [V(z_1)])_0 = [V(z_1, z_2, z_3^3)]_0 = 3;
$$

$$
\lambda_{f,z}^0 = ([\Lambda_{f,z}^0] \cdot \mathbb{C}^3)_0 = 6 + 12 = 18.
$$

6.2. Calculation using Theorem [4.1.](#page-7-0) Consider a polynomial function

$$
f_1(z_1, z_2, z_3) := f(z_1, z_2, z_3) + z_1^{\alpha_1}
$$

such that $\alpha_1 > \max\{2, \rho_{f,z}(0), m(f)\}\)$. Since f is a homogeneous polynomial of degree 4, the maximum polar ratio $\rho_{f,z}(0)$ for f at 0 with respect to the coordinates z is 4 (see Section [2.3\)](#page-2-2), and clearly, we can take $m(f) = 4$. So, let us take for instance $\alpha_1 = 5$. Clearly, $\Gamma_-(f_1)$ is the union of two tetrahedra $\{O, A, C, E\}$ and $\{O, A, B, C\}$. For each subset $I \subseteq \{1, 2, 3\}$, take the "natural" simplicial decomposition Ξ_I of $\Gamma_-(f_1)^I$ generated by the vertices of the set $\{O, A, B, C, E\} \cap (\mathbb{R}^n_+)^I$ as suggested in Figure [1](#page-9-0) (at this level, we ignore the point D mentioned in the figure). For example, $\Xi_{\{1,2\}}$ is defined by the simplexes $\{O, A, E\}$ and $\{O, A, B\}$. By Theorem [4.1,](#page-7-0)

$$
\lambda_{f,z}^0(0) = (-1)^3 + \nu_0(f_1) + \tilde{\nu}_1(f_1) \text{ and } \lambda_{f,z}^1(0) = (-1)^0 \tilde{\nu}_1(f_1).
$$

The data to compute the modified Newton numbers $\nu_0(f_1)$ and $\tilde{\nu}_1(f_1)$ are given in Table [1.](#page-10-0) In this table, $O = (0, 0, 0), D = (1, 0, 0),$ and A, B, C are as above. Each pair in the third and fourth columns of the table consists of a simplex together with its volume. For example, in the first row of the third column, the pair $(\{O, D\}; 1)$ consists of the simplex $\{O, D\} \in \Xi_{\{1\},\{1\},1}$ and its volume $\text{Vol}_1(\{O, D\}) = 1$. The calculation shows that $\nu_0(f_1) = 16$ and $\tilde{\nu}_1(f_1) = 3$, and therefore the Lê numbers are given by

$$
\lambda_{f,z}^0(0) = 18
$$
 and $\lambda_{f,z}^1(0) = 3$.

6.3. Euler characteristic. Since the coordinates $z = (z_1, z_2, z_3)$ are prepolar for f, Corollary [5.4](#page-8-1) says that to calculate the reduced Euler characteristic $\widetilde{\chi}(F_{f,0})$ of the Milnor fibre $F_{f,0}$ of f at 0, it suffices to compute the special modified Newton number $\nu_0(f_1)$. Precisely, $\widetilde{\chi}(F_{f,0})$ is given by

$$
\widetilde{\chi}(F_{f,0}) = (-1)^2 (\nu_0(f_1) + (-1)^3) = 15.
$$

\overline{I}		$\left \overline{(-1)^{3- I } I !}\right (S\in \widetilde{\Xi}_{I,\{1\},1};{\rm Vol}_{ I }(S))\left (S\in \Xi_{I,\{1\},0};{\rm Vol}_{ I }(S))\right $	
${1}$		$({O,D};1)$	$(\emptyset;0)$
${2}$	1	$(\emptyset;0)$	$({O, B}; 4)$
$\{3\}$	1	$(\emptyset;0)$	$({O, C}; 4)$
$\{1,2\}$	-2	$({O, A, D}; 1)$	$({O, A, B}; 4)$
$\{1,3\}$	-2	$({O, C, D}; 2)$	$(\emptyset;0)$
$\{2,3\}$	-2	$(\emptyset;0)$	$({O, B, C}; 8)$
$\{1,2,3\}$	6	$({O, A, C, D}; \frac{4}{3})$	$({O, A, B, C}; \frac{16}{3})$

