SOBOLEV HOMEOMORPHIC EXTENSIONS

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Abstract. Let $\mathbb{X}$ and $\mathbb{Y}$ be ℓ-connected Jordan domains, $\ell \in \mathbb{N}$, with rectifiable boundaries in the complex plane. We prove that any boundary homeomorphism $\varphi \colon \partial\mathbb{X} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ admits a Sobolev homeomorphic extension $h \colon \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ in $\mathscr{W}^{1,1}(\mathbb{X}, \mathbb{C})$. If instead $\mathbb{X}$ has s-hyperbolic growth with $s > p-1$, we show the existence of such an extension lies in the Sobolev class $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{C})$ for $p \in (1,2)$. Our examples show that the assumptions of rectifiable boundary and hyperbolic growth cannot be relaxed. We also consider the existence of $\mathscr{W}^{1,2}$-homeomorphic extensions subject to a given boundary data.

1. Introduction

Throughout this text $\mathbb{X}$ and $\mathbb{Y}$ are ℓ-connected Jordan domains, $\ell = 1, 2, \dots$, in the complex plane $\mathbb{C}$. Their boundaries $\partial\mathbb{X}$ and $\partial\mathbb{Y}$ are thus a disjoint union of ℓ simple closed curves. If $\ell = 1$, these domains are simply connected and will just be called Jordan domains. In the simply connected case, the Jordan-Schönflies theorem states that every homeomorphism $\varphi \colon \partial\mathbb{X} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ admits a continuous extension $h \colon \overline{\mathbb{X}} \to \overline{\mathbb{Y}}$ which takes $\mathbb{X}$ homeomorphically onto $\mathbb{Y}$. In the first part of this paper we focus on a Sobolev variant of the Jordan-Schönflies theorem. The most pressing demand for studying such variants comes from the variational approach to Geometric Function Theory [4, 18, 32] and Nonlinear Elasticity [2, 5, 8]. Both theories share the compilation ideas to determine the infimum of a given energy functional

$$\mathsf{E}_{\mathbb{X}}[h] = \int_{\mathbb{X}} \mathbf{E}(x, h, Dh)\, \mathrm{d}x \,, \tag{1.1}$$

among orientation preserving homeomorphisms $h \colon \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ in the Sobolev space $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$ with given boundary data $\varphi \colon \partial\mathbb{X} \xrightarrow{\text{onto}} \partial\mathbb{Y}$. We denote such a class of mappings by $\mathscr{H}^{1,p}_{\varphi}(\mathbb{X}, \mathbb{Y})$. Naturally, a fundamental question to raise then is whether the class $\mathscr{H}^{1,p}_{\varphi}(\mathbb{X}, \mathbb{Y})$ is non-empty.

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Question 1.1. Under what conditions does a given boundary homeomorphism $\varphi\colon \partial\mathbb{X} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ admit a homeomorphic extension $h\colon \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ of Sobolev class $\mathscr{W}^{1,p}(\mathbb{X},\mathbb{C})$?

A necessary condition is that the mapping φ is the Sobolev trace of some (possibly non-homeomorphic) mapping in $\mathscr{W}^{1,p}(\mathbb{X},\mathbb{C})$. Hence to solve Question 1.1 one could first study the following natural sub-question:

Question 1.2. Suppose that a homeomorphism $\varphi : \partial\mathbb{X} \to \partial\mathbb{Y}$ admits a Sobolev $\mathscr{W}^{1,p}$-extension to $\mathbb{X}$. Does it then follow that φ also admits a homeomorphic Sobolev $\mathscr{W}^{1,p}$-extension to $\mathbb{X}$?

Our main results, Theorem 1.8 and its multiply connected variant (Theorem 1.11), give an answer to these questions when $p \in [1,2)$. The construction of such extensions is important not only to ensure the well-posedness of the related variational questions, but also for example due to the fact that various types of extensions were used to provide approximation results for Sobolev homeomorphisms, see [15, 17]. We touch upon the variational topics in Section 7, where we provide an application for one of our results. Apart from Theorem 1.11 and its proof (§6), the rest of the paper deals with the simply connected case.

Let us start considering the above questions in the well-studied setting of the Dirichlet energy, corresponding to $p = 2$ above. The Radó [31], Kneser [25] and Choquet [7] theorem asserts that if $\mathbb{Y} \subset \mathbb{R}^2$ is a convex domain then the harmonic extension of a homeomorphism $\varphi\colon \partial\mathbb{X} \to \partial\mathbb{Y}$ is a univalent map from $\mathbb{X}$ onto $\mathbb{Y}$. Moreover, by a theorem of Lewy [28], this univalent harmonic map has a non-vanishing Jacobian and is therefore a real analytic diffeomorphism in $\mathbb{X}$. However, such an extension is not guaranteed to have finite Dirichlet energy in $\mathbb{X}$. The class of boundary functions which admit a harmonic extension with finite Dirichlet energy was characterized by Douglas [9]. The *Douglas condition* for a function $\varphi\colon \partial\mathbb{D} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ reads as

$$\int_{\partial\mathbb{D}}\int_{\partial\mathbb{D}} \left|\frac{\varphi(\xi)-\varphi(\eta)}{\xi-\eta}\right|^2 |\mathrm{d}\xi|\,|\mathrm{d}\eta| < \infty\,. \tag{1.2}$$

The mappings satisfying this condition are exactly the ones that admit an extension with finite $\mathscr{W}^{1,2}$-norm. Among these extensions is the harmonic extension of φ, which is known to have the smallest Dirichlet energy among all extensions.

Note that the Dirichlet energy is also invariant with respect to a conformal change of variables in the domain $\mathbb{X}$. Therefore thanks to the Riemann Mapping Theorem, when considering Question 1.1 in the case $p = 2$, we may assume that $\mathbb{X} = \mathbb{D}$ without loss of generality. Now, there is no challenge to answer Question 1.1 when $p = 2$ and $\mathbb{Y}$ is Lipschitz. Indeed, for any Lipschitz domain there exists a global bi-Lipschitz change of variables $\Phi\colon \mathbb{C} \to \mathbb{C}$ for which $\Phi(\mathbb{Y})$ is the unit disk. Since the finiteness of the Dirichlet energy is preserved under a bi-Lipschitz change of variables in the target, we may reduce Question 1.1 to the case when $\mathbb{X} = \mathbb{Y} = \mathbb{D}$, for which the Radó-Kneser-Choquet theorem and the Douglas condition provide an answer. In other words, if $\mathbb{Y}$ is Lipschitz then the following are equivalent for a boundary homeomorphism $\varphi\colon \partial\mathbb{D} \to \partial\mathbb{Y}$

(1) φ admits a $\mathscr{W}^{1,2}$-Sobolev homeomorphic extension $h\colon \overline{\mathbb{D}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$
(2) φ admits $\mathscr{W}^{1,2}$-Sobolev extension to $\mathbb{D}$
(3) φ satisfies the Douglas condition (1.2)

In the case when $1 \leqslant p < 2$, the problem is not invariant under a conformal change of variables in $\mathbb{X}$. However, when $\mathbb{X}$ is the unit disk and $\mathbb{Y}$ is a convex domain, a complete answer to Question 1.1 was provided by the following result of Verchota [38].

Proposition 1.3. *Let $\mathbb{Y}$ be a convex domain, and let $\varphi : \partial\mathbb{D} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ be any homeomorphism. Then the harmonic extension of φ lies in the Sobolev class $\mathscr{W}^{1,p}(\mathbb{D},\mathbb{C})$ for all $1 \leqslant p < 2$.*

This result was further generalized in [19] and [23]. The case $p > 2$ will be discussed in Section 2.3. Our main purpose is to provide a general study of Question 1.1 in the case when $1 \leqslant p < 2$.

Considering now the endpoint case $p = \infty$, we find that Question 1.1 is equivalent to the question of finding a homeomorphic Lipschitz map extending the given boundary data φ. In this case the Kirszbraun extension theorem [24] shows that a boundary map $\varphi\colon \partial\mathbb{D} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ admits a Lipschitz extension if and only if φ is a Lipschitz map itself. In the case when $\mathbb{X}$ is the unit disk, a positive answer to Question 1.2 is then given by the following recent result by Kovalev [26].

Theorem 1.4. ($p = \infty$) *Let $\varphi\colon \partial\mathbb{D} \to \mathbb{C}$ be a Lipschitz embedding. Then φ admits a homeomorphic Lipschitz extension to the whole plane $\mathbb{C}$.*

Let us return to the case of the Dirichlet energy, see (1)-(3) above. The equivalence of a $\mathscr{W}^{1,2}$-Sobolev extension and a $\mathscr{W}^{1,2}$-Sobolev homeomorphic extension for non-Lipschitz targets is a more subtle question. In this perspective, a slightly more general class of domains is the class of inner chordarc domains studied in Geometric Function Theory [16, 30, 34, 36, 37]. By definition [36], a rectifiable Jordan domain $\mathbb{Y}$ is *inner chordarc* if there exists a constant C such that for every pairs of points $y_1, y_2 \in \partial\mathbb{Y}$ there exists an open Jordan arc $\gamma \subset \mathbb{Y}$ with endpoints at y_1 and y_2 such that the shortest connection from y_1 to y_2 along $\partial\mathbb{Y}$ has length at most $C \cdot \text{length}(\gamma)$. For example, an inner chordarc domain allows for inward cusps as oppose to Lipschitz domains. According to a result of Väisälä [36] the inner chordarc condition is equivalent with the requirement that there exists a homeomorphism $\Psi\colon \overline{\mathbb{Y}} \xrightarrow{\text{onto}} \overline{\mathbb{D}}$, which is $\mathscr{C}^1$-diffeomorphic in $\mathbb{Y}$, such that the norms of both the gradient matrices $D\Psi$ and $(D\Psi)^{-1}$ are bounded from above.

Surprisingly, the following example shows that, unlike for Lipschitz targets, the answer to Question 1.2 for $p = 2$ is in general negative when the target is only inner chordarc.

Example 1.5. There exists an inner chordarc domain $\mathbb{Y}$ and a homeomorphism $\varphi\colon \partial\mathbb{D} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ satisfying the Douglas condition (1.2) which does not admit a homeomorphic extension $h\colon \overline{\mathbb{D}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ in $\mathscr{W}^{1,2}(\mathbb{D},\mathbb{Y})$.

