REALIZATION OF TENSOR-PRODUCT AND OF TENSOR-FACTORIZATION OF RATIONAL FUNCTIONS

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Abstract. We here first study the state space realization of a tensor-product of a pair of rational functions. At the expense of "inflating" the dimensions, we recover the classical expressions for realization of a regular product of rational functions. Then, under an additional assumption that the limit at infinity of a given rational function exists and is equal to identity, an explicit formula for a *tensor-factorization* of this function, is introduced.

1. Introduction

The problem of minimal factorization of matrix-valued rational functions of one complex variable has along history; see for instance [1, 2, 6]. Less known seems to be the counterpart of this problem when matrix product is replaced by tensor product. More precisely, we study the following two problems: First, given two rational matrix-valued functions R_1 and R_2 analytic at infinity, write a realization of the tensor product $R_1 \otimes R_2$ in terms of realizations of R_1 and R_2. Next, given a matrix-valued rational function R analytic at infinity, find its representations as $R_1 \otimes R_2$ where R_1 and R_2 are rational and analytic at infinity.

To provide some motivation we note the following. Tensor products play an important role in mathematics and quantum mechanics. In the latter case, a first example (see e.g. [4, p. 162]) is the product of two wave functions, each belonging to a given Hilbert space, which belongs to the tensor product of the given Hilbert spaces; see e.g. [8, Proposition 6.2, p. 111] for the latter. Another example is the case of quantum states (positive matrices with trace equal to 1; see e.g. [9]). Given two states $M_1 \in \mathbb{C}^{N_1 \times N_1}$ and $M_2 \in \mathbb{C}^{N_2 \times N_2}$, of possibly different sizes, the tensor product $M_1 \otimes M_2$ is still a state. Note that if $M = M_1 \otimes M_2$, one can recover M_1 and M_2 uniquely via the formula

$$d_1^* M_1 c_1 = \sum_{k=1}^{N_2} (d_1 \otimes f_k)^* M (c_1 \otimes f_k), \quad c_1, d_1 \in \mathbb{C}^{N_1}, \tag{1.1}$$

where $f_1, \ldots, f_{N_2}$ denotes an orthonormal basis for $\mathbb{C}^{N_2}$, and similarly for M_2,

$$d_2^* M_2 c_2 = \sum_{k=1}^{N_1} (e_k \otimes d_2)^* M (e_k \otimes c_2), \quad c_2, d_2 \in \mathbb{C}^{N_2}, \tag{1.2}$$

where now $e_1, \ldots, e_{N_1}$ is an orthonormal basis for $\mathbb{C}^{N_1}$. See e.g. [9, eq. (9.2.1) p. 97].

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If one starts from an arbitrary state $M \in \mathbb{C}^{N_1N_2\times N_1N_2}$ the matrices defined by (1.1) and (1.2) will be states, called marginal states, but their tensor product need not be equal to M.

One can consider similar problems in the setting of functions. We focus the discussion on rational functions. If $R(z)$ is a $\mathbb{C}^{N\times N}$-valued rational function and if $N = N_1N_2$, formulas (1.1) and (1.2) now define two rational functions R_A and R_B, respectively $\mathbb{C}^{N_1\times N_1}$ and $\mathbb{C}^{N_1\times N_1}$ -valued, via

$$\begin{aligned} d_1^*R_A(z)c_1 &= \sum_{k=1}^{N_2}(d_1\otimes f_k)^*R(z)(c_1\otimes f_k),\\ d_2^*R_B(z)c_2 &= \sum_{k=1}^{N_2}(d_2\otimes f_k)^*R(z)(c_2\otimes f_k), \end{aligned} \tag{1.3}$$

If $R = R_1\otimes R_2$ where R_1 is $\mathbb{C}^{N_1\times N_1}$-valued and R_2 is $\mathbb{C}^{N_2\times N_2}$-valued, then these equations can be rewritten as

$$\begin{aligned} R_A(z) &= R_1(z)\cdot(\operatorname{Tr} R_2(z))\\ R_B(z) &= R_2(z)\cdot(\operatorname{Tr} R_1(z)) \end{aligned} \tag{1.4}$$

and so these equations basically solve the tensor factorization problem.

The purpose of this work is in a somewhat different direction; we would like to express both tensor multiplication and tensor factorization of matrix-valued rational functions using state space representations.

In the rest of this section we cite some known results. Let z_l, z_r (the subscript stands for "left" and "right") be a pair of complex variables, and let $F_l(z_l)$, $F_r(z_r)$ be a pair of $p_l\times m_l$, $p_r\times m_r$-valued rational functions, respectively. Assume that neither has poles at infinity and denote by n_l, n_r the respective McMillan degrees. Thus, one can write the rational functions and the respective realization as

$$\begin{aligned} F_l(z_l) &= D_l + C_l(z_lI_{n_l}-A_l)^{-1}B_l \qquad & F_r(z_r) &= D_r + C_r(z_rI_{n_r}-A_r)^{-1}B_r\\ R_{F_l} &= \left(\begin{array}{c|c} A_l & B_l\\ \hline C_l & D_l\end{array}\right) & R_{F_r} &= \left(\begin{array}{c|c} A_r & B_r\\ \hline C_r & D_r\end{array}\right). \end{aligned} \tag{1.5}$$

Recall that whenever $m_l = p_r$ the product $F_l(z_l)F_r(z_r)$ is well-defined and its realization is given by[1] (see e.g. [3, Section 2.5])

$$R_{F_lF_r} = \left(\begin{array}{cc|c} A_l & B_lC_r & B_lD_r\\ 0 & A_r & B_r\\ \hline C_l & D_lC_r & D_lD_r\end{array}\right) = \left(\begin{array}{c|c} A & B\\ \hline C & D\end{array}\right) = \begin{pmatrix} A_l & 0 & B_l\\ 0 & I_{n_r} & 0\\ C_l & 0 & D_l\end{pmatrix}\begin{pmatrix} I_{n_l} & 0 & 0\\ 0 & A_r & B_r\\ 0 & C_r & D_r\end{pmatrix}, \tag{1.6}$$

in the sense that

$$F_1(z_1)F_2(z_2) = D_1D_2 + \begin{pmatrix} C_1 & D_1C_2\end{pmatrix}\left(\begin{pmatrix} z_1I_{n_1} & 0\\ 0 & z_2I_{n_2}\end{pmatrix} - \begin{pmatrix} A_1 & 0\\ 0 & A_2\end{pmatrix}\right)^{-1}\begin{pmatrix} B_1D_2\\ B_2\end{pmatrix}. \tag{1.7}$$