TABLE 1. Data to compute $\nu_0(f_1)$ and $\tilde{\nu}_1(f_1)$

7. Proof of Theorem [4.1](#page-7-0)

Applying the Iomdine-Lê-Massey formula (see $[12,$ Theorem 4.5) successively to f, f_1, \ldots, f_{d-1} shows that for any $0 \le q \le d-1$:

- (1) $\Sigma f_{q+1} = \Sigma f \cap V(z_1, \ldots, z_{q+1})$ in a neighbourhood of the origin;
- (2) dim₀ $\Sigma f_{q+1} = d (q+1);$
- (3) the Lê numbers $\lambda_{f_{q+1},z^{(q+1)}}^k(0)$ of f_{q+1} at 0 with respect to the rotated coordinates

$$
z^{(q+1)} = (z_{q+2}, \ldots, z_n, z_1, \ldots, z_{q+1})
$$

exist for all $0 \leq k \leq d - (q + 1)$ and are given by

$$
\begin{cases} \lambda^0_{f_{q+1},z^{(q+1)}}(0)=\lambda^0_{f_q,z^{(q)}}(0)+(\alpha_{q+1}-1)\lambda^1_{f_q,z^{(q)}}(0);\\ \lambda^k_{f_{q+1},z^{(q+1)}}(0)=(\alpha_{q+1}-1)\lambda^{k+1}_{f_q,z^{(q)}}(0) \quad \text{for} \quad 1\leq k\leq d-(q+1); \end{cases}
$$

where $\lambda_{f_q,z^{(q)}}^k(0)$ is the kth Lê number of f_q at 0 with respect to the rotated coordinates

$$
z^{(q)}=(z_{q+1},\ldots,z_n,z_1,\ldots,z_q),
$$

and where α_{q+1} is an integer satisfying

$$
\alpha_{q+1} > \max\{2, \rho_{f_q, z^{(q)}}(0), m(f_q)\}.
$$

In particular (see [\[12,](#page-16-3) Corollary 4.6]) f_d has an isolated singularity at 0 and its Milnor number $\mu_{f_d}(0)$ (which, in this case, coincides with its 0th Lê number $\lambda_{f_d,z^{(d)}}^0(0)$) is given by

(7.1)
$$
\mu_{f_d}(0) = \lambda_{f,z}^0(0) + \sum_{k=1}^d \left(\prod_{q=1}^k (\alpha_q - 1) \right) \lambda_{f,z}^k(0).
$$

Let $\{i_1, \ldots, i_p\}$ be the subset of $\{1, \ldots, n\} \setminus \{1, \ldots, d\}$ consisting of all indices i for which $\Gamma(f_d)$ does not meet the ith coordinate axis of \mathbb{R}^n_+ . Then, by $[1,$ Lemmas 3.6–3.8 and Corollary 3.11] and $[7,$ Théorème I], for any $0 \ll \alpha_1 \ll \cdots \ll \alpha_d \ll \alpha_{i_1} \ll \cdots \ll \alpha_{i_p}$ sufficiently large, the function

$$
f'_d(z) := \underbrace{f(z) + z_1^{\alpha_1} + \dots + z_d^{\alpha_d}}_{f_d(z)} + z_{i_1}^{\alpha_{i_1}} + \dots + z_{i_p}^{\alpha_{i_p}}
$$

is non-degenerate, convenient, and the following equalities hold true:

(7.2)
$$
\mu_{f_d}(0) = \mu_{f'_d}(0) = \nu(f'_d) = \nu(f_d).
$$

The expression [\(7.1\)](#page-11-1) for the Milnor number $\mu_{f_d}(0)$ can be viewed as a polynomial in the variables $\alpha_1, \ldots, \alpha_d$. Its linear part is given by

(7.3)
$$
\sum_{k=0}^{d} (-1)^k \lambda_{f,z}^k(0) + \sum_{i=1}^{d} \left(\alpha_i \sum_{k=i}^d (-1)^{k-1} \lambda_{f,z}^k(0) \right).
$$

Now we need the following lemma.

