## On Berinde's method for comparing iterative processes

Constantin Zalinescu\*

#### Abstract

In the literature there are several methods for comparing two convergent iterative processes for the same problem. In this note we have in view mostly the one introduced by Berinde in [Picard iteration converges faster than Mann iteration for a class of quasicontractive operators, Fixed Point Theory and Applications 2, 97–105 (2004)] because it seems to be very successful. In fact, if IP1 and IP2 are two iterative processes converging to the same element, then IP1 is faster than IP2 in the sense of Berinde. The aim of this note is to prove this almost obvious assertion and to discuss briefly several papers that cite the mentioned Berinde's paper and use his method for comparing iterative processes.

MSC 41A99

**Keywords**: faster convergence; better convergence; Berinde's method for comparing iterative processes

#### 1 Introduction

In the literature there are several methods for comparing two convergent iterative processes for the same problem. In this note we have in view mostly the one introduced by Berinde in [12, Definition 2.7] because it seems to be very successful. This was pointed out by Berinde himself in [15]: "This concept turned out to be a very useful and versatile tool in studying the fixed point iterative schemes and hence various authors have used it". However, it was pointed out by Popescu, using [57, Example 3.4], that Berinde's method is not consistent. The inconsistency of Berinde's method is mentioned also by Qing & Rhoades in [58, page 2]. Moreover, referring to Berinde's method, Phuengrattana & Suantai say in [55, page 218]: "It seem not to be clear if we use above definition for comparing the rate of convergence". In fact, if IP<sub>1</sub> and IP<sub>2</sub> are two (arbitrary) iterative processes converging to the same element, then  $IP_1$  is faster than  $IP_2$  (and vice-versa) in the sense of Berinde ([12, Definition 2.7]).

The aim of this note is to prove this almost obvious assertion and to discuss briefly several papers that cite [12] and use Berinde's method for comparing iterative processes.

#### 2 Definitions and the main assertion

First, we quote from [12, pages 99, 100] the text containing the definitions which we have in view; these are reproduced in many papers from our bibliography.

"Definition 2.5. Let  $\{a_n\}_{n=0}^{\infty}$ ,  $\{b_n\}_{n=0}^{\infty}$  be two sequences of real numbers that converge to a and b, respectively, and assume that there exists  $l = \lim_{n \to \infty} \left| \frac{a_n - a}{b_n - b} \right|$ .

(a) If l = 0, then it can be said that  $\{a_n\}_{n=0}^{\infty}$  converges faster to a than  $\{b_n\}_{n=0}^{\infty}$  to b.

<sup>\*</sup>Octav Mayer Institute of Mathematics of Romanian Academy, Iasi, Romania, email: zalinesc@uaic.ro.

(b) If  $0 < l < \infty$ , then it can be said that  $\{a_n\}_{n=0}^{\infty}$ , and  $\{b_n\}_{n=0}^{\infty}$  have the same rate of convergence."

"Suppose that for two fixed point iteration procedures  $\{u_n\}_{n=0}^{\infty}$  and  $\{v_n\}_{n=0}^{\infty}$ , both converging to the same fixed point p, the error estimates

$$||u_n - p|| \le a_n, \ n = 0, 1, 2, \dots$$
 (2.7)  
 $||v_n - p|| \le b_n, \ n = 0, 1, 2, \dots$  (2.8)

are available, where  $\{a_n\}_{n=0}^{\infty}$  and  $\{b_n\}_{n=0}^{\infty}$  are two sequences of positive numbers (converging to zero).

Then, in view of Definition 2.5, we will adopt the following concept.

Definition 2.7. Let  $\{u_n\}_{n=0}^{\infty}$  and  $\{v_n\}_{n=0}^{\infty}$  be two fixed point iteration procedures that converge to the same fixed point p and satisfy (2.7) and (2.8), respectively. If  $\{a_n\}_{n=0}^{\infty}$  converges faster than  $\{b_n\}_{n=0}^{\infty}$ , then it can be said that  $\{u_n\}_{n=0}^{\infty}$  converges faster than  $\{v_n\}_{n=0}^{\infty}$  to p."

