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SERRE DUALITY FOR THE COHOMOLOGY OF LANDAU-GINZBURG MODELS

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Abstract. Let V and F be holomorphic bundles over a complex manifold M, and s be a holomorphic section of V. We study the cohomology associated to the Koszul complex induced by s, and prove a generalized Serre duality theorem for them.

1. Introduction

The Serre duality theorem is a fundamental result in complex manifold, which establishes a duality between the cohomology of a complex manifold and the cohomology of with compact supports, provided that the $\overline{\partial}$ operator has closed range in appropriate degrees. In this paper we extend the Serre duality to the cohomology of Landau-Ginzburg models.

Let V be a holomorphic bundle over a complex manifold (usually noncompact) M with $\operatorname{rank} V = \dim M = n$, and s be a holomorphic section of V with compact zero loci $Z := (s^{-1}(0))$. Let $V^\vee$ be the dual bundle of V, then s induced the following Koszul complex

$$0 \to \wedge^n V^\vee \xrightarrow{\iota_s} \cdots \xrightarrow{\iota_s} \wedge^2 V^\vee \xrightarrow{\iota_s} V^\vee \xrightarrow{\iota_s} \mathbb{C} \to 0, \tag{1.1}$$

where ι_s is the contraction operator induced by s.

Let F be another holomorphic bundle over M, we have the following complex from (1.1)

$$0 \to \wedge^n V^\vee \otimes F \xrightarrow{\iota_s \otimes 1} \cdots \xrightarrow{\iota_s \otimes 1} \wedge^2 V^\vee \otimes F \xrightarrow{\iota_s \otimes 1} V^\vee \otimes F \xrightarrow{\iota_s \otimes 1} F \to 0. \tag{1.2}$$

Denote by $\mathbb{H}^\bullet(M; V, F)$ the hypercohomology associated to the above complex. We will, somewhat abusively, write $\iota_s \otimes 1$ as ι_s. Because (1.2) is exact outside the compact set Z, the cohomology $\mathbb{H}^\bullet(M; V, F)$ is finite dimensional over $\mathbb{C}$. The study of this type cohomology origins to the mathematical interpretation of the Landau-Ginzburg models, which had been widely studied in the following papers [1–3, 5, 7–9].

Let $\psi \in \Gamma(M, \det V \otimes \det \Omega_M)$ be a holomorphic section, where Ω_M is the holomorphic cotangent bundle of M. There is a canonical pairing

$$(-,-)_\psi : \mathbb{H}^\bullet(M; V, F) \times \mathbb{H}^\bullet(M; V, F^\vee) \to \mathbb{C},$$

see (3.24). Then we have the following duality theorem.

Theorem 1.1. *Let V, F be holomorphic bundles over the complex manifold M with $\operatorname{rank} V = \dim M = n$, and s be a holomorphic section of V with compact zero* *loci* $Z = s^{-1}(0)$. *Assume that* $\psi \in \Gamma(M, \det V \otimes \det \Omega_M)$ *is nowhere vanishing. Then the above pairing* $(-,-)_\psi$ *is non-degenerate. Thus for* $-n \leq k \leq n$,

$$\mathbb{H}^k(M;V,F) \cong \mathbb{H}^{-k}(M;V,F^\vee)^\vee.$$

This is a generalization of the non-degenerate theorem of [5, Theorem A] and [10, Theorem 1.2].

Let $V = T_M$ be the holomorphic tangent bundle of the compact complex manifold M, and F be a holomorphic bundle over M. Let s be the zero section of T_M and $\psi = c \in \Gamma(M, \mathcal{O}_M) \cong \Gamma(M, \det T_M \otimes \det \Omega_M)$ be a nonzero constant, then we recover the classical Serre duality theorem.

Corollary 1.2. *Let* F *be a holomorphic bundle over a compact complex manifold* M*, then*

$$\mathrm{H}^{p,q}(M,F) \cong \mathrm{H}^{n-p,n-q}(M,F^\vee)^\vee.$$

Let $V = \Omega_M$ be the holomorphic cotangent bundle of a small open ball $M = \{z \in \mathbb{C}^n \| z| < \epsilon\}$, and $F = \mathcal{O}_M$. Let $s = df$ be a holomorphic section of V, where f is a holomorphic function on M. Let $\psi = dz_1 \wedge \cdots \wedge dz_n \otimes e_1 \wedge \cdots \wedge e_n$, where $\{z_i\}$ is the coordinate of $\mathbb{C}^n$ and $\{e_i\}$ is the holomorphic frame of Ω_M. Assume that $s = df = f_1 e_1 + \cdots + f_n e_n$ and $Z = s^{-1}(0) = 0$, then using Theorem 1.1 and the proof of [10, Theorem 1.3] we have

$$\mathbb{H}^0(M;V,F) \cong \Gamma(M,\mathcal{O}_M)/(f_1,\cdots,f_n); \qquad \mathbb{H}^k(M;V,F) = 0, \quad k \neq 0.$$