Lemma 7.1. The function f has no term of the form $c_1z_1^{a_1}, \ldots, c_dz_d^{a_d}$, where $c_i \in \mathbb{C} \setminus \{0\}, a_i \in \mathbb{Z}_{>0}.$

We postpone the proof of this lemma to the end of this section, and we first complete the proof of Theorem [4.1.](#page-7-0)

Since \hat{f} has no term of the form $c_1 z_1^{a_1}, \ldots, c_d z_d^{a_d}$, the Newton number $\nu(f'_d)$ can be viewed as a polynomial in the variables $\alpha_1, \ldots, \alpha_d$ and $\alpha_{i_1}, \ldots, \alpha_{i_p}$. Its linear part with respect to $\alpha_1, \ldots, \alpha_d$ has the form

$$
(7.4) \qquad P_0(\alpha_{i_1},\ldots,\alpha_{i_p}) + \alpha_1 P_1(\alpha_{i_1},\ldots,\alpha_{i_p}) + \cdots + \alpha_d P_d(\alpha_{i_1},\ldots,\alpha_{i_p}),
$$

where $P_i(\alpha_{i_1}, \ldots, \alpha_{i_p})$ are polynomials in $\alpha_{i_1}, \ldots, \alpha_{i_p}$. Taking the difference $\mu_{f_d}(0) - \nu(f'_d)$ gives a polynomial

$$
Q(\alpha_1,\ldots,\alpha_d,\alpha_{i_1},\ldots,\alpha_{i_p}) := \mu_{f_d}(0) - \nu(f'_d)
$$

in the variables $\alpha_1, \ldots, \alpha_d, \alpha_{i_1}, \ldots, \alpha_{i_p}$. Then it follows from [\(7.2\)](#page-11-2) that for any $0 \ll \alpha_1 \ll \cdots \ll \alpha_d \ll \alpha_{i_1} \ll \cdots \ll \alpha_{i_p}$ sufficiently large (equivalently, for any $(\alpha_1, \ldots, \alpha_d, \alpha_{i_1}, \ldots, \alpha_{i_p})$ in the set $Z(d+p)$ which appears in Lemma [A.1](#page-15-0) of the appendix, with the appropriate coefficients c_1 and $c_{\ell}(\alpha_1,\ldots,\alpha_{\ell-1})$ for $2 \leq \ell \leq d+p$, we have

$$
Q(\alpha_1,\ldots,\alpha_d,\alpha_{i_1},\ldots,\alpha_{i_p})=0.
$$

Thus applying Lemma [A.1](#page-15-0) shows that Q identically vanishes. In particular, comparing the coefficients of the linear parts [\(7.3\)](#page-11-3) and [\(7.4\)](#page-12-0) of $\mu_{f_d}(0)$ and $\nu(f'_d)$, respectively, shows that the polynomials $P_i := P_i(\alpha_{i_1}, \ldots, \alpha_{i_p})$ are independent of $\alpha_{i_1}, \ldots, \alpha_{i_p}$ (i.e., P_i is constant) and are given by

$$
\begin{cases}\nP_0 = \sum_{k=0}^d (-1)^k \lambda_{f,z}^k(0); \\
P_i = \sum_{k=i}^d (-1)^{k-1} \lambda_{f,z}^k(0) \quad \text{for} \quad 1 \le i \le d.\n\end{cases}
$$

Theorem [4.1](#page-7-0) is now an immediate consequence of the following lemma.