Practically, the text above is reproduced in [15, pages 30, 31], getting so Definitions 1.1 and 1.2. The only differences are: "(2.7)" and "(2.8) are available, where" are replaced by "(1.7)" and "(1.8) are available (and these estimates are the best ones available), where", respectively.

Immediately after [15, Definition 1.2] it is said:

"This concept turned out to be a very useful and versatile tool in studying the fixed point iterative schemes and hence various authors have used it, see [1]-[5], [18], [22], [23], [28], [32]-[34], [37]-[41], [40], [43]-[46], [55]-[57], [66], [68]-[72], [74], [78]-[81], to cite just an incomplete list." <sup>1</sup>

Note that Definition 9.1 from [13] is equivalent to Definition 2.5 from [12]; replacing  $u_n$ ,  $v_n$ , p,  $||u_n - p||$  and  $||v_n - p||$  with  $x_n$ ,  $y_n$ ,  $x^*$ ,  $d(x_n, x^*)$  and  $d(y_n, x^*)$  in (2.7), (2.8) and Definition 2.7 from [12], one obtains relations (5), (6) from [13, page 201] and an equivalent formulation of [13, Definition 9.2], respectively. Note that these definitions from Berinde's book [13] are presented in the lecture [14].

Because of the parentheses in "(converging to zero)" in the preamble of [12, Definition 2.7] (and [13, Definition 9.2], [15, Definition 1.2]), the convergence to 0 of  $(a_n)$  and  $(b_n)$  seems to be optional. This is probably the reason for the absence of this condition in [25, page 3]; note that  $(a_n)$  is a constant sequence in [68].

In the next result we use the version for metric spaces of [12, Definition 2.7] (see [13, Definition 9.2]).

**Proposition 1** Let (X, d) be a metric space and  $(x_n)_{n\geq 1}$ ,  $(y_n)_{n\geq 1}$  be two sequences from X converging to  $x^* \in X$ . Then  $(x_n)$  converges faster than  $(y_n)$  to  $x^*$ .

Proof. For each  $n \geq 1$  let us consider

$$0 < a_n := d(x_n, x^*) + d(y_n, x^*) + \frac{1}{n}, \quad 0 < b_n := \begin{cases} \sqrt{a_n} & \text{if } a_n \le 1, \\ d(y_n, x^*) & \text{otherwise.} \end{cases}$$

<sup>&</sup>lt;sup>1</sup>Throughout this paper the references mentioned in the quoted texts are those in the works from where the texts are taken.

It follows that  $a_n \to 0$ ,  $b_n \to 0$ ,

$$d(x_n, x^*) \le a_n, \quad d(y_n, x^*) \le b_n, \quad \forall n \ge 1,$$

and  $a_n/b_n = \sqrt{a_n}$  for sufficiently large n; it follows that  $\lim_{n\to\infty} a_n/b_n = \lim_{n\to\infty} \sqrt{a_n} = 0$ . Therefore,  $(x_n)$  converges faster to  $x^*$  than  $(y_n)$  does.

From our point of view, the preceding result shows that Berinde's notion of rapidity for fixed point iterative schemes, recalled above, is not useful, even if Berinde in [15, page 35] sustains that "Of all concepts of rapidity of convergence presented above for numerical sequences, the one introduced by us in Definition 1.2 [14] appears to be the most suitable in the study of fixed point iterative methods". Berinde (see [15, page 36]) mentions that he "tacitly admitted in Definition 1.2 that the estimates (1.7) and (1.8) taken into consideration are the best possible". Clearly, "the estimates are the best ones available" and "the estimates ... are the best possible" are very different in meaning.<sup>2</sup>

Of course, the best possible estimates in relations (1.7) and (1.8) from [15] (that is in relations (2.7) and (2.8) from [12] recalled above) are

$$a_n := \|u_n - p\|, \quad b_n := \|v_n - p\| \quad (n \ge 0).$$
 (1)

Assuming that  $d(x_n, x^*) \to 0$ , getting (better) upper estimates for  $d(x_n, x^*)$  depends on the proof, including the author's ability to majorize certain expressions. Surely, the best available estimates are exactly those obtained by the authors in their proofs.

The use of Berinde's method for comparing the speeds of convergence is very subjective. It is analogue to deciding that  $a/b \le c/d$  knowing only that  $0 < a \le c$  and  $0 < b \le d$ !