If $z_l = z_r$ the sought realization in (1.6) is of McMillan degree

$$n_l + n_r\ .$$

[1] Strictly speaking, in the references it was formulated for $z_l = z_r = z$ i.e. for $F_l(z)F_r(z)$

when minimal (roughly speaking when there is no pole-zero cancelation). We next address ourselves to the *tensor product*[2] of $F_l(z_l)$ and $F_r(z_r)$, resulting in $F_l \otimes F_r$, a $p_l p_r \times m_l m_r$-valued rational function. Tensor product of rational functions is discussed in [5, Section 5.2].

So far for known results. In the next section we focus on $R_{F_l \otimes F_r}$, the state space realization of $F_l \otimes F_r$. In Section 3 we set the framework for the main result, which is the factorization result presented in Section 4.

2. Realization of a tensor-product of rational functions

We start with technicalities: We denote by boldface characters, "inflated version" of the original ones, i.e.

$$\begin{array}{ll} \mathbf{A_l} := A_l \otimes I_{p_r} & \mathbf{A_r} := I_{m_l} \otimes A_r \\ \mathbf{B_l} := B_l \otimes I_{p_r} & \mathbf{B_r} := I_{m_l} \otimes B_r \\ \mathbf{C_l} := C_l \otimes I_{p_r} & \mathbf{C}_r := I_{m_l} \otimes C_r \\ \mathbf{D_l} := D_l \otimes I_{p_r} & \mathbf{D_r} := I_{m_l} \otimes D_r \\ \mathbf{F_l}(z_l) := \mathbf{C_l}\left(z_l I_{n_l p_r} - \mathbf{A_l}\right)^{-1}\mathbf{B_l} + \mathbf{D_l} & \mathbf{F_r}(z_r) := \mathbf{C_r}\left(z_l I_{m_l n_r} - \mathbf{A_r}\right)^{-1}\mathbf{B_r} + \mathbf{D_r}\ . \end{array} \tag{2.8}$$

We then show that at the expense of "inflating" the dimensions one can replace a *tensor product* by a *usual product*.

Proposition 2.1. *Let $F_l(z_r)$, $F_r(z_r)$ be a pair of $p_l \times m_l$, $p_r \times m_r$-valued rational functions, of McMillan degree n_l, n_r, respectively, whose realization is given in Eq.* (1.5)*. Following Eqs.* (1.6) *and* (2.8)*, one has that,*

$$R_{F_l \otimes F_r} = R_{\mathbf{F_l F_r}}\ . \tag{2.9}$$

In order to go into details we shall repeatedly use the fact, see e.g. [7, Lemma 4.2.10], that for matrices $T \in \mathbb{C}^{n \times m}$, $X \in \mathbb{C}^{m \times l}$, $Y \in \mathbb{C}^{l \times p}$, $Z \in \mathbb{C}^{p \times q}$ one has that

$$TX \otimes YZ = (T \otimes Y)(X \otimes Z). \tag{2.10}$$

We now explicitly compute the tensor product of $F_l(z_l)$ and $F_r(z_r)$,

$$\begin{array}{rl} F_l \otimes F_r = & \Big(D_l + C_l(z_l I_{n_l} - A_l)^{-1}B_l\Big) \otimes \Big(D_r + C_r(z_r I_{n_r} - A_r)^{-1}B_r\Big) \\ = & D_l \otimes D_r + D_l \otimes \Big(C_r(z_r I_{n_r} - A_r)^{-1}B_r\Big) + \Big(C_l(z_l I_{n_l} - A_l)^{-1}B_l\Big) \otimes D_r + \Big(C_l(z_l I_{n_l} - A_l)^{-1}B_l\Big) \otimes \Big(C_r(z_r I_{n_r} - A_r)^{-1}B_r\Big) \end{array}$$

We next separately examine each block

$$\begin{array}{rl} D_l \otimes \Big(C_r(z_r I_{n_r} - A_r)^{-1}B_r\Big) = & D_l I_{m_l} \otimes \Big(C_r(z_r I_{n_r} - A_r)^{-1}B_r\Big) \\ = & \Big(D_l \otimes \Big(C_r(z_r I_{n_r} - A_r)^{-1}\Big)\Big)\Big(I_{m_l} \otimes B_r\Big) \\ = & \Big(D_l I_{m_l} \otimes \Big(C_r(z_r I_{n_r} - A_r)^{-1}\Big)\Big)\Big(I_{m_l} \otimes B_r\Big) \\ = & (D_l \otimes C_r)\Big(I_{m_l} \otimes \Big((z_r I_{n_r} - A_r)^{-1}\Big)\Big)\Big(I_{m_l} \otimes B_r\Big) \\ = & (D_l \otimes I_{p_r})\Big(I_{m_l} \otimes C_r\Big)\Big(I_{m_l} \otimes \Big((z_r I_{n_r} - A_r)^{-1}\Big)\Big)\Big(I_{m_l} \otimes B_r\Big) \\ = & \underbrace{(D_l \otimes I_{p_r})}_{\mathbf{D}_l}\underbrace{(I_{m_l} \otimes C_r)}_{\mathbf{C}_r}\left(z_r I_{m_l n_r} - \underbrace{I_{m_l} \otimes A_r}_{\mathbf{A_r}}\right)^{-1}\underbrace{(I_{m_l} \otimes B_r)}_{\mathbf{B}_r} \\ = & \mathbf{D_l C_r}\Big(z_r I_{m_l n_r} - \mathbf{A_r}\Big)^{-1}\mathbf{B_r} \end{array}$$

[2]In matrix theory circles known as the "Kronecker product ", see e.g. [7, Section 4.2].