Lemma 7.2. For each non-empty subset $I \subseteq \{1, \ldots, n\}$, choose a simplicial decomposition

$$
\Xi_I':=\{S_{I,r}'\}_{1\leq r\leq r_I'}
$$

of $\Gamma_-(f_d')^I$ as in Section [3.4](#page-4-1) such that its restriction to $\Gamma_-(f_d)^I$ coincides with the simplicial decomposition Ξ_I . (We can always achieve this condition by taking $\alpha_{i_1}, \ldots, \alpha_{i_p}$ sufficiently large.) Write $\Xi' := {\{\Xi'_I\}}_{I \subseteq {\{1,\ldots,n\}}, I \neq \emptyset}$ and set $J := \{1, \ldots, d, i_1, \ldots, i_p\}$. Then the following equalities hold true:

$$
\begin{cases}\nP_{i_0} = \widetilde{\nu}_{\Xi', J, i_0}(f'_d) = \widetilde{\nu}_{i_0}(f_d) \text{ for } 1 \le i_0 \le d; \\
P_0 = \nu_{\Xi', J, 0}(f'_d) + (-1)^n = \nu_0(f_d) + (-1)^n.\n\end{cases}
$$

To complete the proof of Theorem [4.1,](#page-7-0) it remains to prove Lemmas [7.1](#page-11-4) and [7.2.](#page-12-1) We start with the proof of Lemma [7.2.](#page-12-1)

Proof of Lemma [7.2.](#page-12-1) By [\(3.1\)](#page-4-0) and [\(3.2\)](#page-5-1), the Newton number $\nu(f'_d)$ is (up to coefficients of the form $(-1)^{n-|I|}$ |I|!) a sum of volumes of the form $\text{Vol}_{|I|}(S'_{I,r})$, where $\emptyset \neq I \subseteq \{1,\ldots,n\}$ and $S'_{I,r}$ is a simplex of Ξ'_{I} with maximal dimension $|I|$, plus the number

$$
(-1)^{n-|\emptyset|} |\emptyset|! \operatorname{Vol}_{|\emptyset|}(\Gamma_-(f'_d)^{\emptyset}) = (-1)^n,
$$

which corresponds to $I = \emptyset$ in the definition of $\nu(f'_d)$ (see Section [3.3\)](#page-4-2). If for any $1 \leq i_0 \leq d$ the matrix used to compute the volume $Vol_{|I|}(S'_{I,r})$ (see [\(3.3\)](#page-5-0)) does not have any column of the form

$$
\left(\begin{matrix} \beta_1 & \cdots & \beta_{i_0-1} & \alpha_{i_0} & \beta_{i_0+1} & \cdots & \beta_{|I|} & 1 \end{matrix}\right)^T,
$$

then $Vol_{|I|}(S'_{I,r})$ contributes to the term P_0 which appears in [\(7.4\)](#page-12-0). (Here, the letter T'' stands for the transposed matrix.) On the other hand, if it contains such a column for some $i_0 \in \{1, ..., d\}$, then necessarily the β_i 's are zero, and the column is of the form

$$
C_{i_0}:=\begin{pmatrix}0&\cdots&0&\alpha_{i_0}&0&\cdots&0&1\end{pmatrix}^T
$$

