CHECKING REAL ANALYTICITY ON SURFACES

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Abstract. We prove that a real-valued function (that is not assumed to be continuous) on a real analytic manifold is analytic whenever all its restrictions to analytic submanifolds homeomorphic to $\mathbb{S}^2$ are analytic. This is a real analog for the classical theorem of Hartogs that a function on a complex manifold is complex analytic iff it is complex analytic when restricted to any complex curve.

By a theorem of Hartogs, a function $f : \mathbb{C}^n \to \mathbb{C}$ (that is not assumed to be continuous) is analytic iff it is analytic when restricted to any translate of the coordinate axes; see [Har1906] or [BM48, p.140]. An analogous claim does not hold for real analytic functions; see Example 10 for a simple non-continuous function $f\colon \mathbb{R}^3 \to \mathbb{R}$ that is real analytic on every translate of the coordinate hyperplanes and a more complicated non-continuous function $g\colon \mathbb{R}^3 \to \mathbb{R}$ that is real analytic on every smooth analytic curve in $\mathbb{R}^3$.

Bochnak and Siciak proved in 1971 [BS71, BS18] that a function $f : \mathbb{R}^n \to \mathbb{R}$ is real analytic iff its restriction to any 2-plane is analytic. They also conjectured that a similar result holds on any real analytic manifold using restrictions to 2-dimensional compact analytic submanifolds.

The aim of this note is to prove a stronger variant of this, using an improvement of [BS18, Thm.1] that uses fewer 2-planes.

Theorem 1. *Let M be a real analytic manifold of dimension $n \geq 3$ and $f : M \to \mathbb{R}$ a (not necessarily continuous) function. Assume that $f|_N$ is real analytic for every real analytic submanifold $N \subset M$ that is homeomorphic to the 2-sphere $\mathbb{S}^2$. Then f is real analytic on M.*

Working on local coordinate charts, this is an immediate consequence of the following more precise version.

Theorem 2. *Let $B \subset \mathbb{R}^n$ be the open unit ball and $f : B \to \mathbb{R}$ a (not necessarily continuous) function. Assume that $n \geq 3$ and the restriction $f|_{S^2}$ is real analytic for every 2-sphere $S^2 \subset B$ passing through the origin. Then f is real analytic.*

Here we use 2-sphere in the most restrictive sense, that is, the intersection of an $(n-1)$-sphere $\left(\sum_i x_i^2 = \sum_i c_i x_i\right)$ with a vector subspace of dimension 3.

Terminology. From now on by a function we mean an arbitrary real-valued function and real analytic is shortened to analytic. To simplify notation, we let $g|_Y$ denote the restriction of g to $Y \cap (\text{domain of } g)$.

Before we start the proof we need some preparation. A subset $C \subset \mathbb{R}^n$ is a *cone* if $\mathbb{R}C = C$. (This is slightly more convenient for us than the more usual variant $\mathbb{R}^+C = C$.) A cone C is called open if $C \setminus \{0\}$ is open in $\mathbb{R}^n \setminus \{0\}$. A cone-neighborhood of a set W is an open cone that contains W.

We will work both with vector subspaces $V \subset \mathbb{R}^n$ and affine-linear subspaces $Q \subset \mathbb{R}^n$. A property $\mathcal{P}$ holds for all *vector subspaces near* V if there is a cone-neighborhood $C \supset V$ such that $\mathcal{P}$ holds for all vector subspaces (of the same dimension) contained in C. A property $\mathcal{P}$ holds for all *affine-linear subspaces near* Q if there is a choice of the origin $0 \in Q$, a cone-neighborhood $C \supset Q$ and $\epsilon > 0$ such that $\mathcal{P}$ holds for all affine-linear subspaces (of the same dimension) contained in $C \cup B(\epsilon)$, where $B(\epsilon)$ denotes the ball of radius ϵ centered at 0. For a cone W and $\epsilon > 0$ we write $W(\epsilon) := W \cap B(\epsilon)$. We need the following obvious fact.

Claim 3. Let $C \subset \mathbb{R}^n$ be an open cone and $V \subset C$ a vector 2-plane and let $C_V \subset C$ be the union of all vector 2-planes $V' \subset C$ for which $\dim(V' \cap V) \geq 1$. Then C_V is an open subcone of C that contains V. □

By Lagrange interpolation, a 1-variable polynomial of degree d is uniquely determined by $d+1$ of its values. Equivalently, a 2-variable homogeneous polynomial of degree d on $\mathbb{R}^2$ is uniquely determined by its restriction to $d+1$ vector lines.

More generally, the following is easy to prove by induction on d.