In [3] it was, as a part of studies of mappings with smallest mean distortion, proved that for $\mathscr{C}^1$-smooth $\mathbb{Y}$ the Douglas condition (1.2) can be equivalently formulated in terms of the inverse mapping $\varphi^{-1}\colon \partial\mathbb{Y} \xrightarrow{\text{onto}} \partial\mathbb{D}$,

$$\int_{\partial\mathbb{Y}}\int_{\partial\mathbb{Y}} \big|\log|\varphi^{-1}(\xi)-\varphi^{-1}(\eta)|\big|\,|\mathrm{d}\xi|\,|\mathrm{d}\eta| < \infty\,. \tag{1.3}$$

It was recently shown that for inner chordarc targets this condition is necessary and sufficient for φ to admit a $\mathscr{W}^{1,2}$-homeomorphic extension, see [27]. We extend this result both to cover rectifiable targets and to give a global homeomorphic extension as follows.

Theorem 1.6. ($p=2$) *Let $\mathbb{X}$ and $\mathbb{Y}$ be Jordan domains, $\partial\mathbb{Y}$ being rectifiable. Every $\varphi\colon \partial\mathbb{X} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ satisfying* (1.3) *admits a homeomorphic extension $h\colon \mathbb{C}\to\mathbb{C}$ of Sobolev class $\mathscr{W}^{1,2}_{loc}(\mathbb{C},\mathbb{C})$.*

Without the rectifiability of $\partial\mathbb{Y}$, Question 1.2 will in general admit a negative answer for all $p \leqslant 2$. This follows from the following example of Zhang [39].

Example 1.7. There exists a Jordan domain $\mathbb{Y}$ and a homeomorphism $\varphi\colon \partial\mathbb{D} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ which admits a $\mathscr{W}^{1,2}$-Sobolev extension to $\mathbb{D}$ but does not admit any homeomorphic extension to $\mathbb{D}$ in the class $\mathscr{W}^{1,1}(\mathbb{D},\mathbb{C})$.

We now return to the case when $1\leqslant p<2$. In this case it is natural to ask under which conditions on the domains $\mathbb{X}$ and $\mathbb{Y}$ does any homeomorphism $\varphi : \partial\mathbb{X} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ admit a $\mathscr{W}^{1,p}$-Sobolev homeomorphic extension. Proposition 1.3 already implies that this is the case for $\mathbb{X}=\mathbb{D}$ and $\mathbb{Y}$ convex. Example 1.7, however, will imply that this result does not hold in general for nonrectifiable targets $\mathbb{Y}$. A general characterization is provided by the following two theorems.

Theorem 1.8. ($1\leqslant p<2$) *Let $\mathbb{X}$ and $\mathbb{Y}$ be Jordan domains in the plane with $\partial\mathbb{Y}$ rectifiable. Let $\varphi\colon \partial\mathbb{X} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ be a given homeomorphism. Then there is a homeomorphic extension $h\colon \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ such that*

(1) $h\in\mathscr{W}^{1,1}(\mathbb{X},\mathbb{C})$, provided $\partial\mathbb{X}$ is rectifiable, and
(2) $h\in\mathscr{W}^{1,p}(\mathbb{X},\mathbb{C})$ for $1<p<2$, provided $\mathbb{X}$ has s-hyperbolic growth *with $s>p-1$.*

Definition 1.9. Let $\mathbb{X}$ be a domain in the plane. Choose and fix a point $x_0\in\mathbb{X}$. We say that $\mathbb{X}$ has *s-hyperbolic growth*, $s\in(0,1)$, if the following condition holds

$$h_{\mathbb{X}}(x_0,x) \leqslant C\left(\frac{\operatorname{dist}(x_0,\partial\mathbb{X})}{\operatorname{dist}(x,\partial\mathbb{X})}\right)^{1-s} \qquad \text{for all } x\in\mathbb{X}\,. \tag{1.4}$$

Here $h_{\mathbb{X}}$ stands for the quasihyperbolic metric on $\mathbb{X}$ and $\operatorname{dist}(x, \partial\mathbb{X})$ is the Euclidean distance of x to the boundary. The constant C is allowed to depend on everything except the point x.

It is easily verified that this definition does not depend on the choice of x_0. Recall that if Ω is a domain, the quasihyperbolic metric h_Ω is defined by [13]

$$h_\Omega(x_1, x_2) = \inf_{\gamma \in \Gamma} \int_\gamma \frac{1}{\operatorname{dist}(x, \partial\mathbb{X})} \, |\mathrm{d}x| \,, \qquad x_1, x_2 \in \Omega \tag{1.5}$$

where Γ is the family of all rectifiable curves in Ω joining x_1 and x_2.

Definition 1.9 is motivated by the following example. For $s \in (0,1)$ we consider the Jordan domain $\mathbb{X}_s$ whose boundary is given by the curve

$$\Gamma_s = \{(x,y) \in \mathbb{C} : -1 \leqslant x \leqslant 1,\ y = |x|^s\} \cup \{z \in \mathbb{C} : |z - i| = 1,\ \operatorname{Im}(z) \geqslant 1\}.$$

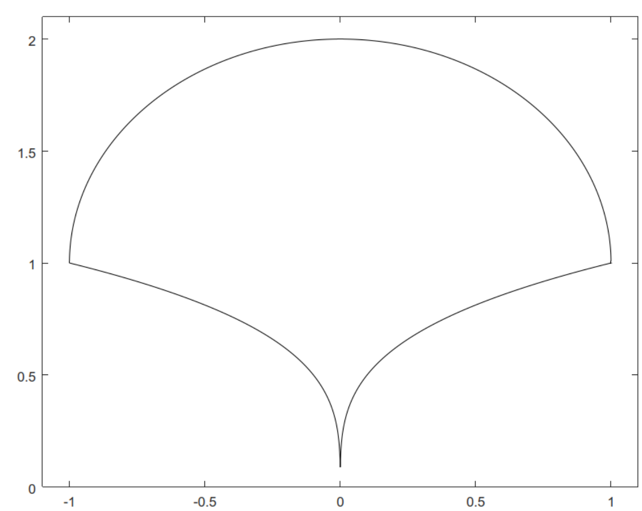

FIGURE 1. The Jordan domain $\mathbb{X}_s$.

In particular, the boundary of $\mathbb{X}_s$ is locally Lipschitz apart from the origin. Near to the origin the boundary of $\mathbb{X}_s$ behaves like the graph of the function $|x|^s$. Then one can verify that the boundary of $\mathbb{X}_s$ has t-hyperbolic growth for every $t \geqslant s$. Note that smaller the number s sharper the cusp is.

The results of Theorem 1.8 are sharp, as described by the following result.

Theorem 1.10.

(1) There exists a Jordan domain $\mathbb{X}$ with nonrectifiable boundary and a homeomorphism $\varphi : \partial\mathbb{X} \to \partial\mathbb{D}$ such that φ does not admit a continuous extension to $\mathbb{X}$ in the Sobolev class $\mathscr{W}^{1,1}(\mathbb{X}, \mathbb{C})$.

(2) *For every* $p \in (1,2)$ *there exists a Jordan domain* $\mathbb{X}$ *which has* s*-hyperbolic growth, with* $p-1 = s$*, and a homeomorphism* $\varphi : \partial\mathbb{X} \to \partial\mathbb{D}$ *such that* φ *does not admit a continuous extension to* $\mathbb{X}$ *in the Sobolev class* $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{C})$*.*

To conclude, as promised earlier, we extend our main result to the case where the domains are not simply connected. The following generalization of Theorem 1.8 holds.

Theorem 1.11. *Let* $\mathbb{X}$ *and* $\mathbb{Y}$ *be multiply connected Jordan domains with* $\partial\mathbb{Y}$ *rectifiable. Let* $\varphi\colon \partial\mathbb{X} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ *be a given homeomorphism which maps the outer boundary component of* $\mathbb{X}$ *to the outer boundary component of* $\mathbb{Y}$*. Then there is a homeomorphic extension* $h\colon \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ *such that*

(1) $h \in \mathscr{W}^{1,1}(\mathbb{X}, \mathbb{C})$*, provided* $\partial\mathbb{X}$ *is rectifiable, and*
(2) $h \in \mathscr{W}^{1,p}(\mathbb{X}, \mathbb{C})$ *for* $1 < p < 2$*, provided* $\mathbb{X}$ *has* s-hyperbolic growth *with* $s > p - 1$*.*

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2. Preliminaries

2.1. **The Dirichlet problem.** Let Ω be a bounded domain in the complex plane. A function $u\colon \Omega \to \mathbb{R}$ in the Sobolev class $\mathscr{W}^{1,p}_{\text{loc}}(\Omega)$, $1 < p < \infty$, is called *p-harmonic* if

$$\text{div}|\nabla u|^{p-2}\nabla u = 0. \tag{2.1}$$

We call 2-harmonic functions simply *harmonic*.

There are two formulations of the Dirichlet boundary value problem for the p-harmonic equation (2.1). We first consider the variational formulation.

Lemma 2.1. *Let* $u_\circ \in \mathscr{W}^{1,p}(\Omega)$ *be a given Dirichlet data. There exists precisely one function* $u \in u_\circ + \mathscr{W}^{1,p}_\circ(\Omega)$ *which minimizes the p-harmonic energy:*

$$\int_\Omega |\nabla u|^p = \inf \left\{ \int_\Omega |\nabla w|^p \colon w \in u_\circ + \mathscr{W}^{1,p}_\circ(\Omega) \right\}.$$

The variational formulation coincides with the classical formulation of the Dirichlet problem.

Lemma 2.2. *Let* $\Omega \subset \mathbb{C}$ *be a bounded Jordan domain and* $u_\circ \in \mathscr{W}^{1,p}(\Omega) \cap \mathscr{C}(\overline{\Omega})$*. Then there exists a unique p-harmonic function* $u \in \mathscr{W}^{1,p}(\Omega) \cap \mathscr{C}(\overline{\Omega})$ *such that* $u_{|\partial\Omega} = u_{\circ|\partial\Omega}$*.*

For a reference for proofs of these facts we refer to [17].