$$
\begin{aligned}
\big(C_l(z_lI_{n_l}-A_l)^{-1}B_l\big)\otimes D_r &= \big(C_l(z_lI_{n_l}-A_l)^{-1}B_l\big)\otimes I_{p_r}D_r\\
&= (C_l\otimes I_{p_r})\Big(\big((z_lI_{n_l}-A_l)^{-1}B_l\big)\otimes D_r\Big)\\
&= (C_l\otimes I_{p_r})\Big(\big((z_lI_{n_l}-A_l)^{-1}B_l\big)\otimes I_{p_r}D_r\Big)\\
&= (C_l\otimes I_{p_r})\Big(\big((z_lI_{n_l}-A_l)^{-1}\big)\otimes I_{p_r}\Big)(B_l\otimes D_r)\\
&= (C_l\otimes I_{p_r})\Big(\big((z_lI_{n_l}-A_l)^{-1}\big)\otimes I_{p_r}\Big)(B_l\otimes I_{p_r})\big(I_{m_l}\otimes D_r\big)\\
&= \underbrace{(C_l\otimes I_{p_r})}_{\mathbf{C}_l}\Bigg(z_lI_{n_lp_r}-\underbrace{A_l\otimes I_{p_r}}_{\mathbf{A}_l}\Bigg)^{-1}\underbrace{(B_l\otimes I_{p_r})}_{\mathbf{B}_l}\underbrace{(I_{m_l}\otimes D_r)}_{\mathbf{D}_r}\\
&= \mathbf{C_l}\big((z_lI_{n_lp_r}-\mathbf{A_l}\big)^{-1}\mathbf{B_l}\mathbf{D_r}
\end{aligned}
$$

$$
\begin{aligned}
\big(C_l(z_lI_{n_l}-A_l)^{-1}B_l\big)\otimes\big(C_r(z_rI_{n_r}-A_r)^{-1}B_r\big) &= \big(C_l(z_lI_{n_l}-A_l)^{-1}B_lI_{m_l}\big)\otimes\big(I_{p_r}C_r(z_rI_{n_r}-A_r)^{-1}B_r\big)\\
&= (C_l\otimes I_{p_r})\big((z_lI_{n_l}-A_l)^{-1}B_l\big)\otimes\big(C_r(z_rI_{n_r}-A_r)^{-1}\big)\big(I_{m_l}\otimes B_r\big)\\
&= (C_l\otimes I_{p_r})\big((z_lI_{n_l}-A_l)^{-1}B_lI_{m_l}\big)\otimes\big(I_{p_r}C_r(z_rI_{n_r}-A_r)^{-1}\big)\big(I_{m_l}\otimes B_r\big)\\
&= (C_l\otimes I_{p_r})\big((z_lI_{n_l}-A_l)^{-1}\otimes I_{p_r}\big)(B_l\otimes C_r)\Big(I_{m_l}\otimes\big((z_rI_{n_r}-A_r)^{-1}\big)\Big)\big(I_{m_l}\otimes B_r\big)\\
&= (C_l\otimes I_{p_r})\big((z_lI_{n_lp_r}-A_l\otimes I_{p_r}\big)^{-1}(B_l\otimes I_{p_r})\big(I_{m_l}\otimes C_r\big)\big((z_rI_{m_ln_r}-I_{m_l}\otimes A_r\big)^{-1}\big(I_{m_l}\otimes B_r\big)\\
&= \mathbf{C_l}\big((z_lI_{n_lp_r}-\mathbf{A_l}\big)^{-1}\mathbf{B_l}\mathbf{C_r}\big((z_rI_{m_ln_r}-\mathbf{A_r}\big)^{-1}\mathbf{B_r}
\end{aligned}
$$

Thus, one can write

$$
\begin{aligned}
F_l\otimes F_r &= \underbrace{D_l\otimes D_r}_{\mathbf{D}}+\begin{pmatrix}\mathbf{C_l} & \mathbf{D_l}\mathbf{C_r}\end{pmatrix}\begin{pmatrix}\big((z_lI_{n_lp_r}-\mathbf{A_l}\big)^{-1} & \big(z_lI_{n_lp_r}-\mathbf{A_l}\big)^{-1}\mathbf{B_l}\mathbf{C_r}\big(z_rI_{m_ln_r}-\mathbf{A_r}\big)^{-1}\\ 0 & \big(z_rI_{m_ln_r}-\mathbf{A_r}\big)^{-1}\end{pmatrix}\begin{pmatrix}\mathbf{B_l}\mathbf{D_r}\\ \mathbf{B_r}\end{pmatrix}\\
&= \mathbf{D}+\begin{pmatrix}\mathbf{C_l} & \mathbf{D_l}\mathbf{C_r}\end{pmatrix}\left(\begin{pmatrix}z_lI_{n_lp_r} & 0\\ 0 & z_rI_{m_ln_r}\end{pmatrix}-\begin{pmatrix}\mathbf{A_l} & \mathbf{B_l}\mathbf{C_r}\\ 0 & \mathbf{A_r}\end{pmatrix}\right)^{-1}\begin{pmatrix}\mathbf{B_l}\mathbf{D_r}\\ \mathbf{B_r}\end{pmatrix}.
\end{aligned}
$$

Note that in particular

$$
D_l\otimes D_r=(D_lI_{m_l})\otimes(I_{p_r}D_r)=\underbrace{(D_l\otimes I_{p_r})}_{\mathbf{D}_l}\underbrace{(I_{m_l}\otimes D_r)}_{\mathbf{D}_r}=\mathbf{D_l}\mathbf{D_r}=\mathbf{D}.
$$

