Stochastic Heat Equations for infinite strings with Values in a Manifold*

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Abstract

In the paper, we construct conservative Markov processes corresponding to the martingale solutions to the stochastic heat equation on $\mathbb{R}^+$ or $\mathbb{R}$ with values in a general Riemannian maifold, which is only assumed to be complete and stochastic complete. This work is an extension of the previous paper [55] on finite volume case.

Moveover, we also obtain some functional inequalities associated to these Markov processes. This implies that on infinite volume case, the exponential ergodicity of the solution if the Ricci curvature is strictly positive and the non-ergodicity of the process if the sectional curvature is negative.

Keywords: Stochastic heat equation; Ricci Curvature; Functional inequality; Quasi-regular Dirichlet form; Infinite volume

*Supported in part by NSFC (11371099, 11501361, 11671035, 11771037). Financial support by the DFG through the CRC 1283 " Taming uncertainty and profiting from randomness and low regularity in analysis, stochastics and their applications" is acknowledged.

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1 Introduction

This work is the continuity of [55], which is motivated by Tadahisa Funaki's pioneering work [33] and Martin Hairer's recent work [42]. Let M be a n-dimensional compact Riemannian manifold. In [42] Hairer considered the stochastic heat equation associated to the energy

$$E(u)=\frac{1}{2}\int_{S^1} g_{u(x)}(\partial_x u(x),\partial_x u(x))\mathrm{d}x,$$

for smooth functions $u: S^1\to M$, and wrote the equation in the local coordinates formally:

$$\partial_t u^\alpha=\partial_x^2 u^\alpha+\Gamma^\alpha_{\beta\gamma}(u)\partial_x u^\beta\partial_x u^\gamma+\sigma_i^\alpha(u)\xi_i, \tag{1.1}$$

where Einsteins convention of summation over repeated indices is implied and $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols for the Levi-Civita connection of (M,g), σ_i are vector fields on M and ξ_i are independent space-time white noise. This equation may be also looked as certain kind of multi-component version of the KPZ equation. By the theory of regularity structure recently developed in [41, 10, 13], local well-posedness of (1.1) has been obtained in [42] (see more recent work [9]).

By the Andersson-Driver's approximation of Wiener measure in [6], we know that there exists an explicit relation between the Langevin energy $E(u)$ and Wiener (Brownian bridge) measure. In particular, it has been obtained in [6] that Wiener (Brownian bridge) measure μ on $C([0,1];M)$ could been interpreted as the limit of a natural approximation of the measure $\exp(-E(u))\mathscr{D}u$, where $\mathscr{D}u$ denotes a 'Lebesgue' like measure on path space. Based on the above connection, one may think the solution to the stochastic heat equation (1.1) may have μ as an invariant (even symmetrizing) measure.

In [55], starting from the Wiener measure (or Brownian bridge measure) μ on $C([0,1],M)$ we use the theory of Dirichlet forms to construct a natural evolution which admits μ as an invariant measure. Moreover, the relation between the evolution constructed in [55] and (1.1) has also been discussed in [55] by using the Andersson-Driver approximation. It is conjectured in [55] that the Markov processes constructed by Dirichlet form in [55] have the same law as the solution to (1.1). Since we consider the Wiener measure on $C([0,1],M)$ in [55], the evolution corresponds to the stochastic heat equation on $[0,1]$ for different boundary conditions with values in a compact Riemannian manifold. In the paper, we extend the results in [55] from finite volume $[0,1]$ to the half line $\mathbb{R}^+$ or the real line $\mathbb{R}$.

When $M=\mathbb{R}^n$ it is well-known that the law of Brownian motion on $C([0,\infty);\mathbb{R}^n)$ is an invariant measure of the following stochastic heat equatioin

$$\partial_t X=\frac{1}{2}\partial_x^2 X+\xi,\quad X(t,0)=0,$$

on $[0,\infty)\times[0,\infty)$. Here ξ is space-time white noise. By similar calculation as that in [35] we easily know that the distribution of a two-sided Brownian motion with a shift given by Lebesgue measure is invariant under the following stochastic heat equation

$$\partial_t X = \frac{1}{2}\partial_x^2 X + \xi,$$

on $[0,\infty)\times\mathbb{R}$. This suggests us to use the law of Brownian motion on $C([0,\infty);M)$ or the law of two sided Brownian motion on $C(\mathbb{R};M)$ to construct the corresponding stochastic heat equation on $\mathbb{R}^+$ or $\mathbb{R}$ with values in a Riemannian manifold.

Similarly as in [55], we construct the solution to stochastic heat equation by using the following L^2-Dirichlet form with the reference measure $\mu =$ the law of Brownian motion on M/the law of two sided Brownian motion on M:

$$\mathscr{E}(F,G) := \frac{1}{2}\int \langle DF, DG\rangle_{\mathbf{H}} \mathrm{d}\mu = \frac{1}{2}\sum_{k=1}^{\infty}\int D_{h_k}F D_{h_k}G \mathrm{d}\mu; \quad F,G\in\mathscr{F}C_b,$$

where $\{h_k\}_{k\geq 1}$ is an orthonormal basis in $\mathbf{H} := L^2(\mathbb{R}^+;\mathbb{R}^d)/L^2(\mathbb{R};\mathbb{R}^d)$, and $\mathscr{F}C_b$ and DF are the set of all cylinder functions and L^2-gradient respectively(refer to the definitions in Section 2). In this case, the associated Dirichlet-Form $\mathscr{E}$ is called $\mathbf{L^2}$**-Dirichlet form**.

For the half line case: we consider the reference measure as the law of Brownian motion for the half line $\mathbb{R}^+$ on Riemannian path space $C([0,\infty);M)$ and choose the state space as some weighted L^2-space (see Section 2). By using a general integration by parts formula from [16] (see also appendix) we can construct a martingale solution to the stochastic heat equation with values in a general Riemannian manifold, which is complete and stochastic complete.

For the whole line case: we first construct the two sided Brownian motion $\hat{x}$ on M with $\hat{x}(0)=o$ by an independent copy of Brownian motion on M. We consider the reference measure given by the law $\mu_{\mathbb{R}}^o$ of $\hat{x}$. By this we derive an integration by parts formula by using the stochastic horizontal lift for independent copy (see Proposition 3.2 for the reason we choose it in this way). We also emphasize that the L^2-Dirichlet form is independent of the stochastic horizontal lift (see Remark 2.1), which can be seen as a tool to obtain the integration by parts formula and the closablity of the associated bilinear form (see Remark 3.1). Moreover, we also consider the reference measure as $\mu_{\mathbb{R}}^\nu := \int \mu_{\mathbb{R}}^x \nu(\mathrm{d}x)$ with some Randon measure ν satisfying (3.19), which could be the volume measure on M under some mild curvature condition (see Remark 3.9 below). As mentioned before, the process corresponds to the stochastic heat equation on $\mathbb{R}$ without any boundary condition for ν given by the volume measure on M. Here we mainly concentrate on the more complicated case that ν and the reference measure have infinite mass. We use a cut-off technique to find suitable test functions and prove the quasi-regularity of the L^2-Dirichlet form (see Theorem 3.12), which gives a

Markov process as a martingale solution to the stochastic heat equation on $[0,\infty)\times\mathbb{R}$ with values in a Riemannian manifold. It is not easy to obtain that the process is conservative in this case, since 1 is not in the domain of the Dirichlet form. Under mild curvature condition we find suitable approximation functions in the domain of the L^2-Dirichlet form and obtain that the Markov process is conservative in the sense that the life time is infinity (see Theorem 3.13).

We also emphasize that the construction of the conservative Markov processes on general manifold with reference measure having infinite mass still holds for the finite volume case, especially for the free loop case with the reference measure $e^{c\int_0^1 \mathrm{Scal}(\gamma(s))\mathrm{d}s}\tilde{\mu}^\nu(\gamma)$, which is conjectured to be invariant measure for (1.1) in [9]. Here $c\in\mathbb{R}, \tilde{\mu}^\nu := \int \tilde{\mu}^x\nu(\mathrm{d}x)$, with $\nu = p_1(x,x)\mathrm{d}x$ and $\tilde{\mu}^x$ given by the Brownian bridge measure and p_t is the heat kernel for $\frac{1}{2}\Delta$ and Scal denotes the scalar curvature. For more details we refer to Remark 3.11 and Remark 3.14.

In the final part of this paper, we study functional inequalities associated to L^2-Dirichlet-Form, which implies the long time behavior of the solutions to the stochastic heat equations for infinite string. In this case, the L^2-Dirichlet form is not comparable with the O-U Dirichlet form constructed in [22], we refer readers to [1, 3, 4, 6, 12, 15, 16, 19, 22, 23, 24, 26, 34, 38, 31, 50, 52, 56, 59, 60, 62, 63, 64, 65] and references therein for various study about O-U Dirichlet form on path and loop space.

As we explained before, this case corresponds to SPDEs on infinite volume. The ergodicity property is different from that for the finite volume case (see [55]). For different manifolds we have ergodicity or non-ergodicity for the associated Markov processes. We establish the log-Sobolev inequality for the corresponding L^2-Dirichlet form if $\mathrm{Ric}\geq K>0$ for some constant K and Poincaré inequality for compact Riemannian manifold with some suitable curvature condition (see Theorem 4.1), which implies the L^2-exponential ergodicity in this case; When $M=\mathbb{R}^n$, ergodicity still holds but the Poincaré inequality does not hold in this case (see Theorem 4.3); When M is not a Liouville manifold, the associated Dirichlet form $\mathscr{E}$ is reducible, which means that the solutions to the stochastic heat equation are not ergodic.

Notations: In this paper we use C_c^m to denote C^m-differentiable functions with compact support. We use C_b^m to denote C^m-differentiable functions with bounded derivatives. For Hilbert space H we also use $|\cdot|_H$ to denote the norm of it.

The rest of this paper is as follow: In Section 2, we will construct the stochastic heat equation for the half line case on general Riemannian manifold M. The stochastic heat equation for the whole line will be established in Section 3, and the ergodicity or non-ergodicity property of the processes will be obtained in Section 4.

2 The case of half line $\mathbb{R}^+$

Throughout the article, suppose that M is a complete and stochastic complete Riemannian manifold with dimension n, and ρ is the Riemannian distance on M. In this section, we will construct the stochastic heat process on half line. We first introduce some notions. Fix $o \in M$, the path space over M is defined by

$$W^o_{\mathbb{R}^+}(M) := \{\gamma \in C([0,\infty); M) : \gamma(0) = o\}.$$

Then $W^o_{\mathbb{R}^+}(M)$ is a Polish (separable metric) space under the following uniform distance

$$d_\infty(\gamma,\sigma) := \sum_{k=1}^\infty \frac{1}{2^k} \sup_{t\in[0,k]} \Big(\rho(\gamma(t),\sigma(t)) \wedge 1\Big), \quad \gamma,\sigma \in W^o_{\mathbb{R}^+}(M).$$

In order to construct Dirichlet forms associated to stochastic heat equations for infinite strings on Riemannian path space, we also define the following weighted L^1-distance:

$$\tilde{d}(\gamma,\eta) := \sum_{k=1}^\infty \frac{1}{2^k} \int_{k-1}^k \tilde{\rho}(\gamma(s),\eta(s))\mathrm{d}s, \quad \gamma,\eta \in W^o_{\mathbb{R}^+}(M), \tag{2.1}$$

where $\tilde{\rho} = \rho \wedge 1$. Obviously we have $\tilde{d} \le d_\infty$. Let $\mathbf{E}^o_{\mathbb{R}^+}(M)$ be the closure of $W^o_{\mathbb{R}^+}(M)$ with respect to the distance $\tilde{d}$, then $\mathbf{E}^o_{\mathbb{R}^+}(M)$ is a Polish space.

Let $O(M)$ be the orthonormal frame bundle over M, we consider the following SDE,

$$\begin{cases} \mathrm{d}U_t = \sum_{i=1}^n H_i(U_t)\circ \mathrm{d}W^i_t, \quad t\ge 0 \\ U_0 = u_o, \end{cases} \tag{2.2}$$

where $\{H_i\}_{i=1}^n$ is a canonical orthonormal basis of horizontal vector fields $O(M)$, u_o is a fixed orthonormal basis of T_oM and $(W^i_t)_{t\ge0}$, $1\le i\le n$ is a standard $\mathbb{R}^n$-valued Brownian motion defined on a probability space $(\Omega,\mathscr{F},\mathbb{P})$. Note that M is stochastically complete, so U_t is well defined for all $t\ge 0$. Let $\pi : O(M)\to M$ denote the canonical projection, then $x_t := \pi(U_t),\ t\ge 0$ is the Brownian motion on M with initial point o, and $U_\cdot$ is the (stochastic) horizontal lift along $x_\cdot$. Let $\mu^o_{\mathbb{R}^+}$ be the law of $x_{[0,\infty)}$, then $\mu^o_{\mathbb{R}^+}$ is a probability measure on $W^o_{\mathbb{R}^+}(M)$, and the (stochastic) horizontal lift $(U_t(\gamma))_{t\in[0,\infty)}$ is well defined for $\mu^o_{\mathbb{R}^+}$-a.s. $\gamma \in W^o_{\mathbb{R}^+}(M)$, (whose distribution is the same as that of $(U_t)_{t\in[0,\infty)}$ under $\mathbb{P}$). Therefore $\mu^o_{\mathbb{R}^+}$ can be seen as a probability measure on $\mathbf{E}^o_{\mathbb{R}^+}(M)$ with support contained in $W^o_{\mathbb{R}^+}(M)$, and $(U_t(\gamma))_{t\in[0,\infty)}$ is also well defined for $\mu^o_{\mathbb{R}^+}$-a.s. $\gamma\in \mathbf{E}^o_{\mathbb{R}^+}(M)$.

Let $\mathscr{F}C^1_b$ be the space of C^1_b cylinder functions on $\mathbf{E}^o_{\mathbb{R}^+}(M)$ defined as follows: for every $F\in\mathscr{F}C^1_b$, there exist some $m\ge 1,\ m\in\mathbb{N}^+,\ f\in C^1_b(\mathbb{R}^m)$, $g_j \in C^{0,1}_b([0,\infty)\times M)$,

$T_j \in [0,\infty)$, $j=1,...,m$, such that

(2.3)

$$F(\gamma) = f\left(\int_0^{T_1} g_1(s,\gamma(s))\mathrm{d}s, \int_0^{T_2} g_2(s,\gamma(s))\mathrm{d}s, ..., \int_0^{T_m} g_m(s,\gamma(s))\mathrm{d}s\right), \quad \gamma \in \mathbf{E}^o_{\mathbb{R}^+}(M).$$

Here $C_b^{0,1}([0,\infty)\times M)$ denotes the bounded functions which are continuous w.r.t. the first variable and C_b^1- differentiable w.r.t. the second variable. It is easy to see that F is well defined for $\gamma \in \mathbf{E}^o_{\mathbb{R}^+}(M)$, $\mathscr{F}C_b^1$ is dense in $L^2(\mathbf{E}^o_{\mathbb{R}^+}(M);\mu^o_{\mathbb{R}^+}) = L^2(W^o_{\mathbb{R}^+}(M);\mu^o_{\mathbb{R}^+})$. For any $F \in \mathscr{F}C_b^1$ of the form (2.3) and $h \in \mathbf{H}_+ := L^2([0,\infty)\to\mathbb{R}^n;\mathrm{d}s) = \{h : [0,\infty)\to\mathbb{R}^n; \int_0^\infty |h(s)|^2\mathrm{d}s < \infty\}$, the directional derivative of F with respect to h is ($\mu^o_{\mathbb{R}^+}$-a.s.) defined by

$$D_hF(\gamma) = \sum_{j=1}^m \hat{\partial}_j f(\gamma)\int_0^{T_j} \left\langle U_s^{-1}(\gamma)\nabla g_j(s,\gamma(s)), h(s)\right\rangle \mathrm{d}s, \quad \gamma \in \mathbf{E}^o_{\mathbb{R}^+}(M),$$

where

$$\hat{\partial}_j f(\gamma) := \partial_j f\bigg(\int_0^{T_1} g_1(s,\gamma(s))\mathrm{d}s, \int_0^{T_2} g_2(s,\gamma(s))\mathrm{d}s, ..., \int_0^{T_m} g_m(s,\gamma(s))\mathrm{d}s\bigg).$$

and ∇g_j denotes the gradient w.r.t. the second variable. By the Riesz representation theorem, there exists a gradient operator $DF(\gamma) \in \mathbf{H}_+$ such that $\langle DF(\gamma), h\rangle_{\mathbf{H}_+} = D_hF(\gamma)$, $\mu^o_{\mathbb{R}^+}$-a.s.$\gamma \in \mathbf{E}^o_{\mathbb{R}^+}, h \in \mathbf{H}_+$. In particular, for $\gamma \in W^o_{\mathbb{R}^+}(M)$,

$$DF(\gamma)(s) = \sum_{j=1}^m \hat{\partial}_j f(\gamma) U_s^{-1}(\gamma)\nabla g_j(s,\gamma(s)) 1_{(0,T_j]}(s). \tag{2.4}$$

We define the (Cameron-Martin) subspace $\mathbb{H}_+^\infty$ of $\mathbf{H}_+$ as follows

$$\mathbb{H}_+^\infty := \left\{ h \in C_c^1([0,\infty);\mathbb{R}^d) \middle| h(0)=0, \int_0^\infty |h'(s)|^2\mathrm{d}s < \infty \right\}. \tag{2.5}$$

Fix a sequence of elements $\{h_k\}_{k=1}^\infty \subset \mathbb{H}_+^\infty$ such that it is an orthonormal basis in $\mathbf{H}_+$, we define the following symmetric quadratic form as follows

$$\mathscr{E}^o_{\mathbb{R}^+}(F,G) := \frac{1}{2}\int_{\mathbf{E}^o_{\mathbb{R}^+}(M)} \langle DF, DG\rangle_{\mathbf{H}_+}\mathrm{d}\mu^o_{\mathbb{R}^+}, \quad F,G \in \mathscr{F}C_b^1. \tag{2.6}$$

Then it is obvious that

$$\mathscr{E}^o_{\mathbb{R}^+}(F,G) = \frac{1}{2}\sum_{k=1}^\infty \int_{\mathbf{E}^o_{\mathbb{R}^+}(M)} D_{h_k}F D_{h_k}G \mathrm{d}\mu^o_{\mathbb{R}^+}; \quad F,G \in \mathscr{F}C_b^1.$$

Remark 2.1. *Although the stochastic horizontal lift $(U_t(\gamma))_{t\in[0,\infty)}$ is applied in the definition of $(\mathscr{E}^o_{\mathbb{R}+}, \mathscr{F}C_b^1)$, the value of $\mathscr{E}^o_{\mathbb{R}+}(F,F)$ is independent of $(U_t(\gamma))_{t\in[0,\infty)}$. In particular, by the definition (2.4) of the gradient, we have*

$$\mathscr{E}^o_{\mathbb{R}+}(F,G) = \frac{1}{2}\int_{\mathbf{E}^o_{\mathbb{R}+}(M)} \sum_{i=1}^m \sum_{j=1}^l \hat{\partial}_i f_1(\gamma)\hat{\partial}_j f_2(\gamma) \int_0^{T_i\wedge T_j} \langle \nabla g_i^1(s,\gamma(s)), \nabla g_j^2(s,\gamma(s))\rangle \mathrm{d}s\mathrm{d}\mu^o_{\mathbb{R}+}$$

for any $F, G \in \mathscr{F}C_b^1$ with

$$\begin{aligned} F(\gamma) &= f_1\left(\int_0^{T_1} g_1^1(s,\gamma(s))\mathrm{d}s, \int_0^{T_2} g_2^1(s,\gamma(s))\mathrm{d}s, ..., \int_0^{T_m} g_m^1(s,\gamma(s))\mathrm{d}s\right) \\ G(\gamma) &= f_2\left(\int_0^{T_1} g_1^2(s,\gamma(s))\mathrm{d}s, \int_0^{T_2} g_2^2(s,\gamma(s))\mathrm{d}s, ..., \int_0^{T_l} g_l^2(s,\gamma(s))\mathrm{d}s\right), \quad \gamma \in \mathbf{E}^o_{\mathbb{R}+}(M), \end{aligned}$$

for $f_1 \in C_b^1(\mathbb{R}^m)$, $f_2 \in C_b^1(\mathbb{R}^l)$, $g_j^i \in C_b^{0,1}([0,\infty)\times M)$ $i = 1, 2$, $j = 1, ..., m$. This implies the quadratic form $\mathscr{E}^o_{\mathbb{R}+}$ is independent of $(U_t(\gamma))_{t\in[0,\infty)}$.

Theorem 2.2. *The quadratic form $(\mathscr{E}^o_{\mathbb{R}+}, \mathscr{F}C_b^1)$ is closable and its closure $(\mathscr{E}^o_{\mathbb{R}+}, \mathscr{D}(\mathscr{E}^o_{\mathbb{R}+}))$ is a quasi-regular Dirichlet form on $L^2(\mathbf{E}^o_{\mathbb{R}+}(M); \mu^o_{\mathbb{R}+})$.*

By using the theory of Dirichlet form (refer to [51]), we obtain the following associated diffusion process.

Theorem 2.3. *There exists a conservative (Markov) diffusion process $\mathbf{M} = (\Omega, \mathscr{F}, \mathscr{M}_t, (X(t))_{t\geq 0}, (\mathbf{P}^z)_{z\in\mathbf{E}^o_{\mathbb{R}+}(M)})$ on $\mathbf{E}^o_{\mathbb{R}+}(M)$ having $\mu^o_{\mathbb{R}+}$ as an invariant measure and* properly associated with *$(\mathscr{E}^o_{\mathbb{R}+}, \mathscr{D}(\mathscr{E}^o_{\mathbb{R}+}))$, i.e. for $u \in L^2(\mathbf{E}^o_{\mathbb{R}+}(M); \mu^o_{\mathbb{R}+}) \cap \mathscr{B}_b(\mathbf{E}^o_{\mathbb{R}+}(M))$, the transition semigroup $P_t u(z) := \mathbb{E}^z[u(X(t))]$ is an $\mathscr{E}^o_{\mathbb{R}+}$-quasi-continuous version of $T_t u$ for all $t > 0$, where T_t is the semigroup associated with $(\mathscr{E}^o_{\mathbb{R}+}, \mathscr{D}(\mathscr{E}^o_{\mathbb{R}+}))$.*

Here for the notion of $\mathscr{E}^o_{\mathbb{R}+}$-quasi-continuity we refer to [51, Definition III-3.2]. By Fukushima decomposition we have

Theorem 2.4. *There exists a* properly $\mathscr{E}^o_{\mathbb{R}+}$-exceptional set *$S \subset \mathbf{E}^o_{\mathbb{R}+}(M)$, i.e. $\mu^o_{\mathbb{R}+}(S) = 0$ and $\mathbf{P}^z[X(t) \in \mathbf{E}^o_{\mathbb{R}+}(M) \setminus S, \forall t \geq 0] = 1$ for $z \in \mathbf{E}^o_{\mathbb{R}+}(M)\backslash S$, such that $\forall z \in \mathbf{E}^o_{\mathbb{R}+}(M)\backslash S$ under $\mathbf{P}^z$, the sample paths of the associated process $\mathbf{M} = (\Omega, \mathscr{F}, \mathscr{M}_t, (X(t))_{t\geq 0}, (\mathbf{P}^z)_{z\in\mathbf{E}^o_{\mathbb{R}+}(M)})$ on $\mathbf{E}^o_{\mathbb{R}+}(M)$ satisfy the following for $u \in \mathscr{D}(\mathscr{E}^o_{\mathbb{R}+})$*

$$u(X_t) - u(X_0) = M_t^u + N_t^u \quad \mathbf{P}^z - a.s., \tag{2.7}$$

where M^u is a martingale with quadratic variation process given by $\int_0^t |Du(X_s)|^2_{\mathbf{H}_+} ds$ and N_t is zero quadratic variation process. In particular, for $u \in D(L)$, $N_t^u = \int_0^t Lu(X_s)ds$, where L is the generator of $(\mathscr{E}^o_{\mathbb{R}+}, \mathscr{D}(\mathscr{E}^o_{\mathbb{R}+}))$.

Remark 2.5. *If we choose* $u(\gamma) = \int_{r_1}^{r_2} u^\alpha(\gamma(s))ds \in \mathscr{F}C_b^1$*, with* u^α *is a local coordinate on* M*, then the quadratic variation process for* M^u *is the same as that for the martingale part in (1.1).*

To prove Theorem 2.2, the crucial ingredient is the local integration by parts formula in [16]. To do that, we need to introduce some notations. In the following, we first introduce another cylinder functions set, every element in which only depends on finite times:

$$\hat{\mathscr{F}}C_b := \Big\{ W^o_{\mathbb{R}^+}(M) \ni \gamma \mapsto f(\gamma(t_1), \cdots, \gamma(t_m)) : \ m \geq 1, \\ 0 < t_1 < t_2 \cdots < t_m < \infty, f \in C_{b,Lip}(M^m) \Big\},$$

where $C_{b,Lip}(M^m)$ denotes the collection of bounded Lipschitz continuous functions on M^m.