2.2. **The Radó-Kneser-Choquet Theorem.**

Lemma 2.3. *Consider a Jordan domain $\mathbb{X} \subset \mathbb{C}$ and a bounded convex domain $\mathbb{Y} \subset \mathbb{C}$. Let $h \colon \partial\mathbb{X} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ be a homeomorphism and $H \colon \mathbb{U} \to \mathbb{C}$ denote its harmonic extension. Then H is a $\mathscr{C}^\infty$-diffeomorphism of $\mathbb{X}$ onto $\mathbb{Y}$.*

For the proof of this lemma we refer to [11, 20]. The following p-harmonic analogue of the Radó-Kneser-Choquet Theorem is due to Alessandrini and Sigalotti [1], see also [21].

Proposition 2.4. *Let $\mathbb{X}$ be a Jordan domain in $\mathbb{C}$, $1 < p < \infty$, and $h = u + iv \colon \overline{\mathbb{X}} \to \mathbb{C}$ be a continuous mapping whose coordinate functions are p-harmonic. Suppose that $\mathbb{Y}$ is convex and that $h \colon \partial\mathbb{X} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ is a homeomorphism. Then h is a diffeomorphism from $\mathbb{X}$ onto $\mathbb{Y}$.*

2.3. **Sobolev homeomorphic extensions onto a Lipschitz target.** Combining the results in this section allows us to easily solve Question 1.2 for convex targets.

Proposition 2.5. *Let $\mathbb{X}$ and $\mathbb{Y}$ be Jordan domains in the plane with $\mathbb{Y}$ convex, and let p be given with $1 < p < \infty$. Suppose that $\varphi \colon \partial\mathbb{X} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ is a homeomorphism. Then there exists a continuous $g \colon \overline{\mathbb{X}} \to \mathbb{C}$ in $\mathscr{W}^{1,p}(\mathbb{X},\mathbb{C})$ such that $g(x) = \varphi(x)$ on $\partial\mathbb{X}$ if and only if there exists a homeomorphism $h \colon \overline{\mathbb{X}} \to \overline{\mathbb{Y}}$ in $\mathscr{W}^{1,p}(\mathbb{X},\mathbb{C})$ such that $h(x) = \varphi(x)$ on $\partial\mathbb{X}$.*

Now, replacing the convex $\mathbb{Y}$ by a Lipschitz domain offers no challenge. Indeed, this follows from a global bi-Lipschitz change of variables $\Phi \colon \mathbb{C} \to \mathbb{C}$ for which $\Phi(\mathbb{Y})$ is the unit disk. If the domain in Proposition 2.5 is the unit disk $\mathbb{D}$, then the existence of a finite p-harmonic extension can be characterized in terms of a Douglas type condition. If $1 < p < 2$, then such an extension exists for an arbitrary boundary homeomorphism (Proposition 1.3) and if $2 \leqslant p < \infty$ the extension exists if and only the boudary homeomorphism $\varphi \colon \partial\mathbb{D} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ satisfies the following condition,

$$\int_{\partial\mathbb{D}}\int_{\partial\mathbb{D}} \left|\frac{\varphi(\xi)-\varphi(\eta)}{\xi-\eta}\right|^p |\mathrm{d}\xi|\,|\mathrm{d}\eta| < \infty\,. \tag{2.2}$$

For the proof of the latter fact we refer to [33, p. 151-152].

2.4. **A Carleson measure and the Hardy space H^p.** Roughly speaking, a Carleson measure on a domain $\mathbb{G}$ is a measure that does not vanish at the boundary of $\mathbb{G}$ when compared to the Hausdorff 1-measure on $\partial\mathbb{G}$. We will need the notion of Carleson measure only on the unit disk $\mathbb{D}$.

Definition 2.6. Let μ be a Borel measure on $\mathbb{D}$. Then μ is a *Carleson measure* if there is a constant $C > 0$ such that

$$\mu(S_\epsilon(\theta)) \leqslant C\epsilon$$

for every $\epsilon > 0$. Here

$$S_\epsilon(\theta) = \{re^{i\alpha} : 1-\epsilon < r < 1, \theta-\epsilon < \alpha < \theta+\epsilon\}\,.$$

Carleson measures have many applications in harmonic analysis. A celebrated result by L. Carleson [6], also see Theorem 9.3 in [10], tells us that a Borel measure μ on $\mathbb{D}$ is a bounded Carleson measure if and only if the injective mapping from the Hardy space $H^p(\mathbb{D})$ into a the measurable space $L^p_\mu(\mathbb{D})$ is bounded:

Proposition 2.7. *Let μ be a Borel measure on the unit disk $\mathbb{D}$. Let $0 < p < \infty$. Then in order that there exist a constant $C > 0$ such that*

$$\left(\int_{\mathbb{D}} |f(z)|^p \, d\mu(z) \right)^{\frac{1}{p}} \leqslant C ||f||_{H^p(\mathbb{D})} \quad \text{for all } f \in H^p(\mathbb{D})$$

it is necessary and sufficient that μ be a Carleson measure.

Recall that the Hardy space $H^p(\mathbb{D})$, $0 < p < \infty$, is the class of holomorphic functions f on the unit disk satisfying

$$||f||_{H^p(\mathbb{D})} := \sup_{0 \leqslant r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{\frac{1}{p}} < \infty \, .$$

Note that $|| \cdot ||_{H^p(\mathbb{D})}$ is a norm when $p \geqslant 1$, but not when $0 < p < 1$.

3. Sobolev integrability of the harmonic extension

At the end of this section we prove our main result in the simply connected case, Theorem 1.8. The proof will be based on a suitable reduction of the target domain to the unit disk, and the following auxiliary result which concerns the regularity of harmonic extensions.

Theorem 3.1. *Let $\mathbb{X}$ be a Jordan domain and $\varphi : \partial\mathbb{X} \to \partial\mathbb{D}$ be an arbitrary homeomorphism. Let h denote the harmonic extension of φ to $\mathbb{X}$, which is a homeomorphism from $\overline{\mathbb{X}}$ to $\overline{\mathbb{D}}$. Then the following hold.*

(1) If the boundary of $\mathbb{X}$ is rectifiable, then $h \in \mathscr{W}^{1,1}(\mathbb{X}, \mathbb{C})$.
(2) If $\mathbb{X}$ has s-hyperbolic growth, then $h \in \mathscr{W}^{1,p}(\mathbb{X}, \mathbb{C})$ for $p = s - 1$.

This theorem will be a direct corollary of the following theorem and the two propositions after it.

Theorem 3.2. *Let $\mathbb{X}$ be a Jordan domain, and denote by $g : \mathbb{D} \to \mathbb{X}$ a conformal map onto $\mathbb{X}$. Let $1 \leqslant p < 2$. Suppose that the condition*

$$\sup_{\omega \in \partial\mathbb{D}} \int_{\mathbb{D}} \frac{|g'(z)|^{2-p}}{|\omega - z|^p} \, dz \leqslant M < \infty \tag{3.1}$$

holds. Then the harmonic extension $h : \mathbb{X} \to \mathbb{D}$ of any boundary homeomorphism $\varphi : \partial\mathbb{X} \to \partial\mathbb{D}$ lies in the Sobolev space $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{C})$, with the estimate

$$||h||_{\mathscr{W}^{1,p}(\mathbb{X},\mathbb{C})} \leqslant cM. \tag{3.2}$$

Proposition 3.3. *Let $\mathbb{X}$ be a Jordan domain with rectifiable boundary. Then the condition (3.1) holds with $p = 1$.*

Proposition 3.4. *Let $\mathbb{X}$ be a Jordan domain which has s-hyperbolic growth, with $s \in (0,1)$. Then condition* (3.1) *holds for all $p > 1$ with $p - 1 < s$.*

Proof of Theorem 3.2. First, since $\mathbb{X}$ is a Jordan domain according to the classical Carathéodory's theorem the conformal mapping $g \colon \mathbb{D} \to \mathbb{X}$ extends continuously to a homeomorphism from the unit circle onto $\partial\mathbb{X}$. Second, since a conformal change of variables preserves harmonicity, we find that the map $H := h \circ g \colon \mathbb{D} \to \mathbb{D}$ is a harmonic extension of the boundary homeomorphism $\psi := \varphi \circ g|_{\partial\mathbb{D}}$.

We will now assume that H is smooth up to the boundary of $\mathbb{D}$. The general result will then follow by an approximation argument. Indeed, for each $r < 1$, we may take the preimage of the disk $B(0,r)$ under H, and letting $\psi_r : \mathbb{D} \to H^{-1}(B(0,r))$ be the conformal map onto this preimage we may define $H_r := H \circ \psi_r$. Then H_r is harmonic, smooth up to the boundary of $\mathbb{D}$, and will converge to H locally uniformly along with its derivatives as $r \to 1$. Hence the general result will follow once we obtain uniform estimates for the Sobolev norm under the assumption of smoothness up to the boundary.