The realization of $F_l(z_l)\otimes F_r(z_r)$ can be compactly written as

$$
R_{F_l\otimes F_r}=\left(\begin{array}{cc|c}\mathbf{A_l} & \mathbf{B_l}\mathbf{C_r} & \mathbf{B_l}\mathbf{D_r}\\ 0 & \mathbf{A_r} & \mathbf{B_r}\\ \hline \mathbf{C_l} & \mathbf{D_l}\mathbf{C_r} & \mathbf{D_l}\mathbf{D_r}\end{array}\right)=\left(\begin{array}{c|c}\mathbf{A_o} & \mathbf{B_o}\\ \hline \mathbf{C_o} & \mathbf{D}\end{array}\right)=\mathbf{R},\tag{2.11}
$$

which is indeed in form of (1.6), (2.8). If $z_l = z_r$ and there is no pole-zero cancelation, the sought realization in (2.11) is of McMillan degree

$$
n_lp_r+m_ln_r\ .
$$

Note now that in a way similar to (1.6), one can factorize the realization in (2.11) as follows,

$$
\mathbf{R}=\left(\begin{array}{cc|c}\mathbf{A_l} & \mathbf{B_l}\mathbf{C_r} & \mathbf{B_l}\mathbf{D_r}\\ 0 & \mathbf{A_r} & \mathbf{B_r}\\ \hline \mathbf{C_l} & \mathbf{D_l}\mathbf{C_r} & \mathbf{D}\end{array}\right)=\begin{pmatrix}\mathbf{A_l} & 0 & \mathbf{B_l}\\ 0 & I_{m_ln_r} & 0\\ \mathbf{C_l} & 0 & \mathbf{D_l}\end{pmatrix}\begin{pmatrix}I_{n_lp_r} & 0 & 0\\ 0 & \mathbf{A_r} & \mathbf{B_r}\\ 0 & \mathbf{C_r} & \mathbf{D_r}\end{pmatrix}.\tag{2.12}
$$

We conclude this section by pointing out that Proposition 2.1 can be easily extended to more elaborate cases like

$$
F_a(z_a)\otimes F_b(z_b)\otimes F_c(z_c)\cdots
$$

3. Realization of the inverse of a tensor product of rational functions

For future reference, in this section we examine the realization of the inverse of rational functions of the form $F_l(z_l)\otimes F_r(z_r)$ studied in the previous section.

We first recall, see e.g. [3, Theorem 2.4], in the realization of the inverse a rational function: Namely if

$$R_F=\left(\begin{array}{c|c} A & B \\ \hline C & D\end{array}\right),$$

is a realization of a square matrix-valued rational function $F(z)$, then whenever D is non-singular, $(F(z))^{-1}$ is well-defined almost everywhere, and a corresponding realization is given by,

$$R_{F^{-1}}=\left(\begin{array}{c|c} A^{\times} & B^{\times} \\ \hline C^{\times} & D^{\times}\end{array}\right)=\left(\begin{array}{c|c} A-BD^{-1}C & -BD^{-1} \\ \hline D^{-1}C & D^{-1}\end{array}\right). \tag{3.13}$$

Next, whenever the above $F_l(z)$ and $F_r(z)$ are so that

$$m_l=p_r \qquad \text{and} \qquad p_l=m_r$$

the product $F_l(z)F_r(z)$ is square, and whenever D_lD_r is non-singular[3], $(F_l(z)F_r(z))^{-1}$ is well-defined almost everywhere, and by combining (1.6) together with (3.13) a corresponding realization is given by

$$R_{(F_lF_r)^{-1}}=\left(\begin{array}{cc|c} A_l^{\times} & 0 & B_l^{\times} \\ B_r^{\times}C_l^{\times} & A_r^{\times} & B_r^{\times}D_l^{-1} \\ \hline D_r^{-1}C_l^{\times} & C_r^{\times} & D_r^{-1}D_l^{-1}\end{array}\right)=\begin{pmatrix} I_{n_l} & 0 & 0 \\ 0 & A_r^{\times} & B_r^{\times} \\ 0 & C_r^{\times} & D_r^{-1}\end{pmatrix}\begin{pmatrix} A_l^{\times} & 0 & B_l^{\times} \\ 0 & I_{n_r} & 0 \\ C_l^{\times} & 0 & D_l^{-1}\end{pmatrix}. \tag{3.14}$$

Similarly, whenever

$$m_lm_r=p_lp_r,$$

the rational function $F_l(z)\otimes F_r(z)$ is square and if $D_l\otimes D_r=\mathbf{D_lD_r}=\mathbf{D}$ is non-singular, then D_l, D_r are square, i.e.

$$m_l=p_l \qquad m_r=p_r$$

and non-singular, see e.g. [7, Theorem 4.2.15]. Thus, we shall denote hereafter by $m_l\times m_l$, $m_r\times m_r$ the dimensions of F_l, F_r, respectively.

Under these conditions, the $m_lm_r\times m_lm_r$-valued rational function, $(F_l(z)\otimes F_r(z))^{-1}$ is almost everywhere defined. (2.11), we next compute the realization of $(F_l\otimes F_r)^{-1}$,

$$R_{(F_l\otimes F_r)^{-1}}=\begin{pmatrix} (A_l\otimes I_{p_r})-(B_l\otimes D_r)(D_l\otimes D_r)^{-1}(C_l\otimes I_{p_r}) & B_l\otimes C_r-(B_l\otimes D_r)(D_l\otimes D_r)^{-1}(D_l\otimes C_r) & -(B_l\otimes D_r)(D_l\otimes D_r)^{-1} \\ -\left(I_{m_l}\otimes B_r\right)(D_l\otimes D_r)^{-1}(C_l\otimes I_{p_r}) & \left(I_{m_l}\otimes A_r\right)-\left(I_{m_l}\otimes B_r\right)(D_l\otimes D_r)^{-1}(D_l\otimes C_r) & -\left(I_{m_l}\otimes B_r\right)(D_l\otimes D_r)^{-1} \\ (D_l\otimes D_r)^{-1}(C_l\otimes I_{p_r}) & (D_l\otimes D_r)^{-1}(D_l\otimes C_r) & (D_l\otimes D_r)^{-1}\end{pmatrix}.$$

[3]this implies that $m_l=p_r\geq \operatorname{rank}(D_lD_r)=p_l=m_r$.