For a fixed $o \in M$, since M is complete, there exists a C^∞ non-negative smooth function $\hat{\rho} : M \to \mathbb{R}$ with the property that $0 < |\nabla \hat{\rho}(z)| \leq 1$ and

$$\left| \hat{\rho}(z) - \frac{1}{2}\rho(o, z) \right| < 1, \quad z \in M.$$

For every non-negative m, define

$$D_m := \{z \in M : \hat{\rho}(z) < m\}, \quad \tau_m(\gamma) := \inf\{s \geq 0 : \ \gamma(s) \notin D_m\}. \tag{2.8}$$

We first introduce the following two results in [16] and [17], for convenience of readers we will give the proof of them in the Appendix

Lemma 2.6. [Chen-Li-Wu [16]] *For any* $m \in \mathbb{N}^+$ *and* $T \in \mathbb{R}^+$*, there exists a stochastic process(vector fields)* $l_{m,T} : [0, \infty) \times W^o_{\mathbb{R}^+}(M) \to [0, 1]$ *such that*

(1) $l_{m,T}(t, \gamma) = \begin{cases} 1, & t < \tau_{m-1}(\gamma) \wedge T \\ 0, & t > \tau_m(\gamma) \end{cases}$.

(2) Given any $o \in D_m$*,* $l_{m,T}(t, \gamma)$ *is* $\mathscr{F}_t^\gamma := \sigma\{\gamma(s); s \in [0, t]\}$*-adapted and* $l_{m,T}(\cdot, \gamma)$ *is absolutely continuous for* $\mu^o_{\mathbb{R}^+}$*-a.s.* $\gamma \in W^o_{\mathbb{R}^+}(M)$*.*

(3) For any positive integers $k, p, m \in \mathbb{Z}_+$ *and* $t \in \mathbb{R}^+$*, we have*

$$\sup_{o \in D_m} \int_{W^o_{\mathbb{R}^+}(M)} \int_0^t |l'_{k,T}(s, \gamma)|^p \mathrm{d}s \mu^o_{\mathbb{R}^+}(\mathrm{d}\gamma) \leq C_1(m, k, p, T) \tag{2.9}$$

for some positive constant $C_1(m, k, p, T)$ *(which may depends on* m*,* T*,* p *and* k*).*

Lemma 2.7. [Chen-Li-Wu [17]] *Let $l_{m,T}$ be the cut-off process constructed in Lemma 2.6, then for every $F \in \hat{\mathscr{F}}C_b$, $m \in \mathbb{Z}^+$, $T \in \mathbb{R}^+$, $h \in \mathbb{H}_+^\infty$ (see (2.5)), the following integration by parts formula holds*

$$\begin{aligned}(2.10)\qquad &\int_{W^o_{\mathbb{R}^+}(M)} (\mathrm{d}F(U.l_{m,T}(\cdot)h(\cdot)))\, \mu^o_{\mathbb{R}^+}(\mathrm{d}\gamma)\\ &= \int_{W^o_{\mathbb{R}^+}(M)} \left(F \int_0^\infty \left\langle (l_{m,T}h)'(s) + \frac{1}{2}\mathrm{Ric}_{U_s}\left(l_{m,T}(s)h(s)\right), \mathrm{d}\beta_s \right\rangle\right) \mu^o_{\mathbb{R}^+}(\mathrm{d}\gamma),\end{aligned}$$

where β_t denotes the anti-development of $\gamma(\cdot)$, whose distribution is a standard $\mathbb{R}^n$-valued Brownian motion under $\mu^o_{\mathbb{R}^+}$.

Based on the above Lemma 2.7, and using an approximation procedure, it is not difficult to obtain the following integration by parts formula.

Lemma 2.8. *Let $l_{m,T}$ be mentioned in Lemma 2.7, then for every $F \in \mathscr{F}C_b^1$, $m \in \mathbb{Z}_+$, $T \in \mathbb{R}^+$, $h \in \mathbb{H}_+^\infty$, the following integration by parts formula holds*

$$\begin{aligned}(2.11)\qquad &\int_{\mathbf{E}^o_{\mathbb{R}^+}(M)} (\mathrm{d}F(U.l_{m,T}(\cdot)h(\cdot)))\, \mu^o_{\mathbb{R}^+}(\mathrm{d}\gamma)\\ &= \int_{\mathbf{E}^o_{\mathbb{R}^+}(M)} \left(F \int_0^\infty \left\langle (l_{m,T}h)'(s) + \frac{1}{2}\mathrm{Ric}_{U_s}\left(l_{m,T}(s)h(s)\right), \mathrm{d}\beta_s \right\rangle\right) \mu^o_{\mathbb{R}^+}(\mathrm{d}\gamma),\end{aligned}$$

where β_t denotes the anti-development of $\gamma(\cdot)$, which is a Brownian motion under $\mu^o_{\mathbb{R}^+}$. Here $\mu^o_{\mathbb{R}^+}$ can be seen as a probability measure on $\mathbf{E}^o_{\mathbb{R}^+}(M)$ with support contained in $W^o_{\mathbb{R}^+}(M)$, and $l_{m,T}(t,\gamma)$ is also well defined for $\mu^o_{\mathbb{R}^+}$-a.s. $\gamma \in \mathbf{E}^o_{\mathbb{R}^+}(M)$.

Proof. In fact, it suffices to check the result holds for $F(\gamma) = f(\int_0^t g(s,\gamma(s))\mathrm{d}s) \in \mathscr{F}C_b^1$ with arbitrarily pre-fixed $t \in \mathbb{R}^+$, and the general case can be handled similarly. For any $k \geq 1$, defining

$$F_k(\gamma) = f\left(\frac{1}{k}\sum_{i=1}^{[kt]} g(i/k, \gamma(i/k))\right).$$

Fix a time $T > t > 0$, then

$$\begin{aligned}(2.12)\qquad &\int_{\mathbf{E}^o_{\mathbb{R}^+}(M)} (\mathrm{d}F_k)(U.l_{m,T}(\cdot)h(\cdot))\, \mu^o_{\mathbb{R}^+}(\mathrm{d}\gamma)\\ &= \int_{\mathbf{E}^o_{\mathbb{R}^+}(M)} \left(F_k \int_0^\infty \left\langle (l_{m,T}h)'(s) + \frac{1}{2}\mathrm{Ric}_{U_s}\left(l_{m,T}(s)h(s)\right), \mathrm{d}\beta_s \right\rangle\right) \mu^o_{\mathbb{R}^+}(\mathrm{d}\gamma),\end{aligned}$$

where we used $\mathrm{supp}(\mu^o_{\mathbb{R}^+}) \subset W^o_{\mathbb{R}^+}(M)$ and (2.10).

By the dominated convergence theorem, it is easy to see that $F_k \to F$ in $L^2(\mathbf{E}^o_{\mathbb{R}^+}(M);\mu^o_{\mathbb{R}^+})$ as $k\to\infty$. According to the definition of directional derivative, we have

$$\mathrm{d}F(U.l_{m,T}(\cdot)h(\cdot)) = \langle DF, l_{m,T}h\rangle_{\mathbf{H}_+} = \hat{\partial}f(\gamma)\int_0^t \left\langle U_s^{-1}(\gamma)\nabla g(s,\gamma(s)),(l_{m,T}h)(s)\right\rangle_{\mathbb{R}^d}\mathrm{d}s$$

$$\mathrm{d}F_k(U.l_{m,T}(\cdot)h(\cdot)) = \langle DF_k, l_{m,T}h\rangle_{\mathbf{H}_+} = \frac{1}{k}\hat{\partial}f_k(\gamma)\sum_{i=1}^{[kt]}\left\langle U_{i/k}^{-1}(\gamma)\nabla g(i/k,\gamma(i/k)),(l_{m,T}h)(i/k)\right\rangle_{\mathbb{R}^n},$$

with $\hat{\partial}f = f'\left(\int_0^t g(s,\gamma(s))\mathrm{d}s\right)$ and $\hat{\partial}f_k = f'\left(\frac{1}{k}\sum_{i=1}^k g(i/k,\gamma(i/k))\right)$. By our assumptions for f and g (especially ∇g is bounded) we know that

$$\langle DF_k, l_{m,T}h\rangle_{\mathbf{H}_+} \to \langle DF, l_{m,T}h\rangle_{\mathbf{H}_+} \text{ in } L^2(\mathbf{E}^o_{\mathbb{R}^+}(M);\mu^o_{\mathbb{R}^+}),\quad k\to\infty.$$

By using the above argument, we get (2.11) by taking $k\to\infty$ on both sides of the equation (2.12) .

□

In the following we prove Theorem 2.2 by using the above integration by parts formula.

Proof of Theorem 2.2. (a) **Closablity:** Let $\{F_m\}_{m=1}^\infty \subseteq \mathscr{F}C_b^1$ be a sequence of cylinder functions with

$$\lim_{m\to\infty}\mu^o_{\mathbb{R}^+}\left(F_m^2\right)=0,\quad \lim_{k,m\to\infty}\mathscr{E}^o_{\mathbb{R}^+}\left(F_k-F_m,F_k-F_m\right)=0. \tag{2.13}$$

Thus $\{DF_m\}_{m=1}^\infty$ is a Cauchy sequence in $L^2\left(\mathbf{E}^o_{\mathbb{R}^+}(M)\to\mathbf{H}_+;\mu^o_{\mathbb{R}^+}\right)$, for which there exists a limit Φ. It only suffices to prove that $\Phi=0$. Suppose that $\{h_i\}_{i=1}^\infty\subset\mathbb{H}_+^\infty\cap C_c^1([0,\infty);\mathbb{R}^n)$ is an orthonormal basis of $\mathbf{H}_+$. By Lemma 2.8, for each $G\in\mathscr{F}C_b^1$ and any positive integers $k,m,i\geq 1$, we have

$$\begin{aligned}
&\mu^o_{\mathbb{R}^+}\left(\langle DF_k, l_{m,T}h_i\rangle_{\mathbf{H}_+}G\right)\\
&=\mu^o_{\mathbb{R}^+}\left(\langle D\left(F_kG\right), l_{m,T}h_i\rangle_{\mathbf{H}_+}\right)-\mu^o_{\mathbb{R}^+}\left(\langle DG, l_{m,T}h_i\rangle_{\mathbf{H}_+}F_k\right)\\
&=\mu^o_{\mathbb{R}^+}\left(F_kG\int_0^\infty\left\langle (l_{m,T}h_i)'(s)+\frac{1}{2}\mathrm{Ric}_{U_s}\left(l_{m,T}(s)h_i(s)\right),\mathrm{d}\beta_s\right\rangle\right)\\
&\quad-\mu^o_{\mathbb{R}^+}\left(\langle DG, l_{m,T}h_i\rangle_{\mathbf{H}_+}F_k\right).
\end{aligned}\tag{2.14}$$

In particular, for each $h_i\in C_c^1([0,\infty);\mathbb{R}^n)$, by (2.9) and the compact property of $\mathrm{supp}(h_i)$, we have

$$\int_0^\infty\left\langle (l_{m,T}h_i)'(s)+\frac{1}{2}\mathrm{Ric}_{U_s}\left(l_{m,T}(s)h_i(s)\right),\mathrm{d}\beta_s\right\rangle\in L^2(\mathbf{E}^o_{\mathbb{R}^+}(M);\mu^o_{\mathbb{R}^+}).$$

Since G and DG are bounded, and $F_k \to 0$, $|DF_k - \Phi|_{\mathbf{H}_+} \to 0$ in $L^2(\mathbf{E}^o_{\mathbb{R}+}(M); \mu^o_{\mathbb{R}+})$, we let $k \to \infty$ in (2.14) and obtain that for every $m, T, i \in \mathbb{N}^+$,

$$\mu^o_{\mathbb{R}+}\big(\langle \Phi, l_{m,T} h_i\rangle_{\mathbf{H}_+} G\big) = 0, \quad \forall\, G \in \mathscr{F}C_b^1.$$

Therefore we could find a $\mu^o_{\mathbb{R}+}$-null set $\Delta_i \subset \mathbf{E}^o_{\mathbb{R}+}(M)$, such that

$$\langle \Phi(\gamma), l_{m,T}(\gamma) h_i\rangle_{\mathbf{H}_+} = 0, \quad \forall\, m, T \in \mathbb{N}^+,\ \gamma \notin \Delta_i. \tag{2.15}$$

For a fixed $h_i \in \mathbb{H}^\infty_+$, there exists a positive integer $T_i \in \mathbb{N}^+$ (which may depend on h_i) such that $\mathrm{supp}(h_i) \subset [0, T_i]$. Since $\gamma(\cdot)$ is non-explosive, there is a $\mu^o_{\mathbb{R}+}$-null set $\Delta_0 \subset \mathbf{E}^o_{\mathbb{R}+}(M)$ such that for every $\gamma \notin \Delta_0$, there exists $m_i(\gamma) \in \mathbb{N}^+$ satisfying

$$\gamma(t) \in D_{m_i - 1}, \quad \text{for all } \ t \in [0, T_i],$$

where D_{m_i-1} is introduced by (2.8). Hence $l_{m_i,T_i}(t,\gamma) = 1$ for all $t \in [0, T_i]$. Combining this with (2.15) we know

$$\langle \Phi(\gamma), h_i\rangle_{\mathbf{H}_+} = 0, \quad i \geq 1, \gamma \notin \Delta_i \cup \Delta_0,$$

which implies that $\Phi(\gamma) = 0$, $\forall\, \gamma \notin \Delta = \cup_{i=0}^\infty \Delta_i$. So $\Phi = 0$, $\mu^o_{\mathbb{R}+}$-a.s., and $(\mathscr{E}^o_{\mathbb{R}+}, \mathscr{F}C_b^1)$ is closable. By the standard method, we show easily that its closure $(\mathscr{E}^o_{\mathbb{R}+}, \mathscr{D}(\mathscr{E}^o_{\mathbb{R}+}))$ is a Dirichlet form.

(*b*) **Quasi-Regularity:** In order to prove the quasi-regularity of $(\mathscr{E}^o_{\mathbb{R}+}, \mathscr{D}(\mathscr{E}^o_{\mathbb{R}+}))$, we need to verify conditions (i)-(iii) in [51, Definition IV-3.1].

It is easy to see that each $G \in \mathscr{F}C_b^1$ is continuous in (Polish space) $(\mathbf{E}^o_{\mathbb{R}+}(M), \tilde{d})$, and $\mathscr{F}C_b^1$ is dense in $\mathscr{D}(\mathscr{E}^o_{\mathbb{R}+})$ under the $(\mathscr{E}^o_{\mathbb{R}+,1})^{1/2}$-norm with

$$\mathscr{E}^o_{\mathbb{R}+,1}(\cdot,\cdot) := \mathscr{E}^o_{\mathbb{R}+}(\cdot,\cdot) + \|\cdot\|^2_{L^2(\mathbf{E}^o_{\mathbb{R}+}(M), \mu^o_{\mathbb{R}+})}.$$

So (ii) of [51, Definition IV-3.1] holds.

Since the metric space $(\mathbf{E}^o_{\mathbb{R}+}(M), \tilde{d})$ is separable, we can choose a fixed countable dense subset $\{\xi_m | m \in \mathbb{N}^+\} \subset W^o_{\mathbb{R}+}(M)$. Next, we prove the tightness of the capacity for $(\mathscr{E}^o_{\mathbb{R}+}, \mathscr{D}(\mathscr{E}^o_{\mathbb{R}+}))$ which ensures (i) of [51, Definition IV-3.1].

Let $\varphi \in C_b^\infty(\mathbb{R})$ be an increasing function satisfying with

$$\varphi(t) = t, \quad \forall\, t \in [-1, 1] \ \text{ and } \ \|\varphi'\|_\infty \leq 1.$$

For each $m \geq 1$, the function $v_m : \mathbf{E}^o_{\mathbb{R}+}(M) \to \mathbb{R}$ is given by

$$v_m(\gamma) = \varphi(\tilde{d}(\gamma, \xi_m)), \quad \gamma \in \mathbf{E}^o_{\mathbb{R}+}(M),$$

with $\tilde{d}$ defined in (2.1). By Lemma 2.9 below $v_m \in \mathscr{D}(\mathscr{E}^o_{\mathbb{R}+})$. We claim that

$$w_k := \inf_{m \leq k} v_m, k \in \mathbb{N}^+, \text{ converges } \mathscr{E}^o_{\mathbb{R}+} - \text{quasi-uniformly to zero on } \mathbf{E}^o_{\mathbb{R}+}(M). \tag{2.16}$$

Then for every $i \in \mathbb{N}^+$ there exists a closed set K_i such that $\mathrm{Cap}(K_i^c) < \frac{1}{i}$ and $w_k \to 0$ uniformly on K_i as $k \to \infty$. Here Cap is the capacity associated to $(\mathscr{E}^o_{\mathbb{R}^+}, \mathscr{D}(\mathscr{E}^o_{\mathbb{R}^+}))$ (see [51, Section III.2]). Hence for every $0 < \varepsilon < 1$ there exists $k \in \mathbb{N}^+$ such that $w_k < \varepsilon$ on K_i, by using the definitions of v_m and w_k, we obtain that $K_i \subset \cup_{m=1}^k B(\xi_m, \varepsilon)$, where $B(\xi_m, \varepsilon) := \{\gamma \in \mathbf{E}^o_{\mathbb{R}^+}(M); \tilde{d}(\xi_m, \gamma) < \varepsilon\}$. Consequently, for every $i \geq 1$, K_i is totally bounded, hence compact. Combining this with the fact $\lim_{i\to\infty} \mathrm{Cap}(K_i^c) = 0$ we know the capacity for $(\mathscr{E}^o_{\mathbb{R}^+}, \mathscr{D}(\mathscr{E}^o_{\mathbb{R}^+}))$ is tight.

Now it only remains to show the claim (2.16). For each fixed $m \geq 1$, by (2.19) in Lemma 2.9 below we obtain

$$Dv_m(\gamma)(s) = \varphi'(\tilde{d}(\gamma, \xi_m)) \cdot \Big(\sum_{k=1}^{\infty} \frac{1}{2^k} U_s^{-1} \nabla_1 \tilde{\rho}(\gamma(s), \xi_m(s)) 1_{(k-1,k]}(s) \Big),$$

where $\nabla_1 \tilde{\rho}$ is the gradient of $\tilde{\rho}$ with respect to the first variable. By the definition (2.6) of the quadratic form $\mathscr{E}^o_{\mathbb{R}^+}$, we have

(2.17)

$$\begin{aligned}
\mathscr{E}^o_{\mathbb{R}^+}(v_m, v_m) &= \frac{1}{2} \int_{\mathbf{E}^o_{\mathbb{R}^+}(M)} \big| Dv_m(\gamma) \big|^2_{\mathbf{H}_+} \mathrm{d}\mu^o_{\mathbb{R}^+}(\gamma) \\
&= \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{2^{2k}} \int_{\mathbf{E}^o_{\mathbb{R}^+}(M)} |\varphi'(\tilde{d}(\gamma, \xi_m))|^2 \cdot \Big(\int_{k-1}^{k} \big| \nabla_1 \tilde{\rho}(\gamma(s), \xi_m(s)) \big|^2_{T_{\gamma(s)}M} \mathrm{d}s \Big) \mathrm{d}\mu^o_{\mathbb{R}^+}(\gamma) \\
&\leq \|\varphi'\|_\infty \cdot \Big(\sum_{k=1}^{\infty} \frac{1}{2^{2k+1}} \Big) \leq C, \ \forall\, m \in \mathbb{N}^+,
\end{aligned}$$

where $C > 0$ is a constant independent of m, and in the first inequality above we applied the property that $|\nabla_1 \rho| \leq 1$.

Since $\{\xi_m | m \in \mathbb{N}\}$ is dense in $(\mathbf{E}^o_{\mathbb{R}^+}(M); \tilde{d})$, it is easy to verify that $w_k \downarrow 0$ $\mu^o_{\mathbb{R}^+}$-a.s. on $\mathbf{E}^o_{\mathbb{R}^+}(M)$ hence in $L^2(\mathbf{E}^o_{\mathbb{R}^+}(M); \mu^o_{\mathbb{R}^+})$. By (2.17) we arrive at

$$\mathscr{E}^o_{\mathbb{R}^+}(w_k, w_k) \leq C, \quad \forall k \in \mathbb{N}^+,$$

where C is independent of k.

Based on this and [51, I.2.12, III.3.5] we obtain that a subsequence of the Cesaro mean of some subsequence of w_k converges to zero $\mathscr{E}^o_{\mathbb{R}^+}$-quasi-uniformly. But since $\{w_k\}_{k\in\mathbb{N}^+}$ is decreasing, (2.16) follows. Now tightness in (i) of [51, Definition IV-3.1] follows.

For any $\gamma, \eta \in \mathbf{E}^o_{\mathbb{R}^+}(M)$ with $\varepsilon := \tilde{d}(\gamma, \eta) > 0$, there exists certain ξ_N such that $\tilde{d}(\xi_N, \eta) < \frac{\varepsilon}{4}$ and $\tilde{d}(\xi_N, \gamma) > \frac{\varepsilon}{4}$. Take $\{F_m(\gamma) := \varphi(\tilde{d}(\xi_m, \gamma)), m \in \mathbb{N}\}$ for φ as above, (iii) of [51, Definition IV-3.1] follows.

□

For a locally Lipschitz continuous function $g : M \to \mathbb{R}$, by Radamacher's theorem, it is well known that the gradient $\nabla g(x)$ of g exists for all $x \in M/S$ with some Lebesgue null set $S \subset M$. For convenience, let us define $\nabla g(x) = 0$ for any $x \in S$. Also note that $\mu^o_{\mathbb{R}^+}\big(\gamma(s) \in S\big) = 0$ for each $s > 0$, hence $\nabla g(\gamma(s))$ is $\mu^o_{\mathbb{R}^+}$-a.s. well defined for every $s > 0$.

Let $C_{b,Lip}([0,\infty)\times M)$ be the set of all functions f on the product space $[0,\infty)\times M$ and each function $g(t,x)$ is bounded and continuous with respect to the first variable $t \in [0,\infty)$, and uniformly Lipschitz continuous with respect to the second variable $x \in M$.

Lemma 2.9. *(1) For each fixed function $F(\gamma) := f(\int_0^t g(s,\gamma(s))\mathrm{d}s)$ with some fixed $t > 0, g \in C_{b,Lip}([0,\infty)\times M)$ and $f \in C_b^1(\mathbb{R})$. Then $F \in \mathscr{D}(\mathscr{E}^o_{\mathbb{R}^+})$ and we have*

$$DF(\gamma)(s) = f'\left(\int_0^t g(r,\gamma(r))\mathrm{d}r\right)\cdot\Big(U_s^{-1}(\gamma)\nabla g(s,\gamma(s))1_{(0,t]}(s)\Big) \tag{2.18}$$

for $\mathrm{d}s \times \mu^o_{\mathbb{R}^+} - a.s.(s,\gamma) \in [0,\infty)\times \mathbf{E}^o_{\mathbb{R}^+}(M)$.

(2) For a fixed $\sigma \in W^o_{\mathbb{R}^+}(M)$, let $G(\gamma) := f(\tilde{d}(\gamma,\sigma))$ with $f \in C_b^1(\mathbb{R})$ and $\tilde{d}$ defined by (2.1). Then $G \in \mathscr{D}(\mathscr{E}^o_{\mathbb{R}^+})$ and we have

$$DG(\gamma) = f'(\tilde{d}(\gamma,\sigma))\cdot\Big(\sum_{k=1}^{\infty}\frac{1}{2^k}U_s^{-1}(\gamma)\nabla_1\tilde{\rho}(\gamma(s),\sigma(s))1_{(k-1,k]}(s)\Big) \tag{2.19}$$

for $\mathrm{d}s \times \mu^o_{\mathbb{R}^+} - a.s.(s,\gamma) \in [0,\infty)\times \mathbf{E}^o_{\mathbb{R}^+}(M)$, where $\nabla_1\tilde{\rho}(\cdot,x)$ denotes the gradient with respect to the first variable of $\tilde{\rho}(\cdot,\cdot)$.

Proof. **Step (i)** First we suppose that $g \in C_{b,Lip}([0,\infty)\times M)$ and there exist a constant $L \in \mathbb{R}^+$ and a compact set $K \subset M$ such that

$$\begin{aligned}&|g(s,x) - g(s,y)| \le L\rho(x,y), \quad \forall x,y \in M, s \in [0,\infty)\\ &\text{and } \mathrm{supp}(g(s,\cdot)) \subset K, \quad \forall s \in [0,\infty).\end{aligned}$$

Consider a local coordinate system $\{U,\varphi_U\}$ on M, i.e. for any $x \in M$, there exists a (bounded) neighborhood U of x and a C^∞ diffeomorphism $\varphi_U : U \to V$, where V is a (bounded open) subset in $\mathbb{R}^n$. Without loss of generality, we may assume that $\mathrm{supp}g \subset [0,\infty)\times K$. According to the unit decomposition theorem on manifold, there exist $N \in \mathbb{N}^+$, $U_i \in \{U,\varphi_U\}, 1 \le i \le N$, and non-negative smooth functions $\alpha_i, 1 \le i \le N$ such that

$$\sum_{i=1}^N \alpha_i\Big|_K \equiv 1,\ K \subset \bigcup_{i=1}^N U_i \text{ and } \mathrm{supp}(\alpha_i) \subset U_i, 1 \le i \le N. \tag{2.20}$$

Define $g_i := g\alpha_i$ for each $1 \le i \le N$. Then, from (2.20), we know $\mathrm{supp}(g_i) \subset [0,\infty)\times U_i$. Let $V_i := \varphi_{U_i}(U_i) \subset \mathbb{R}^n$ and $\tilde{g}_i : [0,\infty)\times V_i \to \mathbb{R}$ denoted by $\tilde{g}_i(s,y) :=$

$g_i(s,\varphi_{U_i}^{-1}(y))$ for $s\in[0,\infty), y\in V_i$. We can easily check that $\tilde g_i(s,\cdot)$ is Lipschitz continuous with support contained in V_i for all $s\in[0,\infty)$.