Taking  $a_n$  and  $b_n$  defined by (1) in [12, Definition 2.7] one obtains Definition 3.5 of Popescu from [57]. Popescu's definition is used explicitly by Rhodes & Xue (see [60, page 3]), but they wrongly attribute it to [12]; this attribution is wrong because [57, Definition 3.5] reduces to [12, Definition 2.5] only in the case in which the involved normed vector space is  $\mathbb{R}$ . Note that Rhoades knew about Popescu's definition because [57] is cited in [58, page 2].

Notice that Popescu's definition is extended to metric spaces by Berinde, Khan & Păcurar in [17, page 8], as well as by Fukhar-ud-din & Berinde in [27, page 228]; also observe that Popescu's paper [57] is not cited in [17] and [27].

Even if in [12] it is not defined when two iteration schemes have the same rate of convergence, Dogan & Karakaya obtain that "the iteration schemes  $\{k_n\}_{n=0}^{\infty}$  and  $\{l_n\}_{n=0}^{\infty}$  have the same rate of convergence to p of  $\wp$ " in [23, Theorem 2.4]. The proof of [23, Theorem 2.4] is based on the fact that one obtained two sequences  $(a_n)$  and  $(b_n)$  converging to 0 such that  $||k_{n+1} - p|| \le a_n$ ,  $||l_{n+1} - p|| \le b_n$  for  $n \ge 0$  and  $\lim_{n \to \infty} a_n/b_n = 1$ .

Accepting such an argument, and taking  $a_n := b_n := d(x_n, x^*) + d(y_n, x^*) + \frac{1}{n}$  in the proof of Proposition 1, one should obtain that any pair of sequences  $(x_n)_{n\geq 1}$ ,  $(y_n)_{n\geq 1} \subset (X, d)$  with the same limit  $x^* \in X$  have the same rate of convergence.

Recall that Rhoades in [59, pages 742, 743] says that having " $\{x_n\}$ ,  $\{z_n\}$  two iteration schemes which converge to the same fixed point p, we shall say that  $\{x_n\}$  is better than  $\{z_n\}$  if  $|x_n - p| \le |z_n - p|$  for all n". It seems that this definition is too restrictive (see for instance [12, Example 2.8]). In this context we propose the following definition.

<sup>&</sup>lt;sup>2</sup>Among the 19 papers from our bibliography published in 2017 and 2018, our reference [15] is mentioned only in [24] and [31].

**Definition 2** Let (X,d) be a metric space, and let  $(x_n)_{n\geq 1}$ ,  $(y_n)_{n\geq 1} \subset (X,d)$  and  $x,y \in X$  be such that  $x_n \to x$ ,  $y_n \to y$ . One says that  $(x_n)$  converges better to x than  $(y_n)$  to y if there exists some  $\alpha > 0$  such that  $d(x_n, x) \leq \alpha d(y_n, y)$  for sufficiently large n; one says that  $(x_n)$  and  $(y_n)$  have the same rate of convergence if  $(x_n)$  converges better to x than  $(y_n)$  to y, and  $(y_n)$  converges better to y than  $(x_n)$  to x.

Using the conventions  $\frac{0}{0} := 1$  and  $\frac{\alpha}{0} := \infty$  for  $\alpha > 0$ ,  $[(x_n)$  converges better to x than  $(y_n)$  to y] if and only if  $\limsup_{n \to \infty} \frac{d(x_n, x)}{d(y_n, y)} < \infty$ ; consequently,  $[(x_n)$  and  $(y_n)$  have the same rate of convergence] (in the sense of Definition 2) if and only if  $0 < \liminf_{n \to \infty} \frac{d(x_n, x)}{d(y_n, y)} \le \limsup_{n \to \infty} \frac{d(x_n, x)}{d(y_n, y)} < \infty$ .