Taking into account the fact that D_l and D_r are square and non-singular, the realization $R_{(F_l\otimes F_r)^{-1}}$ takes the form

$$\begin{aligned}
R_{(F_l\otimes F_r)^{-1}} &= \left(\begin{array}{cc|c} (A_l - B_l D_l^{-1} C_l)\otimes I_{p_r} & 0 & (-B_l D_l^{-1})\otimes I_{p_r} \\ (D_l^{-1}C_l)\otimes(-B_r D_r^{-1}) & I_{m_l}\otimes(A_r - B_r D_r^{-1} C_r) & D_l^{-1}\otimes(-B_r D_r^{-1}) \\ \hline D_l^{-1}C_l\otimes D_r^{-1} & I_{m_l}\otimes D_r^{-1}C_r & (D_l\otimes D_r)^{-1} \end{array}\right) \\
&= \left(\begin{array}{cc|c} A_l^\times\otimes I_{p_r} & 0 & B_l^\times\otimes I_{p_r} \\ C_l^\times\otimes B_r^\times & I_{m_l}\otimes A_r^\times & D_l^{-1}\otimes B_r^\times \\ \hline C_l^\times\otimes D_r^{-1} & I_{m_l}\otimes C_r^\times & (D_l\otimes D_r)^{-1} \end{array}\right) \\
&= \left(\begin{array}{cc|c} A_l^\times\otimes I_{p_r} & 0 & (B_l^\times\otimes I_{p_r}) \\ (I_{n_l}\otimes B_r^\times)(C_l^\times\otimes I_{p_r}) & I_{m_l}\otimes A_r^\times & (I_{m_l}\otimes B_r^\times)(D_l^{-1}\otimes I_{n_r}) \\ \hline (I_{n_l}\otimes D_r^{-1})(C_l^\times\otimes I_{p_r}) & I_{m_l p_r}\otimes(I_{m_l}\otimes C_r^\times) & (I_{m_l}\otimes D_r^{-1})(D_l^{-1}\otimes I_{p_r}) \end{array}\right) \\
&= \left(\begin{array}{cc|c} \mathbf{A_l^\times} & 0 & \mathbf{B_l^\times} \\ \mathbf{B_r^\times C_l^\times} & \mathbf{A_r^\times} & \mathbf{B_r^\times D_l}^{-1} \\ \hline \mathbf{D_r^{-1}C_l^\times} & \mathbf{C_r^\times} & \mathbf{D_r^{-1}D_l^{-1}} \end{array}\right) = \left(\begin{array}{c|c} \mathbf{A_o}^\times & \mathbf{B_o}^\times \\ \hline \mathbf{C_o}^\times & \mathbf{D}^\times \end{array}\right) = \mathbf{R}^\times,
\end{aligned}$$

where the boldface entries are given by

$$\begin{array}{ll} \mathbf{A_l^\times} := A_l^\times\otimes I_{p_r} & \mathbf{A_r^\times} := I_{m_l}\otimes A_r^\times \\ \mathbf{B_l^\times} := B_l^\times\otimes I_{p_r} & \mathbf{B_r^\times} := I_{m_l}\otimes B_r^\times \\ \mathbf{C_l^\times} := C_l^\times\otimes I_{p_r} & \mathbf{C_r^\times} := I_{m_l}\otimes C_r^\times \\ \mathbf{D_l^{-1}} = D_l^{-1}\otimes I_{p_r} & \mathbf{D_r}^{-1} = I_{m_l}\otimes D_r^{-1}. \end{array} \tag{3.15}$$

One can conclude that

$$R_{(F_l\otimes F_r)^{-1}} = R_{(\mathbf{F_l F_r})^{-1}},$$

and in a way similar to (2.12), one can factorize the above realization as follows,

$$\mathbf{R}^\times = \left(\begin{array}{cc|c} \mathbf{A_l^\times} & 0 & \mathbf{B}_l^\times \\ \mathbf{B_r^\times C_l^\times} & \mathbf{A_r^\times} & \mathbf{B_r^\times D_l^{-1}} \\ \hline \mathbf{D_r^{-1}C_l^\times} & \mathbf{C_r^\times} & \mathbf{D_r^{-1}D}_l^{-1} \end{array}\right) = \begin{pmatrix} I_{n_l p_r} & 0 & 0 \\ 0 & \mathbf{A_r^\times} & \mathbf{B_r^\times} \\ 0 & \mathbf{C_r^\times} & \mathbf{D_r^{-1}} \end{pmatrix}\begin{pmatrix} \mathbf{A_l^\times} & 0 & \mathbf{B_l^\times} \\ 0 & I_{m_l n_r} & 0 \\ \mathbf{C_l^\times} & 0 & \mathbf{D_l^{-1}} \end{pmatrix}. \tag{3.16}$$

4. Tensor-factorization of rational functions

We now address a more challenging question: Given $\mathbf{F}(z)$ and $(\mathbf{F}(z))^{-1}$ (assuming that $\det\mathbf{F}(z)\not\equiv 0$), under what conditions and how, can it be "tensor-factorized" to *some* $F_l(z)$ and $F_r(z)$, namely the following relation holds,

$$\mathbf{F}(z) = F_l(z)\otimes F_r(z). \tag{4.1}$$

Note that here, we confine the discussion to a single complex variable, i.e. $z_l = z_r = z$.

Note also that if (4.1) holds, this is true up to complex scaling i.e.,

$$F_l(z)\otimes F_r(z) = c(z)F_l(z)\otimes \tfrac{1}{c(z)}F_r(z) \qquad 0\neq c(z)\in\mathbb{C}.$$

We shall use this degree of freedom in the sequel.

We next recall in the following fact from matrix theory.

Let Π_α, Π_β be a pair of supporting projections of the space $\mathbb{C}^{(\alpha+\beta)\times(\alpha+\beta)}$, i.e.

$$\begin{array}{ll} \Pi_\alpha^2 = \Pi_\alpha & \Pi_\alpha\Pi_\beta = 0_{\alpha+\beta} = \Pi_\beta\Pi_\alpha \\ \\ \Pi_\beta^2 = \Pi_\beta & \Pi_\alpha + \Pi_\beta = I_{\alpha+\beta}\ . \end{array} \tag{4.2}$$

Such a pair of projections can be obtained by partitioning an arbitrary non-singular $T \in \mathbb{C}^{(\alpha+\beta)\times(\alpha+\beta)}$ as follows.

$$
\begin{array}{rcl}
T^{-1}\left(\begin{smallmatrix} I_\alpha & 0 \\ 0 & 0_\beta \end{smallmatrix}\right)T & := & \Pi_\alpha \\
T^{-1}\left(\begin{smallmatrix} 0_\alpha & 0 \\ 0 & I_\beta \end{smallmatrix}\right)T & = & \Pi_\beta \ .
\end{array} \tag{4.3}
$$

By using an isometry-like relation, we next offer a simple way to "deflate" matrix dimensions.

