Spectral gaps for the O-U/Stochastic heat processes on path space over a Riemannian manifold with boundary*

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Abstract

Fang-Wu[15] presented a explicit spectral gap for the O-U process on path space over a Riemannian manifold without boundary under the bounded Ricci curvature conditions. In this paper, we will extend these results to the case of the Riemannian manifold with boundary. Moreover, we also derive the similar results for the stochastic heat process.

Keywords: Functional inequality; Ricci Curvature; Second fundamental form; Diffusion process; Path space.

1 Introduction

sect1

Functional inequality is an important tool to study the spectral gaps for some diffusion operators in the analysis/stochastic analysis field, especially, for the case of infinite dimensional Riemannian path space. For the manifold without boundary, Fang[12] first established the Poincaré inequality for the O-U operator on Riemanian path space by the Clark-Ocean formula, after that the log-Sobolev inequality/(weak)Poincaré inequality have also been established for the O-U Dirichlet form, see e.g. [1, 3, 9, 18, 19, 23, 4, 21, 27, 28, 5] and references therein. Recently Naber[21] gave some characterizations of the uniform bounds of Ricci curvature by the analysis of the path space. Motiviated by this work, Fang and Wu[15] gave the explicit spectral gap of the O-U operator on

*Supported in part by NNSFC (11371099).

path space under the Ricci curvature condition that $K_2 \leq \mathrm{Ric}^Z (:= \mathrm{Ric} + \nabla Z) \leq K_1$ and $K_2 + K_1 \geq 0$. This condition $K_2 + K_1 \geq 0$ is removed by Cheng-Thalimaier[5].

For the manifold with boundary, Wang[26] proved the damped log-Sobolev inequality for the O-U process on path space, but some geometric informations are hidden in this inequality. In this article, our main aim is to present a estimate of the spectral gap for the O-U operator on path space over a manifold (possible with boundary) under the curvature and the second fundamental form conditions

eq1.1
$$K_2 \leq \mathrm{Ric}^Z \leq K_1, \quad \sigma_2 \leq \mathbb{I} \leq \sigma_1 \tag{1.1}$$

for some constants $K_2, K_1, \sigma_2, \sigma_1 \in \mathbb{R}$. In particular, our results cover Fang-Wu's results and Cheng-Thalmaier's results. Moreover, we also obtain the estimate of the spectral gap for the stochastic heat process.

To state our main results, we need to introduce some notation. Let M be a d-dimensional complete Riemannian manifold possibly with a boundary ∂M and N be the inward unit normal vector field of ∂M. Let $L = \frac{1}{2}\Delta + Z$ be the diffusion operator for some C^1 vector field Z, where Δ is the Laplace operator on M.

Denote by the Riemannian path space:

$$W_x^T(M) = \{\gamma \in C([0,T]; M) : \gamma_0 = x\}.$$

Let ρ be the Riemannian distance on M. Then $W_x^T(M)$ is a Polish space under the uniform distance

$$\rho_\infty(\gamma, \sigma) := \sup_{t \in [0,T]} \rho(\gamma_t, \sigma_t), \quad \gamma, \sigma \in W_x^T(M).$$

Let $O(M)$ be the orthonormal frame bundle over M and $\pi : O(M) \to M$ be the canonical projection. Furthermore, we choose a canonical orthonormal basis $\{e_i\}_{1 \leq i \leq d}$ on $\mathbb{R}^d$ and a standard orthonormal basis $H_i(u) := (H_{ue_i})_{1 \leq i \leq d}$ for $u \in O(M)$ of horizontal vector fields on $O(M)$. Then the horizontal reflecting diffusion process is the unique solution to the SDEs:

eq1.2
$$\mathrm{d}U_t^x = H_i(U_t^x) \circ \mathrm{d}W_t + H_Z(U_t^x)\mathrm{d}t + H_N(U_t^x)\mathrm{d}l_t^x, \quad U_0^x \in O_x(M), \tag{1.2}$$

where W_t is the d-dimensional Brwonian motion on a complete filtration probability space $(\Omega, \{\mathscr{F}_t\}_{t \geq 0}, \mathbb{P})$, H_Z and H_N are the horizontal lift of Z and N respectively, and l_t^x is an adapted increasing process which increases only when $X_t^x := \pi U_t^x \in \partial M$ which is called the local time of X_t^x on ∂M. Then it is easy to know that X_t^x solves the equation

eq1.3
$$\mathrm{d}X_t^x = U_t^x \circ dW_t + Z(X_t^x)\mathrm{d}t + N(X_t)\mathrm{d}l_t^x, \quad X_0^x = x \tag{1.3}$$

up to the life time ζ(the maximal time of the solution).

Let $\mathscr{F}C_T^\infty$ be the space of bounded Lipschitz continuous cylinder functions on $W_x^T(M)$, i.e. for every $F \in \mathscr{F}C_T^\infty$, there exist some $N \geq 1$ and $0 < t_1 < t_2 \cdots < t_N \leq T$, $f \in C_{Lip}(M^N)$ such that $F(\gamma) = f(\gamma_{t_1}, \cdots, \gamma_{t_N}), \gamma \in W_x^T(M)$, where $C_{Lip}(M^N)$ is the collection of bounded Lipschitz continuous functions on M^N. Suppose $\mathbb{H}$ is the standard Cameron-Martin space for $C([0,T];\mathbb{R}^d)$, i.e.

$$\mathbb{H} = \left\{ h \in C([0,T];\mathbb{R}^d) : h(0) = 0, \|h\|_{\mathbb{H}_\mathbb{T}}^2 := \int_0^T |h_s'|^2 \mathrm{d}s < \infty \right\}.$$