For $g, h \in \Gamma(M, \mathcal{O}_M)$, let $\psi' = gh\psi$. By (3.31), we have

$$(g,h)_\psi \quad = \quad (-1)^{\lfloor \frac{n+3}{2} \rfloor + \frac{n(n+1)}{2}} (-2\pi i)^n \operatorname{Res} \frac{\psi'}{s},$$

where $\operatorname{Res} \frac{\psi'}{s}$ is the virtual residue associated to ψ' and s, the symbol $\lfloor \frac{n+3}{2} \rfloor$ is the greatest integer less than or equal to $\frac{n+3}{2}$. The virtual residue, which had been constructed by Chang and the author in [3], coincides with the Grothendieck residue up to a sign. Therefore $\operatorname{Res} \frac{\psi'}{s}$ equals to the Grothendieck residue $\operatorname{res}_s(g,h) = \int_{|f_i|=\epsilon_i} \frac{ghdz_1 \wedge \cdots \wedge dz_n}{f_1 \cdots f_n}$ up to a sign, see formula (4.6). Thus we recover the following local duality theorem, see [6, Page 659].

Corollary 1.3. *Let* $V = \Omega_M$ *be the holomorphic cotangent bundle of a small open ball* $M = \{z \in \mathbb{C}^n \| z| < \epsilon\}$*, and* $F = \mathcal{O}_M$*. Let* $s = df$ *be a holomorphic section of* V*, where* f *is a holomorphic function on* M*. Let* $\psi = dz_1 \wedge \cdots \wedge dz_n \otimes e_1 \wedge \cdots \wedge e_n$*, where* $\{z_i\}$ *is the coordinate of* $\mathbb{C}^n$ *and* $\{e_i\}$ *is the holomorphic frame of* Ω_M*. Assume that* $s = df = f_1 e_1 + \cdots + f_n e_n$ *and* $Z = s^{-1}(0) = 0$*. Then*

$$\operatorname{res}_s : \Gamma(M,\mathcal{O}_M)/(f_1,\cdots,f_n) \times \Gamma(M,\mathcal{O}_M)/(f_1,\cdots,f_n) \to \mathbb{C}$$

is non-degenerate.

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2. Cohomology with compact support

In this section, we study different types of hypercohomology associate to the exact sequence (1.2). It is similar to Section 2 in [10]. As before let V, F be holomorphic bundles over a complex manifold M with $\operatorname{rank} V = \dim M = n$, and s is a holomorphic section of V with compact zero loci $Z = s^{-1}(0)$. Let $V^\vee$ be the dual bundle of V.

Let $\mathcal{A}^{i,j}(\wedge^l V^\vee \otimes F)$ be the sheaf of smooth (i,j) forms on M with value in $\wedge^l V^\vee \otimes F$. Let $\Omega^{(i,j)}(\wedge^l V^\vee \otimes F) := \Gamma(M, \mathcal{A}^{i,j}(\wedge^l V^\vee \otimes F))$ and assign its element α to have degree $\sharp\alpha = i + j - l$. Let

$$\Omega_c^{(i,j)}(\wedge^l V^\vee \otimes F) := \{\alpha | \alpha \in \Gamma(M, \mathcal{A}^{i,j}(\wedge^l V^\vee \otimes F)) \quad \text{with compact support}\}.$$

Then

$$\mathbf{B} := \oplus_{i,j,l} \Omega^{(i,j)}(\wedge^l V^\vee \otimes F)$$

is a graded commutative algebra with the (wedge) product uniquely extending wedge products in $\Omega^\bullet, \wedge^\bullet V^\vee$ and mutual tensor products. Denote

$$\mathcal{C}_M^k = \underset{i-j=k}{\oplus} C_M^{i,j} \subset \mathbf{B} \qquad \text{with} \quad C_M^{i,j} := \Omega^{(0,i)}(\wedge^j V^\vee \otimes F) = \Gamma(M, \mathcal{A}^{0,i}(\wedge^j V^\vee \otimes F)),$$

and

$$\mathcal{C}_{c,M}^k = \underset{i-j=k}{\oplus} C_{c,M}^{i,j} \qquad \text{with} \quad C_{c,M}^{i,j} := \{\alpha \in C_M^{i,j} |\ \alpha \text{ has compact support}\}.$$