(because $\Gamma(f'_d)$ intersects the *i*₀th coordinate axis of \mathbb{R}^n_+ precisely at the point $(0, \ldots, 0, \alpha_{i_0}, 0, \ldots, 0)$ by Lemma [7.1\)](#page-11-4). If the matrix has two columns C_{i_0} and $C_{i'_0}$ of the above form, with $i_0, i'_0 \in \{1, ..., d\}$ and $i_0 \neq i'_0$, then $Vol_{|I|}(S'_{I,r})$ is not involved in the linear part [\(7.4\)](#page-12-0) of $\nu(f_d')$. Now, if it has one column C_{i_0} for some $i_0 \in \{1, ..., d\}$ and no any other column $C_{i'_0}$ for $i'_0 \in \{1, ..., d\} \setminus \{i_0\}$, then $\text{Vol}_{|I|}(S'_{I,r})$ contributes to the term $\alpha_{i_0} P_{i_0}$ which appears in [\(7.4\)](#page-12-0). Note that in the latter case, the matrix cannot have any column of the form C_{i_j} with $i_j \in \{i_1, \ldots, i_p\}$ (as otherwise the constant polynomial P_{i_0} would depend on α_{i_j}). Altogether, for any $1 \leq i_0 \leq d$, the volume $\text{Vol}_{|I|}(S'_{I,r})$ contributes to the term $\alpha_{i_0}P_{i_0}$ if and only if the corresponding matrix has a column of the form C_{i_0} and no column of the form $C_{i'_0}$ for any other $i'_0 \in \{1, \ldots, d, i_1, \ldots, i_p\} \setminus \{i_0\} = J \setminus \{i_0\}$. In other words, $\text{Vol}_{|I|}(S'_{I,r})$ contributes to the term $\alpha_{i_0} P_{i_0}$ if and only if $S'_{I,r} \in \Xi'_{I,J,i_0}$. Thus,

$$
\alpha_{i_0} P_{i_0} = \sum_{I \subseteq \{1, ..., n\}, I \ni i_0} \left(\sum_{S'_{I,r} \in \Xi'_{I,J,i_0}} (-1)^{n-|I|} |I|! \operatorname{Vol}_{|I|}(S'_{I,r}) \right)
$$

=
$$
\sum_{I \subseteq \{1, ..., n\}, I \ni i_0} \alpha_{i_0} \left(\sum_{\widetilde{S}'_{I,r} \in \widetilde{\Xi}'_{I,J,i_0}} (-1)^{n-|I|} |I|! \operatorname{Vol}_{|I|}(\widetilde{S}'_{I,r}) \right)
$$

=
$$
\alpha_{i_0} \widetilde{\nu}_{\Xi',J,i_0}(f'_d),
$$

where $\widetilde{S}'_{I,r}$ denotes the reduced simplex associated to $S'_{I,r}$ (see Section [3.4\)](#page-4-1). Since $\Gamma(f_d)$ is obtained from $\Gamma(f_d)$ only by "adding" the vertices

$$
v_{i_j} := (0, \ldots, 0, \alpha_{i_j}, 0, \ldots, 0)
$$

(with α_{i_j} at the *i*_jth place) for large α_{i_j} ($1 \leq j \leq p$), if a simplex $S'_{I,r}$ of Ξ'_I with maximal dimension is not a simplex of Ξ_I (in particular this is the case if $\Xi_I = \emptyset$), then necessarily it intersects the *i_j*th coordinate axis of \mathbb{R}^n_+ for some j $(1 \leq j \leq p)$. It follows that

$$
\Xi_{I,\{1,\ldots,d\},i_0} = \Xi'_{I,J,i_0},
$$

and hence,

$$
\widetilde{\nu}_{\Xi,i_0}(f_d) := \widetilde{\nu}_{\Xi,\{1,\dots,d\},i_0}(f_d) = \widetilde{\nu}_{\Xi',J,i_0}(f_d') = P_{i_0}.
$$

Since the choice of Ξ is arbitrary and P_{i_0} is a constant independent of Ξ , it follows that the modified Newton number $\widetilde{\nu}_{\Xi,i_0}(f_d)$ is also independent of Ξ .
The notation $\widetilde{\nu}_{\Xi}(f_a)$ is the independent relaxant The notation $\widetilde{\nu}_{i_0}(f_d) := \widetilde{\nu}_{\Xi, i_0}(f_d)$ is therefore quite relevant.