In order to construct O-U process on path space by the theory of Dirichlet form, we first introduce the damped Mallavin gradient given by Wang[25]. To do that, we will recommend a multiplicative functional $Q_{s,t}^x$, which is first introduced by Hsu [20] to investigate gradient estimate on P_t. For any fixed $s \geq 0$, $(Q_{s,t}^x)_{t \geq s}$ is an adapted right-continuous process on $\mathbb{R}^d \otimes \mathbb{R}^d$ such that $Q_{s,t}^x P_{U_t^x} = 0$ if $X_t^x \in \partial M$ and

eq1.4
$$Q_{s,t}^x = \left(I - \int_s^t Q_{s,r}^x \left\{ \mathrm{Ric}_Z(U_r^x)\mathrm{d}s + \mathbb{I}(U_r^x)\mathrm{d}l_r^x \right\} \right) \left(I - 1_{\{X_t^x \in \partial M\}} P_{U_t^x} \right), \tag{1.4}$$

where $P_u : \mathbb{R}^d \to \mathbb{R}^d$ is the projection along $u^{-1}N$, i.e.

$$\langle P_u a, b \rangle := \langle ua, N \rangle \langle ub, N \rangle, \quad a, b \in \mathbb{R}^d, u \in \cup_{x \in \partial M} O_x(M).$$

For every $F \in \mathscr{F}C_T^\infty$, by (4.2.1) in [25], the damped gradient is defined by

eq1.5
$$\tilde{D}_t F(X_{[0,T]}^x) = \sum_{i: t_i > t} Q_{t,t_i}^x U_{t_i}^{-1} \nabla_i f(X_{t_1}^x, \cdots, X_{t_N}^x), \quad t \in [0,T]. \tag{1.5}$$

Thus, the associated Mallavin gradient will defined as follows:

eq1.6
$$D_t F(X_{[0,T]}^x) = \sum_{i: t_i > t} \left(I - 1_{\{X_{t_i}^x \in \partial M\}} P_{U_{t_i}^x} \right) U_{t_i}^{-1} \nabla_i f(X_{t_1}^x, \cdots, X_{t_N}^x), \quad t \in [0,T]. \tag{1.6}$$

For any constants $K_2, K_1, \sigma_2, \sigma_1$ with $K_2 \leq K_1, \sigma_2 \leq \sigma_1$and each $t \in [0,T]$, let μ be the random measure on $[0,T]$ given by

$$\begin{aligned} \mu_t(\mathrm{d}r) &= \exp\left[-K_2(r-t) - \sigma_2(l_r^x - l_t^x)\right] \{(|K_1| \vee |K_2|)\mathrm{d}r + (|\sigma_1| \vee |\sigma_2|)\mathrm{d}l_r^x\} \\ &=: \varphi_1(t, r, K_1, K_2, \sigma_1, \sigma_2)\mathrm{d}r + \varphi_1(t, r, K_1, K_2, \sigma_1, \sigma_2)\mathrm{d}l_r^x \end{aligned}$$

Denote by two measurable functions on $W_x^T(M)$

eq1.7
$$\begin{aligned} A_t &= \Big(1 + \mu([t,T])\Big) + \int_0^t \Big(1 + \mu([r,T])\Big)\varphi_1(r, t, K_1, K_2, \sigma_1, \sigma_2)\mathrm{d}r \\ B_t &= \int_0^t \Big(1 + \mu([r,T])\Big)\varphi_2(r, t, K_1, K_2, \sigma_1, \sigma_2)\mathrm{d}r. \end{aligned} \tag{1.7}$$

Throughout the article, we assume that the (reflecting if ∂M exists) L-diffusion process is non-explosive. Let $\mathbb{P}_x$ be the distribution of the L-diffusion process X^x starting from a fixed point x up to some fixed time $T > 0$. Then $\mathbb{P}_x$ is a probability measure on the Riemannian path space $W_x^T(M)$. Define the following quadratic form by

eq1.8
$$\mathscr{E}_{\sigma_1,\sigma_2}^{K_1,K_2}(F,F) := \int_{W_x^T(M)} \int_0^T |D_t F|^2 (A_t \mathrm{d}t + B_t \mathrm{d}l_t) \mathrm{d}\mathbb{P}_x. \tag{1.8}$$

The following Logarithmic Sobolev inequality is the main result of this paper.

T1.1
Theorem 1.1. *Assume that $K_2 \le \mathrm{Ric}^Z \le K_1$ and $\sigma_2 \le \mathbb{I} \le \sigma_1$. Then the following Logarithmic Sobolev inequality holds*

eq1.9
$$\mathbb{E}\left(F^2 \log \frac{F^2}{\|F\|_{L^2}^2}\right) \le \mathscr{E}_{\sigma_1,\sigma_2}^{K_1,K_2}(F,F), \quad F \in \mathscr{F}C_T^\infty. \tag{1.9}$$

By the above Theorem 1.1, we obtain the following Corollary for two special cases.

C1.2
Corollary 1.2. (*a*) *Assume that M is a Riemannian manifold without a boundary and $K_1 \le \mathrm{Ric}^Z \le K_2$, then the Logarithmic Sobolev inequality holds*

eq1.10
$$\mathbb{E}\left(F^2 \log \frac{F^2}{\|F\|_{L^2}^2}\right) \le C(T,K_1,K_2) \int_{W_x^T(M)} \int_0^T |D_t F|^2 \mathrm{d}t \mathrm{d}\mathbb{P}_x, \quad F \in \mathscr{F}C_b^\infty(M). \tag{1.10}$$

In particular, when $K_2 < 0$, we have

eq1.11
$$Spect(L)^{-1} \le \frac{1}{2} + \frac{1}{2}\left(1 + \frac{|K_1| \vee |K_2|}{K_2}\left[1 - \mathrm{e}^{-K_2 T}\right]\right)^2. \tag{1.11}$$

When $K_2 > 0$, we have

eq1.12
$$SG(L)^{-1} \le (1+\beta)^2 - 2\sqrt{\left(\beta + \frac{\beta^2}{2}\right)\left(\beta + \beta^2 - \frac{\beta^2}{2}\mathrm{e}^{-K_2 T}\right)}\, \mathrm{e}^{-\frac{K_2 T}{2}}, \tag{1.12}$$

where $\beta = \frac{|K_1| \vee |K_2|}{K_2}$.