Let $\phi\in C_c^\infty(\mathbb{R}^n)$ be a polishing function satisfying that $\mathrm{supp}(\phi)\subset B_1(0)$ and $\int_{\mathbb{R}^n}\phi(x)\mathrm{d}x=1$, where $B_1(0)$ is the 0-centered unit ball in $\mathbb{R}^n$. Note that $\mathrm{supp}(\tilde g_i(s,\cdot))\subset V_i$, then for each $1\le i\le N$, there exists a constant $\varepsilon_i>0$ such that for every $\varepsilon\in(0,\varepsilon_i)$, the following $\tilde g_i^\varepsilon(s,\cdot)$ is well defined on V_i,

$$\tilde g_i^\varepsilon(s,u):=\tilde g_i*\phi_\varepsilon(s,u)=\int_{\mathbb{R}^d}\tilde g_i(s,v)\phi_\varepsilon(u-v)\mathrm{d}v,\quad\forall\ (s,u)\in[0,\infty)\times V_i,$$

and $\mathrm{supp}\tilde g_i^\varepsilon(s,\cdot)\subset V_i$, where $\phi_\varepsilon(u):=\varepsilon^{-n}\phi\big(\frac{u}{\varepsilon}\big)$. It is easy to verify

$$\lim_{\varepsilon\downarrow0}\sup_{v\in V_i}|\tilde g_i^\varepsilon(s,v)-\tilde g_i(s,v)|=0,\quad s\in[0,\infty). \tag{2.21}$$

Since the Lipschitz constant of $\tilde g_i(s,\cdot)$ is independent of s, we also have for any $p>0$,

$$\begin{aligned}&\sup_{\varepsilon\in(0,\varepsilon_i),v\in V_i,s\in[0,\infty)}|\nabla\tilde g_i^\varepsilon(s,v)|\le C_i,\quad 1\le i\le N,\\&\lim_{\varepsilon\downarrow0}\int_{V_i}|\nabla\tilde g_i^\varepsilon(s,v)-\nabla\tilde g_i(s,v)|^p\mathrm{d}v=0,\ \forall\ s\in[0,\infty)\end{aligned}\tag{2.22}$$

for some constants $C_i>0,\quad 1\le i\le N$, where ∇ is the gradient w.r.t. the second variable.

Define $g_i^\varepsilon:=\tilde g_i^\varepsilon\circ\varphi_{U_i}$, and we extend g_i^ε to the whole product space $[0,\infty)\times M$ by letting $g_i^\varepsilon|_{[0,\infty)\times U_i^c}=0$. Since $\mathrm{supp}(\tilde g_i^\varepsilon)\subset[0,\infty)\times V_i$ for all $\varepsilon\in(0,\varepsilon_i)$ implies that $\mathrm{supp}g_i^\varepsilon\subset[0,\infty)\times U_i$ for every $\varepsilon\in(0,\varepsilon_i)$, it is not difficult to see $g_i^\varepsilon\in C_b^{0,1}([0,\infty)\times M)$. Taking $\varepsilon_0:=\inf_{1\le i\le N}\varepsilon_i$, then for every $\varepsilon\in(0,\varepsilon_0)$ we could define $g^\varepsilon:=\sum_{i=1}^N g_i^\varepsilon$. By (2.20), (2.21) and (2.22) we know for all $p>0$,

$$\begin{aligned}&\lim_{\varepsilon\downarrow0}\sup_{y\in M}|g^\varepsilon(s,y)-g(s,y)|=0,\quad s\in[0,\infty),\\&\sup_{\varepsilon\in(0,\varepsilon_0),y\in M,s\in[0,\infty)}|\nabla g^\varepsilon(s,y)|\le C,\\&\lim_{\varepsilon\downarrow0}\int_M|\nabla g^\varepsilon(s,y)-\nabla g(s,y)|^p\mathrm{d}y=0,\quad s\in[0,\infty)\end{aligned}\tag{2.23}$$

for some constant $C>0$.

Define $F^\varepsilon(\gamma):=f(\int_0^t g^\varepsilon(s,\gamma(s))\mathrm{d}s)\in\mathscr{F}C_b^1$, then from (2.4) it is easy to obtain

$$DF^\varepsilon(\gamma)(s)=f'\left(\int_0^t g^\varepsilon(r,\gamma(r))\mathrm{d}s\right)\cdot\Big(U_s^{-1}(\gamma)\nabla g^\varepsilon(s,\gamma(s))1_{(0,t]}(s)\Big),\quad s\in[0,\infty).$$

Combining this and (2.23) we have

$$\sup_{\varepsilon\in(0,\varepsilon_0)}\mathscr{E}(F^\varepsilon,F^\varepsilon)<\infty,$$

and

$$\lim_{\varepsilon\downarrow 0}\mu^o_{\mathbb{R}^+}\Big(\big|F^\varepsilon(\gamma)-F(\gamma)\big|^2\Big)=0.$$

By [51, Chap. I Lemma 2.12] we know that $F\in\mathscr{D}(\mathscr{E}^o_{\mathbb{R}^+})$. Moreover, (2.23) ensures

$$\lim_{\varepsilon\downarrow 0}DF^\varepsilon(\gamma)(s)=f'\left(\int_0^t g(r,\gamma(r))\mathrm{d}s\right)\cdot\Big(U_s^{-1}(\gamma)\nabla g(s,\gamma(s))1_{(0,t]}(s)\Big)$$

for $\mathrm{d}s\times\mu^o_{\mathbb{R}^+}$-a.s. $(s,\gamma)\in[0,\infty)\times\mathbf{E}^o_{\mathbb{R}^+}(M)$. Combining this with the dominated convergence theorem yields

$$\lim_{\varepsilon\downarrow 0}\int_{\mathbf{E}^o_{\mathbb{R}^+}(M)}\int_0^\infty\big|DF^\varepsilon(\gamma)(s)-\lim_{\varepsilon\downarrow 0}DF^\varepsilon(\gamma)(s)\big|^2\mathrm{d}s\mathrm{d}\mu^o_{\mathbb{R}^+}=0,$$

which implies (2.18) immediately.

Step (ii) Now we consider the general case : $g\in C_{b,Lip}([0,\infty)\times M)$. By the Greene-Wu approximation theorem in [39], there exists a smooth function $\eta:M\to\mathbb{R}^+$ such that for every $R>0$, $\{x\in M;\eta(x)\le R\}$ is compact and $\sup_{x\in M}|\nabla\eta(x)|\le C$. Choose $h_R:\mathbb{R}^+\to[0,1]$, $h_R\in C^\infty(\mathbb{R}^+)$ with

$$h_R(x)=1,\forall\ x\in[0,R],\ \ h_R(x)=0,\forall\ x>R+1,\quad\text{and}\quad\|h_R'\|_\infty\le 2.$$

For each $(s,x)\in[0,\infty)\times M$, define $g_R(s,x):=g(s,x)h_R(\eta(x))$, $F_R(\gamma):=f(\int_0^t g_R(s,\gamma(s))\mathrm{d}s)$. Based on the fact that $\sup_{x\in M}|\nabla\eta(x)|\le C$ it is easy to verify that $g_R(s,\cdot):M\to\mathbb{R}$ is Lipschitz continuous and with uniform compact support and with uniform Lipschitz constant.

From **Step (i)** of the proof we know $F_R\in\mathscr{D}(\mathscr{E}^o_{\mathbb{R}^+})$ and it is not difficult to show

$$\begin{aligned}&\mathscr{E}^o_{\mathbb{R}^+}(F_R,F_R)\le C\|f'\|_\infty^2\|\nabla g_R\|_\infty^2\le C\|f'\|_\infty^2(\|\nabla g\|_\infty+\|g\|_\infty)^2,\\&\lim_{R\to\infty}\mu^o_{\mathbb{R}^+}\Big(\big|F_R(\gamma)-F(\gamma)\big|^2\Big)=0,\\&\lim_{R\to\infty}DF_R(\gamma)(s)=DF(\gamma)(s)\text{ for }\mathrm{d}s\times\mu^o_{\mathbb{R}^+}-a.s.(s,\gamma)\in[0,\infty)\times\mathbf{E}^o_{\mathbb{R}^+}(M).\end{aligned}$$

Combining this with the same arguments as in **Step (i)** we know $F\in\mathscr{D}(\mathscr{E}^o_{\mathbb{R}^+})$ with DF given by (2.18).

Step (iii) By similar arguments as above we can easily check that for F given as in (2.3) with g_i as in (1) the results in (1) follow. Let $G_N(\gamma):=f\Big(\tilde d_N\big(\gamma,\sigma\big)\Big)$, where $\tilde d_N\big(\gamma,\sigma\big):=\sum_{k=1}^N\frac{1}{2^k}\int_{k-1}^k\tilde\rho\big(\gamma(s),\sigma(s)\big)\mathrm{d}s$. Hence according to the conclusion in **Step (i),(ii)** we obtain $G_N\in\mathscr{D}(\mathscr{E}^o_{\mathbb{R}^+})$ and

$$DG_N(\gamma)(s)=f'\Big(\tilde d_N\big(\gamma,\sigma\big)\Big)\cdot\Big(\sum_{k=1}^N\frac{1}{2^k}U_s^{-1}(\gamma)\nabla_1\tilde\rho\big(\gamma(s),\sigma(s)\big)1_{(k-1,k]}(s)\Big)$$

for $\mathrm{d}s \times \mu^o_{\mathbb{R}+} - a.s.(s,\gamma) \in [0,\infty) \times \mathbf{E}^o_{\mathbb{R}+}(M)$. By this and the same arguments as in **Step (i)** (by the dominated convergence theorem) it is easy to prove

$$\lim_{N\to\infty} \mu^o_{\mathbb{R}+}\big(|G_N(\gamma) - G(\gamma)|^2\big) = 0,$$
$$\lim_{N\to\infty} \mu^o_{\mathbb{R}+}\big(|DG_N(\gamma) - DG(\gamma)|^2_{\mathbf{H}_+}\big) = 0,$$

which implies $G \in \mathscr{D}(\mathscr{E}^o_{\mathbb{R}+})$ and DG has the expression (2.19). □

Remark 2.10. (Finite Volume Case) *Let μ^o_T be the distribution of the Brownian motion starting from o on $C([0,T];M)$. Similar to the above argument, we can obtain Theorems 2.2-2.4 and Lemma 2.8 hold with $\mu^o_{\mathbb{R}+}$ be replaced by μ^o_T. These extend the results in [55, Section 2] to general Riemannian manifold.*

3 The case of whole line

Fix $o \in M$, the path space $W^o_{\mathbb{R}}(M)$ over M is defined by

$$W^o_{\mathbb{R}}(M) := \{\gamma \in C(\mathbb{R}; M) : \gamma(0) = o\}.$$

Then $W^o_{\mathbb{R}}(M)$ is a separable metric space with respect to the distance d_∞ as follows

$$d_\infty(\gamma,\sigma) := \sum_{n=1}^{\infty} \frac{1}{2^n} \sup_{s\in[-n,n]} \big(\tilde{\rho}(\gamma(s),\sigma(s))\big), \quad \gamma,\sigma \in W^o_{\mathbb{R}}(M). \tag{3.1}$$

where $\tilde{\rho} = \rho \wedge 1$. Similar as in Section 2, we define the following L^1-distance:

$$\tilde{d}(\gamma,\eta) := \sum_{k=1}^{\infty} \Big(\frac{1}{2^k}\int_{k-1}^{k} \tilde{\rho}(\gamma(s),\eta(s))\mathrm{d}s + \frac{1}{2^k}\int_{-k}^{-k+1} \tilde{\rho}(\gamma(s),\eta(s))\mathrm{d}s\Big), \quad \gamma,\eta \in W^o_{\mathbb{R}}(M). \tag{3.2}$$

Obviously we have $\tilde{d} \le 2d_\infty$. Let $\mathbf{E}^o_{\mathbb{R}}(M)$ be the closure of $W^o_{\mathbb{R}}(M)$ with respect to the distance $\tilde{d}$, then $\mathbf{E}^o_{\mathbb{R}}(M)$ is a Polish space.

Let $\bar{W}$ be an n-dimensional Brownian motion independent of W and let $\bar{U}$ be the solution to (2.2) with W replaced by $\bar{W}$. Set $\bar{x}_t := \pi(\bar{U})$. Then $\bar{x}_\cdot$ is a Brownian motion with initial point o on M , and independent of x. Define

$$\hat{x}_t := \begin{cases} x_t, & t \ge 0 \\ \bar{x}_{-t}, & t < 0 \end{cases}.$$

Denote by $\mu^o_{\mathbb{R}}$ the distribution of $\hat{x}$ on $W^o_{\mathbb{R}}(M)$, then $\mu^o_{\mathbb{R}}$ is also a probability measure on $\mathbf{E}^o_{\mathbb{R}}(M)$ whose support is contained in $W^o_{\mathbb{R}}(M)$. Moreover, we can easily check that $\mu^o_{\mathbb{R}}$

is the unique probability measure such that for $F(\gamma) = f\big(\gamma(-\bar{t}_k), ..., \gamma(-\bar{t}_1), \gamma(t_1), ..., \gamma(t_m)\big)$, $f \in C_b(M^{k+m})$,

$$\int_{\mathbf{E}^o_{\mathbb{R}}(M)} F(\gamma)\mathrm{d}\mu^o_{\mathbb{R}} = \int_{M^{k+m}} \prod_{i=1}^k p_{\Delta_i \bar{t}}(\bar{y}_{i-1}, \bar{y}_i) \prod_{i=1}^m p_{\Delta_i t}(y_{i-1}, y_i)$$
$$f(\bar{y}_k, ..., \bar{y}_1, y_1, ..., y_m)\mathrm{d}\bar{y}_1...\mathrm{d}\bar{y}_k \mathrm{d}y_1...\mathrm{d}y_m, \quad y_0 = \bar{y}_0 = o,$$

where p_t is the heat kernel corresponding to $\frac{1}{2}\Delta$ and $-\bar{t}_k < ... < -\bar{t}_1 < \bar{t}_0 = 0 = t_0 < t_1 < ... < t_m$, $\Delta_i t = t_i - t_{i-1}$ and $\Delta_i \bar{t} = \bar{t}_i - \bar{t}_{i-1}$.

Similar to Section 2, in order to construct Dirichlet forms associated to stochastic heat equations in Riemannian path space, we consider the collection $\mathscr{F}C_b$ of all cylinder functions on $\mathbf{E}^o_{\mathbb{R}}(M)$ as follows: for every $F \in \mathscr{F}C_b$, there exist some $m, k \in \mathbb{N}$, $f \in C_b^1(\mathbb{R}^{m+k})$, $g_i \in C_b^{0,1}([0,\infty) \times M)$, $\bar{g}_j \in C_b^{0,1}((-\infty, 0] \times M)$, $T_i, \bar{T}_j \in [0, \infty)$, $i = 1, ..., m$, $j = 1, ..., k$, such that

(3.3)
$$F(\gamma) = f\left(\int_0^{T_1} g_1(s, \gamma(s))\mathrm{d}s, ..., \int_0^{T_m} g_m(s, \gamma(s))\mathrm{d}s, \int_{-\bar{T}_1}^0 \bar{g}_1(s, \gamma(s))\mathrm{d}s, ..., \int_{-\bar{T}_k}^0 \bar{g}_k(s, \gamma(s))\mathrm{d}s\right).$$

For $\gamma \in \mathbf{E}^o_{\mathbb{R}}(M)$, define $\tilde{\gamma}(s) := \gamma(s), s \geq 0$ and $\bar{\gamma}(s) := \gamma(-s), s \geq 0$ respectively, then $\tilde{\gamma}, \bar{\gamma} \in \mathbf{E}^o_{\mathbb{R}^+}(M)$. Thus we could decompose $\gamma = (\tilde{\gamma}, \bar{\gamma})$, in particular, under $\mu^o_{\mathbb{R}}$, $\tilde{\gamma}(\cdot)$ and $\bar{\gamma}(\cdot)$ are two independent Brownian motions on M. We also define

$$U_s(\gamma) := \begin{cases} U_s(\tilde{\gamma}), & s \geq 0 \\ U_{-s}(\bar{\gamma}), & s < 0, \end{cases} \tag{3.4}$$

where $U_s(\tilde{\gamma}) : \mathbb{R}^n \to T_{\tilde{\gamma}(s)}M$ is the stochastic horizontal lift along $\tilde{\gamma}(\cdot)$ defined via (2.2). By the above argument, for $F \in \mathscr{F}C_b$ with form (3.3) we have

$$\begin{aligned} &\int_{\mathbf{E}^o_{\mathbb{R}}(M)} F(\gamma)\mathrm{d}\mu^o_{\mathbb{R}} \\ &= \int_{\mathbf{E}^o_{\mathbb{R}^+}(M)} \int_{\mathbf{E}^o_{\mathbb{R}^+}(M)} f\bigg(\int_0^{T_1} g_1(s, \tilde{\gamma}(s))\mathrm{d}s, ..., \int_0^{T_m} g_m(s, \tilde{\gamma}(s))\mathrm{d}s, \\ &\qquad \int_{-\bar{T}_1}^0 \bar{g}_1(s, \bar{\gamma}(-s))\mathrm{d}s, ..., \int_{-\bar{T}_k}^0 \bar{g}_k(s, \bar{\gamma}(-s))\mathrm{d}s\bigg)\mathrm{d}\mu^o_{\mathbb{R}^+}(\tilde{\gamma})\mathrm{d}\mu^o_{\mathbb{R}^+}(\bar{\gamma}), \end{aligned} \tag{3.5}$$

where $\mu^o_{\mathbb{R}^+}$ is introduced in Section 2. It is easy to see that $\mathscr{F}C_b$ is dense in $L^2(\mathbf{E}^o_{\mathbb{R}}(M); \mu^o_{\mathbb{R}})$.

Set:

$$\mathbf{H} := L^2(\mathbb{R} \to \mathbb{R}^n; \mathrm{d}s) = \left\{h : \mathbb{R} \to \mathbb{R}^n; \int_{-\infty}^{\infty} |h(s)|^2 \mathrm{d}s < \infty\right\}.$$

For every $h \in \mathbf{H}$ and each $F \in \mathscr{F}C_b$ of the form (3.3), the directional derivative of F with respect to h is ($\mu^o_{\mathbb{R}}$-a.s.) given by
(3.6)

$$\begin{aligned} D_hF(\gamma) = &\sum_{j=1}^m \hat{\partial}_j f(\gamma) \int_0^{T_j} \left\langle U_s^{-1}(\gamma)\nabla g_j(s,\gamma(s)), h(s)\right\rangle \mathrm{d}s \\ &+ \sum_{j=1}^k \hat{\partial}_{m+j} f(\gamma) \int_{-\bar{T}_j}^0 \left\langle U_s^{-1}(\gamma)\nabla \bar{g}_j(s,\gamma(s)), h(s)\right\rangle \mathrm{d}s, \quad \gamma \in \mathbf{E}^o_{\mathbb{R}}(M),\ h \in \mathbf{H}, \end{aligned}$$

where

$$\hat{\partial}_j f(\gamma) := \partial_j f\left(\int_0^{T_1} g_1(s,\gamma(s))\mathrm{d}s, ..., \int_0^{T_m} g_m(s,\gamma(s))\mathrm{d}s, \int_{-\bar{T}_1}^0 \bar{g}_1(s,\gamma(s))\mathrm{d}s, ..., \int_{-\bar{T}_k}^0 \bar{g}_k(s,\gamma(s))\mathrm{d}s\right),$$

$U_s(\gamma)$ is defined by (3.4), and ∇g_j denotes the gradient w.r.t. the second variable. By the Riesz representation theorem, there exists a gradient operator $DF(\gamma) \in \mathbf{H}$ such that $\langle DF(\gamma), h\rangle_{\mathbf{H}} = D_hF(\gamma)$ for every $h \in \mathbf{H}$. In particular, for the above F, $\gamma \in W^o_{\mathbb{R}}(M)$

$$\begin{aligned} DF(\gamma)(s) = &\sum_{j=1}^m \hat{\partial}_j f(\gamma) U_s^{-1}(\gamma)\nabla g_j(s,\gamma(s)) 1_{(0,T_j]}(s) \\ &+ \sum_{j=1}^n \hat{\partial}_{j+m} f(\gamma) U_s^{-1}(\gamma)\nabla \bar{g}_j(s,\gamma(s)) 1_{[-\bar{T}_j,0)}(s). \end{aligned} \tag{3.7}$$

Set

$$\mathbb{H}^\infty := \left\{h \in C_c^1(\mathbb{R};\mathbb{R}^n) \middle| h(0) = 0, \int_{\mathbb{R}} |h'(s)|^2 \mathrm{d}s < \infty\right\}.$$

Fix a sequence of elements $\{h_k\} \subset \mathbb{H}^\infty$ such that it is an orthonormal basis in $\mathbf{H}$, we define the following symmetric quadratic form

$$\mathscr{E}^o_{\mathbb{R}}(F,G) := \frac{1}{2}\int_{\mathbf{E}^o_{\mathbb{R}}(M)} \langle DF, DG\rangle_{\mathbf{H}} \mathrm{d}\mu^o_{\mathbb{R}}; \quad F, G \in \mathscr{F}C_b.$$

Remark 3.1. *We deduce the integration by parts formula by using the above stochastic horizontal lift U below. There are other ways to define the stochastic horizontal lift such that it is adapted to the filtration generated by γ. However, as mentioned in Section 2, the L^2-Dirichlet form is independent of the stochastic horizontal lift, which can be seen as a tool to obtain the integration by parts formula and the closablity of the associated bilinear form.*

Set $\tilde{\beta}_\cdot$, $\bar{\beta}_\cdot$ as the anti-development of $\tilde{\gamma}$ and $\bar{\gamma}$ respectively (whose distribution under $\mu^o_{\mathbb{R}}$ are two independent $\mathbb{R}^n$-valued Brownian motions). Let $l_{m,T} : [0,\infty) \times W^o_{\mathbb{R}^+}(M) \to$

$[0,1]$ be the vector fields constructed in Lemma 2.6 and we define $\hat{l}_{m,T} : \mathbb{R} \times W^o_{\mathbb{R}}(M) \to [0,1]$ as follows,

$$\hat{l}_{m,T}(t,\gamma) = \begin{cases} l_{m,T}(t,\tilde{\gamma}), & t \in [0,\infty), \\ l_{m,T}(-t,\bar{\gamma}), & t \in (-\infty,0). \end{cases} \tag{3.8}$$

Proposition 3.2. *For each $F \in \mathscr{F}C_b$ and $h \in \mathbb{H}^\infty$, and for each $\hat{l}_{m,T}$ defined by (3.8), we have*

$$\int_{\mathbf{E}^o_{\mathbb{R}}(M)} \langle DF, \hat{l}_{m,T} h\rangle_{\mathbf{H}} \mathrm{d}\mu^o_{\mathbb{R}} = \int_{\mathbf{E}^o_{\mathbb{R}}(M)} F\Theta^{m,T}_h \mathrm{d}\mu^o_{\mathbb{R}}, \tag{3.9}$$

where

$$\begin{aligned} \Theta^{m,T}_h(\gamma) = \Theta^{m,T}_h(\tilde{\gamma},\bar{\gamma}) = &\int_0^{+\infty} \left\langle \frac{1}{2}\mathrm{Ric}_{U_s(\tilde{\gamma})} h_{m,T}(s,\tilde{\gamma}) + h'_{m,T}(s,\tilde{\gamma}), \mathrm{d}\tilde{\beta}_s \right\rangle \\ &+ \int_0^{+\infty} \left\langle \frac{1}{2}\mathrm{Ric}_{U_s(\bar{\gamma})} h_{m,T}(s,\bar{\gamma}) + h'_{m,T}(s,\bar{\gamma}), \mathrm{d}\bar{\beta}_s \right\rangle. \end{aligned} \tag{3.10}$$

Here $h_{m,T}(s,\tilde{\gamma}) := h(s) l_{m,T}(s,\tilde{\gamma})$ and $h_{m,T}(s,\bar{\gamma}) := h(-s) l_{m,T}(s,\bar{\gamma})$ for all $s \in [0,\infty)$.

Proof. By (3.5), (3.6) we have

(3.11)

$$\begin{aligned} &\int_{\mathbf{E}^o_{\mathbb{R}}(M)} \langle DF, \hat{l}_{m,T} h\rangle_{\mathbf{H}} \mathrm{d}\mu^o_{\mathbb{R}} \\ &= \int_{\mathbf{E}^o_{\mathbb{R}^+}(M)} \int_{\mathbf{E}^o_{\mathbb{R}^+}(M)} \sum_{j=1}^m \hat{\partial}_j f(\tilde{\gamma},\bar{\gamma}) \int_0^{T_j} \left\langle U_s^{-1}(\tilde{\gamma}) \nabla g_j(s,\tilde{\gamma}(s)), l_{m,T}(s,\tilde{\gamma}) h(s) \right\rangle \mathrm{d}s \mathrm{d}\mu^o_{\mathbb{R}^+}(\tilde{\gamma}) \mathrm{d}\mu^o_{\mathbb{R}^+}(\bar{\gamma}) \\ &+ \int_{\mathbf{E}^o_{\mathbb{R}^+}(M)} \int_{\mathbf{E}^o_{\mathbb{R}^+}(M)} \sum_{j=1}^k \hat{\partial}_{m+j} f(\tilde{\gamma},\bar{\gamma}) \int_0^{\bar{T}_j} \left\langle U_s^{-1}(\bar{\gamma}) \nabla \bar{g}_j(-s,\bar{\gamma}(s)), l_{m,T}(s,\bar{\gamma}) h(-s) \right\rangle \mathrm{d}s \mathrm{d}\mu^o_{\mathbb{R}^+}(\tilde{\gamma}) \mathrm{d}\mu^o_{\mathbb{R}^+}(\bar{\gamma}) \\ &:= I + II, \end{aligned}$$

where $\hat{\partial}_j f(\tilde{\gamma},\bar{\gamma}) := \hat{\partial}_j f(\gamma)$.