The harmonic extension $H := h \circ g \colon \mathbb{D} \to \mathbb{D}$ of $\psi := \varphi \circ g|_{\partial\mathbb{D}}$ is given by the Poisson integral formula [11],

$$(h \circ g)(z) = H(z) = \frac{1}{2\pi} \int_{\partial\mathbb{D}} \frac{1 - |z|^2}{|z - \omega|} \psi(\omega)\, d\omega\, .$$

Differentiating this, we find the formula

$$(h \circ g)_z = \int_{\partial\mathbb{D}} \frac{\psi(\omega)}{(z-\omega)^2}\, d\omega = \int_0^{2\pi} \frac{\psi(e^{it})}{(z - e^{it})^2} ie^{it}\, dt = \int_0^{2\pi} \frac{\psi'(e^{it})}{z - e^{it}} ie^{it}\, dt,$$

where we have used integration by parts to arrive at the last equality. The change of variables formula now gives

$$\begin{aligned} \int_{\mathbb{X}} |h_z(\tilde{z})|^p \, d\tilde{z} &= \int_{\mathbb{D}} |(h \circ g)_z(z)|^p |g'(z)|^{2-p}\, dz \\ &= \int_{\mathbb{D}} \left| \int_0^{2\pi} \frac{\psi'(e^{it})}{z - e^{it}} ie^{it}\, dt \right|^p |g'(z)|^{2-p}\, dz, \end{aligned}$$

We now apply Minkowski's integral inequality to find that

$$\begin{aligned} &\left(\int_{\mathbb{D}} \left| \int_0^{2\pi} \frac{\psi'(e^{it})}{z - e^{it}} ie^{it}\, dt \right|^p |g'(z)|^{2-p}\, dz \right)^{\frac{1}{p}} \\ &\qquad \leqslant \int_0^{2\pi} |\psi'(e^{it})| \left(\int_{\mathbb{D}} \frac{|g'(z)|^{2-p}}{|z - e^{it}|^p}\, dz \right)^{\frac{1}{p}} dt \\ &\qquad \leqslant M \int_0^{2\pi} |\psi'(e^{it})|\, dt \\ &\qquad = 2\pi M \end{aligned}$$

This gives the uniform bound $||h_z||_{L^p(\mathbb{X})} \leqslant 2\pi M$. An analogous estimate for the L^p-norm of $h_{\overline{z}}$ now proves the theorem. □

Proof of Proposition 3.3. Since $\partial\mathbb{X}$ is rectifiable, the derivative g' of a conformal map from $\mathbb{D}$ onto $\mathbb{X}$ lies in the Hardy space $H^1(\mathbb{D})$ by Theorem 3.12 in [10]. By rotational symmetry it is enough to verify condition (3.1) for $\omega = 1$ and $g : \mathbb{D} \to \mathbb{X}$ an arbitrary conformal map. By Proposition 2.7, it suffices to verify that the measure $\mu(z) = \frac{dz}{|1-z|}$ is a Carleson measure, see Definition 2.6, to obtain the estimate

$$\int_{\mathbb{D}} \frac{|g'(z)|}{|1-z|}\, dz \leqslant C||g'||_{H^1(\mathbb{D})},$$

which will imply that the proposition holds. Let us hence for each ϵ define the set $S_\epsilon(\theta) = \{re^{i\alpha} : 1-\epsilon < r < 1, \theta - \epsilon < \alpha < \theta + \epsilon\}$. We then estimate for small ϵ that

$$\mu(S_\epsilon(0)) \leqslant \mu(B(1,2\epsilon)) = \int_{B(1,2\epsilon)} \frac{dz}{|1-z|} = \int_0^{2\pi}\int_0^{2\epsilon} \frac{1}{r}\, r\, dr d\alpha = 4\pi\epsilon.$$

It is clear that for any other angles θ the μ-measure of $S_\epsilon(\theta)$ is smaller than for $\theta = 0$. Hence μ is a Carleson measure and our proof is complete. □

Proof of Proposition 3.4. Recall that g denotes the conformal map from $\mathbb{D}$ onto $\mathbb{X}$. Since $\mathbb{X}$ has s-hyperbolic growth, we may apply Definition 1.4 with $x_0 = g(0)$ to find the estimate

$$h_{\mathbb{X}}(g(0), g(z)) \leqslant C\left(\frac{1}{\operatorname{dist}(g(z), \partial\mathbb{X})}\right)^{1-s} \qquad \text{for all } z \in \mathbb{D}, \tag{3.3}$$

Since $\mathbb{X}$ is simply connected, the quasihyperbolic distance is comparable to the hyperbolic distance $\rho_{\mathbb{X}}$. By conformal invariance of the hyperbolic distance we find that

$$C_1 h_{\mathbb{X}}(g(0), g(z)) \geqslant \rho_{\mathbb{X}}(g(0), g(z)) = \rho_{\mathbb{D}}(0, z) = \log\frac{1}{1-|z|^2}.$$

Now by the Koebe $\frac{1}{4}$-theorem we know that the expression $\operatorname{dist}(g(z), \partial\mathbb{X})$ is comparable to $(1-|z|)|g'(z)|$ by a universal constant. Combining these observations with (3.3) leads to the estimate

$$\log\frac{1}{1-|z|^2} \leqslant C\left(\frac{1}{(1-|z|)|g'(z)|}\right)^{1-s},$$

which we transform into

$$|g'(z)| \leqslant \frac{C}{(1-|z|)\log^{1/(1-s)}\frac{1}{1-|z|}}, \tag{3.4}$$

Let us denote $\beta = (2-p)/(1-s)$ so that $\beta > 1$ by assumption. We now apply the estimate (3.4) to find that

$$\int_{\mathbb{D}} \frac{|g'(z)|^{2-p}}{|1-z|^p}\, dz \leqslant C \int_{\mathbb{D}\setminus\frac{1}{2}\mathbb{D}} \frac{1}{(1-|z|)^{2-p}|1-z|^p \log^\beta \frac{1}{1-|z|}}\, dz + \int_{\frac{1}{2}\mathbb{D}} \frac{|g'(z)|^{2-p}}{|1-z|^p}\, dz. \tag{3.5}$$

It is enough to prove that the quantity on the right hand side above is finite as then rotational symmetry will imply that the estimate (3.1) holds for all ω. The second term is easily seen to be finite, as the integrand is bounded on the set $\frac{1}{2}\mathbb{D}$. To estimate the first integral we will cover the annulus $\mathbb{D}\setminus\frac{1}{2}\mathbb{D}$ by three sets S_1, S_2 and S_3 defined by

$$S_1 = \{1 + re^{i\theta} : r \leqslant 3/4,\ 3\pi/4 \leqslant \theta \leqslant 5\pi/4\}$$
$$S_2 = \{(x, y) \in \mathbb{D} : -1/\sqrt{2} \leqslant y \leqslant 1/\sqrt{2},\ x \leqslant 1,\ x \geqslant 1 - |y|\}$$
$$S_3 = \{re^{i\theta} : 1/2 \leqslant r \leqslant 1,\ \pi/4 \leqslant \theta \leqslant 7\pi/4\}$$

See Figure 2 for an illustration of these sets. Since the sets S_1, S_2 and S_3 cover the set in question, it will be enough to see that the first integral on the right hand side of equation (3.5) is finite when taken over each of these sets. On the set S_1, one may find by geometry that the estimate $1 - |z| \geqslant c|1-z|$

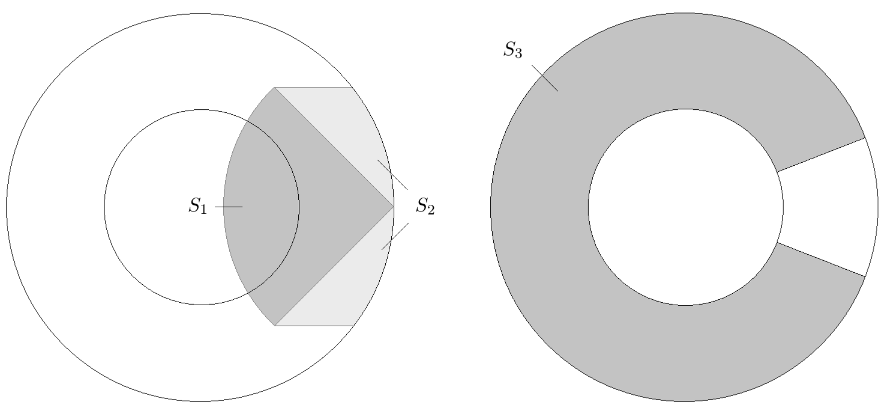

FIGURE 2. The sets S_i, $i = 1, 2, 3$.

holds for some constant c. Hence we may apply polar coordinates around the point $z = 1$ to find that

$$\int_{S_1} \frac{1}{(1-|z|)^{2-p}|1-z|^p \log^\beta \frac{1}{1-|z|}}\, dz \leqslant C \int_{3\pi/4}^{5\pi/4} \int_0^{3/4} \frac{1}{r \log^\beta \frac{1}{r}}\, dr d\theta < \infty.$$

On the set S_3, the expression $|1-z|$ is bounded away from zero. Hence bounding this term and the logarithm from below and changing to polar coordinates around the origin yields that

$$\int_{S_3} \frac{1}{(1-|z|)^{2-p}|1-z|^p \log^\beta \frac{1}{1-|z|}}\, dz \leqslant C \int_{\pi/4}^{7\pi/4} \int_{1/2}^{1} \frac{r}{(1-r)^{2-p}}\, dr d\theta < \infty.$$

On the set S_2, we change to polar coordinates around the origin. For each angle θ, we let R_θ denote the intersection of the ray with angle θ starting from the origin and the set S_2. On each such ray, we find that the expression $|1-z|$ is comparable to the size of the angle θ. Since $1-|z| < |1-z|$, we may also replace $1-|z|$ by $|1-z|$ inside the logarithm, in total giving us the estimate

$$\frac{1}{|1-z|^p \log^\beta \frac{1}{1-|z|}} \leqslant \frac{C}{|\theta|^p \log^\beta \frac{1}{|\theta|}}, \qquad z \in R_\theta. \tag{3.6}$$

On each of the segments R_θ and small enough θ, the modulus $r = |z|$ ranges from a certain distance $\rho(\theta)$ to 1. This distance is found by applying the sine theorem to the triangle with vertices $0, 1$ and $\rho(\theta)e^{i\theta}$, giving us the equation

$$\frac{\rho(\theta)}{\sin(\pi/4)} = \frac{1}{\sin(\pi - \pi/4 - \theta)} = \frac{1}{\sin(\pi/4+\theta)}.$$

From this one finds that the expression $1-\rho(\theta) = \frac{\sin(\pi/4+\theta)-\sin(\pi/4)}{\sin(\pi/4+\theta)}$, which also denotes the length of the segment R_θ, is comparable to $|\theta|$. Using this and (3.6) we now estimate that

$$\begin{aligned}
&\int_{S_2} \frac{1}{(1-|z|)^{2-p}|1-z|^p \log^\beta \frac{1}{1-|z|}}\, dz \\
&\leqslant C \int_{-\pi/4}^{\pi/4} \frac{1}{|\theta|^p \log^\beta \frac{1}{|\theta|}} \int_{\rho(\theta)}^{1} \frac{1}{(1-r)^{2-p}}\, dr d\theta \\
&= C \int_{-\pi/4}^{\pi/4} \frac{1}{|\theta|^p \log^\beta \frac{1}{|\theta|}} \frac{(1-\rho(\theta))^{p-1}}{p-1}\, dr d\theta \\
&\leqslant C \int_{-\pi/4}^{\pi/4} \frac{1}{|\theta| \log^\beta \frac{1}{|\theta|}}\, dr d\theta \\
&< \infty.
\end{aligned}$$

This finishes the proof. □

Proof of Theorem 1.8. Since $\mathbb{Y}$ is a rectifiable Jordan domain, there exists a constant speed parametrization $\gamma : \partial\mathbb{D} \to \partial\mathbb{Y}$. Such a parametrization is then automatically a Lipschitz embedding of $\partial\mathbb{D}$ to $\mathbb{C}$, and hence Theorem 1.4 implies that there exists a homeomorphic Lipschitz extension $G : \overline{\mathbb{D}} \to \overline{\mathbb{Y}}$ of γ.