**Example 3** Consider the sequences  $(x_n)_{n\geq 1}$ ,  $(y_n)_{n\geq 1}\subset \mathbb{R}$  defined by

$$x_n := \begin{cases} n^{-1} & \text{if } n \text{ is odd,} \\ (2n)^{-1} & \text{if } n \text{ is even,} \end{cases} \quad y_n := \begin{cases} (2n)^{-1} & \text{if } n \text{ is odd,} \\ n^{-1} & \text{if } n \text{ is even.} \end{cases}$$

Clearly  $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$ , and it is very natural to consider that they have the same rate of convergence; this is confirmed using Definition 2. It is obvious that neither  $(x_n)$  is better (faster) than  $(y_n)$ , nor  $(y_n)$  is better (faster) than  $(x_n)$  in the senses of Rhoades ([59]), or Berinde [12], or Popescu [57], or Berinde, Khan & Păcurar ([17]), or Fukhar-ud-din & Berinde ([27]).

# 3 Remarks on the use of Berinde and Popescu's notions in papers citing [12]

Practically, all the papers mentioned in the sequel were found on internet when searching, with Google Scholar, the works citing Berinde's article [12].

First we give the list of articles, mentioning their authors and results, in which Berinde's Definition 2.7 from [12] is used (even if not said explicitly):

Berinde & Berinde – [16, Theorem 3.3]; Babu & Prasad – [10, Theorem 2.1], [11, Theorems 3.1, 3.3]; Olaleru – [52, Theorem 1], [53, Theorems 1, 2]; Sahu – [61, Theorem 3.6]; Akbulut & Özdemir – [3, Theorem 2.3]; Hussain et al. – [35, Theorems 18, 19]; Karahan & Ozdemir – [42, Theorem 1]; Abbas & Nazir – [1, Theorem 3]; Gürsoy & Karakaya – [33, Theorem 3]; Kadioglu & Yildirim – [38, Theorem 5]; Karakaya et al. – [40, Theorem 3], [41, Theorem 2.2]; Kumar – [47, Theorem 3.1]; Öztürk Çeliker – [54, Theorem 8]; Thakur et al. – [65, Theorem 2.3], [66, Theorem 3.1]; Chugh et al. – [20, Theorem 3.1], [21, Theorem 13]; Fathollahi et al. – [25, Propositions 3.1, 3.2, Theorem 3.1, Lemmas 3.2–3.4, Theorems 4.1–4.4]; Gursoy – [29, Theorem 3]; Jamil & Abed – [37, Theorems 3.1–3.4]; Yadav – [71, Example 2]; Abed & Abbas, [2, Theorem (3.8)]; Asaduzzaman et al. – [9, Theorem 3.3]; Mogbademu – [50, Theorem 2.1]; Sintunavarat & Pitea – [64, Theorem 2.1]; Verma et al. – [68]; Alecsa – [7, Theorems 3.1, 3.3–3.12]; Okeke & Abbas – [51, Proposition 2.1]; Sharma & Imdad – [62, Remark 4.8]; Yildirim &

<sup>&</sup>lt;sup>3</sup>One appreciates here that "In recent years, Definition 2.2 has been used as a standard tool to compare the fastness of two fixed point iterations", Definition 2.2 being [12, Definition 2.7].

<sup>&</sup>lt;sup>4</sup>See the estimates (23) and (24), as well as the very strange arguments to get the conclusion on page SMC<sub>-</sub>2016 001606.

Abbas – [73, Theorem 2]; Akhtar & Khan – [5, Theorem 3.1–3.3]; Alagoz et al. – [6, Theorem 2.1]; Ertürk & Gürsoy – [24, Theorem 2.3]; Fathollahi & Rezapour – [26, Propositions 2.1–2.3, 3.1, Theorem 3.2]; Garodia & Uddin – [28, Theorem 3.1]; Gürsoy et al. – [31, Theorem 6]<sup>5</sup>; Kosol – [46, Theorem 2.2]; Kumar & Chauhan – [48, Theorems 1, 2]; Piri et al. [56, Lemmas 3.1, 3.2, Theorem 3.3]; Yildirim – [72, Theorem 2].

As mentioned in Section 2, Dogan & Karakaya obtain that "the iteration schemes  $\{k_n\}_{n=0}^{\infty}$  and  $\{l_n\}_{n=0}^{\infty}$  have the same rate of convergence to p of  $\wp$ " in [23, Theorem 2.4] because  $\lim_{n\to\infty} a_n/b_n = 1$ , where the sequences  $(a_n)$  and  $(b_n)$  are such that  $||k_{n+1} - p|| \leq a_n$ ,  $||l_{n+1} - p|| \leq b_n$  for  $n \geq 0$ .