(*b*) *Let M be Ricci flat Riemannian manifold with boundary, and we assume that the second fundamental form satisfies $\sigma_2 \le \mathbb{I} \le \sigma_1$, then*

When $\sigma_2 \ge 0$, we get that for any $F \in \mathscr{F}C_T^\infty$

eq1.13
$$\begin{aligned} \mathbb{E}\left(F^2 \log \frac{F^2}{\|F\|_{L^2}^2}\right) &\le \int_{W_x^T(M)} (1+\sigma_1 l_T^x) \int_0^T |D_t F|^2 \mathrm{d}t \mathrm{d}\mathbb{P}_x \\ &\quad + \int_{W_x^T(M)} (\sigma_1(1+\sigma_1 l_T^x)T) \int_0^T |D_t F|^2 \mathrm{d}l_t \mathrm{d}\mathbb{P}_x. \end{aligned} \tag{1.13}$$

When $\sigma_2 < 0$, we get that for any $F \in \mathscr{F}C_T^\infty$

eq1.14 (1.14)

$$\mathbb{E}\left(F^2 \log \frac{F^2}{\|F\|_{L^2}^2}\right) \le \int_{W_x^T(M)} (1 + (|\sigma_1| \vee |\sigma_2|) \exp\left[(-\sigma_2 + \varepsilon) l_T^x\right]) \int_0^T |D_t F|^2 \mathrm{d}t \mathrm{d}\mathbb{P}_x + \int_{W_x^T(M)} \left(2(|\sigma_1| \vee |\sigma_2|)^2 \exp\left[(-\sigma_2 + \varepsilon) l_T^x\right]\right) \int_0^T |D_t F|^2 \mathrm{d}l_t \mathrm{d}\mathbb{P}_x$$

for some constant $\varepsilon > 0$.

Remark 1.3. (1) *Wang [25] proved that the damped Logarithmic Sobolev inequality.*

(2) *When M is a Riemannian manifold without boundary, and $K_2 \le \mathrm{Ric}^Z \le K_1, K_2 + K_1 \ge$, Fang-Wu [15] first proved* (1.10)*, later , this result had been extended to the general case of K_2 and K_1 by Cheng-Thalimaier[5].*

The rest of this paper is organized as follows: In Section 2, we will prove Theorem 1.1 and Corollary 1.2. The estimate of the spectral gap for the stochastic heat process will be presented in Section 3.

2 Proofs of Theorem 1.1 and Corollary 1.2

2.1 Proof of Theorem 1.1

Proof of Theorem 1.1. By Theorem 4.4 in [25], we know that the following damped logarithmic Sobolev inequality holds

eq2.1 (2.1)

$$\mathbb{E}\left(F^2 \log \frac{F^2}{\|F\|_{L^2}^2}\right) \le 2 \int_{W_x^T(M)} \int_0^T |\tilde{D}_t F|^2 \mathrm{d}t \mathrm{d}\mathbb{P}_x.$$

Therefore it suffices to show that

eq2.2 (2.2)

$$\int_0^T |\tilde{D}_t F|^2 \mathrm{d}t \le \int_0^T |D_t F|^2 (A_t \mathrm{d}t + B_t \mathrm{d}l_t).$$

By using the assumptions of $K_2 \le \mathrm{Ric}^Z \le K_1$ and $\sigma_2 \le \mathbb{I} \le \sigma_1$, we have

$$\|\mathrm{Ric}^Z\| \le |K_1| \vee |K_2|, \quad \|\mathbb{I}\| \le |\sigma_1| \vee |\sigma_2|,$$

Combining this with (1.14)

$$Q_{s,t}^x = \left(I - \int_s^t Q_{s,r}^x \left\{\mathrm{Ric}^Z(U_r^x)\mathrm{d}s + \mathbb{I}(U_r^x)\mathrm{d}l_r^x\right\}\right)\left(I - 1_{\{X_t^x \in \partial M\}} P_{U_t^x}\right).$$

and [26, Theorem 3.2.1], it is easy to derive that

eq2.3 (2.3)

$$\left\|Q_{t,r}^x\right\| \le \exp\left[-K_2(r-t) - \sigma_2(l_r^x - l_t^x)\right].$$

By the definition of the damped gradient, we get

eq2.4 (2.4)

$$\begin{aligned}&\tilde{D}_tF(X^x_{[0,T]})=\sum_{i:t_i>t}Q^x_{t,t_i}U^{-1}_{t_i}\nabla_i f(X^x_{t_1},\cdots,X^x_{t_N})=D_tF(X^x_{[0,T]})-\\
&\sum_{i:t_i>t}\int_t^{t_i}Q^x_{t,r}\big\{\mathrm{Ric}^Z(U^x_r)\mathrm{d}s+\mathbb{I}(U^x_r)\mathrm{d}l^x_r\big\}\Big(I-1_{\{X^x_t\in\partial M\}}P_{U^x_{t_i}}\Big)U^{-1}_{t_i}\nabla_i f(X^x_{t_1},\cdots,X^x_{t_N})\\
&=D_tF(X^x_{[0,T]})-\int_t^T Q^x_{t,r}\big\{\mathrm{Ric}^Z(U^x_r)D_rF(X^x_{[0,T]})\mathrm{d}r+\mathbb{I}(U^x_r)D_rF(X^x_{[0,T]})\mathrm{d}l^x_r\big\}.\end{aligned}$$

Then we have

eq2.5 (2.5)

$$\begin{aligned}&|\tilde{D}_tF|(X^x_{[0,T]})\le|D_tF|(X^x_{[0,T]})\\
&+\int_t^T\exp\left[-K_2(r-t)-\sigma_2(l^x_r-l^x_t)\right]\{(|K_1|\vee|K_2|)\mathrm{d}r+(|\sigma_1|\vee|\sigma_2|)\mathrm{d}l^x_r\}|D_rF|(X^x_{[0,T]})\}\\
&=|D_tF|(X^x_{[0,T]})+\int_t^T|D_rF|\mu_t(\mathrm{d}r).\end{aligned}$$