Observation 4.1. *Given $M \in \mathbb{C}^{s\times q}$, denote*

$$\mathbf{M_l} := M \otimes I_p \qquad \mathbf{M_r} := I_m \otimes M.$$

For arbitrary $u \in \mathbb{C}^p$, $v \in \mathbb{C}^m$ normalized so that $u^*u = 1$, $v^*v = 1$, one has that

$$(I_s \otimes u^*)\,\mathbf{M_l}\,(I_q \otimes u) = M \qquad \text{and} \qquad (v^* \otimes I_s)\,\mathbf{M_r}\,(v \otimes I_q) = M.$$

Indeed, by twice applying (2.10) one obtains,

$$
\begin{array}{rcccl}
(I_s \otimes u^*)\underbrace{(M \otimes I_p)}_{\mathbf{M_l}}(I_q \otimes u) & = & (I_s M I_q) \otimes \underbrace{(u^* I_p u)}_{=1} & = & M \\
(v^* \otimes I_s)\underbrace{(I_m \otimes M)}_{\mathbf{M_r}}(v \otimes I_q) & = & \underbrace{(v^* I_m v)}_{=1} \otimes (I_s M I_q) & = & M.
\end{array}
$$

We next apply the last observation to the variables here.

Corollary 4.2. *For $u \in \mathbb{C}^{p_r}$, $v \in \mathbb{C}^{m_l}$, normalized so that $u^*u = 1$ and $v^*v = 1$, the boldface characters in* (2.8) *satisfy*

$$
\begin{array}{rclcrcl}
A_l & = & (I_{n_l} \times u^*)\mathbf{A_l}(I_{n_l} \otimes u) & \qquad & A_r & = & (v^* \otimes I_{n_l})\mathbf{A_r}(v \otimes I_{n_r}) \\
B_l & = & (I_{n_l} \times u^*)\mathbf{B_l}(I_{m_l} \otimes u) & & B_r & = & (v^* \otimes I_{n_l})\mathbf{B_r}(v \otimes I_{m_r}) \\
C_l & = & (I_{p_l} \times u^*)\mathbf{C_l}(I_{n_l} \otimes u) & & C_r & = & (v^* \otimes I_{p_l})\mathbf{C_r}(v \otimes I_{n_r}) \\
D_l & = & (I_{p_l} \times u^*)\mathbf{D_l}(I_{m_l} \otimes u) & & D_r & = & (v^* \otimes I_{p_l})\mathbf{D_r}(v \otimes I_{m_r}).
\end{array}
$$

We now return to the problem of "tensor-factorization" in (4.1). We note that in place of $\mathbf{R}$ in (2.11) and $\mathbf{R}^\times$ in (3.16), the realization arrays associated with $\mathbf{F}$ and $\mathbf{F^{-1}}$, are known only *up to a coordinate transformation*, i.e. there exists, a non-singular matrix $T \in \mathbb{C}^{(n_l p_r + m_l n_r)\times(n_l p_r + m_l n_r)}$ namely in (4.2) and (4.3)

$$\alpha = n_l p_r \quad \text{and} \quad \beta = m_l n_r \ ,$$

so that the actual realization array is given by

$$\left(\begin{smallmatrix} T & 0 \\ 0 & I_{p_l m_r} \end{smallmatrix}\right)^{-1}\mathbf{R}\left(\begin{smallmatrix} T & 0 \\ 0 & I_{p_l m_r} \end{smallmatrix}\right) = \left(\begin{array}{c|c} T^{-1}\mathbf{A_o}T & T^{-1}\mathbf{B_o} \\ \hline \mathbf{C_o}T & \mathbf{D} \end{array}\right) = \left(\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right), \tag{4.4}$$

and

$$\left(\begin{smallmatrix} T & 0 \\ 0 & I_{p_l m_r} \end{smallmatrix}\right)^{-1}\mathbf{R}^\times\left(\begin{smallmatrix} T & 0 \\ 0 & I_{p_l m_r} \end{smallmatrix}\right) = \left(\begin{array}{c|c} T^{-1}\mathbf{A_o}^\times T & T^{-1}\mathbf{B_o}^\times \\ \hline \mathbf{C_o^\times} T & \mathbf{D^{-1}} \end{array}\right) = \left(\begin{array}{c|c} \mathbf{A}^\times & \mathbf{B}^\times \\ \hline \mathbf{C}^\times & \mathbf{D^{-1}} \end{array}\right). \tag{4.5}$$

As in reality, the specific coordinate transformation, T in (4.4) and (4.5) is *unknown* one can conclude that to extract $F_l(z)$ and $F_r(z)$ from (4.1) along with the realization arrays in (4.4), (4.5), additional conditions are needed.