Let $\mathcal{C}_M := \oplus_k \mathcal{C}_M^k$ and $\mathcal{C}_{c,M} := \oplus_k \mathcal{C}_{c,M}^k$. For $\alpha \in \mathcal{C}_M$ we denote $\alpha_{i,j}$ to be its component in $C_M^{i,j}$. Clearly, $\mathcal{C}_M$ is a bi-graded $C^\infty(M)$-module. Under the operations

$$\overline{\partial} : C_M^{i,j} \longrightarrow C_M^{i+1,j} \quad \text{and} \quad \iota_s : C_M^{i,j} \longrightarrow C_M^{i,j-1}$$

the space $C_M^{\bullet,\bullet}$ becomes a double complex and $C_{c,M}^{\bullet,\bullet}$ is a subcomplex. We shall study the cohomology of $\mathcal{C}_M^\bullet$ and $\mathcal{C}_{c,M}^\bullet$ with respect to the following coboundary operator

$$\overline{\partial}_s := \overline{\partial} + \iota_s.$$

One checks $\overline{\partial}_s^2 = 0$ using Leibniz rule of $\overline{\partial}$ and $\overline{\partial} s = 0$. Denote by

$$\mathbb{H}^k(M; V, F) = \mathrm{H}^k(\mathcal{C}_M^\bullet),$$

and

$$\mathbb{H}_c^k(M; V, F) = \mathrm{H}^k(\mathcal{C}_{c,M}^\bullet).$$

Fix a Hermitian metric h_V on V. For a nonzero s on $U := M \setminus Z$, we can form the following smooth section

$$\bar{s} := \frac{(*, s)_{h_V}}{(s,s)_{h_V}} \in \Gamma(U, \mathcal{A}^{0,0}(V^\vee \otimes F)).$$

It associates a map

$$\bar{s}\wedge : \Gamma(U, \mathcal{A}^{0,i}(\wedge^j V^\vee \otimes F)) \to \Gamma(U, \mathcal{A}^{0,i}(\wedge^{j+1} V^\vee \otimes F)).$$

To distinguish it in later calculation, we denote $\mathcal{T}_s := \bar{s}\wedge : C_U^{\bullet,\bullet} \longrightarrow C_U^{\bullet,\bullet+1}$, where $C_U^{\bullet,\bullet} := \Gamma(U, \mathcal{A}^{0,\bullet}(\wedge^\bullet V^\vee \otimes F))$.

The injection $j : U \to M$ induces the restriction $j^* : C_M^{\bullet,\bullet} \to C_U^{\bullet,\bullet}$. Let ρ be a smooth cut-off function on M such that $\rho|_{U_1} \equiv 1$ and $\rho|_{M \setminus U_2} \equiv 0$ for some relatively compact open neighborhoods $U_1 \subset \overline{U}_1 \subset U_2$ of Z in M.

We define the degree of an operator to be its change on the total degree of elements in $\mathcal{C}_M(\mathcal{C}_U)$. Then $\overline{\partial}$ and $\mathcal{T}_s$ are of degree 1 and -1 respectively, and

$[\overline{\partial}, \mathcal{T}_s] = \overline{\partial}\mathcal{T}_s + \mathcal{T}_s\overline{\partial}$ is of degree 0. Consider two operators introduced in [3, (3.1), (3.2)] or [12, page 11]

$$T_\rho : \mathcal{C}_M \to \mathcal{C}_{c,M} \qquad T_\rho(\alpha) := \rho\alpha + (\overline{\partial}\rho)\mathcal{T}_s \frac{1}{1+[\overline{\partial}, \mathcal{T}_s]}(j^*\alpha) \tag{2.1}$$

and

$$R_\rho : \mathcal{C}_M \to \mathcal{C}_M \qquad R_\rho(\alpha) := (1-\rho)\mathcal{T}_s \frac{1}{1+[\overline{\partial}, \mathcal{T}_s]}(j^*\alpha). \tag{2.2}$$

Here as an operator

$$\frac{1}{1+[\overline{\partial}, \mathcal{T}_s]} := \sum_{k=0}^{\infty}(-1)^k[\overline{\partial}, \mathcal{T}_s]^k$$

is well-defined since $[\overline{\partial}, \mathcal{T}_s]^k(\alpha) = 0$ whenever $k > n$. Clearly T_ρ is of degree zero and R_ρ is of degree by -1. Also $R_\rho(\mathcal{C}_{c,M}) \subset \mathcal{C}_{c,M}$ by definition.

Lemma 2.1. $[\overline{\partial}_s, R_\rho] = 1 - T_\rho$ *as operators on* $\mathcal{C}_M$.