The Hölder's inequality implies that

eq2.6

$$|\tilde{D}_tF|^2(X^x_{[0,T]})\le\Big(1+\mu_t([t,T])\Big)\Big(|D_tF|^2+\int_t^T|D_rF|^2\mu_t(\mathrm{d}r)\Big)\tag{2.6}$$

Thus, we obtain

eq2.7 (2.7)

$$\begin{aligned}&\int_0^T|\tilde{D}_tF|^2\mathrm{d}t\le\int_0^T\Big(1+\mu([t,T])\Big)|D_tF|^2\mathrm{d}t+\int_0^T\int_t^T\Big(1+\mu([t,T])\Big)|D_rF|^2\mu_t(\mathrm{d}r)\mathrm{d}t\\
&=\int_0^T\Big(1+\mu([t,T])\Big)|D_tF|^2\mathrm{d}t+\int_0^T\int_t^T\Big(1+\mu([t,T])\Big)|D_rF|^2\varphi_1(t,r,K_1,K_2,\sigma_1,\sigma_2)\mathrm{d}r\mathrm{d}t\\
&+\int_0^T\int_t^T\Big(1+\mu([t,T])\Big)|D_rF|^2\varphi_2(t,r,K_1,K_2,\sigma_1,\sigma_2)\mathrm{d}l_r\mathrm{d}t\\
&=\int_0^T\Big(1+\mu([t,T])\Big)|D_tF|^2\mathrm{d}t+\int_0^T\int_0^r\Big(1+\mu([t,T])\Big)|D_rF|^2\varphi_1(t,r,K_1,K_2,\sigma_1,\sigma_2)\mathrm{d}r\mathrm{d}t\\
&+\int_0^T\int_0^r\Big(1+\mu([t,T])\Big)|D_rF|^2\varphi_2(t,r,K_1,K_2,\sigma_1,\sigma_2)\mathrm{d}l_r\mathrm{d}t\\
&=\int_0^T\Big(1+\mu([t,T])\Big)|D_tF|^2\mathrm{d}t+\int_0^T\int_0^t\Big(1+\mu([r,T])\Big)\varphi_1(r,t,K_1,K_2,\sigma_1,\sigma_2)\mathrm{d}r|D_tF|^2\mathrm{d}t\\
&+\int_0^T\int_0^t\Big(1+\mu([r,T])\Big)\varphi_2(r,t,K_1,K_2,\sigma_1,\sigma_2)\mathrm{d}r|D_tF|^2\mathrm{d}l_t\\
&=\int_0^T|D_tF|^2(A_t\mathrm{d}t+B_t\mathrm{d}l_t).\end{aligned}$$

Up to now, we complete the proof. □

2.2 Proof of Corollary 1.2

To prove Corollary 1.2, we need some preparations. Let $\beta = \frac{|K_1|\vee|K_2|}{K_2}$ and define

eq2.8

$$\begin{aligned}\Lambda(t,T) &= 1+\beta\Big[1-\exp\left[-K_2(T-t)\right]\Big] + \left(\beta+\beta^2\right)\Big[1-\exp\left(-K_2 t\right)\Big] \\ &\quad + \frac{\beta^2}{2}\left[\exp\left(-K_2(t+T)\right)-\exp\left(-K_2(T-t)\right)\right]\end{aligned} \tag{2.8}$$

and

$$C(T,K_1,K_2) := \sup_{t\in[0,T]} \Lambda(t,T).$$

Similar to the proof of Proposition 3.3 in Fang-Wu[15] for the case of $K_1+K_2\geq 0$, in the following we will discuss monotonicity of the function $\Lambda(\cdot,T)$.

p2.1

Proposition 2.1. (1) *If $K_2<0$, then $t\to\Lambda(t,T)$ is strictly increasing over $[0,T]$.* (2) *If $K_2\geq 0$, then the maximum is attained at a point t_0 in $(0,T)$.*

Proof. According to the definition (2.8) of $\Lambda(t,T)$, we get

$$\Lambda(0,T) = 1+\beta\left(1-\mathrm{e}^{-K_2T}\right)$$

and

$$\begin{aligned}\Lambda(T,T) &= 1+\left(\beta+\beta^2\right)\Big[1-\exp\left(-K_2T\right)\Big] + \frac{\beta^2}{2}\left[\mathrm{e}^{-2K_2T}-1\right] \\ &= \frac{1}{2}+\frac{1}{2}\Big[1+\beta\left(1-\mathrm{e}^{-K_2T}\right)\Big]^2 = \frac{1}{2}+\frac{1}{2}\Lambda^2(0,T).\end{aligned}$$

In particular, the second in the above implies that $\Lambda(T,T)\geq\Lambda(0,T)$.

Next, we take the derivative of $\Lambda(t,T)$ with respect to t,

$$\Lambda'(t,T) = -\beta K_2\mathrm{e}^{-K_2(T-t)} + \left(\beta+\beta^2\right)K_2\mathrm{e}^{-K_2t} - \frac{\beta^2}{2}K_2\left[\mathrm{e}^{-K_2(t+T)}+\mathrm{e}^{-K_2(T-t)}\right].$$

Then we have

eq2.9

$$\begin{aligned}\Lambda'(0,T) &= -\beta K_2\mathrm{e}^{-K_2T} + \left(\beta+\beta^2\right)K_2 - \beta^2K_2\mathrm{e}^{-K_2T} \\ &= \beta K_2(1+\beta)(1-\mathrm{e}^{-K_2T})\geq 0;\end{aligned} \tag{2.9}$$

and

eq2.10

$$\begin{aligned}\Lambda'(T,T) &= -\beta K_2 + \left(\beta+\beta^2\right)K_2\mathrm{e}^{-K_2T} - \frac{\beta^2}{2}K_2\left[\mathrm{e}^{-2K_2T}+1\right] \\ &= -\beta K_2\left(1-(1+\beta)\,\mathrm{e}^{-K_2T}+\frac{\beta}{2}\left[\mathrm{e}^{-2K_2T}+1\right]\right) \\ &= -\beta K_2[1-\mathrm{e}^{-K_2T}] - \frac{\beta^2K_2}{2}[1-\mathrm{e}^{-K_2T}]^2.\end{aligned} \tag{2.10}$$