Let now $\varphi : \partial\mathbb{X} \to \partial\mathbb{Y}$ be a given boundary homeomorphism. We define a boundary homeomorphism $\varphi_0 : \partial\mathbb{X} \to \partial\mathbb{D}$ by setting $\varphi_0 := \varphi \circ \gamma^{-1}$. Let h_0 denote the harmonic extension of φ_0 to $\mathbb{X}$, so that by the RKC-theorem (Lemma 2.3) the composed map $h := G \circ h_0 : \overline{\mathbb{X}} \to \overline{\mathbb{Y}}$ gives a homeomorphic extension of the boundary map φ. If the map h_0 lies in the Sobolev space $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{C})$, then so does the map h since the Sobolev integrability is preserved under a composition by a Lipschitz map. Hence Theorem 1.8 now follows from Theorem 3.1. □

4. Sharpness of Theorem 1.8

In this section we prove Theorem 1.10. We handle the two claims of this theorem separately.

Example (1). In this example we construct a nonrectifiable Jordan domain $\mathbb{X}$ and a boundary map $\varphi : \partial\mathbb{X} \to \partial\mathbb{D}$ which does not admit a continuous extension in the Sobolev class $\mathscr{W}^{1,1}(\mathbb{X}, \mathbb{C})$. The domain $\mathbb{X}$ will be defined as the following "spiral" domain.

Let R_k, $k = 1, 2, 3, \ldots$, be a set of disjoint rectangles in the plane such that their bottom side lies on the x-axis. Each rectangle has width w_k so that $\sum_{k=1}^{\infty} w_k < \infty$ and the rectangles are sufficiently close to each other so that the collection stays in a bounded set. The heights h_k satisfy $\lim_{k\to\infty} h_k = 0$ and $\sum_{k=1}^{\infty} h_k = \infty$.

We now join these rectangles into a spiral domain as in Figure 3, and add a small portion of boundary to the bottom end of R_1. The exact way these rectangles are joined is not significant, but it is clear that it may be done in such a way as to produce a nonrectifiable Jordan domain $\mathbb{X}$ for any sequence of rectangles R_k as described above.

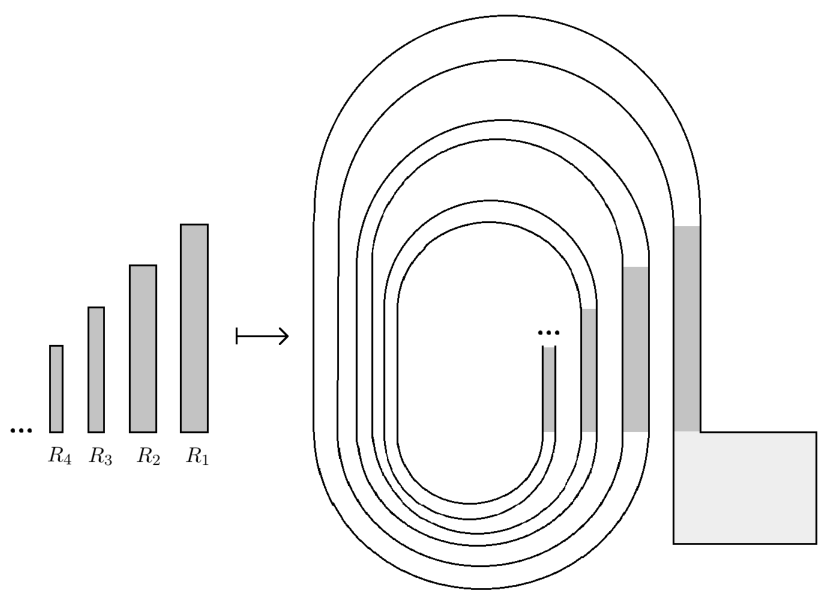

Figure 3. The rectangles R_k joined into the spiral domain $\mathbb{X}$.

Let us now define the boundary homeomorphism φ. The map φ shall map the "endpoint" (i.e. the point on the x-axis to which the rectangles R_k converge) of the spiral domain $\mathbb{X}$ to the point $1 \in \partial\mathbb{D}$. Furthermore, we choose disjoint arcs A_k^+ on the unit circle so that the endpoints of A_k^+ are given by $e^{i\alpha_k}$ and $e^{i\beta_k}$ with

$$\pi/2 > \alpha_1 > \beta_1 > \alpha_2 > \beta_2 > \cdots$$

and $\lim_{k\to\infty} \alpha_k = 0$. We mirror the arcs A_k^+ over the x-axis to produce another set of arcs A_k^-. The arcs are chosen in such a way that the minimal distance between A_k^+ and A_k^- is greater than a given sequence of numbers d_k with $\lim_{k\to\infty} d_k = 0$. It is clear that for any such sequence we may make a choice of arcs as described here.

We now define φ to map the left side of the rectangle R_k to the arc A_k^+, and the right side to A_k^-. On the rest of the boundary $\partial\mathbb{X}$ we may define the map φ in an arbitrary way to produce a homeomorphism $\varphi : \partial\mathbb{X} \to \partial\mathbb{D}$.

Let now H be a continuous $\mathscr{W}^{1,1}$-extension of φ. Let I_k denote any horizontal line segment with endpoints on the vertical sides of R_k. Then by the above construction, H must map the segment I_k to a curve of length at least d_k, as this is the minimal distance between A_k^+ and A_k^-. Hence we find that

$$\int_{R_k} |DH| dz \geqslant \int_0^{h_k} d_k dz = h_k d_k.$$

Summing up, we obtain the estimate

$$\int_{\mathbb{X}} |DH| dz \geqslant \sum_{k=1}^{\infty} h_k d_k.$$

We may now choose, for example, $h_k = 1/k$ and $d_k = 1/\log(1+k)$ to make the above sum diverge, showing that H cannot belong to $\mathscr{W}^{1,1}(\mathbb{X}, \mathbb{C})$. This finishes the proof.

Example (2). Let $1 < p < 2$. Here we construct a Jordan domain $\mathbb{X}$ whose boundary has $(p-1)$-hyperbolic growth and a boundary map $\varphi : \partial\mathbb{X} \to \partial\mathbb{D}$ which does not admit a continuous extension in the Sobolev class $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{C})$. In fact, this domain may be chosen as the domain $\mathbb{X}_s$ defined after Definition 1.4 for $s = p - 1$.

The construction of the boundary map φ is as follows.

We set $\varphi(0) = 1$. Furthermore, we choose two sequences of points p_k^+ and p_k^- belonging to the graph $\{(x, |x|^s) : -1 \leqslant x \leqslant 1\}$ as follows. The points p_k^+ all have positive x-coordinates, their y-coordinates are decreasing in k with limit zero and the difference between the y-coordinates of p_{k-1}^+ and p_k^+ is comparable to a number ϵ_k, for which

$$\sum_{k=1}^{\infty} \epsilon_k < \infty.$$

In fact, for any sequence of numbers ϵ_k satisfying the above conditions one may choose a corresponding sequence p_k^+. We then let p_k^- be the reflection of p_k^+ along the y-axis.

Similarly, we choose points a_k^+ on the unit circle, so that $a_k^+ = e^{i\theta_k}$ for a sequence of angles $\theta_k > 0$ decreasing to zero. Letting a_k^- be the reflection of a_k^+ along the x-axis, we choose the sequence in such a way that the line segment between a_k^+ and a_k^- has length d_k for some decreasing sequence d_k with $\lim_{k\to\infty} d_k = 0$. Again, any such sequence d_k gives rise to a choice of points a_k.

Let Γ_k^+ denote the part of the boundary of $\mathbb{X}_s$ between p_{k-1}^+ and p_k^+. We define the map φ to map Γ_k^- to the arc of the unit circle between a_{k-1}^- and a_k^- with constant speed. We define Γ_k^- and $\varphi|_{\Gamma_k^-}$ similarly. Let now H

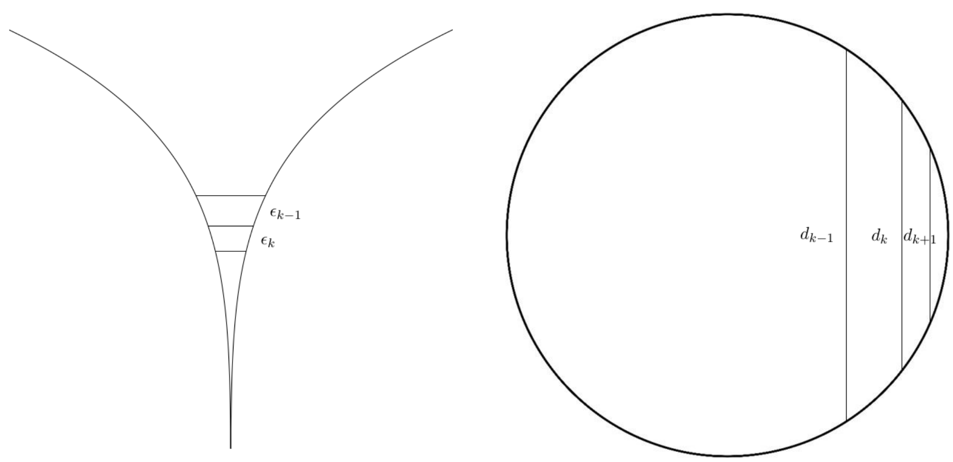

FIGURE 4. The portions of height ϵ_k get mapped onto slices with side length d_k.