According to (2.11) of Lemma 2.8, we get

$$I = \int_{\mathbf{E}^o_{\mathbb{R}^+}(M)} \int_{\mathbf{E}^o_{\mathbb{R}^+}(M)} F(\tilde{\gamma},\bar{\gamma}) \int_0^{+\infty} \left\langle \frac{1}{2}\mathrm{Ric}_{U_s(\tilde{\gamma})} h_{m,T}(s,\tilde{\gamma}) + h'_{m,T}(s,\tilde{\gamma}), \mathrm{d}\tilde{\beta}_s \right\rangle \mathrm{d}\mu^o_{\mathbb{R}^+}(\tilde{\gamma}) \mathrm{d}\mu^o_{\mathbb{R}^+}(\bar{\gamma}),$$

$$II = \int_{\mathbf{E}^o_{\mathbb{R}^+}(M)} \int_{\mathbf{E}^o_{\mathbb{R}^+}(M)} F(\tilde{\gamma},\bar{\gamma}) \int_0^{+\infty} \left\langle \frac{1}{2}\mathrm{Ric}_{U_s(\bar{\gamma})} h_{m,T}(s,\bar{\gamma}) + h'_{m,T}(s,\bar{\gamma}), \mathrm{d}\bar{\beta}_s \right\rangle \mathrm{d}\mu^o_{\mathbb{R}^+}(\tilde{\gamma}) \mathrm{d}\mu^o_{\mathbb{R}^+}(\bar{\gamma}),$$

where $F(\tilde{\gamma},\bar{\gamma}) = F(\gamma)$. Combining this and (3.11), we finish the proof.

□

Similar to the arguments as in the proof of Theorem 2.2 and based on the above integration by parts formula (3.9), we obtain the following:

Theorem 3.3. *The quadratic form* $(\mathscr{E}^o_{\mathbb{R}}, \mathscr{F}C_b)$ *is closable and its closure* $(\mathscr{E}^o_{\mathbb{R}}, \mathscr{D}(\mathscr{E}^o_{\mathbb{R}}))$ *is a quasi-regular Dirichlet form on* $L^2(\mathbf{E}^o_{\mathbb{R}}(M); \mu^o_{\mathbb{R}})$.

Proof. (a) **Closablity:** (I) Suppose that $\{F_k\}_{k=1}^{\infty} \subseteq \mathscr{F}C_b$ is a sequence of cylinder functions with

$$\lim_{m\to\infty} \mu^o_{\mathbb{R}}\left(F_m^2\right) = 0, \quad \lim_{k,m\to\infty} \mathscr{E}^o_{\mathbb{R}}\left(F_k - F_m, F_k - F_m\right) = 0. \tag{3.12}$$

Thus $\{DF_m\}_{m=1}^{\infty}$ is a Cauchy sequence in $L^2\left(\mathbf{E}^o_{\mathbb{R}}(M) \to \mathbf{H}; \mu^o_{\mathbb{R}}\right)$ for which there exists a limit Φ. It suffices to prove that $\Phi = 0$. Given an orthonormal basis $\{h_k\}_{k=1}^{\infty} \subset C_c^{\infty}(\mathbb{R}; \mathbb{R}^n) \cap \mathbb{H}^{\infty}$ of $\mathbf{H}$, by the integration by parts formula (3.9), for every $G \in \mathscr{F}C_b, h_k$ and $k, i, m, T \in \mathbb{N}^+$ we have

$$\begin{aligned} \mu^o_{\mathbb{R}}\left(\langle DF_i, \hat{l}_{m,T} h_k\rangle_{\mathbf{H}} G\right) &= \mu^o_{\mathbb{R}}\left(\langle D\left(F_i G\right), \hat{l}_{m,T} h_k\rangle_{\mathbf{H}}\right) - \mu^o_{\mathbb{R}}\left(\langle DG, \hat{l}_{m,T} h_k\rangle_{\mathbf{H}} F_i\right) \\ &= \mu^o_{\mathbb{R}}\left(F_i G \Theta^{m,T}_{h_k}\right) - \mu^o_{\mathbb{R}}\left(\langle DG, \hat{l}_{m,T} h_k\rangle_{\mathbf{H}} F_i\right). \end{aligned} \tag{3.13}$$

Since G and DG are bounded and $\Theta^{m,T}_{h_k} \in L^2(\mathbf{E}^o_{\mathbb{R}}(M); \mu^o_{\mathbb{R}})$ (due to (2.9) and the fact $h_k \in C_c^1(\mathbb{R}; \mathbb{R}^d)$), by (3.12) we could take the limit $i \to \infty$ under the integral in (3.13) to conclude

$$\mu^o_{\mathbb{R}}\left(\langle \Phi, \hat{l}_{m,T} h_k\rangle_{\mathbf{H}} G\right) = 0, \quad \forall\, G \in \mathscr{F}C_b,\ k, m, T \in \mathbb{N}^+,$$

therefore we could find a $\mu^o_{\mathbb{R}}$-null set $\Delta_k \subset W^o_{\mathbb{R}}(M)$, such that

$$\langle \Phi(\gamma), \hat{l}_{m,T}(\gamma) h_k\rangle_{\mathbf{H}} = 0, \quad \forall\, m, T \in \mathbb{Z}_+,\ \gamma \notin \Delta_k. \tag{3.14}$$

For a fixed h_k, we could find a $T_k \in \mathbb{N}^+$ (which may depend on h_k) satisfying $\mathrm{supp}(h_k) \subset [-T_k, T_k]$. Since the coordinate process $\gamma(\cdot)$ is non-explosive, for every $\gamma \notin \Delta_0$ with some $\mu^o_{\mathbb{R}}$-null set Δ_0, there exists $m_k(\gamma) \in \mathbb{Z}_+$, such that $\tilde{\gamma}(t) \in D_{m_k-1}$ and $\bar{\gamma}(t) \in D_{m_k-1}$ for all $t \in [0, T_k]$, hence $\hat{l}_{m_k,T_k}(t, \gamma) = 1$ for all $t \in [-T_k, T_k]$. Here D_{m_k-1} is defined in (2.8). Combining this with (3.14) we know

$$\langle \Phi(\gamma), h_k\rangle_{\mathbf{H}} = 0, \quad k \geq 1, \gamma \notin \Delta_0 \cup \Delta_k,$$

which implies that $\Phi(\gamma) = 0$, $\forall\, \gamma \notin \Delta := \cup_{k=0}^{\infty} \Delta_k$. So $\Phi = 0$, a.s., and $(\mathscr{E}^o_{\mathbb{R}^+}, \mathscr{F}C_b)$ is closable. By standard procedure, it is not difficult to show that its closure $(\mathscr{E}^o_{\mathbb{R}}, \mathscr{D}(\mathscr{E}^o_{\mathbb{R}}))$ is a Dirichlet form.

(b) **Quasi-Regularity:**

In order to prove the quasi-regularity, we need to verify conditions (i)-(iii) in [51, Definition IV-3.1]. By the same arguments as in the proof of Theorem 2.2, we could check (ii) and (iii) of [51, Definition IV-3.1] for $(\mathscr{E}_{\mathbb{R}}^o, \mathscr{D}(\mathscr{E}_{\mathbb{R}}^o))$, so we omit the proof here.

Since the metric space $(\mathbf{E}_{\mathbb{R}}^o(M); \tilde{d})$ is separable, we can choose a fixed countable dense subset $\{\xi_m | m \in \mathbb{N}^+\} \subset W_{\mathbb{R}}^o(M)$. Let $\varphi \in C_b^\infty(\mathbb{R})$ be an increasing function satisfying with

$$\varphi(t) = t, \quad \forall\, t \in [-1,1] \text{ and } \|\varphi'\|_\infty \le 1.$$

For each $m \ge 1$, the function $v_m : \mathbf{E}_{\mathbb{R}}^o(M) \to \mathbb{R}$ is given by

$$v_m(\gamma) = \varphi(\tilde{d}(\gamma, \xi_m)), \quad \gamma \in \mathbf{E}_{\mathbb{R}}^o(M),$$

where $\tilde{d}$ is defined by (3.2). According to the same procedures as in the proof of Lemma 2.9 we have $v_m \in \mathscr{D}(\mathscr{E}_{\mathbb{R}}^o)$ and

$$\begin{aligned} Dv_m(\gamma)(s) = \varphi'(\tilde{d}(\gamma, \xi_m)) \cdot \Bigg(\sum_{k=1}^\infty \frac{1}{2^k} \Big(& U_s^{-1}(\tilde{\gamma}) \nabla_1 \tilde{\rho}(\tilde{\gamma}(s), \xi_m(s)) 1_{(k-1,k]}(s) \\ & + U_{-s}^{-1}(\bar{\gamma}) \nabla_1 \tilde{\rho}(\bar{\gamma}(-s), \xi_m(s)) 1_{[-k,-k+1)}(s) \Big) \Bigg) \end{aligned}$$

for $\mathrm{d}s \times \mu_{\mathbb{R}}^o - a.s. (s,\gamma) \in \mathbb{R} \times \mathbf{E}_{\mathbb{R}}^o(M)$, where $\nabla_1 \tilde{\rho}(\cdot, x)$ denotes the gradient with respect to the first variable of $\tilde{\rho}(\cdot,\cdot)$. By such expression we arrive at

$$\sup_{m \ge 1} \mathscr{E}_{\mathbb{R}}^o(v_m, v_m) < \infty.$$

Then based on this and repeating the arguments as in the proof of Theorem 2.2 we can show

$$w_k := \inf_{m \le k} v_m, k \in \mathbb{N}^+, \text{ converges } \mathscr{E}_{\mathbb{R}}^o - \text{quasi-uniformly to zero on } \mathbf{E}_{\mathbb{R}}^o(M), \tag{3.15}$$

therefore the capacity associated with $(\mathscr{E}_{\mathbb{R}}^o, \mathscr{D}(\mathscr{E}_{\mathbb{R}}^o))$ is tight. So (i) of [51, Definition IV-3.1] holds. By now we have finished the proof.

□

Remark 3.4. *By the theory of the Dirichlet form, for the case of the whole line, we also derive similarly Theorems 2.3 and 2.4 in Section 2.*

As explained in the introduction, the invariant measure for the stochastic heat equation on the whole line could be the distribution of a two-sided Brownian motion with a shift given by Lebesgue measure, which may not be finite measure. So in our setting it is also natural to consider the reference measure given by $\int_M \mu_{\mathbb{R}}^x(\mathrm{d}\gamma)\nu(\mathrm{d}x)$ with some Randon measure ν (which may not be finite measure). The support of the measure is the paths on M with initial point not fixed.

Let $W_{\mathbb{R}}(M) := C(\mathbb{R}; M)$ be the free path space, then $(W_{\mathbb{R}}(M), d_\infty)$ is also a separable metric space with d_∞ defined by (3.1). Let $\tilde{d}$ be the L^1-distance defined by (3.2), and let $\mathbf{E}_{\mathbb{R}}(M)$ be the closure of $W_{\mathbb{R}}(M)$ under $\tilde{d}$. It is easy to see that $\mathbf{E}_{\mathbb{R}}(M)$ is a Polish space.

For any fixed Radon measure ν (not necessarily finite) on M, we could introduce a measure (not necessarily finite) $\mu_{\mathbb{R}}^{\nu}(\mathrm{d}\gamma) := \int_M \mu_{\mathbb{R}}^{x}(\mathrm{d}\gamma)\nu(\mathrm{d}x)$ on $\mathbf{E}_{\mathbb{R}}(M)$, where $\mu_{\mathbb{R}}^{x}$ is the probability measure defined as $\mu_{\mathbb{R}}^{o}$ with o replaced by x. Then we have that for $F(\gamma) = f\big(\gamma(-\bar{t}_k), ..., \gamma(-\bar{t}_1), \gamma(t_0), \gamma(t_1), ..., \gamma(t_m)\big)$ with $f \in C_c(M^{k+m+1})$, it holds

$$\begin{aligned}(3.16) \qquad \int_{\mathbf{E}_{\mathbb{R}}(M)} F(\gamma)\mathrm{d}\mu_{\mathbb{R}}^{\nu} = \int \prod_{i=1}^{k} p_{\Delta_i \bar{t}}(\bar{y}_{i-1}, \bar{y}_i) \prod_{i=1}^{m} p_{\Delta_i t}(y_{i-1}, y_i) \\ f(\bar{y}_k, ..., \bar{y}_1, y_0, y_1, ..., y_m)\mathrm{d}\bar{y}_1...\mathrm{d}\bar{y}_k \mathrm{d}y_1...\mathrm{d}y_m \nu(\mathrm{d}y_0),\end{aligned}$$

where the variable $y_0 = \bar{y}_0$ and p_t is the heat kernel corresponding to $\frac{1}{2}\Delta$ and $-\bar{t}_k < ... < -\bar{t}_1 < \bar{t}_0 = 0 = t_0 < t_1 < ... < t_m$, $\Delta_i t = t_i - t_{i-1}$ and $\Delta_i \bar{t} = \bar{t}_i - \bar{t}_{i-1}$. Here $C_c(M^{k+m+1})$ denote continuous functions on M^{k+m+1} with compact support.

Remark 3.5. *When M is compact and ν is the normalized volume measure, then $\mu_{\mathbb{R}}^{\nu}$ corresponds to the distribution of stationary M-valued Brownian motion. In the case that ν is given by the volume measure, the Markov process we construct below corresponds to stochastic heat equation on $\mathbb{R}$ with values in M without any boundary conditions.*

Remark 3.6. *If ν is the volume measure (M could be either compact or non-compact), then by expression* (3.16) *we know that $\theta_s^{\sharp}\mu_{\mathbb{R}}^{\nu} = \mu_{\mathbb{R}}^{\nu}$ for any $s \in \mathbb{R}$, where $\theta_s^{\sharp}\mu_{\mathbb{R}}^{\nu}$ denotes the push forward measure for $\mu_{\mathbb{R}}^{\nu}$ by the map $\theta_s : \mathbf{E}_{\mathbb{R}}(M) \to \mathbf{E}_{\mathbb{R}}(M)$ as $\theta_s(\gamma)(t) := \gamma(t+s)$. This means that $\mu_{\mathbb{R}}^{\nu}$ is invariant under any translation on $\mathbb{R}$.*

Remark 3.7. *In [9, 42], the authors studied* (1.1) *with solutions taking values in free loop space $\mathbf{L}(M) := \{\gamma \in C([0,1]; M); \gamma(0) = \gamma(1)\}$. In this case we could also construct the $\mathbf{L^2}$-Dirichlet form $(\tilde{\mathscr{E}}_{\mathbb{R}}^{\nu}, \mathscr{D}(\tilde{\mathscr{E}}_{R}^{\nu}))$ as follows*

$$\begin{aligned}(3.17) \qquad \tilde{\mathscr{E}}_{\mathbb{R}}^{\nu}(F,F) &:= \frac{1}{2}\int_{\mathbf{L}(M)} \langle DF, DF\rangle_{\mathbf{H}} \tilde{\mu}^{\nu}(\mathrm{d}\gamma) \\ &:= \frac{1}{2}\int_M \int_{\mathbf{L}_x(M)} \langle DF, DF\rangle_{\mathbf{H}} \tilde{\mu}^{x}(\mathrm{d}\gamma)\nu(\mathrm{d}x),\end{aligned}$$

where $\mathbf{L}_x(M) := \{\gamma \in C([0,1]; M); \gamma(0) = \gamma(1) = x\}$, $\tilde{\mu}^x$ denotes the Brownian bridge measure on $\mathbf{L}_x(M)$.

For the free loop measure $\tilde{\mu}^{\nu}$ above, if $\nu(\mathrm{d}x) = p_1(x,x)\mathrm{d}x$, then $\tilde{\mu}^{\nu}$ is invariant under any rotation on S^1. In [55], the quasi-regularity of $(\tilde{\mathscr{E}}_{\mathbb{R}}^{\nu}, \mathscr{D}(\tilde{\mathscr{E}}_{R}^{\nu}))$ has been studied under the assumption that M is compact. As explained in Remarks 3.11 and 3.14, by *the method of this paper, we could also obtain the corresponding results for the case that M is non-compact.*

The state space $\mathbf{L}(M)$ corresponds to the spatial variable with values in finite volume, while $\mathbf{E}_{\mathbb{R}}(M)$ and $\mathbf{E}_{\mathbb{R}}^{o}(M)$ correspond to the case that the spatial variable in infinite volume.

Here we only consider the case that ν is an infinite measure, since when ν is a finite measure, the case is simpler and it may be handled similar as in Theorem 4.1.

Next, we assume that ν is infinite, then $\mu_{\mathbb{R}}^{\nu}$ is also an infinite measure on $\mathbf{E}_{\mathbb{R}}(M)$ with support contained in $W_{\mathbb{R}}(M)$. In this case $1 \notin L^2(\mu_{\mathbb{R}}^{\nu})$ and we need to introduce a new class of cut-off functions $\mathbf{E}_{\mathbb{R}}(M)$. Let $\mathscr{F}C_{Lip}$ be the space of bounded Lipschitz continuous functions on $\mathbf{E}_{\mathbb{R}}(M)$, i.e. for every $F \in \mathscr{F}C_{Lip}$, there exist some $m, k \in \mathbb{N}$, $f \in C_b^1(\mathbb{R}^{m+k}), g_i \in C_{Lip}^{0,1}([0,\infty)\times M), \bar{g}_j \in C_{Lip}^{0,1}((-\infty,0]\times M), T_i, \bar{T}_j \in [0,\infty)$, $i = 1, ..., m$, $j = 1, ..., k$, such that

(3.18)
$$F(\gamma) = f\left(\int_0^{T_1} g_1(s,\gamma(s))\mathrm{d}s, ..., \int_0^{T_m} g_m(s,\gamma(s))\mathrm{d}s, \int_{-\bar{T}_1}^0 \bar{g}_1(s,\gamma(s))\mathrm{d}s, ..., \int_{-\bar{T}_k}^0 \bar{g}_k(s,\gamma(s))\mathrm{d}s\right),$$

where $C_{Lip}^{0,1}([0,\infty)\times M)$ denotes the collection of functions $g : [0,\infty)\times M \to \mathbb{R}$ such that g is continuous on $[0,\infty)$ and Lipschitz continuous (not necessarily bounded) on M with the associated Lipschitz constants independent of $s \in [0,\infty)$. Now we fix a point $o \in M$. Let

$$\mathscr{F}C_c := \Big\{ F \in \mathscr{F}C_{Lip}; \text{there exists } R > 0 \text{ such that } F(\gamma) = 0 \text{ for all } \gamma \in \mathbf{E}_{\mathbb{R}}(M) \text{ satisfying } \int_0^1 \rho\big(o,\gamma(s)\big)\mathrm{d}s > R \Big\}.$$

Lemma 3.8. *Suppose that for every $R > 0$, it holds*

$$\int_M \mu_{\mathbb{R}^+}^{x}\Big(\sup_{s\in[0,1]} \rho\big(x,\gamma(s)\big) > \rho(o,x) - R\Big)\nu(\mathrm{d}x) < \infty, \tag{3.19}$$

then $\mathscr{F}C_c$ is a dense subset of $L^2(\mathbf{E}_{\mathbb{R}}(M);\mu_{\mathbb{R}}^{\nu})$.

Proof. **Step (i)** We first show $\mathscr{F}C_c \subset L^2(\mathbf{E}_{\mathbb{R}}(M);\mu_{\mathbb{R}}^{\nu})$. For every $F \in \mathscr{F}C_c$, without loss of generality we may assume that there exist $R > 0$, such that $F(\gamma) = 0$ for all

$\gamma \in \mathbf{E}_{\mathbb{R}}(M)$ satisfying $\int_0^1 \rho\big(o, \gamma(s)\big)\mathrm{d}s > R$. Then we have

$$\begin{aligned}
&\int_{\mathbf{E}_{\mathbb{R}}(M)} |F(\gamma)|^2 \mu_{\mathbb{R}}^{\nu}(\mathrm{d}\gamma) = \int_M \int_{\mathbf{E}_{\mathbb{R}}^x(M)} |F(\gamma)|^2 \mu_{\mathbb{R}}^x(\mathrm{d}\gamma)\nu(\mathrm{d}x) \\
&= \int_{B(o,2R)} \int_{\mathbf{E}_{\mathbb{R}}^x(M)} |F(\gamma)|^2 \mu_{\mathbb{R}}^x(\mathrm{d}\gamma)\nu(\mathrm{d}x) + \int_{B(o,2R)^c} \int_{\mathbf{E}_{\mathbb{R}}^x(M)} |F(\gamma)|^2 \mu_{\mathbb{R}}^x(\mathrm{d}\gamma)\nu(\mathrm{d}x) \\
&\le \|F\|_\infty^2 \Big(\nu\big(B(o,2R)\big) + \int_{B(o,2R)^c} \mu_{\mathbb{R}^+}^x\Big(\sup_{s\in[0,1]} \rho(x,\gamma(s)) > \rho(o,x) - R\Big)\Big)\nu(\mathrm{d}x) < \infty,
\end{aligned}$$

where the third step is due to the fact when $x \notin B(o,2R)$, $F(\gamma) = 0$ for all $\gamma \in \mathbf{E}_{\mathbb{R}}^x(M)$ with $\sup_{s\in[0,1]} \rho(\gamma(s), x) \le \rho(o,x) - R$ (if $\sup_{s\in[0,1]} \rho(\gamma(s), x) \le \rho(o,x) - R$, then $\int_0^1 \rho\big(o, \gamma(s)\big)\mathrm{d}s \ge \inf_{s\in[0,1]} \rho(\gamma(s), o) > R$, hence $F(\gamma) = 0$).

Step (ii) Now we are going to show $\mathscr{F}C_c$ is dense in $L^2(\mathbf{E}_{\mathbb{R}}(M); \mu_{\mathbb{R}}^{\nu})$. It suffices to prove that for every $G(\gamma) := f\Big(\gamma(t_1), \cdots, \gamma(t_m)\Big)$ with some $m \in \mathbb{N}^+$, $t_1 < t_2 \cdots < t_m$ and $f \in C_c^1(M^m)$, there exists a sequence $\{G_{k,R}\}_{k,R} \subset \mathscr{F}C_c$ such that $\lim_{k,R\to\infty} \mu_{\mathbb{R}}^{\nu}\big(|G_{k,R} - G|^2\big) = 0$. Here $C_c^1(M^m)$ denotes the C^1 functions on M^m with compact support.

By Nash isometric imbedding theorem, there is a smooth isometric imbedding $\eta : M \to \mathbb{R}^N$ with some $N \in \mathbb{N}^+$ and we can extend $f \in C_c^1(M^m)$ to $\tilde{f} \in C_c^1(\mathbb{R}^{Nm})$ satisfying $\tilde{f}\big(\eta(x)\big) = f(x)$ for all $x \in M$. Choose $\varphi_R \in C_b^1(\mathbb{R}, \mathbb{R})$, $\phi_R \in C_c^1(\mathbb{R}, \mathbb{R})$ satisfying

$$\varphi_R(x) = \begin{cases} x, & \text{if } |x| \le R, \\ R+1, & \text{if } x > R+1, \\ -R-1, & \text{if } x < -R-1, \end{cases}$$

$$\phi_R(x) = \begin{cases} 1, & \text{if } |x| \le R, \\ \in (0,1), & \text{if } R < |x| < R+1, \\ 0, & \text{if } |x| > R+1. \end{cases}$$

We set $\varphi_{R,N}(x) := \prod_{i=1}^N \varphi_R(x_i)$ for $x = (x_1, ..., x_N)$.

$$G_{k,R}(\gamma) := \phi_R\Big(\int_0^1 \rho\big(o, \gamma(s)\big)\mathrm{d}s\Big)\tilde{f}\Big(k\int_{t_1}^{t_1+\frac{1}{k}} \varphi_{R,N}\circ\eta\big(\gamma(s)\big)\mathrm{d}s, \cdots, k\int_{t_m}^{t_m+\frac{1}{k}} \varphi_{R,N}\circ\eta\big(\gamma(s)\big)\mathrm{d}s\Big),$$

then it is easy to verify that $G_{k,R} \in \mathscr{F}C_c$ for all $k > 0$ and R large enough, and $\lim_{k,R\to\infty} \mu_{\mathbb{R}}^{\nu}\big(|G_{k,R} - G|^2\big) = 0$ ((since $\tilde{f} \in C_c^1(\mathbb{R}^{Nm})$, this could be shown by the dominated convergence theoem). By now we have finished the proof.

□

Now we give some sufficient conditions on the curvature of M for (3.19).