Since the volume $\text{Vol}_{|I|}(\overline{S'_{I,r}})$ contributes to the term P_0 if and only if the simplex $S'_{I,r}$ belongs to $\Xi'_{I,J,0}$ (we recall that P_0 is constant, independent of $\alpha_{i_1}, \ldots, \alpha_{i_p}$, a similar argument shows that

$$
\nu_0(f_d) + (-1)^n = \nu_{\Xi',J,0}(f_d') + (-1)^n = P_0.
$$

Now we prove Lemma [7.1.](#page-11-4)

Proof of Lemma [7.1.](#page-11-4) We argue by contradiction. Suppose that f has a term of the form $c_i z_i^{a_i}$ for some i $(1 \leq i \leq d)$. To simplify, without loss of generality, we may assume $i = 1$ (the other cases are similar). By the Iomdine-Lê-Massey formula again and by Lemmas 3.7 and 3.8 of [\[1\]](#page-16-1), for any $0 \ll b_1 \ll \cdots \ll b_d$ sufficiently large (in particular so that $a_1 < b_1$), the function

$$
g(z) := f(z) + z_1^{b_1} + \dots + z_d^{b_d}
$$

is non-degenerate and has an isolated singularity at 0. Then, by [\[1,](#page-16-1) Corollary 2.9, its support (denoted by supp q) satisfies so-called Kouchnirenko condition (see [\[8\]](#page-16-12) or Section 2 of [\[1\]](#page-16-1) for the definition; see also Section 3 of [\[14\]](#page-16-13) and the references mentioned therein for equivalent formulations and historical comments). Now, since $a_1 < b_1$, the Newton diagrams of g and of the function

$$
g'(z) := g(z) - z_1^{b_1}
$$

coincide. It follows that g' is also non-degenerate and such that its support supp g' satisfies the Kouchnirenko condition. Theorem 3.1 of [\[1\]](#page-16-1) then implies that g' has an isolated singularity at 0. If $d := \dim_0 \Sigma f = 1$, then this is already a contradiction, because in this case $g' = f$. If $d > 1$, then define

$$
Z_{g'} := \left\{ z \in \mathbb{C}^n \, ; \, \frac{\partial g'}{\partial z_i}(z) = 0 \text{ for all } i \in \{1, d+1, \dots, n\} \right\}.
$$

Define Z_f similarly (replacing g' by f). Clearly, $Z_{g'} = Z_f$. Therefore, we have $d := \dim_0 \Sigma f \leq \dim_0 Z_f = \dim_0 Z_{g'}$, but since g' has an isolated singularity at 0, we must also have dim₀ $Z_{g'} = d-1$, a new contradiction. \Box

8. Proof of Corollary [5.1](#page-7-1)

First, we show that $\dim_0 \Sigma f = \dim_0 \Sigma g$. The argument is similar to that given in the proof of Lemma [7.1.](#page-11-4) We argue by contradiction. Put $d := \dim_0 \Sigma f$ and $s := \dim_0 \Sigma g$, and suppose for instance $d \leq s$. By the Iomdine-Lê-Massey formula and by Lemmas 3.7 and 3.8 of [\[1\]](#page-16-1), for any integers $0 \ll \alpha_1 \ll \cdots \ll \alpha_d$ sufficiently large^{[2](#page-14-1)}, the functions

$$
f_d(z) := f(z) + z_1^{\alpha_1} + \dots + z_d^{\alpha_d}
$$
 and $g_d(z) := g(z) + z_1^{\alpha_1} + \dots + z_d^{\alpha_d}$

are non-degenerate, f_d has an isolated singularity at 0, and dim₀ Σg_d = $s - d > 0$. Then, by [\[1,](#page-16-1) Corollary 2.9], the support of f_d satisfies the Kouchnirenko condition. Since f and g have the same Newton diagram, it follows that $\Gamma(f_d) = \Gamma(g_d)$ too. Thus the support of g_d also satisfies the

 $\overline{\text{2} \text{Precisely, } \alpha_p > \max\{2, \rho_{f_{p-1},z^{(p-1)}}(0), \rho_{g_{p-1},z^{(p-1)}}(0), m(f_{p-1}), m(g_{p-1})\}.$

Kouchnirenko condition, and by [1, Theorem 3.1], the function g_d must have an isolated singularity at $0 - a$ contradiction.