Noting that

eq2.11 (2.11)
$$\begin{cases} \Lambda'(T,T) > 0 & if\ K_2 < 0, \\ \Lambda'(T,T) < 0 & if\ K_2 > 0. \end{cases}$$

Now we look for $t \in [0,T]$ such that $\Lambda'(t,T) = 0$. We have

eq2.12 (2.12)
$$\begin{aligned} &\Lambda'(t,T) = 0 \\ \Leftrightarrow \quad & -\mathrm{e}^{-K_2(T-t)} + (1+\beta)\,\mathrm{e}^{-K_2 t} - \frac{\beta}{2}\left[\mathrm{e}^{-K_2(t+T)} + \mathrm{e}^{-K_2(T-t)}\right] = 0 \\ \Leftrightarrow \quad & -\mathrm{e}^{-K_2 T}\mathrm{e}^{2K_2 t} + (1+\beta) - \frac{\beta}{2}\left[\mathrm{e}^{-K_2 T} + \mathrm{e}^{-K_2 T}\mathrm{e}^{2K_2 t}\right] = 0 \\ \Leftrightarrow \quad & \Big(1 + \frac{\beta}{2}\Big)\mathrm{e}^{-K_2 T}\mathrm{e}^{2K_2 t} = 1 + \beta - \frac{\beta}{2}\mathrm{e}^{-K_2 T}. \end{aligned}$$

Therefore there exists at most one t such that $\Lambda'(t,T) = 0$. For the case where $K_2 < 0$, if there exists $t_0 \in (0,T)$ such that $\Lambda(t_0,T) < 0$. Then by (2.9) and (2.11), the equation $\Lambda'(t,T) = 0$ has at least two solutions, it is impossible. Therefore for $K_2 < 0$, $\Lambda'(t,T) \geq 0$. For $K_2 > 0$, we suppose t_0 such that $\Lambda'(t_0,T) = 0$, then by (2.12)

$$\mathrm{e}^{2K_2 t_0} = \Big(1 + \frac{\beta}{2+\beta}\Big(1 - \mathrm{e}^{-K_2 T}\Big)\Big)\mathrm{e}^{K_2 T}.$$

Thus the proof is completed. □

By the above Proposition 2.1, it is easy to obtain the following Proposition 2.2.

p2.2 **Proposition 2.2.** *(i) If $K_2 > 0$,*

eq2.13 (2.13)
$$\begin{aligned} \sup_{t\in[0,T]} \Lambda(t,T) = (1+\beta)^2 - \Big(\beta + \frac{\beta^2}{2}\Big)\sqrt{1 + \frac{\beta}{2+\beta}\Big(1 - \mathrm{e}^{-K_2 T}\Big)}\ \mathrm{e}^{-\frac{K_2 T}{2}} \\ - \frac{\Big(\beta + \beta^2 - \frac{\beta^2}{2}\mathrm{e}^{-K_2 T}\Big)}{\sqrt{1 + \frac{\beta}{2+\beta}\Big(1 - \mathrm{e}^{-K_2 T}\Big)}}\mathrm{e}^{-\frac{K_2 T}{2}}. \end{aligned}$$

(ii) If $K_2 < 0$,

eq2.14 (2.14)
$$\sup_{t\in[0,T]} \Lambda(t,T) = \frac{1}{2} + \frac{1}{2}\left(1 + \frac{|K_1| \vee |K_2|}{K_2}\Big[1 - \mathrm{e}^{-K_2 T}\Big]\right)^2.$$

Proof of Corollary 1.2. (a) Since M is a Riemannian manifold without boundary, then the local time $l_t = 0$, thus we have

$$\mu_t(\mathrm{d}r) := (|K_1| \vee |K_2|) \exp\left[-K_2(r-t)\right] \mathrm{d}r.$$

Then

eq2.15 (2.15)
$$\mu([t,T]) = \frac{(|K_1| \vee |K_2|)}{K_2}\left[1 - \exp\left[-K_2(T-t)\right]\right]. \tag{2.15}$$

Which implies that

eq2.16 (2.16)
$$\begin{aligned} \varphi_1(r,t,K_1,K_2,\sigma_1,\sigma_2) &= (|K_1| \vee |K_2|) \exp\left[-K_2(t-r)\right] \\ \varphi_2(r,t,K_1,K_2,\sigma_1,\sigma_2) &= 0. \end{aligned} \tag{2.16}$$

Then by (2.15) and the first equality of (2.16),

eq2.17 (2.17)
$$\begin{aligned} &\int_0^t \Big(1 + \mu([r,T])\Big)\varphi_1(r,t,K_1,K_2,\sigma_1,\sigma_2)\mathrm{d}r \\ &= (|K_1| \vee |K_2|)\int_0^t \left(1 + \frac{(|K_1| \vee |K_2|)}{K_2}\left[1 - \exp\left[-K_2(T-r)\right]\right]\right)\exp\left[-K_2(t-r)\right]\mathrm{d}r \\ &= \left(\frac{(|K_1| \vee |K_2|)}{K_2} + \frac{(|K_1|^2 \vee |K_2|^2)}{K_2^2}\right)\Big[1 - \exp\left(-K_2 t\right)\Big] \\ &\qquad - \frac{(|K_1|^2 \vee |K_2|^2)}{2K_2^2}\exp\left(-K_2(t+T)\right)\left(\exp\left(2K_2 t\right) - 1\right). \end{aligned} \tag{2.17}$$