Theorem 4.3. *Let* $\mathbf{F}(z)$ *be a given square matrix-valued rational function. Assume that*

$$\lim_{z \longrightarrow \infty} \mathbf{F}(z) = I.$$

Let $\left(\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & I \end{array}\right)$, *see* (4.4), *and* $\left(\begin{array}{c|c} \mathbf{A}^{\times} & \mathbf{B}^{\times} \\ \hline \mathbf{C}^{\times} & I \end{array}\right)$. *see* (4.5), *be realizations of* $\mathbf{F}(z)$ *and of* $(\mathbf{F}(z))^{-1}$, *respectively.*

Substituting in (4.2), $\alpha = n_l m_r$ *and* $\beta = m_l n_r$, *assume also that there exists a pair of supporting projection to* $\mathbb{C}^{n_l m_r + m_l n_r}$ *denoted by* $\Pi_{n_l m_r}$ *and* $\Pi_{m_l n_r}$ *so that*

$$\mathbf{A}\Pi_{n_l m_r} = \Pi_{n_l m_r}\mathbf{A}\Pi_{n_l m_r} \qquad \mathbf{A}^{\times}\Pi_{m_l n_r} = \Pi_{m_l n_r}\mathbf{A}^{\times}\Pi_{m_l n_r} \ . \tag{4.6}$$

Following the definition of the projections $\Pi_{n_l m_r}$ *and* $\Pi_{m_l n_r}$, *see* (4.3) *and* (4.6), *along with Corollary 4.2, for arbitrary* $u \in \mathbb{C}^{m_r}$, $v \in \mathbb{C}^{m_l}$, *normalized so that* $u^*u = 1$ *and* $v^*v = 1$, *we find it convenient to introduce the following related projections*[4],

$$\hat{\Pi}_{n_l m_r} = T^{-1}\begin{pmatrix} I_{n_l}\otimes uu^* & 0 \\ 0 & 0_{m_l n_r}\end{pmatrix} T \qquad \hat{\Pi}_{m_l n_r} = T^{-1}\begin{pmatrix} 0_{n_l m_r} & 0 \\ 0 & vv^*\otimes I_{m_l}\end{pmatrix} T \tag{4.7}$$

Then, using (2.11) *and* (4.4), *one can take in* (4.1) $\mathbf{F} = F_l \otimes F_r$ *where,*

$$\begin{aligned} F_l(z) &= (I_{m_l}\otimes u^*)\mathbf{C}\hat{\Pi}_{n_l m_r}\left(zI_{n_l m_r + m_l n_r} - \mathbf{A}\right)^{-1}\hat{\Pi}_{n_l m_r}\mathbf{B}(I_{m_l}\otimes u) + I_{m_l} \\ F_r(z) &= (v^*\otimes I_{m_r})\mathbf{C}\hat{\Pi}_{m_l n_r}\left(zI_{n_l m_r + m_l n_r} - \mathbf{A}\right)^{-1}\hat{\Pi}_{m_l n_r}\mathbf{B}(v\otimes I_{m_r}) + I_{m_r} \end{aligned}$$

Proof : First, recall (see Section 3) that the assumption that $D_l \otimes D_r = \mathbf{D_l}\mathbf{D_r} = \mathbf{D}$ is square non-singular, it implies that both D_l and D_r are square non-singular. We shall thus denote the dimensions of F_l and F_r, by $m_l \times m_l$ and $m_r \times m_r$, respectively.

The assumption here that $\mathbf{D} = I_{m_l m_r}$ implies (see e.e. [7, Theorem 4.2.12]) that

$$D_l = cI_{m_l} \qquad D_r = \tfrac{1}{c}I_{m_r} \qquad \text{for some non} - \text{zero} \quad c \in \mathbb{C}.$$

As already mentioned after (4.1), to simplify the exposition we shall take $c = 1$.

Next, let T in (4.3), (4.4), (4.5) be the same so that the supporting projections are $\Pi_{n_l m_r}$ and $\Pi_{m_l n_r}$. Next note that substituting (2.11), (3.16), (4.4) and (4.5) in condition (4.6) yields,

$$\begin{aligned} \mathbf{A}\Pi_{n_l m_r} = \ & T^{-1}\begin{pmatrix} \mathbf{A_l} & 0 \\ 0 & 0_{m_l n_r}\end{pmatrix} T \qquad & \Pi_{m_l n_r}\mathbf{A} = \ & T^{-1}\begin{pmatrix} 0_{n_l m_r} & 0 \\ 0 & \mathbf{A_r}\end{pmatrix} T \\ \mathbf{A}^{\times}\Pi_{m_l n_r} = \ & T^{-1}\begin{pmatrix} 0_{n_l m_r} & 0 \\ 0 & \mathbf{A_r^{\times}}\end{pmatrix} T \qquad & \Pi_{n_l m_r}\mathbf{A}^{\times} = \ & T^{-1}\begin{pmatrix} \mathbf{A_l^{\times}} & 0 \\ 0 & 0_{m_l n_r}\end{pmatrix} T \end{aligned}$$

and thus in the sequel we shall use the two upper relations, i.e.

$$\Pi_{n_l m_r}\mathbf{A}\Pi_{n_l m_r} = T^{-1}\begin{pmatrix} \mathbf{A_l} & 0 \\ 0 & 0_{m_l n_r}\end{pmatrix} T \qquad \Pi_{m_l n_r}\mathbf{A}\Pi_{m_l n_r} = T^{-1}\begin{pmatrix} 0_{n_l m_r} & 0 \\ 0 & \mathbf{A_r}\end{pmatrix} T.$$

[4]note that $\hat{\Pi}_{n_l m_r}\Pi_{n_l m_r} = \hat{\Pi}_{n_l m_r}\Pi_{n_l m_r} = \hat{\Pi}_{n_l m_r}$ and $\hat{\Pi}_{m_l n_r}\Pi_{m_l n_r} = \Pi_{m_l n_r}\hat{\Pi}_{m_l n_r} = \hat{\Pi}_{m_l n_r}$.

We are now ready to recover $F_l(z)$,

$$F_l(z) = C_l\left(zI_{n_l} - A_l\right)^{-1} B_l + I_{m_l}$$

$$= \underbrace{(I_{m_l} \otimes u^*)\mathbf{C_l}(I_{n_l} \otimes u)}_{C_l} \underbrace{(I_{n_l} \otimes u^*)\left(zI_{n_l m_r} - \mathbf{A_l}\right)^{-1}(I_{n_l} \otimes u)}_{\left(zI_{n_l} - A_l\right)^{-1}} \underbrace{(I_{n_l} \otimes u^*)\mathbf{B_l}(I_{m_l} \otimes u)}_{B_l} + I_{m_l}$$

$$= (I_{m_l} \otimes u^*)\mathbf{C_l}(I_{n_l} \otimes uu^*)\left(zI_{n_l m_r} - \mathbf{A_l}\right)^{-1}(I_{n_l} \otimes uu^*)\mathbf{B_l}(I_{m_l} \otimes u) + I_{m_l}$$