Now, to show that the Lê numbers of f and g at 0 with respect to the coordinates $z = (z_1, \ldots, z_n)$ are equal, we apply Theorem [4.1.](#page-7-0) By this theorem, these Lê numbers are described in terms of the modified (and special modified) Newton numbers of the functions f_d and g_d . Then the result follows immediately from the equality $\Gamma(f_d) = \Gamma(g_d)$.

Appendix A.

For completeness, in this appendix, we give a proof of a useful elementary lemma which we have used in the proof of Theorem [4.1.](#page-7-0)

Let d be a positive integer. Consider the following system $\mathscr S$ of d integral inequalities with d variables $\alpha_1, \ldots, \alpha_d$:

$$
\begin{cases}\n\alpha_1 \geq c_1, \\
\alpha_2 \geq c_2(\alpha_1), \\
\alpha_3 \geq c_3(\alpha_1, \alpha_2), \\
\dots \\
\alpha_d \geq c_d(\alpha_1, \dots, \alpha_{d-1}).\n\end{cases}
$$

Here, c_1 is a constant, and for $2 \leq \ell \leq d$, $c_{\ell}(\alpha_1,\ldots,\alpha_{\ell-1})$ is a number depending on $\alpha_1, \ldots, \alpha_{\ell-1}$. For each $1 \leq r \leq d$, let $\mathscr{S}(r)$ be the system consisting only of the first r inequalities of the system \mathscr{S} . Finally, let $Z(r) \subseteq$ \mathbb{Z}^r be the set of (integral) solutions of the system $\mathscr{S}(r)$.

Lemma A.1. For any $1 \leq r \leq d$, if $P(x_1, \ldots, x_r)$ is a polynomial function that vanishes on $Z(r)$, then it is identically zero.

Proof. By induction on r. For $r = 1$, the lemma immediately follows from the fundamental theorem of algebra. Now suppose the lemma holds true for some integer $r - 1$ (with $r \geq 2$), and let us show that it also holds true for the integer r. So, let $P(x_1, \ldots, x_r)$ be a polynomial function such that $P(\alpha_1,\ldots,\alpha_r)=0$ for any $(\alpha_1,\ldots,\alpha_r)\in Z(r)$. Note that $(\alpha_1,\ldots,\alpha_r)\in$ $Z(r)$ implies $(\alpha_1, \ldots, \alpha_{r-1}) \in Z(r-1)$. Expand P with respect to the variable x_r :

$$
P(x_1, ..., x_r) = \sum_{k=0}^{\delta} P_k(x_1, ..., x_{r-1}) x_r^k.
$$

(Here, δ denotes the degree of P.) Then for all $(\alpha_1, \ldots, \alpha_r) \in Z(r)$,

$$
\sum_{k=0}^{\delta} P_k(\alpha_1,\ldots,\alpha_{r-1})\,\alpha_r^k=0.
$$

By the fundamental theorem of algebra, it follows that for each $0 \leq k \leq \delta$,

$$
P_k(\alpha_1,\ldots,\alpha_{r-1})=0
$$

for every fixed $(\alpha_1, \ldots, \alpha_{r-1}) \in Z(r-1)$. Now, by the induction hypothesis, this implies that the polynomial P_k identically vanishes. \Box

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Christophe Eyral

Institute of Mathematics, Polish Academy of Sciences Sniadeckich 8, 00-656 Warszawa, Poland ´ E-mail: ch.eyral@impan.pl

Grzegorz Oleksik

Faculty of Mathematics and Computer Science, University of Lódź S. Banacha 22, 90-238 Lódź, Poland E-mail: oleksig@math.uni.lodz.pl

Adam Różycki

Faculty of Mathematics and Computer Science, University of Lódź S. Banacha 22, 90-238 Łódź, Poland E-mail: rozycki@math.uni.lodz.pl