Thus we get

eq2.18 (2.18)
$$\begin{aligned} B_t &= \int_0^t \Big(1 + \mu([r,T])\Big)\varphi_2(r,t,K_1,K_2,\sigma_1,\sigma_2)\mathrm{d}r = 0 \\ A_t &= \Big(1 + \mu([t,T])\Big) + \int_0^t \Big(1 + \mu([r,T])\Big)\varphi_1(r,t,K_1,K_2,\sigma_1,\sigma_2)\mathrm{d}r \\ &= 1 + \frac{(|K_1| \vee |K_2|)}{K_2}\left[1 - \exp\left[-K_2(T-t)\right]\right] \\ &\quad + \left(\frac{(|K_1| \vee |K_2|)}{K_2} + \frac{(|K_1|^2 \vee |K_2|^2)}{K_2^2}\right)\Big[1 - \exp\left(-K_2 t\right)\Big] \\ &\quad + \frac{(|K_1|^2 \vee |K_2|^2)}{2K_2^2}\left[\exp\left(-K_2(t+T)\right) - \exp\left(-K_2(T-t)\right)\right]. \end{aligned} \tag{2.18}$$

From which we have

eq2.19 (2.19)
$$\int_0^T |D_t F|^2 (A_t \mathrm{d}t + B_t \mathrm{d}l_t) \le \int_0^T \Lambda(t,T)|D_t F|^2 \mathrm{d}t. \tag{2.19}$$

Then (1.9), (1.10) and (1.11) come from Theorem 1.1 and Proposition 2.2.

(b) By the assumption of M, we know that

eq2.20 (2.20)
$$\mu_t(\mathrm{d}r) = (|\sigma_1| \vee |\sigma_2|) \exp\left[-\sigma_2(l_r^x - l_t^x)\right] \mathrm{d}l_r^x. \tag{2.20}$$

Thus,

eq2.21 (2.21)
$$\mu([t,T]) = (|\sigma_1| \vee |\sigma_2|) \exp\left[\sigma_2 l_t^x\right] \int_t^T \exp\left[-\sigma_2 l_r^x\right] \mathrm{d}l_r^x. \tag{2.21}$$

In addition, (2.20) implies that

eq2.22 (2.22)
$$\begin{aligned} &\varphi_1(r,t,K_1,K_2,\sigma_1,\sigma_2) = 0 \\ &\varphi_2(r,t,K_1,K_2,\sigma_1,\sigma_2) = (|\sigma_1| \vee |\sigma_2|) \exp\left[-\sigma_2(l_t^x - l_r^x)\right]. \end{aligned} \tag{2.22}$$

Then, by the definition of A_t and B_t,

eq2.23 (2.23)
$$\begin{aligned} A_t &= 1 + (|\sigma_1| \vee |\sigma_2|) \exp\left[\sigma_2 l_t^x\right] \int_t^T \exp\left[-\sigma_2 l_r^x\right] \mathrm{d}l_r^x \\ B_t &= (|\sigma_1| \vee |\sigma_2|) \int_0^t \left(1 + (|\sigma_1| \vee |\sigma_2|) \exp\left[\sigma_2 l_r^x\right] \int_r^T \exp\left[-\sigma_2 l_u^x\right] \mathrm{d}l_u^x\right) \\ &\qquad \times \exp\left[-\sigma_2(l_t^x - l_r^x)\right] \mathrm{d}r. \end{aligned} \tag{2.23}$$

Since l_t^x is a increasing process, we have

eq2.24 (2.24)
$$A_t \le \begin{cases} 1 + \sigma_1 l_T^x, & \text{if } \sigma_2 \ge 0, \\ 1 + (|\sigma_1| \vee |\sigma_2|) \exp\left[(-\sigma_2 + \varepsilon) l_T^x\right], & \text{if } \sigma_2 < 0 \end{cases} \tag{2.24}$$

and

eq2.25 (2.25)
$$B_t \le \begin{cases} \sigma_1(1 + \sigma_1 l_T^x)T, & \text{if } \sigma_2 \ge 0, \\ 2(|\sigma_1| \vee |\sigma_2|)^2 \exp\left[(-\sigma_2 + \varepsilon) l_T^x\right], & \text{if } \sigma_2 < 0 \end{cases} \tag{2.25}$$

By Theorem 1.1, when $\sigma_2 \ge 0$, we get

eq2.26 (2.26)
$$\begin{aligned} &\mathbb{E}\left(F^2 \log \frac{F^2}{\|F\|_{L^2}^2}\right) \\ &\le \int_{W_x^T(M)} (1 + \sigma_1 l_T^x) \int_0^T |D_t F|^2 \mathrm{d}t \mathrm{d}\mathbb{P}_x \\ &+ \int_{W_x^T(M)} (\sigma_1(1 + \sigma_1 l_T^x)T) \int_0^T |D_t F|^2 \mathrm{d}l_t \mathrm{d}\mathbb{P}_x, \end{aligned} \tag{2.26}$$

and when $\sigma_2 < 0$, we get

eq2.27

$$\begin{aligned}
&\mathbb{E}\left(F^2 \log \frac{F^2}{\|F\|_{L^2}^2}\right) \\
&\le \int_{W_x^T(M)} (1 + (|\sigma_1| \vee |\sigma_2|) \exp\left[(-\sigma_2 + \varepsilon) l_T^x\right]) \int_0^T |D_t F|^2 \mathrm{d}t \mathrm{d}\mathbb{P}_x \\
&+ 2 \int_{W_x^T(M)} \big(|\sigma_1| \vee |\sigma_2|\big)^2 \exp\left[(-\sigma_2 + \varepsilon) l_T^x\right]\big) \int_0^T |D_t F|^2 \mathrm{d}l_t \mathrm{d}\mathbb{P}_x.
\end{aligned} \tag{2.27}$$

□

Corollary 2.3. *Let* $M := \big\{x = (x_1, \cdots, x_d) \in \mathbb{R}^d : a_1 x_1 + \cdots + a_d x_d \ge c\big\}$ *for some constant* $c \in \mathbb{R}$*, then* M *is a Riemannian manifold without boundary and* $\mathrm{Ric} = 0$ *with* $\mathbb{I} = 0$*, thus we have*

eq2.28

$$\mathbb{E}\left(F^2 \log \frac{F^2}{\|F\|_{L^2}^2}\right) \le \int_{W_x^T(M)} \int_0^T |D_t F|^2 \mathrm{d}t \mathrm{d}\mathbb{P}_x. \tag{2.28}$$

3 Stochastic heat equation

In this section, we will consider the spectral gap for the stochastic heat equation on a Riemannian manifold with boundary. Before moving on, let's introduce some notation.