Lemma 3.9. *Suppose that*

$$\mathrm{Ric}_x(X,X) \ge -C_1\big(1+\rho(o,x)^\alpha\big), \quad \forall\, x\in M,\ X\in T_xM, |X|=1, \tag{3.20}$$

for some $C_1>0$, $\alpha\in(0,2)$ *and* $o\in M$, *where* Ric_x *denotes the Ricci curvature operator at* $x\in M$. *Then for every Radon measure* $\nu(\mathrm{d}x)=\nu(x)\mathrm{d}x$ *(here* $\mathrm{d}x$ *denotes the volume measure on* M*) such that*

$$|\nu(x)| \le C_2\exp(C_3\rho(o,x)^\beta), \quad \forall x\in M \tag{3.21}$$

with some $C_2, C_3>0$ *and* $\beta\in(0,2)$, (3.19) *holds.*

Proof. Note that (3.20) implies that

$$\mathrm{Ric}_y(Y,Y) \ge -K_1\big(\rho(o,y)\big), \quad \forall\, y\in M,\ Y\in T_yM,\ |Y|=1, \tag{3.22}$$

with $K_1(r):=C_1(1+r^\alpha)$. It is easy to verify that we could find a $c_1>0$ such that

$$c_2 := \sup_{t>0}\big(t\sqrt{(n-1)K_1(t)}-2c_1t^2\big)<\infty. \tag{3.23}$$

Then according to [60, Lemma 2.2] we know that for every $N>0$ and $T>0$,

$$\mu^o_{\mathbb{R}^+}\Big(\sup_{s\in[0,T]}\rho\big(o,\gamma(s)\big)>N\Big) \le e^{n+c_2-\kappa(T)N^2}, \tag{3.24}$$

where $\kappa(T):=\frac{1}{2T}e^{-1-2c_1T}$.

Also note that (3.22) implies for every $x\in M$,

$$\mathrm{Ric}_y(Y,Y)\ge -2K_1\big(\rho(x,y)\big)-2K_1\big(\rho(o,x)\big), \quad \forall\, y\in M,\ Y\in T_yM,$$

Then taking $c_1=1$ and using $2K_1(t)+2K_1\big(\rho(o,x)\big)$ to replace $K_1(t)$ in (3.23), we have $c_2\le c_3(1+\rho(o,x)^\alpha)$. Therefore according to (3.24) we know for all $R>0$ and $x\notin B(o,2R)$,

$$\begin{aligned}
&\mu^x_{\mathbb{R}^+}\Big(\sup_{s\in[0,1]}\rho\big(x,\gamma(s)\big)>\rho(o,x)-R\Big)\\
&\le \exp\Big(n+c_3(1+\rho(o,x)^\alpha)-\kappa(1)(\rho(o,x)-R)^2\Big)\\
&\le \exp\Big(n+c_3(1+\rho(o,x)^\alpha)-\frac{\kappa(1)}{4}\rho(o,x)^2\Big)\\
&\le c_4e^{-c_5\rho(o,x)^2},
\end{aligned} \tag{3.25}$$

where c_4, c_5 are positive constants independent of $x\in M$ and $R>0$. Denote by $\mathrm{Cut}(o)$ the cut-locus of o in M and the exponential map from $o\in M$ by $\exp_o: T_oM\to M$. It is well known that $\exp_o^{-1}: M\setminus \mathrm{Cut}(o) \to \exp_o^{-1}\big(M\setminus \mathrm{Cut}(o)\big)\subset T_oM\simeq \mathbb{R}^+\times \mathbb{S}^{n-1}$ is a diffeomorphism, which induces the geodesic spherical coordinates of M (see e.g. [14, Section III.1] for details). Let $(r,\theta)\in \mathbb{R}^+\times\mathbb{S}^{n-1}$ be the element in geodesic spherical coordinates, then for every $f\in C_c(M)$ we have (see e.g. [14, Theorem III 3.1])

$$\int_M f(x)dx=\int_{\mathbb{R}^+}\int_{\mathbb{S}^{n-1}} f\big((r,\theta)\big)\big|\mathscr{A}(r,\theta)\big|\mathrm{d}r\mathrm{d}\theta,$$

where $\mathscr{A}(r,\theta)$ is a $n\times n$ matrix, $|\mathscr{A}|$ denotes the determinant of $\mathscr{A}$, and $\mathscr{A}$ satisfying the following equation

$$\mathscr{A}''(t,\theta)+\mathscr{R}(t,\theta)\mathscr{A}(t,\theta)=0,\ \ \mathscr{A}(0,\theta)=0,\ \mathscr{A}'(0,\theta)=\mathbf{I}.$$

Here $\mathscr{R}(t,\theta)\in L(\mathbb{R}^n;\mathbb{R}^n)\simeq \mathbb{R}^{n\times n}$ and $\mathscr{R}(t,\theta)\xi:=U_t^{-1}\mathrm{R}\big(\gamma_\theta'(t),U_t\xi\big)\gamma_\theta'(t)$ for all $\xi\in\mathbb{R}^n$ with $\gamma_\theta(t)=\exp_o\big((t,\theta)\big)$, $U_t:\mathbb{R}^n\to T_{\gamma_\theta(t)}M$ is the parallel translation along geodesic $\gamma_\theta(\cdot)$, R denotes the Riemannian curvature operator on M.

Moreover, we have the following estimates for $|\mathscr{A}|$ (see e.g. [14, Theorem III 4.3]),

$$\big|\mathscr{A}(r,\theta)\big|\le\Big(\sqrt{\frac{n-1}{K_1(r)}}\sinh\Big(\sqrt{\frac{K_1(r)}{n-1}}r\Big)\Big)^{n-1}\le c_6e^{c_7r^{1+\alpha/2}},\ \ r>0, \tag{3.26}$$

where c_6,c_7 are positive constants independent of r, $K_1(r)$ is the function in (3.22) and the last step is due to $\frac{\sinh a}{a}\le\cosh a$ and (3.20).

Combining (3.21), (3.25) and (3.26) yields

$$\begin{aligned}&\int_{B(o,2R)^c}\mu^x_{\mathbb{R}^+}\Big(\sup_{s\in[0,T]}\rho\big(x,\gamma(s)\big)>\rho(o,x)-R\Big)\nu(\mathrm{d}x)\\&\le c_8\int_{2R}^\infty\int_{\mathbb{S}^{n-1}}\exp\big(C_3r^\beta+c_7r^{1+\alpha/2}-c_5r^2\big)\mathrm{d}\theta\mathrm{d}r\\&\le c_9\int_{2R}^\infty e^{-c_{10}r^2}\mathrm{d}r\le c_{11}e^{-c_{12}R^2}.\end{aligned} \tag{3.27}$$

Here in the second step of inequality we have applied the fact $\alpha\in(0,2)$ and $\beta\in(0,2)$.

Based on this estimate we could obtain (3.19) immediately. □

Remark 3.10. *By Lemma 3.9 we know that under curvature condition* (3.20)*, the property* (3.19) *holds if ν is the volume measure of M.*

Remark 3.11. *For the free loop measure $\tilde{\mu}^\nu$ defined by* (3.17) *on $\mathbf{L}(M)$, by carefully tracking the proof of Lemma 3.8 we know if for every $R>0$,*

$$\int_M\tilde{\mu}^x\Big(\sup_{s\in[0,\frac{1}{2}]}\rho\big(x,\gamma(s)\big)>\rho(o,x)-R\Big)\nu(\mathrm{d}x)<\infty, \tag{3.28}$$

then $\mathscr{F}C_c$ is dense in $L^2(\mathbf{L}(M);\tilde{\mu}^\nu)$.

When we choose $\nu(\mathrm{d}x)=p_1(x,x)\mathrm{d}x$, it holds.

$$\begin{aligned}
&\int_M \tilde{\mu}^x\Big(\sup_{s\in[0,\frac{1}{2}]}\rho\big(x,\gamma(s)\big)>\rho(o,x)-R\Big)\nu(\mathrm{d}x)\\
&=\int_M \mu_{\mathbb{R}^+}^x\Big(1_{\{\sup_{s\in[0,\frac{1}{2}]}\rho\big(x,\gamma(s)\big)>\rho(o,x)-R\}}(\gamma)p_{\frac{1}{2}}\big(\gamma(\frac{1}{2}),x\big)\Big)\mathrm{d}x \qquad (3.29)\\
&\le \int_M\sqrt{\mu_{\mathbb{R}^+}^x\Big(\sup_{s\in[0,\frac{1}{2}]}\rho\big(x,\gamma(s)\big)>\rho(o,x)-R\Big)}\sqrt{\int_M p_{\frac{1}{2}}(x,y)^3\mathrm{d}y}\mathrm{d}x,
\end{aligned}$$

where the last step is due to Cauchy-Schwartz inequality.

If the curvature condition (3.20) *holds, then we know that* (3.25) *is true. Moreover, suppose* (3.20) *holds and the following lower bound of volume* (3.30) *is satisfied (, which could be viewed as an local volume non-collapsed condition)*

$$\inf_{x;\rho(o,x)\le R} m\Big(B(x,\frac{1}{2})\Big)\ge C_1e^{-C_2R^\beta},\ \forall R>1, \qquad (3.30)$$

where $\beta\in(0,2), C_1, C_2$ are constants, m denotes the volume measure on M and $B(x,r):=\{y\in M;\rho(y,x)\le r\}$ is the geodesic ball on M. Then according to the proof of [7, Corollary 3] (here our curvature condition (3.20) *is a little different from [7]), we have*

$$p_{\frac{1}{2}}(x,y)\le C_3\exp\Big(-C_4\rho(x,y)^2+C_5\big(\rho(o,x)^{\max(\alpha,\beta)}+\rho(o,y)^{\max(\alpha,\beta)}\big)\Big).$$

Putting this into (3.29) *and by the same arguments as in the proof of Lemma 3.9 we could prove* (3.28).

As a result, combining all the above estimates we could prove that if (3.20) *and* (3.30) *are true, then $\mathscr{F}C_c$ is dense in $L^2(\mathbf{L}(M);\tilde{\mu}^\nu)$.*

For $F\in\mathscr{F}C_c$, we still define the directional derivative $D_hF(\gamma)$ along $h\in\mathbf{H}:=L^2(\mathbb{R}\to\mathbb{R}^n;\mathrm{d}s)$ and the gradient operator $DF\in\mathbf{H}$ as in (3.6) and (3.7), respectively. Here as explained before Lemma 2.9 we know that D_hF and DF are well-defined for $\mu_{\mathbb{R}}^\nu$-a.e. γ.

Now for the fixed $o\in M$, as in Lemma 2.6 (although here the initial point will not be fixed, see e.g. [58] or [16]) we could construct a series of relatively compact subset $\{D_m\}_{m=1}^\infty$ of M (with $o\in D_m$ for all m), and a series of adapted vector fields $\{l_{m,T}\}_{m,T=1}^\infty$ such that $l_{m,T}:[0,\infty)\times W_{\mathbb{R}^+}(M)\to[0,1]$, items (1)-(2) in Lemma 2.6 and the following estimates hold

$$\sup_{x\in D_m}\int_{\mathbf{E}_{\mathbb{R}^+}^x(M)}\int_0^t |l'_{k,T}(s,\gamma)|^p\mathrm{d}s\mu_{\mathbb{R}^+}^x(\mathrm{d}\gamma)<\infty,\quad k>m,\ p>0.$$

In particular, by (1) in Lemma 2.6 we have

$$l_{m,T}(s,\gamma) \equiv 0,\ \mu^x_{\mathbb{R}^+} - a.s.\ \gamma \in \mathbf{E}^x_{\mathbb{R}^+}(M)\ \text{ if } x \notin D_m.$$

As before, we split $\gamma \in \mathbf{E}_{\mathbb{R}}(M)$ into $\tilde{\gamma}, \bar{\gamma} \in \mathbf{E}_{\mathbb{R}^+}(M)$ by

$$\tilde{\gamma}(s) := \gamma(s), s \geq 0, \quad \bar{\gamma}(s) := \gamma(-s), s \geq 0,$$

and following the procedures of (3.8) we could extend $l_{m,T}$ to an adapted vector field $\hat{l}_{m,T} : \mathbb{R} \times \mathbf{E}_{\mathbb{R}}(M) \to [0,1]$. Moreover, it holds that

$$\hat{l}_{m,T}(s,\gamma) \equiv 0,\ \mu^x_{\mathbb{R}} - a.s.\ \gamma \in \mathbf{E}^x_{\mathbb{R}}(M)\ \text{ if } x \notin D_m.$$

$$\sup_{x \in D_m} \int_{\mathbf{E}^x_{\mathbb{R}}(M)} \int_0^t |\hat{l}'_{k,T}(s,\gamma)|^p \mathrm{d}s \mu^x_{\mathbb{R}}(\mathrm{d}\gamma) < \infty, \quad \forall\ k > m,\ p > 0. \tag{3.31}$$

By the proof of Theorem 2.8 in [16] (see also appendix), (3.9) holds for $\mu^x_{\mathbb{R}}$ with every $x \in D_q$ with $q < m$, which yields immediately for every $F \in \mathscr{F}C_c,\ h \in \mathbb{H}^\infty,\ m,k,T \in \mathbb{N}^+$ with $k > m$ (note that $h(0) = 0$ for every $h \in \mathbb{H}^\infty$),

$$\int_{D_m} \int_{\mathbf{E}^x_{\mathbb{R}}(M)} \langle DF, \hat{l}_{k,T} h \rangle_{\mathbf{H}} \mathrm{d}\mu^x_{\mathbb{R}} \nu(\mathrm{d}x) = \int_{D_m} \int_{\mathbf{E}^x_{\mathbb{R}}(M)} F \Theta^{k,T}_h \mathrm{d}\mu^x_{\mathbb{R}} \nu(\mathrm{d}x), \tag{3.32}$$

where $\Theta^{k,T}_h$ is defined by (3.10).

Fix a sequence of elements $\{h_k\} \subset \mathbb{H}^\infty$ such that it is an orthonormal basis in $\mathbf{H}$, we define the following symmetric quadratic form

$$\mathscr{E}^\nu_{\mathbb{R}}(F,G) := \frac{1}{2} \int_{\mathbf{E}_{\mathbb{R}}(M)} \langle DF, DG \rangle_{\mathbf{H}} \mathrm{d}\mu^\nu_{\mathbb{R}} = \frac{1}{2} \sum_{k=1}^\infty \int_{\mathbf{E}_{\mathbb{R}}(M)} D_{h_k} F D_{h_k} G \mathrm{d}\mu^\nu_{\mathbb{R}}; \quad F, G \in \mathscr{F}C_c.$$

In particular, by the same arguments as in the proof of Lemma 3.8, we know $\mathscr{E}^\nu_{\mathbb{R}}(F,F) < \infty$ for every $F \in \mathscr{F}C_c$.

Since the reference measure $\mu^\nu_{\mathbb{R}}$ has infinite mass, we use a cut-off technique to prove the quasi-regularity of the associated L^2-Dirichlet form.

Theorem 3.12. *Suppose that* (3.19) *holds. Then the quadratic form* $(\mathscr{E}^\nu_{\mathbb{R}}, \mathscr{F}C_c)$ *is closable and its closure* $(\mathscr{E}^\nu_{\mathbb{R}}, \mathscr{D}(\mathscr{E}^\nu_{\mathbb{R}}))$ *is a quasi-regular Dirichlet form on* $L^2(\mathbf{E}_{\mathbb{R}}(M); \mu^\nu_{\mathbb{R}})$.

Proof. (*a*) **Closablity:** The proof is similar to that of Theorem 3.3. Suppose $\{F_k\}_{k=1}^\infty \subseteq \mathscr{F}C_c$ is a sequence of cylinder functions with

$$\lim_{m\to\infty} \mu^\nu_{\mathbb{R}}\left(F_m^2\right) = 0, \quad \lim_{k,m\to\infty} \mathscr{E}^\nu_{\mathbb{R}}\left(F_k - F_m, F_k - F_m\right) = 0. \tag{3.33}$$

Thus $\{DF_m\}_{m=1}^{\infty}$ is a Cauchy sequence in $L^2\left(\mathbf{E}_{\mathbb{R}}(M)\to\mathbf{H};\mu_{\mathbb{R}}^{\nu}\right)$ for which there exists a limit Φ. It suffices to prove that $\Phi=0$.

Combining (3.33) with (3.31) and (3.32) yields that for all $m,k,T\in\mathbb{N}^+$, $G\in\mathscr{F}C_c$ and the orthonormal basis $\{h_i\}_{i=1}^{\infty}\subset\mathbb{H}^{\infty}$ of $\mathbf{H}$ with $k>m$,

$$\int_{D_m}\int_{\mathbf{E}_{\mathbb{R}}^x(M)} G\langle\Phi,\hat{l}_{k,T}h_i\rangle_{\mathbf{H}}\mathrm{d}\mu_{\mathbb{R}}^x\nu(\mathrm{d}x)=0,$$

which ensures the existence of a $\mu_{\mathbb{R}}^{\nu}$-null set Δ_i such that for all $m,k,T\in\mathbb{N}^+$ with $k>m$,

$$\hat{l}_{k,T}(\gamma)\langle\Phi(\gamma),h_i\rangle_{\mathbf{H}}=0,\quad\forall\,\gamma\notin\Delta_i,\gamma(0)\in D_m. \tag{3.34}$$

For a fixed $h_i\in\mathbb{H}^{\infty}$, we could find $T_i\in\mathbb{N}^+$ (which may depend on h_i) satisfying supp$h_k\subset[-T_i,T_i]$. Since $\gamma(\cdot)$ is non-explosive, for every $\gamma\notin\Delta_0$ with some $\mu_{\mathbb{R}}^{o}$-null set Δ_0, there exist $m_i,k_i\in\mathbb{Z}_+$ (which may depend on γ), such that $k_i>m_i$, $\gamma(0)\in D_{m_i}$, $\gamma(t)\in D_{k_i-1}$ for all $t\in[-T_i,T_i]$, hence $\hat{l}_{k_i,T_i}(t,\gamma)=1$ for all $t\in[-T_i,T_i]$. By this and (3.34) we know

$$\langle\Phi(\gamma),h_i\rangle_{\mathbf{H}}=0,\quad i\geq 1,\gamma\notin\Delta_0\cup\Delta_i,$$

which implies that $\Phi(\gamma)=0$, $\forall\,\gamma\notin\Delta:=\cup_{i=0}^{\infty}\Delta_i$. So $\Phi=0$, a.s., and $(\mathscr{E}_{\mathbb{R}}^{\nu},\mathscr{F}C_c)$ is closable. By standard methods, we show easily that its closure $(\mathscr{E}_{\mathbb{R}}^{\nu},\mathscr{D}(\mathscr{E}_{\mathbb{R}}^{\nu}))$ is a Dirichlet form.

(b) **Quasi-Regularity:**

We first verify (i) of [51, Definition IV-3.1]: Since the metric space $(\mathbf{E}_{\mathbb{R}}(M);\tilde{d})$ ($\tilde{d}$ is defined by (3.2)) is separable, we can choose a fixed countable dense subset $\{\xi_m|m\in\mathbb{N}^+\}\subset W_{\mathbb{R}}(M)$. Let $\varphi\in C_b^{\infty}(\mathbb{R})$ such that φ is an increasing function satisfying

$$\varphi(t)=t,\quad\forall\,t\in[-1,1]\ \text{ and }\ \|\varphi'\|_{\infty}\leq 1.$$

Let $\phi_R\in C_c^{\infty}(\mathbb{R})$ such that $\|\phi_R'\|_{\infty}\leq 2$ and

$$\phi_R(x)=\begin{cases}1, & \text{if } |x|\leq R,\\ \in(0,1), & \text{if } R<|x|\leq R+1,\\ 0, & \text{if } |x|>R+1.\end{cases}$$

For fixed $o\in M$ and each $m,R\in\mathbb{N}^+$, we define $v_{m,R}:\mathbf{E}_{\mathbb{R}}(M)\to\mathbb{R}$ by

$$v_{m,R}(\gamma)=\phi_R\Big(\int_0^1\rho(o,\gamma(s))\mathrm{d}s\Big)\varphi(\tilde{d}(\gamma,\xi_m)),\quad\gamma\in\mathbf{E}_{\mathbb{R}}(M).$$

Then by similar argument as in the proof of Lemma 2.9, it is easy to see that $v_{m,R}\in\mathscr{D}(\mathscr{E}_{\mathbb{R}}^{\nu})$.

Define for closed set $A \subset \mathbf{E}_{\mathbb{R}}(M)$

$$\mathscr{D}_A(\mathscr{E}^{\nu}_{\mathbb{R}}) := \{u \in \mathscr{D}(\mathscr{E}^{\nu}_{\mathbb{R}}) | u = 0 \quad \mu^{\nu}_{\mathbb{R}} - \text{ a.e. on } A^c\},$$

which is a closed subspace of $\mathscr{D}(\mathscr{E}^{\nu}_{\mathbb{R}})$. This implies that $(\mathscr{E}^{\nu}_{\mathbb{R}}, \mathscr{D}_A(\mathscr{E}^{\nu}_{\mathbb{R}}))$ is a Dirichlet form. Now we have $v_{m,R} \in \mathscr{D}_{B_{R+1}}(\mathscr{E}^{\nu}_{\mathbb{R}})$, with $B_R := \{\gamma \in \mathbf{E}_{\mathbb{R}}(M), \int_0^1 \rho(o, \gamma(s))\mathrm{d}s \le R\}$.

Still according to the same procedures as that in the proof of Lemma 2.9 (2) we have for every $m, R \in \mathbb{N}^+$,

$$\begin{aligned}
&Dv_{m,R}(\gamma)(s) \\
=&\phi_R\Big(\int_0^1 \rho(o,\gamma(s))\mathrm{d}s\Big)\varphi'(\tilde{d}(\gamma,\xi_m)) \cdot \left(\sum_{k=1}^{\infty} \frac{1}{2^k}\Big(U_s^{-1}(\tilde{\gamma})\nabla_1\tilde{\rho}(\tilde{\gamma}(s),\xi_m(s))1_{(k-1,k]}(s)\right. \\
&\left. + U_{-s}^{-1}(\bar{\gamma})\nabla_1\tilde{\rho}(\bar{\gamma}(-s),\xi_m(s))1_{[-k,-k+1)}(s)\Big)\right) \\
&+ \phi_R'\Big(\int_0^1 \rho(\gamma(s),o)\mathrm{d}s\Big)\varphi(\tilde{d}(\gamma,\xi_m))\Big(U_s^{-1}(\tilde{\gamma})\nabla_1\rho(\tilde{\gamma}(s),o)\Big)1_{(0,1]}(s)
\end{aligned}$$

for $\mathrm{d}s \times \mu^{\nu}_{\mathbb{R}} - a.s. (s, \gamma) \in \mathbb{R} \times \mathbf{E}^o_{\mathbb{R}}(M)$. Such expression yields that for every fixed $R \in \mathbb{N}^+$,

$$\int \sup_{m \ge 1} |Dv_{m,R}|^2_{\mathbf{H}} \mathrm{d}\mu^{\nu}_{\mathbb{R}} < \infty.$$

Based on this and [51, Lemma I-2.12, Proposition III-3.5, Lemma IV-4.1] we obtain that for every fixed $R > 0$,

(3.35) $\quad w_{k,R} := \inf_{m \le k} v_{m,R}$ converges $\mathscr{E}^{\nu}_{\mathbb{R}}$ − quasi-uniformly to zero on $\mathbf{E}_{\mathbb{R}}(M)$.

Therefore for each $R, N \in \mathbb{N}^+$ there exists a closed set $\hat{F}_{N,R} \subset \mathbf{E}_{\mathbb{R}}(M)$ with

$$\mathrm{Cap}((\hat{F}_{N,R})^c) < \frac{1}{N}, \tag{3.36}$$

and $w_{k,R}$ converges uniformly on $\hat{F}_{N,R}$ to zero as $k \to \infty$. Here Cap denotes the capacity associated to the Dirichlet form $(\mathscr{E}^{\nu}_{\mathbb{R}}, \mathscr{D}(\mathscr{E}^{\nu}_{\mathbb{R}}))$. In particular, for every open set $U \subset \mathbf{E}_{\mathbb{R}}(M)$

$$\mathrm{Cap}(U) := \inf\{\mathscr{E}^{\nu}_{\mathbb{R},1}(w,w) | w \in \mathscr{D}(\mathscr{E}^{\nu}_{\mathbb{R}}), w \ge G_1\psi \quad \mu^{\nu}_{\mathbb{R}}\text{-a.e. on } U\},$$

where $\psi \in L^2(\mathbf{E}_{\mathbb{R}}(M), \mu^{\nu}_{\mathbb{R}})$ with $\psi > 0$ is arbitrarily chosen, for $\beta \in \mathbb{R}^+, w \in \mathscr{D}(\mathscr{E}^{\nu}_{\mathbb{R}})$, $\mathscr{E}^{\nu}_{\mathbb{R},\beta}(w,w) := \mathscr{E}^{\nu}_{\mathbb{R}}(w,w) + \beta\mu^{\nu}_{\mathbb{R}}(w^2)$ and $(G_\alpha)_{\alpha>0}$ is the resolvent associated to the Dirichlet form $(\mathscr{E}^{\nu}_{\mathbb{R}}, \mathscr{D}(\mathscr{E}^{\nu}_{\mathbb{R}}))$ (we refer readers to [51, Chapter III. Defi. 2.4] for more details).