denote any continuous $\mathscr{W}^{1,p}$-extension of φ to $\mathbb{X}$. By the above definition, any horizontal line segment with endpoints on Γ_k^+ and Γ_k^- is mapped into a curve of length at least d_k under H. Such a line segment is of length at most the distance of p_{k-1}^+ to p_{k-1}^-, a distance which is comparable to $\left(\sum_{j=k}^{\infty} \epsilon_j\right)^{1/s}$. If S_k denotes the union of all the horizontal line segments between Γ_k^+ and Γ_k^-, this gives the estimate

$$\int_{S_k} |DH|^p dz \geqslant \frac{\left(\int_{S_k} |DH| dz\right)^p}{|S_k|^{p-1}} \geqslant \frac{c\left(\int_0^{\epsilon_k} d_k dy\right)^p}{\epsilon_k^{p-1}\left(\sum_{j=k}^{\infty} \epsilon_j\right)^{(p-1)/s}} = \frac{c d_k^p \epsilon_k}{\sum_{j=k}^{\infty} \epsilon_j}$$

Let now, for example, $\epsilon_k = 1/k^2$. Then $\sum_{j=k}^{\infty} \epsilon_j$ is comparable to $1/k$, so by summing up we obtain the estimate

$$\int_{\bigcup_k S_k} |DH|^p dz \geqslant c \sum_{k=1}^{\infty} \frac{d_k^p}{k}. \tag{4.1}$$

Choosing a suitably slowly converging sequence d_k such as $d_k = (\log(1+k))^{-1/p}$, we find that the right hand side of (4.1) diverges. It follows that H cannot lie in the Sobolev space $\mathscr{W}^{1,p}(\mathbb{X}_s, \mathbb{C})$, which completes our proof.

5. The case $p = 2$

In this section we address Theorem 1.6 as well as Examples 1.5 and 1.7.

Example 1.5. For this example, let first Φ_τ for any $\tau \in (0,1]$ denote the conformal map

$$\Phi_\tau(z) = \log^{-\tau}\left(\frac{1-z}{3}\right)$$

defined on the unit disk and having target $\mathbb{Y}_\tau := \Phi_\tau(\mathbb{D})$. In fact, the domain $\mathbb{Y}_\tau$ is a domain with smooth boundary apart from one point at which it has an outer cusp of degree $\tau/(1+\tau)$ (i.e. it is bilipschitz-equivalent with the domain $\mathbb{X}_{\tau/(1+\tau)}$ as pictured in Figure 1).

Since Φ_τ is conformal and maps the unit disk into a set of finite measure, it lies in the Sobolev space $\mathscr{W}^{1,2}(\mathbb{D}, \mathbb{C})$. However, it does not admit a homeomorphic extension to the whole plane in the Sobolev class $\mathscr{W}^{1,2}_{loc}(\mathbb{C})$. The reason for this is that there is a modulus of continuity estimate for any homeomorphism in the Sobolev class $\mathscr{W}^{1,2}_{loc}(\mathbb{C})$. Indeed, let $\omega(t)$ denote the the modulus of continuity of $g \colon \mathbb{C} \to \mathbb{C}$; that is,

$$\omega(t) = \operatorname*{osc}_{B(z,t)} g = \sup\{|g(x_1) - g(x_2)| \colon x_1, x_2 \in B(z,t)\}\,.$$

If g is a homeomorphism in $\mathscr{W}^{1,2}_{\text{loc}}(\mathbb{C}, \mathbb{C})$, then

$$\int_0^r \frac{\omega(t)^2}{t} dt < \infty. \tag{5.1}$$

Proof of (5.1). Since g is a homeomorphism we have

$$\operatorname*{osc}_{B(z,t)} g \leqslant \operatorname*{osc}_{\partial B(z,t)} g\,.$$

According to Sobolev's inequality on spheres for almost every $t > 0$ we obtain

$$\operatorname*{osc}_{\partial B(z,t)} g \leqslant C \int_{\partial B(z,t)} |Dg|\,.$$

These together with Hölder's inequality imply

$$\omega(t) = \operatorname*{osc}_{B(z,t)} \leqslant \operatorname*{osc}_{\partial B(z,t)} g \leqslant C \left(t \int_{\partial B(z,t)} |Dg|^2 \right)^{\frac{1}{2}}$$

and, therefore, for almost every $t > 0$ we have

$$\frac{\omega(t)^2}{t} \leqslant C \int_{\partial B(z,t)} |Dg|^2 .$$

Integrating this from 0 to $r > 0$, the claim (5.1) follows. □

Now, since the map Φ_τ for $\tau \leqslant 1$ does not satisfy the modulus of continuity estimate (5.1) at the boundary point $z = 1$, it follows that it is not possible to extend Φ_τ even locally as a $\mathscr{W}^{1,2}$-homeomorphism around the point $z = 1$.

To address the exact claim of Example 1.5, we now define an embedding $\varphi : \partial\mathbb{D} \to \mathbb{C}$ as follows. Fixing $\tau \in (0, 1]$, in the set $\{z \in \partial\mathbb{D} \colon \operatorname{Re}(z) \geqslant 0\}$ we let $\varphi(z) = \Phi_\tau(z)$. We also map the complementary set $\{z \in \partial\mathbb{D} \colon \operatorname{Re}(z) < 0\}$ smoothly into the complement of $\overline{\mathbb{Y}_\tau}$, and in such a way that $\varphi(\partial\mathbb{D})$ becomes the boundary of a Jordan domain $\tilde{\mathbb{Y}}$. See Figure 5 for an illustration. It

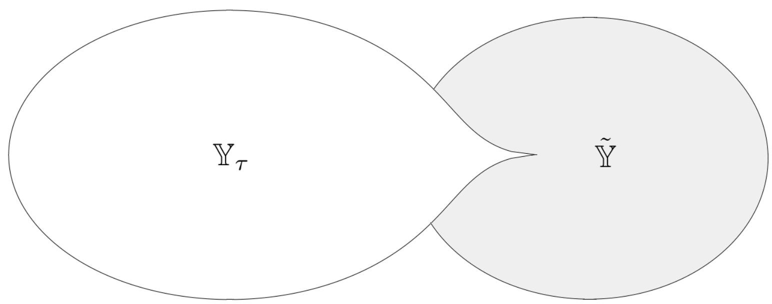

FIGURE 5. The Jordan domains $\mathbb{Y}_\tau$ and $\tilde{\mathbb{Y}}$.

is now easy to see that the map φ satisfies the Douglas condition (1.2). Indeed, since the map Φ_τ is in the Sobolev space $\mathscr{W}^{1,2}(\mathbb{D}, \mathbb{C})$ its restriction to the boundary must necessarily satisfy the Douglas condition. Since the map φ aligns with this boundary map in a neighborhood of the point $z = 1$, verifying the finiteness of the integral in (1.2) poses no difficulty in this neighborhood. On the rest of the boundary of $\partial\mathbb{D}$ we may choose φ to be locally Lipschitz, which shows that (1.2) is necessarily satisfied for φ. Hence we have found a map from $\partial\mathbb{D}$ into the boundary of the chord-arc domain $\tilde{\mathbb{Y}}$ which admits a $\mathscr{W}^{1,2}$-extension to $\mathbb{D}$ but not a homeomorphic one.

Example 1.7. In [39], Zhang constructed an example of a Jordan domain, which we shall denote by $\mathbb{Y}$, so that the conformal map $g : \mathbb{D} \to \mathbb{Y}$ does not admit a $\mathscr{W}^{1,1}$-homeomorphic extension to the whole plane. We shall not repeat this construction here, but will instead briefly show how it relates to our questions.

The domain $\mathbb{Y}$ is constructed in such a way that there is a boundary arc $\Gamma \subset \partial\mathbb{Y}$ over which one cannot extend the conformal map g even locally as a $\mathscr{W}^{1,1}$-homeomorphism. The complementary part of the boundary $\mathbb{Y} \setminus \Gamma$

is piecewise linear. Hence we may employ the same argument as in the previous example. We choose a Jordan domain $\tilde{\mathbb{Y}}$ in the complement of $\mathbb{Y}$ whose boundary consists of the arc Γ and, say, a piecewise linear curve. We then define a boundary map $\varphi : \partial\mathbb{D} \to \partial\tilde{\mathbb{Y}}$ so that it agrees with g in a neighborhood of the set $g^{-1}(\Gamma)$ and is locally Lipschitz everywhere else. With the same argument as before, this boundary map must satisfy the Douglas condition (1.2). Hence this boundary map admits a $\mathscr{W}^{1,2}$-extension to $\mathbb{D}$ but not even a $\mathscr{W}^{1,1}$-homeomorphic extension. Naturally the boundary of the domain $\tilde{\mathbb{Y}}$ is quite ill-behaved, in particular nonrectifiable (though the Hausdorff dimension is still one).

Proof of Theorem 1.6. Let $\gamma : \partial\mathbb{D} \to \partial\mathbb{Y}$ denote a constant speed parametrization of the rectifiable curve $\partial\mathbb{Y}$. Let $G : \mathbb{C} \to \mathbb{C}$ be the homeomorphic Lipschitz extension of γ given by Theorem 1.4. Denoting $f := \varphi^{-1} \circ \gamma$, we find by change of variables that

$$\int_{\partial\mathbb{Y}} \int_{\partial\mathbb{Y}} \big|\log|\varphi^{-1}(\xi) - \varphi^{-1}(\eta)|\big|\, |\mathrm{d}\xi||\mathrm{d}\eta| = \int_{\partial\mathbb{D}} \int_{\partial\mathbb{D}} |\log|f(z) - f(\omega)||\, |\mathrm{d}z||\mathrm{d}\omega|.$$

Now the result of Astala, Iwaniec, Martin and Onninen [3] shows that the inverse map $f^{-1} : \partial\mathbb{X} \to \partial\mathbb{D}$ satisfies the Douglas condition (1.2). Thus f^{-1} extends to a harmonic $\mathscr{W}^{1,2}$-homeomorphism H_1 to $\overline{\mathbb{D}}$ by the RKC-Theorem (Lemma 2.3). Letting $h := G \circ H_1$, we find that h lies in the space $\mathscr{W}^{1,2}(\mathbb{X})$ since G is Lipschitz. Moreover, the boundary values of h are equal to $\gamma \circ (\varphi^{-1} \circ \gamma)^{-1} = \varphi$, giving us a homeomorphic extension of φ in the Sobolev space $\mathscr{W}^{1,2}(\mathbb{X})$.