Proof. It is direct to check that [1]

$$[\iota_s, \mathcal{T}_s] = 1 \quad \text{on } \mathcal{C}_U. \tag{2.3}$$

Moreover,

$$[P, [\overline{\partial}, \mathcal{T}_s]] = 0$$

for P being $\iota_s, \overline{\partial}$ or $\mathcal{T}_s$. Therefore, we have

$$\begin{aligned}
[\overline{\partial}_s, R_\rho] &= [\overline{\partial}_s, 1-\rho]\mathcal{T}_s \frac{1}{1+[\overline{\partial}, \mathcal{T}_s]} j^* + (1-\rho)[\overline{\partial}_s, \mathcal{T}_s]\frac{1}{1+[\overline{\partial}, \mathcal{T}_s]} j^* \\
&= -(\overline{\partial}\rho)\mathcal{T}_s \frac{1}{1+[\overline{\partial}, \mathcal{T}_s]} j^* + (1-\rho)j^* \\
&= -(\overline{\partial}\rho)\mathcal{T}_s \frac{1}{1+[\overline{\partial}, \mathcal{T}_s]} j^* + (1-\rho) = 1 - T_\rho.
\end{aligned}$$

□

Proposition 2.2. *The embedding* $(\mathcal{C}_{c,M}, \overline{\partial}_s) \to (\mathcal{C}_M, \overline{\partial}_s)$ *is a quasi-isomorphism. Thus for* $-n \le k \le n$,

$$\mathbb{H}^k_c(M; V, F) \cong \mathbb{H}^k(M; V, F).$$

Proof. By Lemma 2.1 $\mathrm{H}^*(\mathcal{C}_M/\mathcal{C}_{c,M}, \overline{\partial}_s) \equiv 0$, and thus the proposition follows. □

[1] As a notation convention, we always denote $[,]$ for the graded commutator, that is for operators A, B of degree $|A|$ and $|B|$, the bracket is given by

$$[A, B] = AB - (-1)^{|A||B|}BA.$$

3. Non-dengeneracy

First we recall the definition of operators on wedge products of vector bundles over M, see [3, Appendix]. Let $\Omega^{(i,j)}(\wedge^k V\otimes\wedge^l V^\vee):=\Gamma(M,\mathcal{A}^{i,j}(\wedge^k V\otimes\wedge^l V^\vee))$ be the smooth differential forms valued in $\wedge^k V\otimes\wedge^l V^\vee$. Let

$$\mathfrak{B}:=\oplus_{i,j,k,l}\Omega^{(i,j)}(\wedge^k V\otimes\wedge^l V^\vee)$$

be a graded commutative algebra extending the wedge products of $\Omega^\bullet,\wedge^\bullet V$ and $\wedge^\bullet V^\vee$. The degree of $\alpha\in\Omega^{(i,j)}(\wedge^k V\otimes\wedge^l V^\vee)$ is $\sharp\alpha:=i+j+k-l$. We briefly denote $A^0(\wedge^k V\otimes\wedge^l V^\vee)=\Omega^{(0,0)}(\wedge^k V\otimes\wedge^l V^\vee)$.

Set $\kappa:\mathfrak{B}\to\Omega^\bullet$ which sends $\omega(e\otimes e')$(for $\omega\in\Omega^{(i,j)},e\in\wedge^k V,e'\in\wedge^l V^\vee$) to $\omega\langle e,e'\rangle$, where $\langle\cdot,\cdot\rangle$ is the dual pairing between $\wedge^k V$ and $\wedge^k V^\vee$, and $\langle e,e'\rangle=0$ when $k\neq l$. We further extend the pairing by setting $\langle\alpha,\beta\rangle:=\kappa(\alpha\beta)$ for $\alpha,\beta\in\mathfrak{B}$. It is direct to verify

$$\overline{\partial}\langle\alpha,\beta\rangle=\langle\overline{\partial}\alpha,\beta\rangle+(-1)^{\sharp\alpha}\langle\alpha,\overline{\partial}\beta\rangle. \tag{3.1}$$

We now define different types of contraction maps. Given $u\in\Omega^{(i,j)}(\wedge^k V)$ and $k\geq l$, we define

$$u\lrcorner:\Omega^{(p,q)}(\wedge^l V^\vee)\longrightarrow\Omega^{(p+i,q+j)}(\wedge^{k-l}V) \tag{3.2}$$

where for $\theta\in\Omega^{(p,q)}(\wedge^l V^\vee)$, the $u\lrcorner\theta$ is determined by

$$\langle u\lrcorner\theta,\nu^*\rangle=(-1)^{(i+j)l+(p+q)\sharp u+\frac{l(l-1)}{2}}\langle u,\theta\wedge\nu^*\rangle,\qquad\forall\nu^*\in A^0(\wedge^{k-l}V^\vee).$$

Given $\alpha\in A^0(V)$, we define

$$\iota_\alpha:\Omega^{(i,j)}(\wedge^k V^\vee)\longrightarrow\Omega^{(i,j)}(\wedge^{k-1}V^\vee) \tag{3.3}$$

where for $w\in\Omega^{(i,j)}(\wedge^k V^\vee)$, the $\iota_\alpha(w)$ is determined by

$$\langle\nu,\iota_\alpha(w)\rangle=\langle\alpha\wedge\nu,w\rangle,\qquad\forall\nu\in A^0(\wedge^{k-1}V). \tag{3.4}$$

For above α, θ and w one has $\iota_\alpha(w\wedge\theta)=\iota_\alpha(w)\wedge\theta+(-1)^{\sharp w}w\wedge\iota_\alpha(\theta)$.