Claim 4. Let $H_0, \dots, H_d \subset \mathbb{R}^n$ be vector hyperplanes such that the intersection of any r of them has codimension r for every $r \leq n$. For $i = 0, \dots, d$ let g_i be a degree d homogeneous polynomial on H_i such that $g_i|_{H_i \cap H_j} = g_j|_{H_i \cap H_j}$ for all i, j. Then there is a unique degree d homogeneous polynomial g on $\mathbb{R}^n$ such that $g|_{H_i} = g_i$ for every i. □

Lemma 5. *Let $V \subset \mathbb{R}^n$ be a vector 2-plane, $C \supset V$ a cone-neighborhood and $C_V \subset C$ as in Claim 3. Let $\sigma : C \to \mathbb{R}$ be a function such that $\sigma|_{V'}$ is a degree d homogeneous polynomial for every vector 2-plane $V' \subset C_V$. Then there is a unique degree d homogeneous polynomial g on $\mathbb{R}^n$ such that $g|_{C_V} = \sigma|_{C_V}$.*

Proof. By induction on n, starting with $n = 2$ when there is nothing to prove. For $n \geq 3$ pick any $V' \subset C_V$ such that $\dim(V' \cap V) = 1$ and let H be a vector hyperplane that contains V but does not contain V'. For every $i = 0, \dots, d$ let $V_i \subset H_i$ be a general perturbation of $V \subset H$ such that $V_i \subset C_V$, the intersections $V' \cap V_i$ are distinct lines and $V' \cap H_i = V' \cap V_i$.

By induction, there are degree d homogeneous polynomials g_i defined on H_i such that g_i agrees with σ on $H_i \cap C_V$. By Claim 4 there is thus a unique degree d homogeneous polynomial g on $\mathbb{R}^n$ such that $g|_{H_i} = g_i$ for every i.

Thus $g|_{V'}$ and $\sigma|_{V'}$ are 2 homogeneous polynomials of degree d that agree on the $d+1$ lines $V' \cap H_i$. So $g|_{V'} = \sigma|_{V'}$. □

Lemma 6. *Let $V \subset \mathbb{R}^n$ be a vector 2-plane, $C \subset \mathbb{R}^n$ a cone-neighborhood of V and $C_V \subset C$ as in Claim 3. Let $f : C \to \mathbb{R}$ be a function such that, for every vector 2-plane $V' \subset C_V$, there is an $\epsilon = \epsilon(V') > 0$ such that the restriction of f to $V'(\epsilon)$ is analytic. Then there is an analytic function T in a neighborhood of $0 \in \mathbb{R}^n$ such that, for every vector line $L \subset C_V$, the restrictions $f|_L$ and $T|_L$ agree in a neigborhood of the origin in L.*

Proof. For any vector line $L \subset C_V$ the function $f|_L$ is analytic at the origin. Thus, for $0 \neq p \in L$ we can define

$$\tau_r(p) := \tfrac{d^r f}{dt^r}(tp)\Big|_{t=0} \quad \text{and set} \quad \tau_r(0) = 0.$$

By assumption $f|_{V'}$ is analytic at the origin for every vector 2-plane $V' \subset C_V$, hence $\tau_r|_{V'}$ is a degree r homogeneous polynomial. Furthermore, $f|_L = \sum_r \frac{1}{r!}\tau_r$

near the origin. By Lemma 5 there is a degree r homogeneous polynomial T_r on $\mathbb{R}^n$ that agrees with τ_r on $C_V \subset C$.

Consider the series of homogeneous polynomials

$$T := \sum_r \tfrac{1}{r!} T_r. \tag{6.1}$$

As we noted, this sum converges in some neighborhood of the origin $0 \in L(\epsilon_L) \subset L$ for every vector line $L \subset C_V$. Therefore the series (6.1) defines an analytic function T in a neighborhood $B(\eta) \ni 0$ by [BS18, Lem.3]. Clearly T has the required properties. □

Remark. It should be noted that the convergence of a series of homogeneous polynomials is quite different from the usual convergence of power series. In particular, if the series (6.1) absolutely converges at a point $(p_1, \dots, p_n)$, it does not imply that it also converges at all points $(x_1, \dots, x_n)$ with $|x_i| < |p_i|$. Thus [BS18, Lem.3] is quite a subtle tool.

Now we are ready to prove the following stronger form of [BS18, Thm.1].

Theorem 7. *Let $U \subset \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}$ a function. Let $Q \subset \mathbb{R}^n$ be an affine 2-plane and assume that $f|_{Q' \cap U}$ is analytic for every affine 2-plane Q' near Q. Then f is analytic in a neighborhood of $Q \cap U$.*

Proof. The question is local, so we may assume that $0 \in Q \cap U$ and work near it. Lemma 6 gives a function T that is analytic in a ball $B(\eta)$, and a cone-neighborhood C of Q, such that the restrictions $f|_L$ and $T|_L$ agree in a neigborhood of the origin for every vector line $L \subset C$. We may assume that $B(\eta) \subset U$, thus f and T are both defined and analytic on $L(\eta)$. Hence f and T agree on $C(\eta) = C \cap B(\eta)$.