$$= (I_{m_l} \otimes u^*)\mathbf{C_o}\begin{pmatrix} I_{n_l m_r} \\ 0_{m_l n_r \times n_l m_r}\end{pmatrix}(I_{n_l} \otimes uu^*)\begin{pmatrix} I_{n_l m_r} & 0_{n_l m_r \times m_l n_r}\end{pmatrix}\left(zI_{n_l m_r + m_l n_r} - \mathbf{A_o}\right)^{-1}$$

$$\times \begin{pmatrix} I_{n_l m_r} \\ 0 \end{pmatrix}(I_{n_l} \otimes uu^*)\mathbf{B_l}(I_{m_l} \otimes u) + I_{m_l}$$

$$= (I_{m_l} \otimes u^*)\mathbf{C_o}\begin{pmatrix} I_{n_l}\otimes uu^* & 0 \\ 0 & 0_{m_l n_r}\end{pmatrix}\left(zI_{n_l m_r + m_l n_r} - \mathbf{A_o}\right)^{-1}\begin{pmatrix} I_{n_l}\otimes uu^* & 0 \\ 0 & 0_{m_l n_r}\end{pmatrix}\mathbf{B_o}(I_{m_l} \otimes u) + I_{m_l}$$

$$= (I_{m_l} \otimes u^*)\underbrace{\mathbf{C_o}T}_{\mathbf{C}}\underbrace{T^{-1}\begin{pmatrix} I_{n_l}\otimes uu^* & 0 \\ 0 & 0_{m_l n_r}\end{pmatrix}T}_{\hat{\Pi}_{n_l m_r}}\underbrace{T^{-1}\left(zI_{n_l m_r + m_l n_r} - \mathbf{A_o}\right)^{-1}T}_{\left(zI_{n_l m_r + m_l n_r} - \mathbf{A}\right)^{-1}}\underbrace{T^{-1}\begin{pmatrix} I_{n_l}\otimes uu^* & 0 \\ 0 & 0_{m_l n_r}\end{pmatrix}T}_{\hat{\Pi}_{n_l m_r}}$$

$$\times \underbrace{T^{-1}\mathbf{B_o}}_{\mathbf{B}}(I_{m_l} \otimes u) + I_{m_l}$$

$$= (I_{m_l} \otimes u^*)\mathbf{C}\hat{\Pi}_{n_l m_r}\left(zI_{n_l m_r + m_l n_r} - \mathbf{A}\right)^{-1}\hat{\Pi}_{n_l m_r}\mathbf{B}(I_{m_l} \otimes u) + I_{m_l}\ .$$

Similarly, for $F_r(z)$

$$F_r(z) = C_r(zI_{n_r} - A_r)^{-1}B_r + I_{m_r}$$

$$= \underbrace{(v^* \otimes I_{m_r})\mathbf{C_r}(v \otimes I_{n_r})}_{C_r}\underbrace{(v^* \otimes I_{n_l})(zI_{n_r} - \mathbf{A_r})^{-1}(v \otimes I_{n_r})}_{(zI_{n_r} - A_r)^{-1}}\underbrace{(v^* \otimes I_{n_l})\mathbf{B_r}(v \otimes I_{m_r})}_{B_r} + I_{m_r}$$

$$= (v^* \otimes I_{m_r})\mathbf{C_r}(vv^* \otimes I_{m_l})(zI_{n_r} - \mathbf{A_r})^{-1}(vv^* \otimes I_{m_l})\mathbf{B_r}(v \otimes I_{m_r}) + I_{m_r}$$

$$= (v^* \otimes I_{m_r})\mathbf{C_o}\begin{pmatrix} 0_{n_l m_r \times m_l n_r} \\ I_{m_l n_r}\end{pmatrix}(vv^* \otimes I_{m_l})\begin{pmatrix} 0_{m_l n_r \times n_l m_r} & I_{m_l n_r}\end{pmatrix}\left(zI_{n_l m_r + m_l n_r} - \mathbf{A_o}\right)^{-1}$$

$$\times \begin{pmatrix} 0 \\ I_{m_l n_r}\end{pmatrix}(vv^* \otimes I_{m_l})\begin{pmatrix} 0 & I_{m_l n_r}\end{pmatrix}\mathbf{B_o}(v \otimes I_{m_r}) + I_{m_r}$$

$$= (v^* \otimes I_{m_r})\underbrace{\mathbf{C_o}T}_{\mathbf{C}}\underbrace{T^{-1}\begin{pmatrix} 0_{n_l m_r} & 0 \\ 0 & vv^*\otimes I_{m_l}\end{pmatrix}T}_{\hat{\Pi}_{m_l n_r}}\underbrace{T^{-1}\left(zI_{n_l m_r + m_l n_r} - \mathbf{A_o}\right)^{-1}T}_{\left(zI_{n_l m_r + m_l n_r} - \mathbf{A}\right)^{-1}}\underbrace{T^{-1}\begin{pmatrix} 0_{n_l m_r} & 0 \\ 0 & vv^*\otimes I_{m_l}\end{pmatrix}T}_{\hat{\Pi}_{m_l n_r}}$$

$$\times \underbrace{T^{-1}\mathbf{B_o}}_{\mathbf{B}}(v \otimes I_{m_r}) + I_{m_r}$$

$$= (v^* \otimes I_{m_r})\mathbf{C}\hat{\Pi}_{m_l n_r}\left(zI_{n_l m_r + m_l n_r} - \mathbf{A}\right)^{-1}\hat{\Pi}_{m_l n_r}\mathbf{B}(v \otimes I_{m_r}) + I_{m_r}\ .$$

□

Remark 4.4. At first sight, the assumptions in Theorem 4.3 seem very restrictive. For persective recall that to factorize a given rational function $F(z)$ to $F(z) = F_l(z)F_r(z)$, the assumptions are virtually the same[5], see [3, Section 2.5]).

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[5]There they only assume D is square non-singular, but then only $F_l(z)D_r$ and $D_lF_r(z)$ are obtained.