The stochastic heat equation on Riemannian manifold had been studied detailed by [22](see also [17]). Here they introduced some notation. In particular, the classical cylinder function depending on finite times is not in the domain of generator associated to the stochastic heat equation. Thus, we need to introduce a class of new cylinder function $\mathscr{F}C_b^1$ on $W_x^T(M)$, i.e. for every $F \in \mathscr{F}C_b^1$, there exist some $m \ge 1,\ m \in \mathbb{N},\ f \in C_b^1(\mathbb{R}^m), g_i \in C_b^{0,1}([0,1] \times M)$, $i = 1, ..., m$, such that

eq3.1

$$F(\gamma) = f\left(\int_0^1 g_1(s, \gamma_s)\mathrm{d}s, \int_0^1 g_2(s, \gamma_s)\mathrm{d}s, ..., \int_0^1 g_m(s, \gamma_s)\mathrm{d}s\right), \quad \gamma \in W_x^T(M), \tag{3.1}$$

where $C_b^{0,1}([0,1] \times M)$ denotes the functions which are continuous w.r.t. the first variable and differentiable w.r.t. the second variable with continuous derivatives.

For any $F \in \mathscr{F}C_b^1$ with (2.2) form and $h \in L^2([0,1]; \mathbb{R}^d)$, according to Wang[25], the damped Malliavin gradient of F is given by

$$\dot{\tilde{D}} F(s)(\gamma) := \sum_{j=1}^m \hat{\partial}_j f(\gamma) \int_s^T Q_{s,u} U_u^{-1}(\gamma) \nabla g_j(u, \gamma_u) \mathrm{d}u, \quad \gamma \in W_x^T(M).$$

Let $\tilde{\nabla} F$ be the damped L^2-gradient of F, and since

$$\begin{aligned}
&\int_0^T \left\langle \dot{\tilde{D}}F(s), h_s' \right\rangle \mathrm{d}s = \langle h, \mathbf{D}F\rangle_{\mathbb{H}} = D_h F = \langle h, \mathbf{D}F\rangle\rangle_{L^2} \\
&= \int_0^T \left\langle \tilde{\nabla} F(s), h_s \right\rangle \mathrm{d}s = \int_0^T \left\langle \tilde{\nabla} F(s), \int_0^s h_u' \mathrm{d}u \right\rangle \mathrm{d}s \\
&= \int_0^T \int_0^s \left\langle \tilde{\nabla} F(s), h_u' \right\rangle \mathrm{d}u \mathrm{d}s = \int_0^T \int_u^T \left\langle \tilde{\nabla} F(s), h_u' \right\rangle \mathrm{d}s \mathrm{d}u \\
&= \int_0^T \left\langle \int_s^T \tilde{\nabla} F(u) \mathrm{d}u, h_s' \right\rangle \mathrm{d}s.
\end{aligned}$$

Then, we have

$$\int_s^T \tilde{\nabla} F(u) \mathrm{d}u = \dot{\tilde{D}}F(s).$$

Thus,

$$\tilde{\nabla} F(s) = \sum_{j=1}^m \hat{\partial}_j f(\gamma) Q_{s,T} U_s^{-1}(\gamma) \nabla g_j(s, \gamma_s).$$

The L^2-gradient of F is defined by

$$\nabla F(s) = \sum_{j=1}^m \hat{\partial}_j f(\gamma) U_s^{-1}(\gamma) \nabla g_j(s, \gamma_s).$$

By Lemma 4.3.2 in [25], we have

eq3.2

$$\begin{aligned}
F &= F + \sqrt{2} \int_0^T \langle \dot{\tilde{D}}F(s), \mathrm{d}B_s \rangle \\
&= F + \sqrt{2} \int_0^T \left\langle \int_s^T \tilde{\nabla} F(u) \mathrm{d}u, \mathrm{d}B_s \right\rangle \\
&= F + \sqrt{2} \int_0^T \left\langle \int_s^T Q_{s,u} \nabla F(u) \mathrm{d}u, \mathrm{d}B_s \right\rangle
\end{aligned} \tag{3.2}$$

By the standard the procedure, we have

eq3.3

$$\mathbb{E}\left(F^2 \log \frac{F^2}{\|F\|_{L^2}^2} \right) \le 2 \int_{W_x^T(M)} \int_0^T \left| \int_s^T Q_{s,u} \nabla F(u) \mathrm{d}u \right|^2 \mathrm{d}s \mathrm{d}\mathbb{P}_x. \tag{3.3}$$

By (2.3) and Hölder's inequality, we get

eq3.4
$$\begin{aligned}\left|\int_t^T Q_{s,u}\nabla F(u)\mathrm{d}u\right|^2 &\le \left|\int_s^T \mathrm{e}^{-K(u-s)-\sigma(l_u^x-l_s^x)}\nabla F(u)\mathrm{d}u\right|^2\\ &\le \int_s^T \mathrm{e}^{-K(u-s)-\sigma(l_u^x-l_s^x)}\mathrm{d}u\int_s^T \mathrm{e}^{-K(u-s)-\sigma(l_u^x-l_s^x)}|\nabla F|^2(u)\mathrm{d}u\\ &= \int_s^T \mathrm{e}^{-K(u-s)-\sigma(l_u^x-l_s^x)}\mathrm{d}u\int_s^T \mathrm{e}^{-K(u-s)-\sigma(l_u^x-l_s^x)}|\nabla F|^2(u)\mathrm{d}u.\end{aligned}\tag{3.4}$$