It remains to show that $T = f$ in a neighborhood of 0. Fix an affine line $\ell \subset Q$ not passing through the origin such that $\ell \cap C(\eta)$ is not empty. If $p \in \mathbb{R}^n$ is close to 0 then the affine 2-plane Q_p obtained as the span of p and ℓ is close to Q. Thus T and f restrict to analytic functions on the convex open set $Q_p \cap B(\eta)$ that agree on the nonempty open subset $Q_p \cap C(\eta)$. Hence they agree everywhere in a neigborhood of $Q \cap U$. □

8 (Proof of Theorem 2)**.** Inversion

$$\mu : (x_1, \dots, x_n) \mapsto \left(\tfrac{x_1}{\sum x_i^2}, \dots, \tfrac{x_n}{\sum x_i^2}\right)$$

maps the punctured unit ball $\left(0 < \sum x_i^2 < 1\right)$ to the outside of the unit ball $\left(\sum x_i^2 > 1\right)$ and gives a one-to-one correspondence between the 2-spheres $S^2 \subset B(1)$ that pass through the origin and 2-planes contained in the outside of the unit ball.

For every $p \in \left(\sum x_i^2 > 1\right)$ there is an affine 2-plane $p \in Q_p \subset \left(\sum x_i^2 > 1\right)$ and every affine 2-plane near Q_p is also contained in $\left(\sum x_i^2 > 1\right)$. Thus $f \circ \mu^{-1}$ is analytic by Theorem 7, so f is analytic, except possibly at the origin.

To see what happens at the origin, we use inversion $\mu_{\mathbf{p}}(\mathbf{x}) := \mu(\mathbf{x} - \mathbf{p}) + \mathbf{p}$ centered at another point $0 \neq \mathbf{p} \in B(1)$. This maps $B(1) \setminus \{\mathbf{p}\}$ to an open set $U_{\mathbf{p}} \subset \mathbb{R}^n$. The 2-planes that pass through $\mu_{\mathbf{p}}(0)$ and are contained in $U_{\mathbf{p}}$ are in one-to-one correspondence with the 2-spheres $S^2 \subset B(1)$ passing through both the origin and $\mathbf{p}$. Thus $f \circ \mu_{\mathbf{p}}^{-1}$ is analytic on these planes. We already know that $f \circ \mu_{\mathbf{p}}^{-1}$ is analytic away from $\mu_{\mathbf{p}}(0)$, thus it is also analytic at $\mu_{\mathbf{p}}(0)$ by Theorem 7. So f is also analytic at the origin. □

The following is another variant of Theorem 2.

Proposition 9. *Let M be an analytic manifold of dimension n and $f : M \to \mathbb{R}$ a function. Let $h : M \to \mathbb{R}$ be an analytic function that has an isolated 0 at a point $p \in M$ and let $x_1, \dots, x_n$ local coordinates at p. Assume that $f|_S$ is analytic for every compact, 2-dimensional, smooth submanifold $S \subset M$ of the form*

$$S := \big(h - \ell_1 = \ell_2 = \cdots = \ell_{n-2} = 0\big),$$

where the $\ell_i := \sum_j a_{ij} x_j$ are linear.

Then f is analytic in a punctured neighborhood of $p \in M$.

Proof. We argue as in Paragraph 8, but use the map $\big(\frac{x_1}{h}, \dots, \frac{x_n}{h}\big)$, which sends the submanifolds $S \subset M$ to affine 2-planes in $\mathbb{R}^n$. □

Example 10. The function defined by

$$f := \frac{x_1 \cdots x_n}{x_1^{2n} + \cdots + x_n^{2n}} \quad \text{for} \quad (x_1, \dots, x_n) \neq (0, \dots, 0) \quad \text{and} \quad f(0, \dots, 0) = 0$$

is analytic on every translate of the coordinate hyperplanes, but not even bounded at the origin. Thus we definitely need more planes than in the complex case.

Let $g\colon \mathbb{R}^3 \to \mathbb{R}$ be the function defined by

$$g(x, y, z) = \frac{x^8 + y(x^2 - y^3)^2 + z^4}{x^{10} + (x^2 - y^3)^2 + z^2} \quad \text{for} \quad (x, y, z) \neq (0, 0, 0) \quad \text{and} \quad g(0, 0, 0) = 0.$$

Then f is analytic on every nonsingular analytic curve in $\mathbb{R}^3$, but f is not even continuous at $(0, 0, 0)$. See also [BMP91] for an even stronger example.

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