Set $F_{N,R} := \hat{F}_{N,R} \cap B_R$. Since by definition of $\mathscr{F}C_c$ and $\mathscr{D}(\mathscr{E}_{\mathbb{R}}^{\nu})$, it is easy to verify that $\mathscr{F}C_c \subset \bigcup_{R=1}^{\infty} \mathscr{D}_{B_R}(\mathscr{E}_{\mathbb{R}}^{\nu}) \subset \mathscr{D}(\mathscr{E}_{\mathbb{R}}^{\nu})$, which implies that $\bigcup_{R=1}^{\infty} \mathscr{D}_{B_R}(\mathscr{E}_{\mathbb{R}}^{\nu})$ is dense in $\mathscr{D}(\mathscr{E}_{\mathbb{R}}^{\nu})$ (with respect to $\mathscr{E}_{\mathbb{R},1}^{\nu}$ norm). Then according to [51, Theorem III-2.11] we obtain

$$\lim_{R\to\infty} \mathrm{Cap}(B_R^c) = 0. \tag{3.37}$$

Note that by (3.36)

$$\begin{aligned}\mathrm{Cap}((F_{N,R})^c) \le& \mathrm{Cap}((\hat{F}_{N,R})^c) + \mathrm{Cap}(B_R^c)\\ \le& \frac{1}{N} + \mathrm{Cap}(B_R^c).\end{aligned}$$

Combining this with (3.37) yields

$$\lim_{N,R\to\infty} \mathrm{Cap}((F_{N,R})^c) = 0. \tag{3.38}$$

Moreover, we have $w_{k,R} \to 0$ uniformly on $F_{N,R} \subset B_R$ as $k \to \infty$ and $\phi_R(\int_0^1 \rho(o, \gamma(s))\mathrm{d}s) = 1$ on B_R, therefore due to the definition of $w_{k,R}$ it is not difficult to verify for every fixed $N, R \in \mathbb{N}^+$,

$$\lim_{k\to\infty} \sup_{\gamma\in F_{N,R}} \inf_{m\le k} \varphi\Big(\tilde{d}(\gamma, \xi_m)\Big) = 0.$$

Hence for every $0 < \varepsilon < 1$ there exists $k \in \mathbb{N}^+$ such that $w_{k,R} < \varepsilon$ on $F_{N,R}$, which implies that $F_{N,R} \subset \cup_{m=1}^{k} B(\xi_m, \varepsilon)$, where $B(\xi_m, \varepsilon) := \{\gamma \in \mathbf{E}_{\mathbb{R}}(M); \tilde{d}(\xi_m, \gamma) < \varepsilon\}$ denotes the ball in $(\mathbf{E}_{\mathbb{R}}(M), \tilde{d})$. Consequently, for every $N, R \in \mathbb{N}^+$, $F_{N,R}$ is totally bounded, hence compact.

By now we have shown that $\{F_{N,R}\}_{N,R=1}^{\infty}$ is a compact $\mathscr{E}$-nest. So (i) of [51, Definition IV-3.1] holds

For any $\gamma, \eta \in \mathbf{E}_{\mathbb{R}}(M)$ with $\varepsilon := \tilde{d}(\gamma, \eta) > 0$, then there exist $R \in \mathbb{N}$ and certain ξ_M such that $\tilde{d}(\xi_M, \eta) < \frac{\varepsilon}{4}$ and $\tilde{d}(\xi_M, \gamma) > \frac{\varepsilon}{4}$. Taking a R large enough such that $\phi_R\Big(\int_0^1 \rho(\gamma(s), o)\mathrm{d}s\Big) = \phi_R\Big(\int_0^1 \rho(\eta(s), o)\mathrm{d}s\Big) = 1$, then it is easy to see $v_{M,R}(\gamma) \neq v_{M,R}(\eta)$. Hence $\{v_{m,R}(\gamma), m, R \in \mathbb{N}^+\}$ separate points and (iii) of [51, Definition IV-3.1] follows. Following the same procedures as in the proof of Theorem 2.2 and Theorem 4.1 above, we could check (ii) in [51, Definition IV-3.1]. By now we have finished the proof. □

By using the theory of Dirichlet form (refer to [51]), we obtain the following associated diffusion process. Furthermore, we also obtain that the process is conservative in the sense that the lifetime of the process is infinity. If the reference measure is finite, it is easy to see $1 \in \mathscr{D}(\mathscr{E}_{\mathbb{R}}^{\nu})$ and $\mathscr{E}_{\mathbb{R}}^{\nu}(1,1) = 0$, which implies the processes are conservative and recurrent. However, in this case $1 \notin \mathscr{D}(\mathscr{E}_{\mathbb{R}}^{\nu})$. Motivated by [21] for the finite dimensional case, we construct suitable approximation functions and obtain that the processes are conservative under mild assumptions.

Theorem 3.13. *Suppose that* (3.19) *holds. There exists a (Markov) diffusion process* $\mathbf{M} = (\Omega, \mathscr{F}, \mathscr{M}_t, (X(t))_{t\geq 0}, (\mathbf{P}^z)_{z\in \mathbf{E}_{\mathbb{R}}(M)})$ *on* $\mathbf{E}_{\mathbb{R}}(M)$ *properly associated with* $(\mathscr{E}_{\mathbb{R}}^{\nu}, \mathscr{D}(\mathscr{E}_{\mathbb{R}}^{\nu}))$, *i.e. for* $u \in L^2(\mathbf{E}_{\mathbb{R}}(M); \mu_{\mathbb{R}}^{\nu}) \cap \mathscr{B}_b(\mathbf{E}_{\mathbb{R}}(M))$, *the transition semigroup* $P_t u(z) := \mathbb{E}^z[u(X(t))]$ *is an* $\mathscr{E}_{\mathbb{R}}^{\nu}$*-quasi-continuous version of* $T_t u$ *for all* $t > 0$, *where* T_t *is the semigroup associated with* $(\mathscr{E}_{\mathbb{R}}^{\nu}, \mathscr{D}(\mathscr{E}_{\mathbb{R}}^{\nu}))$. *Moreover, the results in Theorem 2.4 also hold in this case.*

Moreover, if conditions (3.20) *and* (3.21) *hold, then the diffusion process* $\mathbf{M} = (\Omega, \mathscr{F}, \mathscr{M}_t, (X(t))_{t\geq 0}, (\mathbf{P}^z)_{z\in \mathbf{E}_{\mathbb{R}}(M)})$ *is conservative in the sense that* $T_t 1 = 1$ $\mu_{\mathbb{R}}^{\nu}$*-a.e. for all* $t > 0$ *(c.f. [37, Section 1.6 P56]).*

In particular, for $M = \mathbb{R}$, ν *being Lebesgue measure, the diffusion process* $\mathbf{M} = (\Omega, \mathscr{F}, \mathscr{M}_t, (X(t))_{t\geq 0}, (\mathbf{P}^z)_{z\in \mathbf{E}_{\mathbb{R}}(M)})$ *is recurrent in the sense that* $Gf = 0$ *or* ∞ $\mu_{\mathbb{R}}^{\nu}$*-a.e. with* $f \in L^1(\mathbf{E}_{\mathbb{R}}(M); \mu_{\mathbb{R}}^{\nu})$, $f \geq 0$ *(c.f. [37, Section 1.6 P56]). Here* $Gf = \int_0^\infty T_t f \mathrm{d}t$.

Proof. The existence of a diffusion process is the same as that for Theorem 2.4 (due to quasi-regularity of $(\mathscr{E}_{\mathbb{R}}^{\nu}, \mathscr{D}(\mathscr{E}_{\mathbb{R}}^{\nu}))$), so we omit it here.

Step (1) We first prove that the process is conservative.

Choose $\phi_R \in C_c^\infty(\mathbb{R})$ to be the same function as that in the proof of Theorem 3.12. For every $R > 0$, we define $\Phi_R(\gamma) := \phi_R\Big(\int_0^1 \rho(o, \gamma(s))\mathrm{d}s\Big)$. For $N > 0$, choose $F \in L^2(\mathbf{E}_{\mathbb{R}}(M), \mu_{\mathbb{R}}^{\nu}), F \geq 0$ with $F(\gamma) = \phi_N(\int_0^1 \rho(o, \gamma(s))\mathrm{d}s)$. Let $(L, \mathscr{D}(L))$ denote the infinitesimal generator associated with $(\mathscr{E}_{\mathbb{R}}^{\nu}, \mathscr{D}(\mathscr{E}_{\mathbb{R}}^{\nu}))$, then it holds that $u_t := T_t F \in \mathscr{D}(L)$ for all $t > 0$.

Note that

$$D\Phi_R(\gamma)(s) = \phi_R'\Big(\int_0^1 \rho(\gamma(s), o)\mathrm{d}s\Big)\Big(U_s^{-1}(\gamma)\nabla_1 \rho(\gamma(s), o)\Big)1_{[0,1]}(s).$$

Since $D\Phi_R(\gamma) = 0$ for all γ satisfying $\inf_{t\in[0,1]} \rho(\gamma(s), o) > R + 1$, by (3.26) and (3.27) we obtain for all $R > 1$,

(3.39)

$$\begin{aligned}
&\mathscr{E}_{\mathbb{R}}^{\nu}(\Phi_R) = \int_M \int_{\mathbf{E}_{\mathbb{R}}^x(M)} |D\Phi_R(\gamma)|_{\mathbf{H}}^2 \mathrm{d}\mu_{\mathbb{R}}^x \nu(\mathrm{d}x) \\
&\leq \int_{B(o,2R)} \int_{\mathbf{E}_{\mathbb{R}}^x(M)} \mathrm{d}\mu_{\mathbb{R}}^x \nu(\mathrm{d}x) + \int_{B(o,2R)^c} \mu_{\mathbb{R}^+}^x\Big(\sup_{s\in[0,1]} \rho(x, \gamma(s)) \geq \rho(o, x) - R - 1\Big)\nu(\mathrm{d}x) \\
&\leq \nu(B(o, 2R)) + c_1 \exp(-c_2 R^2) \leq c_3 \exp(c_4 R^\zeta),
\end{aligned}$$

where $c_1 - c_4$ are positive constants independent of R, $\zeta := \max\{1 + \frac{\alpha}{2}, \beta\} < 2$ with

$\alpha, \beta \in (0,1)$ being the constants in (3.20) and (3.21). Then we have

(3.40)

$$
\begin{aligned}
\mu_{\mathbb{R}}^{\nu}(F\Phi_R) - \mu_{\mathbb{R}}^{\nu}(u_t\Phi_R) =& - \int_0^t \frac{d}{ds}\mu_{\mathbb{R}}^{\nu}(u_s\Phi_R)\mathrm{d}s = - \int_0^t \mu_{\mathbb{R}}^{\nu}(Lu_s\Phi_R)\mathrm{d}s \\
=& \int_0^t \int \langle Du_s, D\Phi_R\rangle_{\mathbf{H}}\mathrm{d}\mu_{\mathbb{R}}^{\nu}\mathrm{d}s = \int_0^t \int \langle \varphi_{N,R}Du_s, \varphi_{N,R}^{-1}D\Phi_R\rangle_{\mathbf{H}}\mathrm{d}\mu_{\mathbb{R}}^{\nu}\mathrm{d}s \\
\le& \Big(\int_0^t \int |\varphi_{N,R}Du_s|_{\mathbf{H}}^2\mathrm{d}\mu_{\mathbb{R}}^{\nu}\mathrm{d}s\Big)^{1/2}\Big(\int_0^t \int |\varphi_{N,R}^{-1}D\Phi_R|_{\mathbf{H}}^2\mathrm{d}\mu_{\mathbb{R}}^{\nu}\mathrm{d}s\Big)^{1/2},
\end{aligned}
$$

where the operator D on u_s is the closure of D defined in (3.7) and

$$\varphi_{N,R}(\gamma) := \exp\Big(\theta\psi_{N,R}(\int_0^1 \rho(\gamma(s), o)\mathrm{d}s)\Big),$$

for some $\theta > 0$, $R > 2(N+1)$ and $\psi_{N,R} \in C_b^1(\mathbb{R}^+)$ satisfies $\|\psi_{N,R}'\|_\infty \le 2$, $\psi_{N,R}(t) = t$ for $t \in [R, R+1]$ and $\psi_{N,R}(t) = 0$ for $t \in [0, N+1]$. Define $\varphi_{N,R,M} := \varphi_{N,R}\Phi_M$. It is obvious that $\varphi_{N,R,M} \in \mathscr{F}C_c$ and $\lim_{M\to\infty}\varphi_{N,R,M} = \varphi_{N,R}$ $\mu_{\mathbb{R}}^{\nu}$-a.s. γ. By [51, Corollary I-4.15] we know $\varphi_{N,R,M}^2 u_t \in \mathscr{D}(\mathscr{E}_{\mathbb{R}}^{\nu})$. Furthermore we have

$$
\begin{aligned}
&\frac{\partial}{\partial t}\mu_{\mathbb{R}}^{\nu}(\varphi_{N,R,M}^2 u_t^2) = 2\mu_{\mathbb{R}}^{\nu}(\varphi_{N,R,M}^2 Lu_t \cdot u_t) = -2\int \langle Du_t, D(\varphi_{N,R,M}^2 u_t)\rangle_{\mathbf{H}}\mathrm{d}\mu_{\mathbb{R}}^{\nu} \\
=& -2\int \langle Du_t, 2u_t\varphi_{N,R,M}D\varphi_{N,R,M} + \varphi_{N,R,M}^2 Du_t\rangle_{\mathbf{H}}\mathrm{d}\mu_{\mathbb{R}}^{\nu} \\
\le& -2\int |\varphi_{N,R,M}Du_t|_{\mathbf{H}}^2\mathrm{d}\mu_{\mathbb{R}}^{\nu} + 2\Big(\lambda^{-1}\int |\varphi_{N,R,M}Du_t|_{\mathbf{H}}^2\mathrm{d}\mu_{\mathbb{R}}^{\nu} + \lambda\int |u_t D\varphi_{N,R,M}|_{\mathbf{H}}^2\mathrm{d}\mu_{\mathbb{R}}^{\nu}\Big) \\
\le& -2\int |\varphi_{N,R,M}Du_t|_{\mathbf{H}}^2\mathrm{d}\mu_{\mathbb{R}}^{\nu} + 2\Big(\lambda^{-1}\int |\varphi_{N,R,M}Du_t|_{\mathbf{H}}^2\mathrm{d}\mu_{\mathbb{R}}^{\nu} + 8\lambda\theta^2\mu_{\mathbb{R}}^{\nu}(\varphi_{N,R,M}^2 u_t^2) \\
&+ 2\lambda\mu_{\mathbb{R}}^{\nu}(\varphi_{N,R}^2|D\Phi_M|_{\mathbf{H}}^2 u_t^2)\Big).
\end{aligned}
$$

Here the last step is due to the property $|D\varphi_{N,R,M}|_{\mathbf{H}}^2 \le 8\theta^2\varphi_{N,R,M}^2 + 2\varphi_{N,R}^2|D\Phi_M|_{\mathbf{H}}^2$.

Choosing $\lambda = 1$ and using Gronwall's Lemma we obtain that

$$\mu_{\mathbb{R}}^{\nu}(\varphi_{N,R,M}^2 u_t^2) \le \exp(16\theta^2 t)\Big(\mu_{\mathbb{R}}^{\nu}(\varphi_{N,R,M}^2 F^2) + \frac{1}{4\theta^2}\lambda\mu_{\mathbb{R}}^{\nu}(\varphi_{N,R}^2|D\Phi_M|_{\mathbf{H}}^2 u_t^2)\Big).$$

By the dominated convergence theorem we know $\lim_{M\to\infty}\mu_{\mathbb{R}}^{\nu}(\varphi_{N,R}^2|D\Phi_M|_{\mathbf{H}}^2 u_t^2) = 0$. Based on this, choosing $\lambda = 2$ and letting $M \to \infty$ we have

$$\int_0^t |\varphi_{N,R}Du_s|_{\mathbf{H}}^2\mathrm{d}s \le 2e^{16\theta^2 t}\mu_{\mathbb{R}}^{\nu}(\varphi_{N,R}^2 F^2). \tag{3.41}$$

For γ with $D\Phi_R(\gamma) \neq 0$ (i.e. $R \leq \int_0^1 \rho(o,\gamma(s))\mathrm{d}s \leq R+1$) it is easy to see $\varphi_{N,R}(\gamma)^{-1} \leq e^{-\theta R}$. Now combining (3.39), (3.40) and (3.41) yields

$$\mu_{\mathbb{R}}^{\nu}(F\Phi_R) - \mu_{\mathbb{R}}^{\nu}(u_t\Phi_R) \leq \left[2c_3 e^{16\theta^2 t}\mu_{\mathbb{R}}^{\nu}(\varphi_{N,R}^2 F^2)e^{-2\theta R} t e^{c_4 R^{\zeta}}\right]^{1/2},$$

Choosing $\theta = \frac{R}{16t}$ we have

$$\mu_{\mathbb{R}}^{\nu}(F\Phi_R) - \mu_{\mathbb{R}}^{\nu}(u_t\Phi_R) \leq \left[c_5 \mu_{\mathbb{R}}^{\nu}(\varphi_{N,R}^2 F^2) t e^{-\frac{R^2}{16t}+c_4 R^{\zeta}}\right]^{1/2}, \tag{3.42}$$

where c_4, c_5 are independent of F, N and R.

We arrive at for all $R > 2(N+1)$

$$\begin{aligned}\mu_{\mathbb{R}}^{\nu}(\Phi_N\Phi_R) - \mu_{\mathbb{R}}^{\nu}(T_t(\Phi_N)\Phi_R) &\leq \left[c_5 \mu_{\mathbb{R}}^{\nu}(\varphi_{N,R}^2 \Phi_N^2) t e^{-\frac{R^2}{16t}+c_4 R^{\zeta}}\right]^{1/2} \\ &= \left[c_5 \mu_{\mathbb{R}}^{\nu}(\Phi_N^2) t e^{-\frac{R^2}{16t}+c_4 R^{\zeta}}\right]^{1/2},\end{aligned}$$

where the last equality is due to the fact $\Phi_N(\gamma) \neq 0$ only if $\varphi_{N,R}(\gamma) = 1$ since $R > 2(N+1)$. Hence letting $R \to \infty$ we derive for every $N > 0$ and $t > 0$ (note that $\zeta < 2$ here)

$$\int \Phi_N \mathrm{d}\mu_{\mathbb{R}}^{\nu} - \int \Phi_N T_t 1 \mathrm{d}\mu_{\mathbb{R}}^{\nu} = \int \Phi_N \mathrm{d}\mu_{\mathbb{R}}^{\nu} - \int T_t(\Phi_N)\mathrm{d}\mu_{\mathbb{R}}^{\nu} \leq 0.$$

Since it always hold $T_t 1 \leq 1$, the above inequality implies that $T_t 1(\gamma) = 1$ for all $\gamma \in \mathbf{E}_{\mathbb{R}}(M)$ satisfying $\int_0^1 \rho\big(\gamma(s), o\big)\mathrm{d}s \leq N$. Also note that N is arbitrary, we obtain $T_t 1(\gamma) = 1$ for $\mu_{\mathbb{R}}^{\nu}$-a.e. $\gamma \in \mathbf{E}_{\mathbb{R}}(M)$ immediately, therefore the process $\mathbf{M}$ is conservative.

Step (ii) Now we prove the recurrence property. Choosing $\tilde{\phi}_R \in C_c^{\infty}(\mathbb{R}^+)$ satisfying

$$\tilde{\phi}_R(x) = \begin{cases} 1, & \text{if } x \leq R, \\ \in (0,1), & \text{if } R < x < 2R, \\ 0. & \text{if } x > 2R, \end{cases}$$

and $\|\phi_R'\|_{\infty} \leq \frac{1}{R}$. We define $\tilde{\Phi}_R(\gamma) := \tilde{\phi}_R\Big(\int_0^1 \rho(o,\gamma(s))\mathrm{d}s\Big)$. Then we have

$$D\tilde{\Phi}_R(\gamma)(s) = \phi_R'\Big(\int_0^1 \rho(\gamma(s),o)\mathrm{d}s\Big)\Big(U_s^{-1}(\gamma)\nabla_1\rho(\gamma(s),o)\Big)1_{[0,1]}(s).$$

Now it holds $|D\tilde{\Phi}_R|_{\mathbf{H}} \leq \frac{1}{R}$ and $D\tilde{\Phi}_R(\gamma) = 0$ all γ satisfying $\inf_{t\in[0,1]} \rho(\gamma(s),o) > 2R$, then still according to (3.27) we get

$$
\begin{aligned}
&\mathscr{E}^{\nu}_{\mathbb{R}}(\tilde{\Phi}_R)=\int_M\int_{\mathbf{E}^x_{\mathbb{R}}(M)}|D\tilde{\Phi}_R(\gamma)|^2_{\mathbf{H}}\mathrm{d}\mu^x_{\mathbb{R}}\nu(\mathrm{d}x)\\
&\leq\frac{1}{R^2}\int_{B(o,3R)}\nu(\mathrm{d}x)+\int_{B(o,3R)^c}\int_{\mathbf{E}^x_{\mathbb{R}^+}(M)}\mu^x_{\mathbb{R}^+}\Big(\sup_{s\in[0,1]}\rho(x,\gamma(s))\geq\rho(o,x)-2R-1\Big)\nu(\mathrm{d}x)\\
&\leq\frac{c_6}{R}+c_6\exp(-c_7R^2)\to 0, R\to\infty.
\end{aligned}
$$

Therefore we have found as series of $\tilde{\Phi}_R$ such that $\tilde{\Phi}_R\to 1$ $\mu^{\nu}_{\mathbb{R}}$-a.e. as $R\to\infty$ and $\mathscr{E}^{\nu}_{\mathbb{R}}(\tilde{\Phi}_R)\to 0$ as $R\to\infty$, so the recurrence follows by [37, Theorem 1.6.5].

□

Remark 3.14. *By integration by parts formula obtained in [18] and carefully tracking the proof of Theorem 3.12 and 3.13, we could verify that if* (3.20) *and* (3.30) *are true, then the conclusions of Theorems 3.12 and 3.13 still hold for* $(\tilde{\mathscr{E}}^{\nu}_{\mathbb{R}},\mathscr{D}(\tilde{\mathscr{E}}^{\nu}_{R}))$ *with* $\nu(\mathrm{d}x)=p_1(x,x)\mathrm{d}x$. *Here* $(\tilde{\mathscr{E}}^{\nu}_{\mathbb{R}},\mathscr{D}(\tilde{\mathscr{E}}^{\nu}_{R}))$ *is defined in Remark 3.7.*

Furthermore a similar argument implies that the results in Theorems 3.12 and 3.13 also hold for the reference measure given by $e^{c\int_0^1\mathrm{Scal}(\gamma(s))\mathrm{d}s}\tilde{\mu}^{\nu}(\gamma)$, *if* (3.20) *and* (3.30) *are true. Here* $c\in\mathbb{R}$ *and* Scal *denotes the scalar curvature.*

Remark 3.15. (Finite Volume Case for the line) *For each* $A_1,A_2\in[0,\infty)$, *we could also construct Wiener measure on* $C([-A_2,A_1],M)$. *In this case the above results also hold.*

4 Ergodicity/ Non-ergodicity

4.1 Half line

In this section, we study the long time behavior of the Markov process $X(t),t\geq 0$, and the L^2-Dirichlet form $(\mathscr{E}^o_{\mathbb{R}^+},\mathscr{D}(\mathscr{E}^o_{\mathbb{R}^+}))$ constructed in Section 2. In fact, we establish some functional inequalities associated with $(\mathscr{E}^o_{\mathbb{R}^+},\mathscr{D}(\mathscr{E}^o_{\mathbb{R}^+}))$, which gives ergodicity or non-ergodicity of the corresponding Markov process $X(t),t\geq 0$.

4.1.1 M has strictly positive Ricci curvature

Theorem 4.1. *[Log-Sobolev inequality and Poincaré inequality]*

(1) Suppose that Ric $\geq K$ *for* $K>0$, *then the log-Sobolev inequality holds*

$$\mu^o_{\mathbb{R}^+}(F^2\log F^2)\leq 2C(K)\mathscr{E}^o_{\mathbb{R}^+}(F,F),\qquad F\in\mathscr{F}C^1_b,\ \mu^o_{\mathbb{R}^+}(F^2)=1,\tag{4.1}$$

where $C(K):=\frac{4}{K^2}$.

(2) Suppose that M is compact and there exists $\varepsilon \in (0,1)$ such that

$$\delta_\varepsilon := \sup_{T\in[0,\infty)} \delta_\varepsilon(T) < \infty, \tag{4.2}$$

where

(4.3)

$$\delta_\varepsilon(T) := \varepsilon^{-1}\big(1-e^{-\varepsilon T}\big)\int_0^T e^{\varepsilon s}\eta(s)\mathrm{d}s,\ \ \eta(s) := \sup_{x\in M}\mu^x_{\mathbb{R}^+}\Big[\exp\Big(-\int_0^s K(\gamma(r))\mathrm{d}r\Big)\Big],$$

and $K(x) := \inf\{\mathrm{Ric}_x(X,X); X\in T_xM, |X|=1\}$, $x\in M$. Then the following Poincaré inequality holds,

$$\mu^o_{\mathbb{R}^+}(F^2) - \mu^o_{\mathbb{R}^+}(F)^2 \le \delta_\varepsilon \mathscr{E}^o_{\mathbb{R}^+}(F,F), \qquad F\in \mathscr{F}C^1_b, \tag{4.4}$$

where δ_ε is defined by (4.2).