Given $\gamma\in A^0(V^\vee)$, we define

$$\iota_\gamma:\Omega^{(i,j)}(\wedge^k V)\longrightarrow\Omega^{(i,j)}(\wedge^{k-1}V) \tag{3.5}$$

where for $\nu\in\Omega^{(i,j)}(\wedge^k V)$, the $\iota_\gamma(\nu)$ is determined by

$$\langle\iota_\gamma(\nu),w\rangle=\langle\nu,\gamma\wedge w\rangle,\qquad\forall w\in A^0(\wedge^{k-1}V^\vee). \tag{3.6}$$

We have the following identities.

Lemma 3.1 ([3])**.** *Given $u\in\Omega^{(i,j)}(\wedge^n V)$, and θ, α, γ as above, one has*

$$\alpha\wedge(u\lrcorner\theta)=u\lrcorner(\iota_\alpha(\theta)),\qquad\iota_\gamma(u\lrcorner\theta)=u\lrcorner(\gamma\wedge\theta).$$

Lemma 3.2 ([3])**.** *For $u\in\Omega^{(i,j)}(\wedge^k V)$, $\theta\in\Omega^{(p,q)}(\wedge^l V^\vee)$, $k\geq l$ and a smooth form $\alpha\in\Omega^{(a,b)}(M)$, we have*

$$\alpha\wedge(u\lrcorner\theta)=u\lrcorner(\alpha\theta)\quad\text{and}\quad\overline{\partial}(u\lrcorner\theta)=(-1)^{\sharp\theta}(\overline{\partial}u)\lrcorner\theta+u\lrcorner(\overline{\partial}\theta).$$

Denote by $\mathscr{D}^{p,q}(\wedge^l V\otimes F^\vee)$ the space of $\wedge^l V\otimes F^\vee$-valued (p,q)-current, which is the dual of the space $\Omega_c^{n-p,n-q}(\wedge^l V^\vee\otimes F)$. There is a naturally pairing

$$(-,-)_N:\mathscr{D}^{n,n-i}(\wedge^j V\otimes F^\vee)\times\Omega_c^{0,i}(\wedge^j V^\vee\otimes F)\to\mathbb{C}, \tag{3.7}$$

where

$$(\alpha,\beta)_N:=\int_M\langle\alpha,\beta\rangle.$$

Denote

$$\mathcal{D}_M^k = \underset{i+j-n=k}{\oplus} D_M^{i,j} \qquad \text{with} \quad D_M^{i,j} := \mathscr{D}^{(n,i)}(\wedge^j V \otimes F^\vee).$$

The coboundary map δ_s of $\mathcal{D}_M^\bullet$ is defined as follows

$$\delta_s\alpha = \overline{\partial}\alpha + (-1)^{l+1} s \wedge \alpha, \qquad \text{for} \qquad \alpha \in \mathscr{D}^{p,q}(\wedge^l V \otimes F^\vee). \tag{3.8}$$

By (3.1) and (3.4),

$$(\alpha, \overline{\partial}_s\beta)_N + (-1)^{\sharp\alpha}(\delta_s\alpha, \beta)_N = 0. \tag{3.9}$$

Thus $(\mathcal{D}_M^\bullet, (-1)^{\bullet+1}\delta_s)$ is the dual complex of $(\mathcal{C}_{c,M}^\bullet, \overline{\partial}_s)$. Let

$$\mathcal{H}^k(M; V, F^\vee) := \frac{\operatorname{Ker}(\delta_s : \mathcal{D}_M^k \to \mathcal{D}_M^{k+1})}{\operatorname{Im}(\delta_s : \mathcal{D}_M^{k-1} \to \mathcal{D}_M^k)}. \tag{3.10}$$

Because the complex $(\mathcal{D}_M^\bullet, (-1)^{\bullet+1}\delta_s)$ is quasi-isomorphic to the complex $(\mathcal{D}_M^\bullet, \delta_s)$, the pairing (3.7) induces a pairing on the hypercohomologies which we denote by

$$(-,-)_N : \mathcal{H}^\bullet(M; V, F^\vee) \times \mathbb{H}_c^\bullet(M; V, F) \to \mathbb{C}. \tag{3.11}$$