It is worth repeating that Popescu (in [57]) recalls [12, Definition 2.7], mentions its inconsistency, introduces his direct comparison of iterative processes in [57, Definition 3.5], and uses this definition in [57, Theorem 3.7].

Other papers in which [57, Definition 3.5] is used, without citing it (but possibly recalling [12, Definition 2.5 or/and Definition 2.7]), are: Xue [70], Rhodes & Xue [60], Chugh et al. [19], Thong [67], Alotaibi et al. [8], Hussain et al. [35]<sup>6</sup>, Phuengrattana & Suantai [55], Doğan & Karakaya [22], Khan et al. [45, Theorem 3.1], Fukhar-ud-din & Berinde [27], Gürsoy [30], Khan et al. [44, Theorem 3], Gürsoy et al. [32, Theorem 2.3, Corollary 2.4], Ertürk & Gürsoy – [24, Theorem 2.3],

It is also worth noticing that by taking simple examples in  $\mathbb{R}$ , Hussain et al. [36, Example 9], Chugh et al. [19, Example 4.1], Hussain et al. [34, Example 3.1], Kang et al. [43, Example 11], Karakaya et al. [39, Example 4], Kumar et al. [49, Example 9], Chugh et al. [20, Example 14] (see also P. Veeramani's review MR3352138 from Mathematical Reviews), Chauhan et al. [18], Sintunavarat [63], Wahab & Rauf [69, Example 11, Remarks 12–17] and Akewe & Eke [4], "prove" that certain iteration processes are faster than other ones.

### References

- [1] Abbas, M, Nazir, T: A new faster iteration process applied to constrained minimization and feasibility problems. Mat. Vesnik, **66**(2), 223–234 (2014)
- [2] Abed, SS, Abbas RF: S-iteration for general quasi multi valued contraction mappings. Int. J. Appl. Math. Stat. Sci. **5**(4), 9–22 (2016)
- [3] Akbulut, S, Özdemir, M: Picard iteration converges faster than Noor iteration for a class of quasi-contractive operators. Chiang Mai J. Sci. **39**(4), 688–692 (2012)
- [4] Akewe, H, Eke, KS: Convergence speed of some random implicit-Kirk-type iterations for contractive-type random operators. Austr. J. Math. Anal. Appl. **15**(2), Article 15, 1–14, (2018)
- [5] Akhtar, Z, Khan, MAA: Rates of convergence for a class of generalized quasi contractive mappings in Kohlenbach hyperbolic spaces, arXiv:1802.09773v1 [math.FA]

 $<sup>^{5}</sup>$ In [24] and [31] one refers to [15] when adding "and these estimates are the best possible", but without any mention to these "best estimates" in the proofs.

<sup>&</sup>lt;sup>6</sup>Note the strange quantity  $\left\|\frac{JN_{n+1}-p}{JI_{n+1}-p}\right\|$ , the numerator and denominator being in  $(X,\|\cdot\|)$  "an arbitrary Banach space".