Let

$$\varphi(s)=\int_s^T \mathrm{e}^{-K(u-s)-\sigma(l_u^x-l_s^x)}\mathrm{d}u.$$

Then by changing the order of integration we obtain

$$\int_{W_x^T(M)}\int_0^T\left|\int_s^T Q_{s,u}\nabla F(u)\mathrm{d}u\right|^2\mathrm{d}s\mathrm{d}\mathbb{P}_x\le\int_{W_x^T(M)}\int_0^T\varphi(s)\int_s^T \mathrm{e}^{-K(u-s)-\sigma(l_u^x-l_s^x)}|\nabla F|^2(u)\mathrm{d}u\mathrm{d}s\mathrm{d}\mathbb{P}_x$$
$$=\int_{W_x^T(M)}\int_0^T A(s)|\nabla F|^2(s)\mathrm{d}s\mathrm{d}\mathbb{P}_x,$$

where

$$A(s)=\int_0^s\varphi(u)\mathrm{e}^{-K(s-u)-\sigma(l_s^x-l_u^x)}\mathrm{d}u.$$

Thus, we get the following Logarithmic Sobolev inequality.

T3.1 **Theorem 3.1.** *Assume that* $\mathrm{Ric}^Z\ge K$ *and* $\mathbb{I}\ge\sigma$*. Then the following Logarithmic Sobolev inequality holds*

eq3.5
$$\mathbb{E}\left(F^2\log\frac{F^2}{\|F\|_{L^2}^2}\right)\le 2\int_{W_x^T(M)}\int_0^T A(s)|\nabla F|^2(s)\mathrm{d}s\mathrm{d}\mathbb{P}_x,\quad F\in\mathscr{F}C_b^1.\tag{3.5}$$

c3.2 **Corollary 3.2.** (*a*) *Assume that* M *is a Riemannian manifold with a convex boundary and* $K\le\mathrm{Ric}^Z$*, then the Logarithmic Sobolev inequality holds*

eq3.6
$$\mathbb{E}\left(F^2\log\frac{F^2}{\|F\|_{L^2}^2}\right)\le C(T,K)\int_{W_x^T(M)}\int_0^T|\nabla F|^2\mathrm{d}t\mathrm{d}\mathbb{P}_x,\quad F\in\mathscr{F}C_b^\infty(M),\tag{3.6}$$

where

$$C(T,K)=\begin{cases}\frac{1}{2K^2}\left[2-\frac{2-e^{-KT}}{\sqrt{2\mathrm{e}^{KT}-1}}-e^{-KT}\sqrt{2\mathrm{e}^{KT}-1}\right], & \text{if } K\ge 0,\\ \frac{1}{K^2}\left[1-e^{-KT}\right]^2, & \text{if } K<0.\end{cases}$$

Proof. Since the boundary is convex, thus $\mathbb{I} \geq \sigma \geq 0$. Thus

$$\varphi(s) = \int_s^T \mathrm{e}^{-K(u-s)-\sigma(l_u^x - l_s^x)}\mathrm{d}u \leq \int_s^T \mathrm{e}^{-K(u-s)}\mathrm{d}u = \frac{1}{K}\left[1 - \mathrm{e}^{-K(T-s)}\right]$$

and

$$\begin{aligned} A(s) &= \int_0^s \varphi(u)\mathrm{e}^{-K(s-u)-\sigma(l_s^x - l_u^x)}\mathrm{d}u \leq \frac{1}{K}\int_0^s \left[1 - \mathrm{e}^{-K(T-u)}\right]\mathrm{e}^{-K(s-u)}\mathrm{d}u \\ &= \frac{1}{2K^2}\Bigg[2 - 2e^{-Ks} + e^{-K(T+s)} - e^{-K(T-s)}\Bigg]. \end{aligned}$$

Then we get

$$A(0) = 0, \quad A(T) = \frac{1}{2K^2}\left(1 - e^{-K}\right)^2$$

In the following, similar to the argument of Proposition 2.1. Taking the derivative of $s \to A(s)$ gives

$$A'(s) = \frac{1}{2K}\Bigg[2e^{-Ks} - e^{-K(T+s)} - e^{-K(T-s)}\Bigg].$$

Thus,

eq3.7
$$A'(0) = \frac{1}{K}\left[1 - e^{-K}\right] \geq 0, \quad A'(T) = -\frac{1}{2K}\left[e^{-K} - 1\right]^2. \tag{3.7}$$

Noting that

eq3.8
$$\begin{cases} A'(T) > 0 & if\ K < 0, \\ A'(T) < 0 & if\ K > 0. \end{cases} \tag{3.8}$$

Now we look for $s \in [0, T]$ such that $A'(s) = 0$. We have

c3.1
$$\begin{aligned} & A'(s) = 0 \\ \Leftrightarrow \quad & 2e^{-Ks} - e^{-K(T+s)} - e^{-K(T-s)} = 0 \\ \Leftrightarrow \quad & 2 - e^{-KT} - e^{-KT+2Ks} = 0 \\ \Leftrightarrow \quad & e^{2Ks} = 2\mathrm{e}^{KT} - 1. \end{aligned} \tag{3.9}$$

Therefore there exists at most one t such that $A'(s) = 0$. For the case where $K < 0$, if there exists $s_0 \in (0, T)$ such that $A(s_0) < 0$. Then by (3.7) and (3.8), the equation $A'(s) = 0$ has at least two solutions, it is impossible. Therefore for $K < 0$, $A'(s) \geq 0$. For $K > 0$, we suppose s_0 such that $\Lambda'(t_0, T) = 0$, then by (3.9)

$$e^{2Ks_0} = 2\mathrm{e}^{KT} - 1.$$

The proof is completed. $\square$

Corollary 3.3. *Let M be a Ricci-flat Riemannian manifold with a convex boundary, then we have*

$$\mathbb{E}\bigg(F^2\log\frac{F^2}{\|F\|^2_{L^2}}\bigg)\le T^2\int_{W_x^T(M)}\int_0^T|\nabla F|^2\mathrm{d}t\mathrm{d}\mathbb{P}_x.$$

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