To further extend φ into the complement of $\mathbb{X}$, assume first without loss of generality that $0 \in \mathbb{X}$ and $0 \in \mathbb{Y}$. We now let $\tau(z) = 1/\overline{z}$ denote the inversion map, which is a diffeomorphism in $\mathbb{C} \setminus \{0\}$. The map $\psi := \tau \circ \varphi \circ \tau$ is then a homeomorphism from $\partial\tau(\mathbb{X})$ to $\partial\tau(\mathbb{Y})$, and must also satisfy the condition (1.3) due to the bounds on τ. The earlier part of the proof shows that we may extend ψ as a $\mathscr{W}^{1,2}$-homeomorphism $\tilde{h}$ from the Jordan domain bounded by $\partial\tau(\mathbb{X})$ to the Jordan domain bounded by $\partial\tau(\mathbb{Y})$. Hence the map $\tau \circ \tilde{h} \circ \tau$ is a $\mathscr{W}^{1,2}_{loc}$-homeomorphism from the complement of $\mathbb{X}$ to the complement of $\mathbb{Y}$ and equal to φ on the boundary. This concludes the proof. □

6. The multiply connected case, Proof of Theorem 1.11

In this section we consider multiply connected Jordan domains $\mathbb{X}$ and $\mathbb{Y}$ of the same topological type. Any such domains can be equivalently obtained by removing from a simply connected Jordan domain the same number, say $0 \leqslant k < \infty$, of closed disjoint topological disks. If $k = 1$, the obtained doubly connected domain is conformally equivalent with a circular annulus $\mathbb{A} = \{z \in \mathbb{C} \colon r < |z| < 1\}$ with some $0 < r < 1$. In fact, if $k \geqslant 1$ every $(k+1)$-connected Jordan domain can be mapped by a conformal mapping onto a *circular domain*, see [14]. In particular we may consider a $(k+1)$-connected circuilar domain consisting of the domain bounded by the boundary of the unit disk $\mathbb{D}$ and k other circles (including points) in the interior of $\mathbb{D}$. The conformal mappings between multiply connected Jordan domains extends continuously up to the boundaries.

The idea of the proof of Theorem 1.11 is simply to split the multiply connected domains $\mathbb{X}$ and $\mathbb{Y}$ into simply connected parts and apply Theorem 1.8 in each of these parts. Let us consider first the case where $\mathbb{X}$ and $\mathbb{Y}$ are doubly connected.

6.1. **Doubly connected $\mathbb{X}$ and $\mathbb{Y}$.**

Case 1. $p = 1$. Suppose that the boundary of $\mathbb{X}$ is rectifiable. We split the domain $\mathbb{X}$ into two rectifiable simply connected domains as follows. Take a line ℓ passing through any point in the bounded component of $\mathbb{C} \setminus \mathbb{X}$. Then necessarily there exist two open line segments I_1 and I_2 on ℓ such that these segments are contained in $\mathbb{X}$ and their endpoints lie on different components of the boundary of $\mathbb{X}$. These segments split the domain $\mathbb{X}$ into two rectifiable Jordan domains $\mathbb{X}_1$ and $\mathbb{X}_2$.

For $k = 1, 2$, let p_k denote the endpoint of I_k lying on the inner boundary of $\mathbb{X}$ and P_k the endpoint on the outer boundary. We let $q_k = \varphi(p_k)$ and $Q_k = \varphi(P_k)$. We would now simply like to connect q_k with Q_k by a rectifiable curve γ_k inside of $\mathbb{Y}$ such that γ_1 and γ_2 do not intersect. It is quite obvious this can be done but we provide a proof regardless.

Let $\mathbb{Y}_+$ denote the Jordan domain bounded by the outer boundary of $\mathbb{Y}$. Take a conformal map $g_+ : \mathbb{D} \to \mathbb{Y}_+$. Then g_+' is in the Hardy space H^1 since $\partial\mathbb{Y}_1$ is rectifiable, and we find by Theorem 3.13 in [10] that g_+ maps the segment $[0, g_+^{-1}(Q_k)]$ into a rectifiable curve in $\mathbb{Y}_+$. Let γ_k^+ denote the image of the segment $[(1-\epsilon)g_+^{-1}(Q_k), g_+^{-1}(Q_k)]$ under g_+ for a sufficiently small ϵ. Hence we have a rectifiable curve γ_k^+ connecting Q_k to an interior point Q_k^+ of $\mathbb{Y}$ if ϵ is small enough. With a similar argument, possibly adding a Möbius transformation to the argument to invert the order of the boundaries, one finds a rectifiable curve γ_k^- connecting q_k to an interior point q_k^-. For small enough ϵ the four curves constructed here do not intersect.

If Γ denotes the union of these four curves, we may now use the path-connectivity of the domain $\mathbb{Y} \setminus \Gamma$ to join the points Q_1^+ and q_1^- with a smooth simple curve inside $\mathbb{Y}$ that does not intersect Γ. By adding the curves γ_1^+ and γ_1^- one obtains a rectifiable simple curve γ_1 connecting Q_1 and q_1. Using the fact that $\mathbb{Y} \setminus \Gamma$ is doubly connected, we may now join Q_2^+ and q_2^- with a smooth curve that does not intersect γ_1 nor Γ. This yields a rectifiable simple curve γ_2 connecting Q_2 and q_2. This proves the existence of the curves γ_k with the desired properties. These curves split $\mathbb{Y}$ into two simply connected Jordan domains $\mathbb{Y}_1$ and $\mathbb{Y}_2$.

We may now extend the homeomorphism φ to map the boundary of $\mathbb{X}_k$ to the boundary of $\mathbb{Y}_k$ homeomorphically. The exact parametrization which maps the segments I_k to the curves γ_k does not matter. The rest of the claim follows directly from the first part of Theorem 1.8, giving us a homeomorphic extension of φ in the Sobolev class $\mathscr{W}^{1,1}(\mathbb{X},\mathbb{C})$, as claimed.

Case 2. $1 < p < 2$. Suppose that $\mathbb{X}$ has s-hyperbolic growth. Then we take an annulus $\mathbb{A}$ centered at the origin such that there exists a conformal map $g : \mathbb{A} \to \mathbb{X}$. By a result of Gehring and Osgood [12], the quasihyperbolic metrics $h_{\mathbb{X}}$ and $h_{\mathbb{A}}$ are comparable via the conformal map g. This shows that for any fixed $x_0 \in \mathbb{A}$ and all $x \in \mathbb{A}$ we have

$$h_{\mathbb{A}}(x_0, x) \leqslant C h_{\mathbb{X}}(g(x_0), g(x)) \leqslant \frac{C}{\operatorname{dist}(g(x), \partial\mathbb{X})^{1-s}}. \tag{6.1}$$

Let now $\mathbb{A}_+$ denote the simply connected domain obtained by intersecting $\mathbb{A}$ and the upper half plane. We claim that the the domain $\mathbb{X}_+ := g(\mathbb{A}_+)$ has s-hyperbolic growth as well.

To prove this claim, fix $x_0 \in \mathbb{A}_+$ and take arbitrary $x \in \mathbb{A}$. Let $d = \operatorname{dist}(x, \partial\mathbb{A}_+)$. We aim to establish the inequality

$$h_{\mathbb{A}_+}(x_0, x) \leqslant \frac{C}{\operatorname{dist}(g(x), \partial\mathbb{X}_+)^{1-s}}. \tag{6.2}$$

Note that $\mathbb{A}_+$ is bi-Lipschitz equivalent with the unit disk, implying that $h_{\mathbb{A}_+}(x_0, x)$ is comparable to $\log(1/d)$. Since the boundary of $\mathbb{A}_+$ contains two line segments on the real line, let us denote them by I_1 and I_2. Note that we have the estimate

$$\operatorname{dist}(g(x), \partial\mathbb{X}_+) \leqslant \operatorname{dist}(g(x), \partial\mathbb{X}). \tag{6.3}$$

If it would happen that $d = \operatorname{dist}(x, \partial\mathbb{A})$, meaning that the closest point to x on $\partial\mathbb{A}_+$ is not on I_1 or I_2, then the hyperbolic distances $h_{\mathbb{A}_+}(x_0, x)$ and $h_{\mathbb{A}}(x_0, x)$ are comparable and by the inequalities (6.1) and (6.3) the inequality (6.2) holds. It is hence enough to prove (6.2) in the case when $d = \operatorname{dist}(x, I_1 \cup I_2)$. We may also assume that d is small. Due to the geometry of the half-annulus $\mathbb{A}_+$, the vertical line segment L_x between x and its projection to the real line lies on either I_1 or I_2 and its length is d. Letting D denote the distance of x to $\partial\mathbb{A}_+ \setminus (I_1 \cup I_2)$, we have that $D \geqslant d$.

We may now reiterate the proof of (3.4) to find that

$$|g'(z)| \leqslant \frac{C}{\operatorname{dist}(z, \partial\mathbb{A}) \log^{\frac{1}{1-s}}(\operatorname{dist}(z, \partial\mathbb{A})^{-1})}$$

for $z \in \mathbb{A}$. We should mention that the simply connectedness assumption used in the proof of (3.4) may be circumvented by using the equivalence of the quasihyperbolic metrics under g instead of passing to the hyperbolic metric. Hence

$$\operatorname{dist}(g(x), \partial\mathbb{X}_+) \leqslant \int_{L_x} |g'(z)||dz| \leqslant \frac{Cd}{D \log^{\frac{1}{1-s}}(1/D)}.$$

From this we find that (6.2) is equivalent to

$$\log(1/d) \leqslant C\frac{D^{1-s}\log(1/D)}{d^{1-s}},$$

which is true since $D \geqslant d$. Hence (6.2) holds, and this implies that $\mathbb{X}_+$ has s-hyperbolic growth by reversing the argument that gives (6.1).