Remark 4.2. *Obviously if*

$$\limsup_{t\uparrow\infty}\frac{1}{t}\sup_{x\in M}\log\mu^x_{\mathbb{R}^+}\Big[\exp\Big(-\int_0^s K(\gamma(r))\mathrm{d}r\Big)\Big] < 0, \tag{4.5}$$

then condition (4.2) holds.

Moreover, as explained in [27, 59], condition (4.5) is equivalent to the spectral positivity of the operator $L_0 = -\Delta + K$ (here $L_0 f(x) := \Delta f(x) + K(x)f(x)$). In particular, if $\mathrm{Ric}\ge K$ for some constant $K>0$, then (4.2) holds.

Remark 4.3. *(i) According to [61], the log-Sobolev inequality implies hypercontractivity of the associated semigroup P_t and Poincaré inequality, which derives the L^2-exponential ergodicity of the process: $\|P_tF - \int F d\mu\|_{L^2} \le e^{-t/C(K)}\|F\|_{L^2}$.*

(ii) Poincaré inequality also implies the irreducibility of the Dirichlet form $(\mathscr{E}^o_{\mathbb{R}^+}, \mathscr{D}(\mathscr{E}^o_{\mathbb{R}^+}))$. It is obvious that the Dirichlet form $(\mathscr{E}^o_{\mathbb{R}^+}, \mathscr{D}(\mathscr{E}^o_{\mathbb{R}^+}))$ is recurrent. Combining these two results, by [37, Theorem 4.7.1], for any nearly Borel non-exceptional set B,

$$\mathbf{P}^z(\sigma_B\circ\theta_n<\infty, \forall n\ge 0) = 1, \qquad \text{for q.e. } z\in \mathbf{E}^o_{\mathbb{R}^+}(M).$$

Here $\sigma_B = \inf\{t>0 : X(t)\in B\}$, θ is the shift operator for the Markov process X, and for the definition of any nearly Borel non-exceptional set we refer to [37]. Moreover by [37, Theorem 4.7.3] we obtain the following strong law of large numbers: for $f\in L^1(\mathbf{E}^o_{\mathbb{R}^+}(M), \mu^o_{\mathbb{R}^+})$

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t f(X(s))\mathrm{d}s = \int f\mathrm{d}\mu^o_{\mathbb{R}^+}, \quad \mathbf{P}^z - a.s.,$$

for q.e. $z\in\mathbf{E}^o_{\mathbb{R}^+}(M)$.

Proof of Theorem 4.1. Step (1) By the standard method and the technique in [31](See also [40] and [55] and references therein), it is not difficult to prove (4.1). For the reader's convenience, in the following we give a detailed proof.

By [40] we have the martingale representation theorem, that is, for $F \in \mathscr{F}C_b^1$ with the form

(4.6)

$$F(\gamma) = f\left(\int_0^{T_1} g_1(s,\gamma(s))\mathrm{d}s, \int_0^{T_2} g_2(s,\gamma(s))\mathrm{d}s, ..., \int_0^{T_m} g_m(s,\gamma(s))\mathrm{d}s\right), \quad \gamma \in \mathbf{E}^o_{\mathbb{R}^+}(M),$$

we have

$$F = \mu^o_{\mathbb{R}^+}(F) + \int_0^T \langle H^F_s, \mathrm{d}\beta_s\rangle, \tag{4.7}$$

where $T = \max T_i$, β_s is the anti-development of canonical path $\gamma(\cdot)$ (whose distribution is an $\mathbb{R}^n$-valued Brownian motion under $\mu^o_{\mathbb{R}^+}$) and

$$H^F_s = \mu^o_{\mathbb{R}^+}\left[M_s^{-1}\int_s^T M_r(DF(r))\mathrm{d}r\middle|\mathscr{F}_s\right]. \tag{4.8}$$

Here and in the following $(\mathscr{F}_t)$ is the natural filtration generated by $\gamma(\cdot)$, $\mu^o_{\mathbb{R}^+}[\cdot|\mathscr{F}_t]$ denotes the conditional expectation under $\mu^o_{\mathbb{R}^+}$ and M_t is the solution of the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}M_t + \frac{1}{2}M_t\mathrm{Ric}_{U_t} = 0, \quad M_0 = I. \tag{4.9}$$

Let $F = G^2$ for $G \in \mathscr{F}C_b^1$ being strictly positive and with the form (4.6), consider the continuous version of the martingale $N_s = \mathbb{E}[F|\mathscr{F}_s]$. By the lower bound of the Ricci curvature it is easy to verify for every $0 \le s \le r < \infty$

$$\|M_s^{-1}M_r\| \le \exp\Big(-\frac{1}{2}\int_s^r K(\gamma(t))\mathrm{d}t\Big) \le \exp\Big(-\frac{K(r-s)}{2}\Big), \tag{4.10}$$

where $\|\cdot\|$ denotes the matrix norm. Then we can take the conditional expectation $\mu^o_{\mathbb{R}^+}[\cdot|\mathscr{F}_s]$ in (4.7) to obtain

$$N_s = \mu^o_{\mathbb{R}^+}[F] + \int_0^s \langle H^F_r, \mathrm{d}\beta_r\rangle. \tag{4.11}$$

Now applying Itô's formula to $N_s \log N_s$, we have

$$\begin{aligned} &\mu^o_{\mathbb{R}^+}\left(G^2\log G^2\right) - \mu^o_{\mathbb{R}^+}(G^2)\log\mu^o_{\mathbb{R}^+}(G^2) \\ &= \mu^o_{\mathbb{R}^+}\left(N_T\log N_T\right) - \mu^o_{\mathbb{R}^+}\left(N_0\log N_0\right) = \frac{1}{2}\mu^o_{\mathbb{R}^+}\left[\int_0^T N_s^{-1}|H^F_s|^2\mathrm{d}s\right]. \end{aligned} \tag{4.12}$$

Here and in the following we use $|\cdot|$ to denote the norm in $\mathbb{R}^d$. Note that

$$DF = D(G^2) = 2GDG.$$

Using this relation in the explicit formula (4.8) for H^F, we have

$$H_s^F = 2\mu^o_{\mathbb{R}^+}\left[GM_s^{-1}\int_s^T M_r DG(r)\mathrm{d}r\Big|\mathscr{F}_s\right]. \tag{4.13}$$

By Cauchy-Schwarz inequality in (4.13) and (4.10), we have

$$|H_s^F|^2 \le 4\mu^o_{\mathbb{R}^+}[G^2|\mathscr{F}_s]\mu^o_{\mathbb{R}^+}\left[\left(\int_s^T e^{-K(r-s)/2}|DG(r)|\mathrm{d}r\right)^2\Big|\mathscr{F}_s\right].$$

Thus the right hand side of (4.12) can be controlled by

$$2\mu^o_{\mathbb{R}^+}\left[\int_0^T\left(\int_s^T e^{-K(r-s)/2}|DG(r)|\mathrm{d}r\right)^2\mathrm{d}s\right]. \tag{4.14}$$

By Hölder's inequality we have

$$\left(\int_s^T e^{-K(r-s)/2}|DG(r)|\mathrm{d}r\right)^2 \le \int_s^T e^{-K(r-s)/2}\mathrm{d}r\int_s^T e^{-K(r-s)/2}|DG(r)|^2\mathrm{d}r.$$

Then changing the order of integration we obtain

$$\mu^o_{\mathbb{R}^+}\left(\int_0^T\left(\int_s^T e^{-K(r-s)/2}|DG(r)|\mathrm{d}\tau\right)^2\mathrm{d}s\right) \le \mu^o_{\mathbb{R}^+}\left(\int_0^T J_1(s,T)|DG(s)|^2\mathrm{d}s\right),$$

where

$$\begin{aligned}J_1(s,T) :=& \int_0^s \frac{2}{K}\big(1-e^{-K(T-t)/2}\big)e^{-K(s-t)/2}\mathrm{d}t\\ =&\frac{2}{K^2}\left[2(1-e^{-\frac{Ks}{2}}) - e^{-\frac{K(T-s)}{2}} + e^{-\frac{K(T+s)}{2}}\right] \le \frac{4}{K^2}, \quad \forall\ s\in[0,T]\end{aligned}$$

Hence

$$\mu^o_{\mathbb{R}^+}\left(\int_0^T\left[\left(\int_s^T e^{-K(r-s)/2}|DG(r)|\mathrm{d}r\right)^2\right]\mathrm{d}s\right) \le \frac{4}{K^2}\mathscr{E}^o_{\mathbb{R}^+}(G,G).$$

Combining all above estimates into (4.12), we complete the proof for (4.1).

Step (2) Some proof in this step is inspired by that of [59, Theorem 1]. Still applying Itô formula to N_s^2 (where $N_s = \mu^o_{\mathbb{R}^+}[F|\mathscr{F}_s]$ and $F\in\mathscr{F}C_b^1$ with the form (4.6)) we arrive at

$$\begin{aligned}&\mu^o_{\mathbb{R}^+}(F^2) - \mu^o_{\mathbb{R}^+}(F)^2\\ &= \mu^o_{\mathbb{R}^+}(N_T^2) - \mu^o_{\mathbb{R}^+}(N_0^2) = \mu^o_{\mathbb{R}^+}\Big(\int_0^T |H_s^F|^2\mathrm{d}s\Big).\end{aligned} \tag{4.15}$$

By (4.8), (4.10), Markov property and Cauchy-Schwartz inequality we obtain

$$\begin{aligned}&\left|H_s^F\right|^2 \le \mu^o_{\mathbb{R}^+}\left[\int_s^T \exp\left(-\int_s^r K(\gamma(t))\mathrm{d}t\right)e^{-\varepsilon(T-r)}\mathrm{d}r\Big|\mathscr{F}_s\right]\mu^o_{\mathbb{R}^+}\left[\int_s^T e^{\varepsilon(T-r)}|DF(r)|^2\mathrm{d}r\Big|\mathscr{F}_s\right]\\&\le \left(\int_s^T \sup_{x\in M}\mu^x_{\mathbb{R}^+}\left[\exp\left(-\int_0^{r-s} K(\gamma(t))\mathrm{d}t\right)\right]e^{-\varepsilon(T-r)}\mathrm{d}r\right)\mu^o_{\mathbb{R}^+}\left[\int_s^T e^{\varepsilon(T-r)}|DF(r)|^2\mathrm{d}r\Big|\mathscr{F}_s\right]\\&= \left(\int_s^T \eta(r-s)e^{-\varepsilon(T-r)}\mathrm{d}r\right)\mu^o_{\mathbb{R}^+}\left[\int_s^T e^{\varepsilon(T-r)}|DF(r)|^2\mathrm{d}r\Big|\mathscr{F}_s\right],\end{aligned}$$

where in the second inequality we used the Markov property of the canonical process $\gamma(\cdot)$ and $\eta(t)$ is defined by (4.3). Therefore let $\phi(t) := \int_0^t\left(\int_s^T \eta(r-s)e^{-\varepsilon(T-r)}\mathrm{d}r\right)\mathrm{d}s$, $t\in[0,T]$ it holds

$$\begin{aligned}\mu^o_{\mathbb{R}^+}\Big(\int_0^T |H_s^F|^2\mathrm{d}s\Big) &\le \int_0^T \Big(\int_s^T \eta(r-s)e^{-\varepsilon(T-r)}\mathrm{d}r\Big)\Big(\int_s^T e^{\varepsilon(T-r)}\mu^o_{\mathbb{R}^+}(|DF(r)|^2)\mathrm{d}r\Big)\mathrm{d}s\\&= \int_0^T \phi'(s)\Big(\int_s^T e^{\varepsilon(T-r)}\mu^o_{\mathbb{R}^+}(|DF(r)|^2)\mathrm{d}r\Big)\mathrm{d}s\\&= \mu^o_{\mathbb{R}^+}\Big(\int_0^T \phi(r)e^{\varepsilon(T-r)}|DF(r)|^2\mathrm{d}r\Big).\end{aligned}$$

Since by elementary calculation it is easy to check $\sup_{r\in[0,T]}\phi(r)e^{\varepsilon(T-r)} \le \delta_\varepsilon(T)$, combining all the estimates into (4.15) yields (4.4). □

4.1.2 $M=\mathbb{R}^n$

In this subsection we consider the case that $M=\mathbb{R}^n$ and $o=0\in\mathbb{R}^n$ and we use X_t to denote $X(t)$ for simplicity. As mentioned in the introduction, it is easy to see that the Markov process $(X_t)_{t\ge 0}$ associated with $(\mathscr{E}^o_{\mathbb{R}^+}, \mathscr{D}(\mathscr{E}^o_{\mathbb{R}^+}))$ is the unique solution to the following stochastic heat equations on $\mathbb{R}^+\times\mathbb{R}^+$

$$\begin{aligned}\partial_t X_t =&\frac{1}{2}\Delta X_t + \xi,\ t>0,\\X_t(0) =&0,\ t>0,\\X_0(\cdot) =&\gamma(\cdot)\in \mathbf{E}^o_{\mathbb{R}^+}(\mathbb{R}^n)\end{aligned}\tag{4.16}$$

where ξ denotes an standard $\mathbb{R}^n$-valued space-time white noise on $\mathbb{R}^+\times\mathbb{R}^+$ (on some probability $(\Omega,\mathscr{F},\mathbf{P})$). In the Euclidean space, we have the following ergodicity results. In this case, the exponential ergodicity does not hold any more, which implies that the L^2-spectral gap is zero.

Theorem 4.4. *Suppose $M=\mathbb{R}^n$, then the following statements hold*

(1) For every $F \in L^2(\mathbf{E}^o_{\mathbb{R}^+}(\mathbb{R}^n);\mu^o_{\mathbb{R}^+})$ we have

$$\lim_{t\to\infty} \mu^o_{\mathbb{R}^+}\Big(\big|P_tF(\gamma)-\mu^o_{\mathbb{R}^+}(F)\big|^2\Big)=0, \tag{4.17}$$

where $P_tF(\gamma) := \mathbf{E}\big[F(X_t^\gamma)\big]$, $(X_t^\gamma)_{t\ge 0}$ is the solution to (4.16) with initial value $X_0(\cdot)=\gamma$.

(2) The Poincaré inequality does not hold, i.e. for any $C>0$, there exists $F \in \mathscr{D}(\mathscr{E}^o_{\mathbb{R}^+})$ such that

$$\mu^o_{\mathbb{R}^+}(F^2)-\mu^o_{\mathbb{R}^+}(F)^2 \ge C\mathscr{E}^o_{\mathbb{R}^+}(F,F). \tag{4.18}$$

In particular, the spectral gap

$$C_{\mathbb{R}^+}(SG) := \inf_{F\neq const, F\in \mathscr{D}(\mathscr{E}^o_{\mathbb{R}^+})} \frac{\mathscr{E}^o_{\mathbb{R}^+}(F,F)}{\mu^o_{\mathbb{R}^+}(F^2)-\mu^o_{\mathbb{R}^+}(F)^2}=0,$$

and the exponential ergodicity does not hold in this case.

Proof. **Step (1)** As explained in [36, Page 315], the solution X_t to (4.16) with initial value $X_0(\cdot)=\gamma$ has the following expression,

$$\begin{aligned} X_t^\gamma(x) &= \int_{\mathbb{R}^+} p(t,x,y)\gamma(y)\mathrm{d}y + \int_0^t\int_{\mathbb{R}^+} p(t-s,x,y)\xi(\mathrm{d}s,\mathrm{d}y)\\ &:= U_1(t,x)+U_2(t,x), \end{aligned}$$

where $p(t,x,y)$ is the Dirichlet heat kernel on $\mathbb{R}^+$ with the following expression

$$p(t,x,y)=\frac{1}{\sqrt{2\pi t}}\Big[\exp\big(-\frac{(x-y)^2}{2t}\big)-\exp\big(-\frac{(x+y)^2}{2t}\big)\Big],\quad x,y\in\mathbb{R}^+,\ t>0.$$

By [36, Lemma 4.3] and the law of iterated logarithm (which implies $\lim_{y\to+\infty}\frac{\gamma(y)}{y}=0$ for $\mu^o_{\mathbb{R}^+}$-a.s. $\gamma\in\mathbf{E}^o_{\mathbb{R}^+}(\mathbb{R}^n)$), it is easy to verify that for $\mu^o_{\mathbb{R}^+}$-a.s. $\gamma\in\mathbf{E}^o_{\mathbb{R}^+}(\mathbb{R}^n)$ and every $x\in\mathbb{R}^+$,

$$\lim_{t\to+\infty} U_1(t,x)=0.$$

Note that $U_2(t,\cdot)=\big(U_2^1(t,\cdot),\dots,U_2^n(t,\cdot)\big)$ is a centered Gaussian vector on $L^2(\mathbb{R}^+;e^{-rx}\mathrm{d}x)$, and for every $x,y\in\mathbb{R}^+$ it holds

$$\begin{aligned} &\lim_{t\uparrow\infty}\mathbf{E}[U_2^i(t,x)U_2^j(t,y)]\\ &=\lim_{t\uparrow\infty}\mathbf{E}\Big[\Big(\int_0^t\int_{\mathbb{R}^+}p(t-s,x,z)\xi^i(\mathrm{d}s,\mathrm{d}z)\Big)\Big(\int_0^t\int_{\mathbb{R}^+}p(t-s,y,z)\xi^j(\mathrm{d}s,\mathrm{d}z)\Big)\Big]\\ &=\delta_i^j\lim_{t\uparrow\infty}\int_0^t\int_{\mathbb{R}^+}p(t-s,x,z)p(t-s,y,z)\mathrm{d}z\mathrm{d}s\\ &=\delta_i^j\lim_{t\uparrow\infty}\int_0^t p(2(t-s),x,y)\mathrm{d}s=\delta_i^j\lim_{t\uparrow\infty}\frac{1}{2}\int_0^{2t}p(s,x,y)\mathrm{d}s=\delta_i^j(x\wedge y),\ 1\le i,j\le n, \end{aligned}$$

where the last calculation can be found in [20, Section 2.3], $\delta_i^j = 1$ when $i = j$ and $\delta_i^j = 0$ when $i \neq j$.

This implies that $U_2(t, \cdot)$ converges weakly in $L^2(\mathbb{R}^+; e^{-rx}\mathrm{d}x)$ as $t \uparrow \infty$ to a Gaussian random vector whose distribution is $\mu^o_{\mathbb{R}^+}$. Combining all the estimates above we know that for $\mu^o_{\mathbb{R}^+}$-a.s. $\gamma \in \mathbf{E}^o_{\mathbb{R}^+}(\mathbb{R}^n)$, $X_t^\gamma(\cdot)$ converges weakly on $L^2(\mathbb{R}^+; e^{-rx}\mathrm{d}x)$ as $t \uparrow \infty$ to a Gaussian random vector whose distribution is $\mu^o_{\mathbb{R}^+}$. Thus for $\mu^o_{\mathbb{R}^+}$-a.s. $\gamma \in \mathbf{E}^o_{\mathbb{R}^+}(\mathbb{R}^n)$ and every $F \in \mathscr{F}C_b^1$ we have

$$\lim_{t\to\infty} P_t F(\gamma) = \mu^o_{\mathbb{R}^+}(F).$$

By this and the dominated convergence theorem we obtain (4.17) holds for $F \in \mathscr{F}C_b^1$ immediately. By approximations we can easily check that (4.17) holds for $F \in L^2(\mathbf{E}^o_{\mathbb{R}^+}(\mathbb{R}^n); \mu^o_{\mathbb{R}^+})$, which implies that $\mu^o_{\mathbb{R}^+}$ is ergodic.

Step (2) We first suppose the Poincaré inequality holds, i.e. for $F \in \mathscr{D}(\mathscr{E}^o_{\mathbb{R}^+})$

$$\mu^o_{\mathbb{R}^+}(F^2) - \mu^o_{\mathbb{R}^+}(F)^2 \leq C\mathscr{E}^o_{\mathbb{R}^+}(F, F) \tag{4.19}$$

for some $C > 0$. For a fixed $T > 0$, let $F_T(\gamma) := \int_0^T \gamma_1(s)\mathrm{d}s$, where $\gamma_1(s)$ denotes the first coordinate of process $\gamma(s) := (\gamma_1(s), \cdots, \gamma_n(s))$. By the proof of Lemma 2.9, it is not difficult to verify that $F_T \in \mathscr{D}(\mathscr{E}^o_{\mathbb{R}^+})$.

At the same time, we have for $o = 0 \in \mathbb{R}^n$

$$\begin{aligned}
\mu^o_{\mathbb{R}^+}(F_T^2) &= \mu^o_{\mathbb{R}^+}\Big(\int_0^T \int_0^T \gamma_1(s)\gamma_1(t)\mathrm{d}s\mathrm{d}t\Big) \\
&= \int_0^T \int_0^T \mu^o_{\mathbb{R}^+}\Big(\gamma_1(s)\gamma_1(t)\Big)\mathrm{d}s\mathrm{d}t = \int_0^T \int_0^T (s \wedge t)\mathrm{d}s\mathrm{d}t \\
&\geq \int_0^T \int_0^t s\mathrm{d}s\mathrm{d}t \geq \frac{T^3}{6},
\end{aligned}$$

$$\mu^o_{\mathbb{R}^+}(F_T) = \int_0^T \mu^o_{\mathbb{R}^+}(\gamma_1(s))\mathrm{d}s = 0,$$

and

$$\mathscr{E}^o_{\mathbb{R}^+}(F_T) = \int_{\mathbf{E}^o_{\mathbb{R}^+}(\mathbb{R}^n)} |DF_T(\gamma)|^2_{\mathbf{H}_+}\mathrm{d}\mu^o_{\mathbb{R}^+} \leq T.$$

Here we have applied the property that $|DF_T(\gamma)(s)| \leq 1_{[0,T]}(s)$. Combining all the estimates above and putting F_T into (4.19) we arrive at $\frac{T^3}{6} \leq CT$. Then letting $T \to \infty$ we get $C = +\infty$ and there is a contradiction. So (4.19) does not hold for any $C > 0$. The results for spectral gap follow from [61]. □

Remark 4.5. *By carefully tracking the proof of Theorem 4.4, it is not difficult to verify that the conclusion of Theorem 4.4 still holds for every initial point $o \in \mathbb{R}^n$, not only $o = 0$.*

4.1.3 M is not a Liouville manifold

In this subsection, we prove that when M is not a Liouville manifold, $(\mathscr{E}^o_{\mathbb{R}+}, \mathscr{D}(\mathscr{E}^o_{\mathbb{R}+}))$ is reducible, which by [11, Propsition 2.1.6] implies that the Markov semigroup $(P_t)_{t\geq 0}$ constructed in Theorem 2.3 is non-ergodic in the sense that there exists a non-constant function $F \in \mathscr{D}(\mathscr{E}^o_{\mathbb{R}+})$ such that $P_t F = F$ $\mu^o_{\mathbb{R}+}$-a.s..

Recall that we call a connected Riemannian manifold M a Louville manifold, if there does not exist a non-constant bounded harmonic function on M. In particular, if M is not a Liouville manifold, then there exists a bounded harmonic function $u : M \to \mathbb{R}$ which is not a constant.

Theorem 4.6. *If M is not a Liouville manifold, then $(\mathscr{E}^o_{\mathbb{R}+}, \mathscr{D}(\mathscr{E}^o_{\mathbb{R}+}))$ is reducible. Hence $\mu^o_{\mathbb{R}+}$ is not ergodic for the Markov process associated with $(\mathscr{E}^o_{\mathbb{R}+}, \mathscr{D}(\mathscr{E}^o_{\mathbb{R}+}))$.*

Proof. The following argument follows essentially from [2, Theorem 4.3] and [60, Theorem 1.5]. Since M is not Louville manifold, we could find a non-constant harmonic function $u : M \to \mathbb{R}$. For every fixed $T > 0$, we define $F_T := \frac{1}{T}\int_0^T u(\gamma(t))\mathrm{d}t$. Since u is harmonic, by Itô's formula we obtain

$$u(\gamma(t)) - u(o) = \int_0^t \langle \nabla u(\gamma(s)), U_s(\gamma)\mathrm{d}\beta_s\rangle_{T_{\gamma(s)}M}, \tag{4.20}$$

where β_s denotes the anti-development of $\gamma(\cdot)$, whose law is an $\mathbb{R}^n$-valued Brownian motion under $\mu^o_{\mathbb{R}+}$. Thus $N_t := u(\gamma(t)) - u(o)$ is a bounded martingale, according to the martingale convergence theorem, there is a non-constant random variable N_∞ such that

$$\lim_{t\uparrow\infty} \mu^o_{\mathbb{R}+}\left(\left|N_t - N_\infty\right|^2\right) = 0,$$

which implies immediately

$$\lim_{T\uparrow\infty} \mu^o_{\mathbb{R}+}\left(\left|F_T - N_\infty\right|^2\right) = 0. \tag{4.21}$$

On the other hand, set $F_T^R := \frac{1}{T}\int_0^T \phi_R(\rho(o,\gamma(s)))u(\gamma(s))\mathrm{d}s$, where $o \in M$, ϕ_R is defined as in the proof of Theorem 3.12. Then by Lemma 2.9 it is easy to see that $F_T^R \in \mathscr{D}(\mathscr{E}^o_{\mathbb{R}+})$ for $R, T > 0$. Note that for fixed $T > 0$, $F_T^R \to F_T$ in $L^2(\mathbf{E}^o_{\mathbb{R}+}(M), \mu^o_{\mathbb{R}+})$, as $R \to \infty$. We also have

$$\begin{aligned}\mathscr{E}^o_{\mathbb{R}+}(F_T^R, F_T^R) \leq& \frac{1}{T^2}\int_0^T \mu^o_{\mathbb{R}+}\left(|\nabla u(\gamma(s))|^2\right)\mathrm{d}s + \frac{4}{T^2}\int_0^T \mu^o_{\mathbb{R}+}\left(|u(\gamma(s))|^2\right)\mathrm{d}s \\ \leq& \frac{1}{T^2}\mu^o_{\mathbb{R}+}\Big(\left|u(\gamma(T)) - u(o)\right|^2\Big) + C_1 \leq C,\end{aligned}$$

where C, C_1 are constants independent of R and the second inequality follows from (4.20). This by [51, Lemma I-2.12] implies that $F_T \in \mathscr{D}(\mathscr{E}^o_{\mathbb{R}+})$ and

$$DF_T(\gamma)(s) = \frac{1}{T}\left(U_s(\gamma)^{-1}\nabla u(\gamma(s))\right)1_{[0,T]}(s),$$

hence

$$\lim_{T\uparrow\infty}\mathscr{E}^o_{\mathbb{R}^+}(F_T,F_T)=\lim_{T\uparrow\infty}\frac{1}{T^2}\int_0^T\mu^o_{\mathbb{R}^+}\big(|\nabla u(\gamma(s))|^2\big)\mathrm{d}s$$
$$(4.22)\qquad =\lim_{T\uparrow\infty}\frac{1}{T^2}\mu^o_{\mathbb{R}^+}\Big(\big|u(\gamma(T))-u(o)\big|^2\Big)\le\lim_{T\uparrow\infty}\frac{4\|u\|_\infty}{T^2}=0,$$

where the second equality follows from (4.20).