- [6] Alagoz, O, Gunduz, B, Akbulut, S: Numerical Reckoning Fixed Points for Berinde Mappings via a Faster iteration Process, Facta Universitatis, Ser. Math. Inform. 33(2), 295–305 (2018)
- [7] Alecsa, CD: On new faster fixed point iterative schemes for contraction operators and comparison of their rate of convergence in convex metric spaces. Int. J. Nonlinear Anal. Appl. 8(1), 353–388 (2017)
- [8] Alotaibi, A, Kumar, V, Hussain, N: Convergence comparison and stability of Jungck-Kirk-type algorithms for common fixed point problems. Fixed Point Theory Appl. **2013**:173 (2013)
- [9] Asaduzzaman, M, Khatun, MS, Ali, MZ: On new three-step iterative scheme for approximating the fixed points of non-expansive mappings. JP J. Fixed Point Theory Appl. 11(1), 23–53 (2016)
- [10] Babu, GVR, Prasad, KV: Mann iteration converges faster than Ishikawa iteration for the class of Zamfirescu operators. Fixed Point Theory Appl. 2006, Article ID 49615 (2006); erratum ibid. 2007, Article ID 97986 (2007)
- [11] Babu, GVR, Prasad, KV: Comparison of fastness of the convergence among Krasnoselskij, Mann, and Ishikawa iterations in arbitrary real Banach spaces. Fixed Point Theory Appl. 2006, Article ID 35704 (2006)
- [12] Berinde, V: Picard iteration converges faster than Mann iteration for a class of quasicontractive operators. Fixed Point Theory Appl. 2, 97–105 (2004)
- [13] Berinde, V: Iterative Approximation of Fixed Points. Springer, Berlin (2007)
- [14] Berinde, V: Iterative approximation of fixed points (Approximation itérative des points fixes). CNRS, GT Méthodes Numériques, 18 Juin 2007
- [15] Berinde, V: On a notion of rapidity of convergence used in the study of fixed point iterative methods. Creat. Math. Inform. **25**(1), 29–40 (2016)
- [16] Berinde, V, Berinde, M: The fastest Krasnoselskij iteration for approximating fixed points of strictly pseudo-contractive mappings. Carpathian J. Math. **21**(1-2) (2005), 13–20
- [17] Berinde, V, Khan, AR, Păcurar, M: Analytic and empirical study of the rate of convergence of some iterative methods. J. Numer. Anal. Approx. Theory 44(1), 25–37 (2015)
- [18] Chauhan, SS, Utreja, K, Imdad, M, Ahmadullah, M: Strong convergence theorems for a quasi contractive type mapping employing a new iterative scheme with an application. Honam Math. J. **39**(1), 1–25 (2017)
- [19] Chugh, R, Kumar, V, Kumar, S: Strong convergence of a new three step iterative scheme in Banach spaces. Amer. J. Comput. Math. 2, 345–357 (2012)
- [20] Chugh, R, Malik, P, Kumar, V: On analytical and numerical study of implicit fixed point iterations. Cogent Math. 2, Article ID 1021623 (2015)
- [21] Chugh, R, Malik, P, Kumar, V: On a new faster implicit fixed point iterative scheme in convex metric spaces. J. Funct. Spaces 2015, Article ID 905834 (2015)

- [22] Doğan, K, Karakaya, V: On the convergence and stability results for a new general iterative process. The Scientific World J. **2014**, Article ID 852475 (2014)
- [23] Doğan, K, Karakaya, V: A study in the fixed point theory for a new iterative scheme and a class of generalized mappings. Creat. Math. Inform. **27**(2), 151–160 (2018)
- [24] Ertürk, M, Gürsoy, F: Some convergence, stability and data dependency results for a Picard-S iteration method of quasi-strictly contractive operators. Math. Bohemica, (2018) DOI: 10.21136/MB.2018.0085-17
- [25] Fathollahi, S, Ghiura, A, Postolache, M, Rezapour, S: A comparative study on the convergence rate of some iteration methods involving contractive mappings. Fixed Point Theory Appl. **2015**:234 (2015)
- [26] Fathollahi, S, Rezapour, S: Efficacy of coefficients on rate of convergence of some iteration methods for quasi-contractions, Iran. J. Sci. Tech. Trans. Sci. 42(3), 1517–1523 (2018)
- [27] Fukhar-ud-din, H, Berinde, V: Iterative methods for the class of quasi-contractive type operators and comparsion of their rate of convergence in convex metric spaces. Filomat **30**(1), 223–230 (2016)
- [28] Garodia, C, Uddin, I: Solution of a nonlinear integral equation via new fixed point iteration process. arXiv:1809.03771v1 [math.FA]
- [29] Gürsoy, F: On Huang and Noor's open problem. arXiv:1501.03318v1 [math.FA]
- [30] Gürsoy, F: A Picard-S iterative method for approximating fixed point of weak-contraction mappings. Filomat **30**(10) (2016), 2829–2845
- [31] Gürsoy, F, Eksteen, JJA, Khan, AR, Karakaya, V: An iterative method and its application to stable inversion, Soft Comput (2018). https://doi.org/10.1007/s00500-018-3384-6
- [32] Gürsoy, F, Khan, AR, Fukhar-ud-din, H: Convergence and data dependence results for quasi-contractive type operators in hyperbolic spaces. Hacettepe Journal of Mathematics and Statistics 46(3), 373–388 (2017)
- [33] Gürsoy, F, Karakaya, V: A Picard-S hybrid type iteration method for solving a differential equation with retarded argument. arXiv:1403.2546v2 [math.FA]
- [34] Hussain, N, Chugh, R, Kumar, V, Rafiq, A: On the rate of convergence of Kirk-type iterative schemes. J. Appl. Math. **2012**, Art. ID 526503 (2012)
- [35] Hussain, N, Kumar, V, Kutbi, MA: On rate of convergence of Jungck-type iterative schemes. Abstr. Appl. Anal. **2013**, Article ID 132626 (2013)
- [36] Hussain, N, Rafiq, A, Damjanović, B, Lazović, R: On rate of convergence of various iterative schemes. Fixed Point Theory Appl. **2011**:45 (2011)
- [37] Jamil, ZZ, Abed, MB: On a modified SP-iterative scheme for approximating fixed point of a contraction mapping. Iraqi J. Science, **56**(4B), 3230–3239 (2015)
- [38] Kadioglu, N, Yildirim, I: Approximating fixed points of nonexpansive mappings by a faster iteration process. arXiv:1402.6530v1 [math.FA]