Theorem 3.3. *Let V, F be holomorphic bundles over the complex manifold M with $\operatorname{rank} V = \dim M = n$ and s be a holomorphic section of V with compact zero loci $Z = s^{-1}(0)$. Then the above pairing (3.11) is non-degenerate. Thus for $-n \le k \le n$*

$$\mathbb{H}_c^k(M; V, F) \cong \mathcal{H}^{-k}(M; V, F^\vee)^\vee.$$

Proof. Because $\mathbb{H}_c^k(M; V, F)$ is finite dimensional and the complex $(\mathcal{D}_M^\bullet, (-1)^{\bullet+1}\delta_s)$ is the dual complex of $(\mathcal{C}_{c,M}^\bullet, \overline{\partial}_s)$, by applying [11, Theorem 1.6] and [11, Corollary 1.7], we obtain the theorem. □

Remark 3.4. *By the Dolbeault-Grothendieck Lemma [4, 3.29] for current, $\mathcal{H}^\bullet(M; V, F^\vee)$ (up to a shift on degree) is the hypercohomology of the following complex,*

$$0 \to \det\Omega_M \otimes F^\vee \xrightarrow{-s} \det\Omega_M \otimes V \otimes F^\vee \xrightarrow{(-1)^2 s\wedge} \cdots \xrightarrow{(-1)^n s\wedge} \det\Omega_M \otimes \wedge^n V \otimes F^\vee \to 0, \tag{3.12}$$

which is quasi-isomorphic to the complex

$$0 \to \det\Omega_M \otimes F^\vee \xrightarrow{s} \det\Omega_M \otimes V \otimes F^\vee \xrightarrow{s\wedge} \cdots \xrightarrow{s\wedge} \det\Omega_M \otimes \wedge^n V \otimes F^\vee \to 0. \tag{3.13}$$

Let $(\overline{\mathcal{D}}_M^\bullet, \widetilde{\delta}_s)$ be the complex with

$$\overline{\mathcal{D}}_M^k = \underset{i-j=k}{\oplus} \overline{D}_M^{i,j} \qquad \text{where} \quad \overline{D}_M^{i,j} := \mathscr{D}^{(0,i)}(\wedge^j V^\vee \otimes F^\vee),$$

and the coboundary map $\widetilde{\delta}_s$ is defined as follows

$$\widetilde{\delta}_s\alpha = \overline{\partial}\alpha + (-1)^{n-l+1}\iota_s\alpha, \qquad \text{for} \qquad \alpha \in \mathscr{D}^{p,q}(\wedge^l V^\vee \otimes F^\vee). \tag{3.14}$$

By the Dolbeault-Grothendieck Lemma [4, 3.29] for current, the complex $(\overline{\mathcal{D}}_M^\bullet, \widetilde{\delta}_s)$ is the Dolbeault resolution of the following complex

$$0 \to \wedge^n V^\vee \otimes F^\vee \xrightarrow{-\iota_s} \cdots \xrightarrow{(-1)^n\iota_s} V^\vee \otimes F^\vee \xrightarrow{(-1)^n\iota_s} F^\vee \to 0. \tag{3.15}$$

Denote by

$$\mathrm{H}^k(\overline{\mathcal{D}}_M^\bullet) := \frac{\operatorname{Ker}(\widetilde{\delta}_s : \overline{\mathcal{D}}_M^k \to \overline{\mathcal{D}}_M^{k+1})}{\operatorname{Im}(\widetilde{\delta}_s : \overline{\mathcal{D}}_M^{k-1} \to \overline{\mathcal{D}}_M^k)} \tag{3.16}$$

the cohomology of the complex $(\overline{\mathcal{D}}^\bullet_M, \widetilde{\delta}_s)$.

Let $\psi \in \Gamma(M, \det V \otimes \det \Omega_M)$ be a holomorphic section which is nowhere vanishing. It induces a bundle isomorphism

$$\psi \lrcorner : \wedge^l V^\vee \otimes F^\vee \to \det \Omega_M \otimes \wedge^{n-l} V \otimes F^\vee.$$

Thus

$$\mathscr{D}^{0,q}(\wedge^l V^\vee \otimes F^\vee) \cong \mathscr{D}^{n,q}(\wedge^{n-l} V \otimes F^\vee). \tag{3.17}$$

By Lemma 3.1, Lemma 3.2 and (3.8), the map

$$\psi \lrcorner : \overline{\mathcal{D}}^\bullet_M \to \mathcal{D}^\bullet_M$$

is a complex isomorphism. Therefore for $-n \leq k \leq n$

$$\mathrm{H}^k(\overline{\mathcal{D}}^\bullet_M) \cong \mathcal{H}^k(M; V, F^\vee). \tag{3.18}$$