Combining (4.21), (4.22) with the closbility of $(\mathscr{E}^o_{\mathbb{R}^+},\mathscr{D}(\mathscr{E}^o_{\mathbb{R}^+}))$ yields that N_∞ is not a constant, $N_\infty\in\mathscr{D}(\mathscr{E}^o_{\mathbb{R}^+})$ and $\mathscr{E}^o_{\mathbb{R}^+}(N_\infty,N_\infty)=0$. So $(\mathscr{E}^o_{\mathbb{R}^+},\mathscr{D}(\mathscr{E}^o_{\mathbb{R}^+}))$ is reducible. □

Note that if M is a Cartan-Hadamard manifold with section curvature $-c_1(\rho(o,x)\vee 1)^2\le \mathrm{Sec}_x(X_1,X_2)\le -c_2(\rho(o,x)\vee 1)^{-2}$ for some $c_1,c_2>0$ and every $x\in M$, $X_1,X_2\in T_xM$ with $|X_1|=|X_2|=1$, then M is not a Louville manifold (where Sec_x denotes the sectional Curvature tensor at $x\in M$). So we have the following result immediately.

Corollary 4.7. *If M is a Cartan-Hadamard manifold with section curvature $-c_1(\rho(o,x)\vee 1)^2\le \mathrm{Sec}_x(X_1,X_2)\le -c_2(\rho(o,x)\vee 1)^{-2}$ for some $c_1,c_2>0$ and every $x\in M$, $X_1,X_2\in T_xM$ with $|X_1|=|X_2|=1$, then $(\mathscr{E}^o_{\mathbb{R}^+},\mathscr{D}(\mathscr{E}^o_{\mathbb{R}^+}))$ is reducible. Hence $\mu^o_{\mathbb{R}^+}$ is not ergodic for the Markov process associated with $(\mathscr{E}^o_{\mathbb{R}^+},\mathscr{D}(\mathscr{E}^o_{\mathbb{R}^+}))$ constructed in Theorem 2.3.*

4.2 The whole line

In this section, we will study the functional inequality and ergodic property for the Dirichlet form $(\mathscr{E}^\nu_{\mathbb{R}},\mathscr{D}(\mathscr{E}^\nu_{\mathbb{R}}))$ constructed in Section 3, where $\nu(\mathrm{d}x)=\nu(x)\mathrm{d}x$ is a probability measure on M which is absolutely continuous with respect to volume (Lebesgue) measure on M. The case for $(\mathscr{E}^o_{\mathbb{R}},\mathscr{D}(\mathscr{E}^o_{\mathbb{R}}))$ is similar and we omit the details here.

As in Section 3, for $\gamma\in\mathbf{E}_{\mathbb{R}}(M)$, we could decompose $\gamma=(\tilde\gamma,\bar\gamma)$ with

$$\tilde\gamma(s):=\gamma(s),\ \bar\gamma(s):=\gamma(-s),\ s\ge 0$$

. We also set

$$M_s(\gamma):=\begin{cases}\hat M_s(\tilde\gamma), & s\ge 0,\\ \hat M_{-s}(\bar\gamma), & s<0.\end{cases}$$

Here $\hat M_t(\gamma)$ denotes the solution to (4.9) with $\gamma\in\mathbf{E}_{\mathbb{R}^+}(M)$.

Lemma 4.8. *Suppose M is compact, for every $F\in\mathscr{F}C_b$ with the form* (3.3) *we have*

$$\nabla_x\mu^x_{\mathbb{R}}(F)=\sum_{j=1}^m\mu^x_{\mathbb{R}}\Big[\int_0^{T_j}\hat\partial_j f(\gamma)M_s(\gamma)U_s(\gamma)^{-1}\nabla g_j\big(s,\gamma(s)\big)\mathrm{d}s\Big]$$
$$(4.23)\qquad +\sum_{j=1}^k\mu^x_{\mathbb{R}}\Big[\int_{-\bar T_j}^0\hat\partial_{m+j} f(\gamma)M_s(\gamma)U_s(\gamma)^{-1}\nabla \bar g_j\big(s,\gamma(s)\big)\mathrm{d}s\Big],$$

where $\hat{\partial}_j f(\gamma)$ denotes the same item as that in (3.6) *and $U_s(\gamma)$ is defined in* (3.4).

Proof. For simplicity, we only prove (4.23) for $F = f\Big(\int_0^T g(s,\gamma(s))\mathrm{d}s\Big)$ for some $f \in C_b^1(\mathbb{R})$ and $g \in C_b^{0,1}([0,\infty)\times M)$. Other cases could be tackled similarly (by decomposing into $\gamma = (\tilde{\gamma}, \bar{\gamma})$).

For each $k \in \mathbb{N}^+$, let $F_k(\gamma) := f\Big(\sum_{i=1}^k \frac{T}{k} g\big(t_i, \gamma(t_i)\big)\Big)$ with $t_i = \frac{iT}{k}$, $1 \le i \le k$. Then applying [38, Lemma 3.3] we obtain that

$$\nabla_x \mu_{\mathbb{R}}^x(F_k) = \mu_{\mathbb{R}}^x\Big[\sum_{i=1}^k \frac{T}{k} \hat{\partial} f_k(\gamma) M_{t_i} U_{t_i}(\gamma)^{-1} \nabla g\big(t_i, \gamma(t_i)\big)\Big],$$

where $\hat{\partial} f_k(\gamma) = f'\Big(\sum_{i=1}^k \frac{T}{k} g\big(t_i, \gamma(t_i)\big)\Big)$.

Based on such expression it is easy to verify that

$$\begin{aligned}
&\lim_{k\to\infty}\int_M \Big|\nabla_x \mu_{\mathbb{R}}^x(F_k) - \mu_{\mathbb{R}}^x\Big[\int_0^T \hat{\partial} f(\gamma) M_s U_s(\gamma)^{-1} \nabla g\big(s,\gamma(s)\big)\mathrm{d}s\Big]\Big|^2 \mathrm{d}x = 0,\\
&\lim_{k\to\infty}\int_M \big|\mu_{\mathbb{R}}^x(F_k) - \mu_{\mathbb{R}}^x(F)\big|^2 \mathrm{d}x = 0,
\end{aligned}$$

where $\hat{\partial} f(\gamma) := f'\Big(\int_0^T g\big(s,\gamma(s)\big)\mathrm{d}s\Big)$. According to this we could prove for every smooth vector fields $V \in C^\infty(TM)$,

$$\int_M \Big\langle \mu_{\mathbb{R}}^x\Big[\int_0^T \hat{\partial} f(\gamma) M_s U_s(\gamma)^{-1} \nabla g\big(s,\gamma(s)\big)\mathrm{d}s\Big], V(x)\Big\rangle_{T_xM} \mathrm{d}x = -\int_M \mu_{\mathbb{R}}^x(F)\mathrm{div}V(x)\mathrm{d}x,$$

which means

$$\nabla_x \mu_{\mathbb{R}}^x(F) = \mu_{\mathbb{R}}^x\Big[\int_0^T \hat{\partial} f(\gamma) M_s U_s(\gamma)^{-1} \nabla g\big(s,\gamma(s)\big)\mathrm{d}s\Big].$$

Thus (4.23) holds for $F = f\Big(\int_0^T g(s,\gamma(s))\mathrm{d}s\Big)$ and we have finished the proof. □

Theorem 4.9. *[Log-Sobolev inequality and Poincaré inequality]*

(1) Suppose that Ric $\ge K$ *for $K > 0$ and the following log-Sobolev inequality holds for ν (on M)*

$$(4.24)\qquad \nu\big(f^2\log f^2\big) - \nu(f^2)\log\nu(f^2) \le C_1 \int_M |\nabla f(x)|^2 \nu(\mathrm{d}x), \quad \forall\ f \in C^1(M).$$

Then the log-Sobolev inequality holds

$$(4.25)\qquad \mu_{\mathbb{R}}^\nu(F^2\log F^2) \le \Big(\frac{8}{K^2} + \frac{2C_1}{K}\Big)\mathscr{E}_{\mathbb{R}}^\nu(F,F), \qquad F \in \mathscr{F}C_c.\ \mu_{\mathbb{R}}^\nu(F^2) = 1.$$

(2) Suppose M is compact and the following Poincaré inequality holds

$$\nu(f^2)-\nu(f)^2 \le C_2\int_M |\nabla f(x)|^2\nu(\mathrm{d}x), \quad \forall\ f\in C^1(M), \tag{4.26}$$

and there exists $\varepsilon\in(0,1)$ such that

$$\delta_\varepsilon := \sup_{T\in[0,\infty)} \delta_\varepsilon(T) < \infty, \tag{4.27}$$

and

$$C_0 := \int_0^\infty \eta(s)\mathrm{d}s < \infty,$$

where $\delta_\varepsilon(T)$, $\eta(s)$ are defined by (4.3). Then the following Poincaré inequality holds,

$$\mu_{\mathbb{R}}^\nu(F^2)-\mu_{\mathbb{R}}^\nu(F)^2 \le \big(\delta_\varepsilon + C_0C_2\big)\mathscr{E}_{\mathbb{R}}^\nu(F,F), \qquad F\in\mathscr{F}C_c. \tag{4.28}$$

Remark 4.10. *As explained by [61, Chapter 5], if M is compact and $\nu(\mathrm{d}x)=\nu(x)\mathrm{d}x$ is a probability measure such that $\inf_{x\in M}\nu(x)>0$, then the log-Sobolev inequality (4.24) and Poincaré inequality (4.26) hold. In particular, (4.24) and (4.26) hold for the normalized volume measure when M is compact.*

Proof of Theorem 4.9. Step (1) Let

$$G(\bar\gamma) := \sqrt{\int_{\mathbf{E}_{\mathbb{R}^+}^x(M)} F^2(\tilde\gamma,\bar\gamma)\mu_{\mathbb{R}^+}^x(\mathrm{d}\tilde\gamma)},$$

$$g(x) := \sqrt{\int_{\mathbf{E}_{\mathbb{R}^+}^x(M)}\int_{\mathbf{E}_{\mathbb{R}^+}^x(M)} F^2(\tilde\gamma,\bar\gamma)\mu_{\mathbb{R}^+}^x(\mathrm{d}\tilde\gamma)\mu_{\mathbb{R}^+}^x(\mathrm{d}\bar\gamma)} = \sqrt{\int_{\mathbf{E}_{\mathbb{R}}(M)} F^2(\gamma)\mu_{\mathbb{R}}^x(\mathrm{d}\gamma)}.$$

Then we have for every $F \in \mathscr{F}C_c$ with form (3.3),

(4.29)

$$
\begin{aligned}
&\int_{\mathbf{E}_{\mathbb{R}}(M)} F^2(\gamma)\log F^2(\gamma)\mu_{\mathbb{R}}^{\nu}(\mathrm{d}\gamma)\\
&= \int_M \int_{\mathbf{E}_{\mathbb{R}^+}^x(M)} \int_{\mathbf{E}_{\mathbb{R}^+}^x(M)} F^2(\tilde{\gamma},\bar{\gamma})\log F^2(\tilde{\gamma},\bar{\gamma})\mu_{\mathbb{R}^+}^x(\mathrm{d}\tilde{\gamma})\mu_{\mathbb{R}^+}^x(\mathrm{d}\bar{\gamma})\nu(\mathrm{d}x)\\
&\le 2C(K)\int_M \int_{\mathbf{E}_{\mathbb{R}^+}^x(M)} \int_{\mathbf{E}_{\mathbb{R}^+}^x(M)} |\tilde{D}F(\tilde{\gamma},\bar{\gamma})|_{\mathbf{H}_+}^2 \mu_{\mathbb{R}^+}^x(\mathrm{d}\tilde{\gamma})\mu_{\mathbb{R}^+}^x(\mathrm{d}\bar{\gamma})\nu(\mathrm{d}x)\\
&\quad + \int_M \int_{\mathbf{E}_{\mathbb{R}^+}^x(M)} G^2(\bar{\gamma})\log G^2(\bar{\gamma})\mu_{\mathbb{R}^+}^x(\mathrm{d}\bar{\gamma})\nu(\mathrm{d}x)\\
&\le 2C(K)\int_M \int_{\mathbf{E}_{\mathbb{R}^+}^x(M)} \int_{\mathbf{E}_{\mathbb{R}^+}^x(M)} |\tilde{D}F(\tilde{\gamma},\bar{\gamma})|_{\mathbf{H}_+}^2 \mu_{\mathbb{R}^+}^x(\mathrm{d}\tilde{\gamma})\mu_{\mathbb{R}^+}^x(\mathrm{d}\bar{\gamma})\nu(\mathrm{d}x)\\
&\quad + 2C(K)\int_M \int_{\mathbf{E}_{\mathbb{R}^+}^x(M)} |\bar{D}G(\bar{\gamma})|_{\mathbf{H}_+}^2 \mu_{\mathbb{R}^+}^x(\mathrm{d}\bar{\gamma})\nu(\mathrm{d}x) + \int_M g^2(x)\log g^2(x)\nu(\mathrm{d}x)\\
&\le 2C(K)\int_M \int_{\mathbf{E}_{\mathbb{R}^+}^x(M)} \int_{\mathbf{E}_{\mathbb{R}^+}^x(M)} |\tilde{D}F(\tilde{\gamma},\bar{\gamma})|_{\mathbf{H}_+}^2 \mu_{\mathbb{R}^+}^x(\mathrm{d}\tilde{\gamma})\mu_{\mathbb{R}^+}^x(\mathrm{d}\bar{\gamma})\nu(\mathrm{d}x)\\
&\quad + 2C(K)\int_M \int_{\mathbf{E}_{\mathbb{R}^+}^x(M)} |\bar{D}G(\bar{\gamma})|_{\mathbf{H}_+}^2 \mu_{\mathbb{R}^+}^x(\mathrm{d}\bar{\gamma})\nu(\mathrm{d}x) + C_1\int_M |\nabla g(x)|^2\nu(\mathrm{d}x) + \mu_{\mathbb{R}}^{\nu}(F^2)\log\mu_{\mathbb{R}}^{\nu}(F^2).
\end{aligned}
$$

Here in the second step we applied (4.1) to $F(\cdot,\bar{\gamma})$ (with $\bar{\gamma}$ fixed) with $\tilde{D}F(\tilde{\gamma},\bar{\gamma})$ denoting the L^2 gradient with respect to the variable $\tilde{\gamma} \in \mathbf{E}_{\mathbb{R}^+}(M)$; in the third step we applied (4.1) to $G(\bar{\gamma})$ with $\bar{D}G(\bar{\gamma})$ denoting L^2 gradient with respect to the variable $\bar{\gamma} \in \mathbf{E}_{\mathbb{R}^+}(M)$ and the property $\int_{\mathbf{E}_{\mathbb{R}^+}^o(M)} G^2(\bar{\gamma})\mu_{\mathbb{R}^+}^x(\mathrm{d}\bar{\gamma}) = g^2(x)$; in the last step we applied (4.24) to $g(x)$ and the property $\int_M g^2(x)\nu(\mathrm{d}x) = \mu_{\mathbb{R}}^{\nu}(F^2)$. At the same time, it holds

$$
\begin{aligned}
&\big|\tilde{D}F(\tilde{\gamma},\bar{\gamma})\big|_{\mathbf{H}_+}^2 + \big|\bar{D}F(\tilde{\gamma},\bar{\gamma})\big|_{\mathbf{H}_+}^2 = \big|DF(\gamma)\big|_{\mathbf{H}}^2,\\
&\big|\bar{D}G(\bar{\gamma})\big|_{\mathbf{H}_+}^2 = \frac{\Big|\int_{\mathbf{E}_{\mathbb{R}^+}^x(M)} F(\tilde{\gamma},\bar{\gamma})\bar{D}F(\tilde{\gamma},\bar{\gamma})\mu_{\mathbb{R}^+}^x(\mathrm{d}\tilde{\gamma})\Big|_{\mathbf{H}_+}^2}{\int_{\mathbf{E}_{\mathbb{R}^+}^x(M)} F^2(\tilde{\gamma},\bar{\gamma})\mu_{\mathbb{R}^+}^x(\mathrm{d}\tilde{\gamma})} \le \int_{\mathbf{E}_{\mathbb{R}^+}^x(M)} \big|\bar{D}F(\tilde{\gamma},\bar{\gamma})\big|_{\mathbf{H}_+}^2 \mu_{\mathbb{R}^+}^x(\mathrm{d}\tilde{\gamma}).
\end{aligned}
$$

Meanwhile by (4.23),

$$
|\nabla g(x)|^2 = \frac{\Big|\mu_{\mathbb{R}}^x\Big[F(\gamma)J(\gamma)\Big]\Big|^2}{\int_{\mathbf{E}_{\mathbb{R}}^x(M)} F^2(\gamma)\mu_{\mathbb{R}}^x(\mathrm{d}\gamma)} \le \mu_{\mathbb{R}}^x[|J(\gamma)|^2],
$$

where

$$
\begin{aligned}
J(\gamma) &= \sum_{j=1}^{m} \int_0^{T_j} \hat{\partial}_j f(\gamma) M_s(\gamma) U_s(\gamma)^{-1} \nabla g_j\big(s, \gamma(s)\big) \mathrm{d}s \\
&\quad + \sum_{j=1}^{k} \int_{-\bar{T}_j}^{0} \hat{\partial}_{m+j} f(\gamma) M_s(\gamma) U_s(\gamma)^{-1} \nabla \bar{g}_j\big(s, \gamma(s)\big) \mathrm{d}s \qquad (4.30) \\
&= \int_{-T}^{T} M_s(\gamma) DF(\gamma)(s) \mathrm{d}s
\end{aligned}
$$

with $T := \max\{\max_{1\le j\le m} T_j, \max_{1\le j\le k} \bar{T}_j\}$. Based on the expression of $J(\gamma)$ above we arrive at

$$
\begin{aligned}
|\nabla g(x)|^2 &\le \mu_{\mathbb{R}}^x\Big[\Big(\int_{-T}^{T} \|M_s(\gamma)\|^2 \mathrm{d}s\Big) \cdot \Big(\int_{-T}^{T} |DF(\gamma)|^2(s) \mathrm{d}s\Big)\Big] \\
&\le 2\Big(\int_0^{\infty} e^{-Ks} \mathrm{d}s\Big) \mu_{\mathbb{R}}^x\Big[|DF(\gamma)|_{\mathbf{H}}^2\Big],
\end{aligned}
$$

where the last step follow from the estimates $\|M_s(\gamma)\| \le e^{-\frac{K|s|}{2}}$ for all $s \in \mathbb{R}$.

Finally, combining all the estimates above into (4.29) yields (4.25).

Step (2) Similar as (4.29) (and apply (4.4)) we obtain

$$
\begin{aligned}
&\int_{\mathbf{E}_{\mathbb{R}}(M)} F^2(\gamma) \mu_{\mathbb{R}}^{\nu}(\mathrm{d}\gamma) - \Big(\int_{\mathbf{E}_{\mathbb{R}}(M)} F(\gamma) \mu_{\mathbb{R}}^{\nu}(\mathrm{d}\gamma)\Big)^2 \\
&\le \delta_\varepsilon \int_M \int_{\mathbf{E}_{\mathbb{R}^+}^x(M)} \int_{\mathbf{E}_{\mathbb{R}^+}^x(M)} |\tilde{D}F(\tilde{\gamma}, \bar{\gamma})|_{\mathbf{H}_+}^2 \mu_{\mathbb{R}^+}^x(\mathrm{d}\tilde{\gamma}) \mu_{\mathbb{R}^+}^x(\mathrm{d}\bar{\gamma}) \nu(\mathrm{d}x) \qquad (4.31) \\
&+ \delta_\varepsilon \int_M \int_{\mathbf{E}_{\mathbb{R}^+}^x(M)} |\bar{D}Q(\bar{\gamma})|_{\mathbf{H}_+}^2 \mu_{\mathbb{R}^+}^x(\mathrm{d}\bar{\gamma}) \nu(\mathrm{d}x) + C_2 \int_M |\nabla q(x)|^2 \nu(\mathrm{d}x),
\end{aligned}
$$

where

$$
Q(\bar{\gamma}) := \int_{\mathbf{E}_{\mathbb{R}^+}^x(M)} F(\tilde{\gamma}, \bar{\gamma}) \mu_{\mathbb{R}^+}^x(\mathrm{d}\tilde{\gamma}), \quad q(x) := \int_{\mathbf{E}_{\mathbb{R}}^x(M)} F(\gamma) \mu_{\mathbb{R}}^x(\mathrm{d}\gamma).
$$

Still by the same arguments in **Step (1)** we could show

$$
\begin{aligned}
&|\bar{D}Q(\bar{\gamma})|_{\mathbf{H}_+}^2 = \Big|\int_{\mathbf{E}_{\mathbb{R}^+}^x(M)} \bar{D}F(\tilde{\gamma}, \bar{\gamma}) \mu_{\mathbb{R}^+}^x(\mathrm{d}\tilde{\gamma})\Big|_{\mathbf{H}_+}^2 \le \int_{\mathbf{E}_{\mathbb{R}^+}^x(M)} \big|\bar{D}F(\tilde{\gamma}, \bar{\gamma})\big|_{\mathbf{H}_+}^2 \mu_{\mathbb{R}^+}^x(\mathrm{d}\tilde{\gamma}), \\
&|\nabla q(x)|^2 \le \big|\mu_{\mathbb{R}}^x\big[J(\gamma)\big]\big|^2 \le \mu_{\mathbb{R}}^x\Big[\int_{-T}^{T} \|M_s\|^2 \mathrm{d}s\Big] \cdot \mu_{\mathbb{R}}^x\Big[\int_{-T}^{T} |DF(s)|^2 \mathrm{d}s\Big] \\
&\le 2\Big(\int_0^{\infty} \eta(s) \mathrm{d}s\Big) \cdot \mu_{\mathbb{R}}^x\Big[\big|DF(\gamma)\big|_{\mathbf{H}}^2\Big],
\end{aligned}
$$

where $J(\gamma)$ is defined by (4.30) and the last step above is due to

$$\mu^x_{\mathbb{R}}\Big[\int_{-T}^{T}\|M_s\|^2\mathrm{d}s\Big]=2\mu^x_{\mathbb{R}^+}\Big[\int_0^T\|M_s\|^2\mathrm{d}s\Big]$$
$$\leq 2\mu^x_{\mathbb{R}^+}\Big[\int_0^T\exp(-\int_0^s K(\gamma(r))\mathrm{d}r)\mathrm{d}s\Big]\leq 2\int_0^\infty\eta(s)\mathrm{d}s.$$

Then combining all the above estimates into (4.31) yields (4.28). □

When $M=\mathbb{R}^n$, the Markov process constructed in Section 3 corresponds to the solutions to the stochastic heat equations. The most interesting case is that ν is given by Lebesgue measure, which is related the the stochastic heat equations without any boundary condition. In this case the reference measure has infinite mass. So we do not investigate the long time behavior here.

Following the same procedure in Theorem 4.6 we can still get the following result. So we omit the proof here.

Theorem 4.11. *If M is not a Liouville manifold and ν is a probability measure, then $(\mathscr{E}^\nu_{\mathbb{R}},\mathscr{D}(\mathscr{E}^\nu_{\mathbb{R}}))$ is reducible. Hence $\mu^\nu_{\mathbb{R}}$ is not ergodic for the Markov process associated with $(\mathscr{E}^\nu_{\mathbb{R}},\mathscr{D}(\mathscr{E}^\nu_{\mathbb{R}}))$.*

Acknowledgments

We would like to thank Professor Tadahisa Funaki for helpful suggestions.

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