- [39] Karakaya, V, Doğan, K, Gürsoy, F, Ertürk, M: Fixed point of a new three-step iteration algorithm under contractive-like operators over normed spaces. Abstr. Appl. Anal. 2013, Article ID 560258 (2013)
- [40] Karakaya, V, Gürsoy, F, Ertürk, M: Comparison of the speed of convergence among various iterative schemes. arXiv:1402.6080v1 [math.FA]
- [41] Karakaya, V, Gürsoy, F, Ertürk, M: Some convergence and data dependence results for various fixed point iterative methods. Kuwait J. Sci. **43**(1), 112–128 (2016)
- [42] Karahan, I, Ozdemir, M: A general iterative method for approximation of fixed points and their applications. Adv. Fixed Point Theory 3(3), 510–526 (2013)
- [43] Kang, SM, Ćirić, LB, Rafiq, A, Ali, F, Kwun, YC: Faster multistep iterations for the approximation of fixed points applied to Zamfirescu operators. Abstr. Appl. Anal. 2013, Article ID 464593 (2013)
- [44] Khan, AR, Gürsoy, F, Karakaya, V: Jungck-Khan iterative scheme and higher convergence rate. Int. J. Comput. Math. **93**(12), 2092–2105 (2016)
- [45] Khan, AR, Kumar, V, Hussain, N: Analytical and numerical treatment of Jungck-type iterative schemes. Appl. Math. Comp. **231**, 521–535 (2014)
- [46] Kosol, S: Strong convergence theorem of a new iterative method for weak contractions and comparison of the rate of convergence in Banach space, Adv. Fixed Point Theory, 8(3), 303-312 (2018)
- [47] Kumar, L: On the fastness of the convergence between Mann and Noor iteration for the class of Zamfirescu operators. IOSR J. Math. **10**(5), 48–52 (2014)
- [48] Kumar, N, Chauhan, SS: Analysis of Jungck-Mann and Jungck-Ishikawa iteration schemes for their speed of convergence, AIP Conference Proceedings 2050, 020011 (2018); doi: 10.1063/1.5083598
- [49] Kumar, V, Latif, A, Rafiq, A, Hussain, N: S-iteration process for quasi-contractive mappings. J. Inequal. Appl. 2013:206 (2013)
- [50] Mogbademu, AA: New iteration process for a general class of contractive mappings, Acta Comment. Univ. Tartu. Math. **20**(2), 117–122 (2016)
- [51] Okeke, GA, Abbas, M: A solution of delay differential equations via Picard–Krasnoselskii hybrid iterative process. Arab. J. Math. (Springer) **6**(1), 21–29 (2017)
- [52] Olaleru, JO: A comparison of Picard and Mann iterations for quasi-contraction maps. Fixed Point Theory 8(1), 87–95 (2007)
- [53] Olaleru, JO: On the convergence rates of Picard, Mann and Ishikawa iterations of generalized contractive operators. Stud. Univ. Babeş-Bolyai Math. **54**(4), 103–114 (2009)
- [54] Öztürk Çeliker, F: Convergence analysis for a modified SP iterative method. The Scientific World J. 2014, Article ID 840504 (2014)