The complex (3.15) is quasi-isomorphic to

$$0 \to \wedge^n V^\vee \otimes F^\vee \xrightarrow{\iota_s} \cdots \xrightarrow{\iota_s} V^\vee \otimes F^\vee \xrightarrow{\iota_s} F^\vee \to 0, \tag{3.19}$$

and we denote this quasi-isomorphism by Φ. It induces the following hypercohomology isomorphism

$$H^\bullet(\Phi) : \mathbb{H}^\bullet(M; V, F^\vee) \cong \mathrm{H}^\bullet(\overline{\mathcal{D}}^\bullet_M). \tag{3.20}$$

Thus we have

$$\mathbb{H}^\bullet(M; V, F^\vee) \cong \mathcal{H}^\bullet(M; V, F^\vee). \tag{3.21}$$

Combining (3.11) and (3.21), there is a pairing

$$(-,-)'_\psi : \mathbb{H}^\bullet(M; V, F^\vee) \times \mathbb{H}^\bullet_c(M; V, F) \to \mathbb{C}, \tag{3.22}$$

where $(\alpha, \beta)'_\psi := (\psi \lrcorner (H^\bullet(\Phi)(\alpha)), \beta)_N$.

From (3.21) and Theorem 3.3, we obtain the following statement.

Corollary 3.5. *Let V, F be holomorphic bundles over the complex manifold M with $\operatorname{rank} V = \dim M = n$ and s be a holomorphic section of V with compact zero loci $Z = s^{-1}(0)$. Assume that $\psi \in \Gamma(M, \det V \otimes \det \Omega_M)$ is a holomorphic section which is nowhere vanishing. Then the pairing $(-,-)'_\psi$ is non-degenerate. Thus for $-n \leq k \leq n$*

$$\mathbb{H}^k_c(M; V, F) \cong \mathbb{H}^{-k}(M; V, F^\vee)^\vee. \tag{3.23}$$

We define the following pairing

$$(-,-)_\psi : \mathbb{H}^\bullet(M; V, F^\vee) \times \mathbb{H}^\bullet(M; V, F) \to \mathbb{C}, \tag{3.24}$$

such that $(\alpha, \beta)_\psi := ([\alpha], [\beta])'_\psi$, where $[\alpha]$ and $[\beta]$ are the image of α, β under the isomorphic $\mathbb{H}^\bullet(M; V, F) \cong \mathbb{H}^\bullet_c(M; V, F)$ and $\mathbb{H}^\bullet(M; V, F^\vee) \cong \mathbb{H}^\bullet_c(M; V, F^\vee)$. The pairing (3.24) is well defined because it does not dependent on the compact support representation. Therefore

Corollary 3.6. *Under the conditions as in Corollary 3.5, the pairing $(-,-)_\psi$ is non-degenerate. Thus for $-n \leq k \leq n$*

$$\mathbb{H}^k(M; V, F) \cong \mathbb{H}^{-k}(M; V, F^\vee)^\vee. \tag{3.25}$$

Let $V=\Omega_M$ be the holomorphic cotangent bundle of a small open ball $M=\{z\in\mathbb{C}^n||z|<\epsilon\}$, and $F=\mathcal{O}_M$. Let $s=df$ be a holomorphic section of V, where f is a holomorphic function on M. Let $\psi=dz_1\wedge\cdots\wedge dz_n\otimes e_1\wedge\cdots\wedge e_n$, where $\{z_i\}$ is the coordinate of $\mathbb{C}^n$ and $\{e_i\}$ is the holomorphic frame of Ω_M. Assume that $s=df=f_1e_1+\cdots+f_ne_n$ and $Z=s^{-1}(0)=0$, then using Corollary 3.6 and the proof of [10, Theorem 1.3]

$$\mathbb{H}^0(M;V,F)\cong\Gamma(M,\mathcal{O}_M)/(f_1,\cdots,f_n);\qquad \mathbb{H}^k(M;V,F)=0,\quad k\neq 0.$$

On the other hand s induced the following complex

$$0\to\det\Omega_M\xrightarrow{-s}\det\Omega_M\otimes V\xrightarrow{(-1)^2 s\wedge}\cdots\xrightarrow{(-1)^n s\wedge}\det\Omega_M\otimes\wedge^n V\to 0. \tag{3.26}$$

As in the definition of (3.10), let $\mathcal{H}^\bullet(M;V,\mathcal{O}_M)$ be the hypercohomology of (3.26), and $\mathcal{H}_c^\bullet(M;V,\mathcal{O}_M)$ be its hypercohomology with compact support. By [3, Proposition 3.2]

$$\mathcal{H}^k(M;V,\mathcal{O}_M)\cong\mathcal{H}_c^k(M;V,\mathcal{O}_M), \tag{3.27}$$

for $-n\leq k\leq n$.