We define $\mathbb{X}_-$ similarly. Hence we have split $\mathbb{X}$ into two simply connected domains with s-hyperbolic growth. On the image side, we may split $\mathbb{Y}$ into two simply connected domains with rectifiable boundary as in Case 1. Extending φ in an arbitrary homeomorphic way between the boundaries of these domains and applying part 2 of Theorem 1.8 gives a homeomorphic extension of φ in the Sobolev class $\mathscr{W}^{1,p}(\mathbb{X},\mathbb{C})$ whenever $s > p-1$.

6.2. **The general case.**

Case 3. $p = 1$. Assume that $\mathbb{X}$ and $\mathbb{Y}$ are ℓ-connected Jordan domains with rectifiable boundaries. By induction, we may assume that the result of Theorem 1.11 holds for $(\ell-1)$-connected Jordan domains. Hence we are only required to split $\mathbb{X}$ and $\mathbb{Y}$ into two domains with rectifiable boundary, one which is doubly connected and another which is $(\ell-1)$-connected.

We hence describe how to 'isolate' a given boundary component X_0 from a ℓ-connected Jordan domain $\mathbb{X}$. Let X_{outer} denote the outer boundary component of $\mathbb{X}$. Take a small neighborhood of X_0 inside $\mathbb{X}$. Let γ_0 be a piecewise linear Jordan curve contained in this neighborhood and separating X_0 from the rest of the boundary components of $\mathbb{X}$. Let also γ_1 be a piecewise linear Jordan curve inside $\mathbb{X}$ and in a small enough neighborhood of X_{outer} so that all of the other boundary components of $\mathbb{X}$ are contained inside γ_1. Take y_0 and y_1 on γ_0 and γ_1 respectively, and connect them with a piecewise linear curve α_y not intersecting any boundary components of $\mathbb{X}$. Choose z_0 and z_1 close to y_0 and y_1 respectively so that we may connect z_0 and z_1 by a piecewise linear curve α_z arbitrarily close to α_y but neither intersecting it nor any boundary components of $\mathbb{X}$. Since the region bounded by X_{outer} and γ_1 is doubly connected, by the construction in Case 1 we may connect y_1 and z_1 with any two given points y_2 and z_2 on the boundary X_{outer} via non-intersecting rectifiable curves β_y and β_z lying inside this region.

Let now Γ denote the union of the curves β_y, β_z, α_y, α_z, and the curve γ_0' obtained by taking the curve γ_0 and removing the part between y_0 and z_0. By construction Γ contains two arbitrary points on $\mathbb{X}_{outer}$ and separates the domain $\mathbb{X}$ into a doubly connected domain with inner boundary component X_0 and a $(n-1)$-connected Jordan domain. Since Γ is rectifiable, both of these domains are also rectifiable.

Applying the same construction for $\mathbb{Y}$, we may separate the boundary component $\varphi(X_0)$ of $\mathbb{Y}$ by a rectifiable curve Γ'. Since the boundary points y_2 and z_2 above were arbitrary, we may assume that Γ' intersects the outer boundary of $\mathbb{Y}$ at the points $\varphi(y_2)$ and $\varphi(z_2)$. Extending φ to a homeomorphism from Γ onto Γ' and applying the induction assumptions now gives a homeomorphic extension in the class $\mathscr{W}^{1,1}(\mathbb{X},\mathbb{C})$.

Case 4. $1 < p < 2$. We still have to deal with the case where $\mathbb{X}$ has s-hyperbolic growth and is ℓ-connected. By the same arguments as in the previous case, it will be enough to split $\mathbb{X}$ into a doubly connected and $(\ell - 1)$-connected domain with s-hyperbolic growth.

Since $\mathbb{X}$ is ℓ-connected, there exists a domain Ω such that every boundary component of Ω is a circle and there is a conformal map $g : \Omega \to \mathbb{X}$. Let $\Gamma \subset \Omega$ be a piecewise linear curve separating one of the inner boundary components of $\partial\Omega$. Hence Ω splits into a doubly connected set Ω_1 and a $(\ell - 1)$-connected set Ω_2. We claim that the domains $\mathbb{X}_1 = g(\Omega_1)$ and $\mathbb{X}_2 = g(\Omega_2)$ have s-hyperbolic growth.

The proof of this claim is nearly identical to the arguments in Case 2, so we will summarize it briefly. For $\mathbb{X}_2$, we aim to establish the inequality

$$h_{\Omega_2}(x_0, x) \leqslant \frac{C}{\operatorname{dist}(g(x), \partial\mathbb{X}_2)^{1-s}} \tag{6.4}$$

for fixed $x_0 \in \Omega_2$ and $x \in \Omega_2$. For this inequality, it is only essential to consider x close to $\partial\Omega_2$. If x is closer to the boundary of the original set $\partial\Omega$ than to Γ, then the hyperbolic distance of x_0 and x in Ω_2 is comparable to the distance inside the larger set Ω. Then the s-hyperbolic growth of Ω implies (6.4) as in Case 2. If x is closer to Γ but a fixed distance away from the boundary of Ω, then the smoothness of g in compact subsets of Ω implies the result. If x is closest to a line segment in Γ which has its other endpoint on $\partial\Omega$, then we may employ a similar estimate as in Case 2, using the bound for $|g'(z)|$ in terms of $\operatorname{dist}(z, \partial\Omega)$, to conclude that (6.4) also holds here. This implies that $\mathbb{X}_2$ satisfies (6.4), and hence it has s-hyperbolic growth. The argument for $\mathbb{X}_1$ is the same.

After splitting $\mathbb{X}$ into two domains of smaller connectivity and s-hyperbolic growth, we split the target $\mathbb{Y}$ accordingly into rectifiable parts using the argument from Case 3. Applying induction on n now proves the result. This finishes the proof of Theorem 1.11.

7. Monotone Sobolev minimizers

The classical *harmonic mapping problem* deals with the question of whether there exists a harmonic homeomorphism between two given domains. Of course, when the domains are Jordan such a mapping problem is always solvable. Indeed, according to the Riemann Mapping Theorem there is a conformal mapping $h \colon \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$. Finding a harmonic homeomorphism which coincides with a given boundary homeomorphism $\varphi \colon \partial\mathbb{X} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ is a more subtle question. If $\mathbb{Y}$ is convex, then there always exists a harmonic homeomorphism $h \colon \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ with $h(x) = \varphi(x)$ on $\partial\mathbb{X}$ by Lemma 2.3. For a non-convex target $\mathbb{Y}$, however, there always exists at least one boundary homeomorphism whose harmonic extension takes points in $\mathbb{X}$ beyond $\overline{\mathbb{Y}}$. To find a deformation $h \colon \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ which resembles harmonic homeomorphisms

Iwaniec and Onninen [22] applied the direct method in the calculus of variations and considered minimizing sequences in $\mathscr{H}_\varphi^{1,2}(\overline{\mathbb{X}}, \overline{\mathbb{Y}})$. They called such minimizers *monotone Hopf-harmonics* and proved the existence and uniqueness result in the case when $\mathbb{Y}$ is a Lipschitz domain and the boundary data φ satisfies the Douglas condition. Note that by the Riemann Mapping Theorem one may always assume that $\mathbb{X} = \mathbb{D}$. Theorem 1.6 opens up such studies beyond the Lipschitz targets. Indeed, under the assumptions of Theorem 1.6, the class $\mathscr{H}_\varphi^{1,2}(\overline{\mathbb{D}}, \overline{\mathbb{Y}})$ is non-empty. Furthermore, if $h_\circ \in \mathscr{H}_\varphi^{1,2}(\overline{\mathbb{D}}, \overline{\mathbb{Y}})$, then $h_\circ$ satisfies the uniform modulus of continuity estimate

$$|h_\circ(x_1) - h_\circ(x_2)|^2 \leqslant C \frac{\int_{\mathbb{D}} |Dh_\circ|^2}{\log\left(\frac{1}{|x_1 - x_2|}\right)}$$

for $x_1, x_2 \in \mathbb{D}$ such that $|x_1 - x_2| < 1$. This follows from taking the global $\mathscr{W}^{1,2}_{loc}$-homeomorphic extension given by Theorem 1.6 and applying a standard local modulus of continuity estimate for $\mathscr{W}^{1,2}$-homeomorphisms, see [18, Corollary 7.5.1 p.155]. Now, applying the direct method in the calculus of variations allows us to find a minimizing sequence in $\mathscr{H}_\varphi^{1,2}(\overline{\mathbb{D}}, \overline{\mathbb{Y}})$ for the Dirichlet energy which converges weakly in $\mathscr{W}^{1,2}(\mathbb{D}, \mathbb{C})$ and uniformly in $\overline{\mathbb{D}}$. Being a uniform limit of homeomorphisms the limit mapping $H : \overline{\mathbb{D}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ becomes *monotone*. Indeed, the classical Youngs approximation theorem [35] which asserts that a continuous map between compact oriented topological 2-manifolds (surfaces) is monotone if and only if it is a uniform limit of homeomorphisms. Monotonicity, the concept of Morrey [29], simply means that for a continuous $H : \overline{\mathbb{X}} \to \overline{\mathbb{Y}}$ the preimage $H^{-1}(y_\circ)$ of a point $y_\circ \in \overline{\mathbb{Y}}$ is a continuum in $\overline{\mathbb{X}}$. We have hence just given a proof of the following result.

Theorem 7.1. *Let $\mathbb{X}$ and $\mathbb{Y}$ be Jordan domains and assume that $\partial\mathbb{Y}$ is rectifiable. If $\varphi : \partial\mathbb{X} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ satisfies* (1.3)*, then there exists a monotone Sobolev mapping $H : \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ in $\mathscr{W}^{1,2}(\mathbb{X}, \mathbb{C})$ such that H coincides with φ on $\partial\mathbb{X}$ and*

$$\int_{\mathbb{X}} |DH(x)|^2 \, \mathrm{d}x = \inf_{h \in \mathscr{H}_\varphi^{1,2}(\overline{\mathbb{X}}, \overline{\mathbb{Y}})} \int_{\mathbb{X}} |Dh(x)|^2 \, \mathrm{d}x \, .$$

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