- [55] Phuengrattana, W, Suantai, S: Comparison of the rate of convergence of various iterative methods for the class of weak contractions in Banach spaces. Thai J. Math. **11**(1), 217–226 (2013)
- [56] Piri, H, Daraby, B, Rahrovi, S, Ghasemi, M:Approximating fixed points of generalized  $\alpha$ -nonexpansive mappings in Banach spaces by new faster iteration process. Numerical Algorithms (2018)
- [57] Popescu, O: Picard iteration converges faster than Mann iteration for a class of quasicontractive operators. Math. Commun. **12**(2), 195–202 (2007)
- [58] Qing, Y, Rhoades, BE: Letter to the editor: Comments on the rate of convergence between Mann and Ishikawa iterations applied to Zamfirescu operators. Fixed Point Theory Appl. **2008**, Article ID 387504 (2008)
- [59] Rhoades, BE: Comments on two fixed point iteration methods. J. Math. Anal. Appl. **56**(3), 741–750 (1976)
- [60] Rhoades, BE, Xue, Z: Comparison of the rate of convergence among Picard, Mann, Ishikawa, and Noor iterations applied to quasicontractive maps. Fixed Point Theory Appl. 2010, Article ID 169062 (2010)
- [61] Sahu, DR: Applications of the S-iteration process to constrained minimization problems and split feasibility problems. Fixed Point Theory **12**(1), 187–204 (2011)
- [62] Sharma, A, Imdad, M: Fixed point approximation of generalized nonexpansive multivalued mappings in Banach spaces via new iterative algorithms. Dynamic Syst. Appl. 26, 395–410 (2017)
- [63] Sintunavarat, W: An Iterative Process for Solving Fixed Point Problems for Weak Contraction Mappings, Proceedings of the International MultiConference of Engineers and Computer Scientists 2017 Vol II, IMECS 2017, March 15–17, 2017, Hong Kong, 1019–1023
- [64] Sintunavarat, W, Pitea, A: On a new iteration scheme for numerical reckoning fixed points of Berinde mappings with convergence analysis. J. Nonlinear Sci. Appl. 9, 2553– 2562 (2016)
- [65] Thakur, D, Thakur, BS, Postolache, M: New iteration scheme for numerical reckoning fixed points of nonexpansive mappings. J. Inequal. Appl. **2014**:328 (2014)
- [66] Thakur, BS, Thakur, D, Postolache, M: A new iteration scheme for approximating fixed points of nonexpansive mappings. Filomat **30**(10), 2711–2720 (2016)
- [67] Thong DV: The comparison of the convergence speed between Picard, Mann, shikawa and two-step iterations in Banach spaces. Acta Math. Vietnam. **37**(2), 243–249 (2012)
- [68] Verma, M, Jain, P, Shukla, KK: A new faster first order iterative scheme for sparsity-based multitask learning. 2016 IEEE International Conference on Systems, Man, and Cybernetics (SMC) (2016)

- [69] Wahab, OT, Rauf, K: On faster implicit hybrid Kirk-multistep schemes for contractivetype operators. Intern. J. Anal. 2016, Article ID 3791506 (2016)
- [70] Xue, Z: The comparison of the convergence speed between Picard, Mann, Krasnoselskij and Ishikawa iterations in Banach spaces. Fixed Point Theory Appl. 2008, Article ID 387056 (2008)
- [71] Yadav, MR: Two-step iteration scheme for nonexpansive mappings in Banach space. Math. Morav. **19**(1), 95–105 (2015)
- [72] Yildirim, I: On the rate of convergence of different implicit iterations in convex metric spaces. Konuralp J. Math. **6**(1), 110-116 (2018)
- [73] Yildirim, I, Abbas, M: Convergence rate of implicit iteration process and a data dependence result. arXiv:1703.10357v1 [math.FA]