Because $\psi=dz_1\wedge\cdots\wedge dz_n\otimes e_1\wedge\cdots\wedge e_n$ is nowhere vanishing. It induces a bundle isomorphism

$$\psi\lrcorner:\wedge^l V^\vee\to\det\Omega_M\otimes\wedge^{n-l}V.$$

So is the Dolbeault resolution of the two complex (1.1) and (3.26).

For $g,\ h\in\Gamma(M,\mathcal{O}_M)$, let $[g],\ [h]$ be the images of $g,\ h$ under the isomorphism $\mathbb{H}^0(M;V,F)\cong\mathbb{H}_c^0(M;V,F)$. By (3.27), (3.22) and (3.24) the pairing

$$\begin{aligned}(g,h)_\psi &= \int_M\langle(-1)^{\lfloor\frac{n+3}{2}\rfloor}g\psi,[h]\rangle\\ &= (-1)^{\lfloor\frac{n+3}{2}\rfloor}\int_M\langle gh\psi,[1]\rangle\\ &= (-1)^{\lfloor\frac{n+3}{2}\rfloor+\frac{n(n+1)}{2}}\int_M(gh\psi)\lrcorner[1].\end{aligned}$$

Let ρ be a smooth cut-off function on M such that $\rho|_{U_1}\equiv 1$ and $\rho|_{M\setminus U_2}\equiv 0$ for some relatively compact open neighborhoods $U_1\subset\overline{U}_1\subset U_2$ of 0 in M. By Lemma 2.1, we have

$$[\overline{\partial}_s,R_\rho]=1-T_\rho.$$

Therefore

$$\int_M(gh\psi)\lrcorner[1]=\int_M(gh\psi)\lrcorner[T_\rho 1]. \tag{3.28}$$

For the nonzero s on $U:=M\setminus Z$, the smooth section $\bar{s}:=\frac{(*,s)_{h_V}}{(s,s)_{h_V}}\in\Gamma(U,\mathcal{A}^{0,0}(V^\vee))$ induces a contraction

$$\iota_{\bar{s}}:\Gamma(U,\mathcal{A}^{0,i}(\wedge^j V))\to\Gamma(U,\mathcal{A}^{0,i}(\wedge^{j-1}V)).$$

Denote

$$\widetilde{C}_M^k=\bigoplus_{i+j=k}\widetilde{C}_M^{i,j}\qquad\text{with}\quad \widetilde{C}_M^{i,j}:=\Omega^{(0,i)}(\wedge^j V)=\Gamma(M,\mathcal{A}^{0,i}(\wedge^j V)),$$

and

$$\widetilde{C}_{c,M}^k=\bigoplus_{i+j=k}\widetilde{C}_{c,M}^{i,j}\qquad\text{with}\quad \widetilde{C}_{c,M}^{i,j}:=\{\alpha\in\widetilde{C}_M^{i,j}|\ \alpha\text{ has compact support}\}.$$

Let $\widetilde{\mathcal{C}}_M := \oplus_k \widetilde{\mathcal{C}}_M^k$ and $\widetilde{\mathcal{C}}_{c,M} := \oplus_k \widetilde{\mathcal{C}}_{c,M}^k$. Let $j : U \to M$ be the injection. We can form the following operator by using the contraction $\iota_{\bar{s}}$

$$\widetilde{T}_\rho : \widetilde{\mathcal{C}}_M \to \widetilde{\mathcal{C}}_{c,M} \qquad \widetilde{T}_\rho(\alpha) := \rho\alpha + (\overline{\partial}\rho)\iota_{\bar{s}} \frac{1}{1+[\overline{\partial}, \iota_{\bar{s}}]}(j^*\alpha). \tag{3.29}$$

Because ψ is holomorphic, by Lemma 3.1

$$\int_M (gh\psi) \lrcorner [T_\rho 1] = \int_M \widetilde{T}_\rho(gh\psi). \tag{3.30}$$

Denote $\psi' = gh\psi$, and applying [3, Proposition 3.3] to ψ' and $s = df$, we have

$$\begin{aligned} (g,h)_\psi &= (-1)^{\lfloor \frac{n+3}{2} \rfloor + \frac{n(n+1)}{2}} \int_M (gh\psi) \lrcorner [T_\rho 1] \\ &= (-1)^{\lfloor \frac{n+3}{2} \rfloor + \frac{n(n+1)}{2}} \int_M \widetilde{T}_\rho(gh\psi) \\ &= (-1)^{\lfloor \frac{n+3}{2} \rfloor + \frac{n(n+1)}{2}} (-2\pi i)^n \operatorname{Res} \frac{\psi'}{s}, \end{aligned} \tag{3.31}$$

where $\operatorname{Res} \frac{\psi'}{s}$ is the virtual residue associated to ψ' and s, it coincides with the Grothendieck residue $\operatorname{res}_s(g,h) = \int_{|f_i|=\epsilon_i} \frac{gh dz_1 \wedge \cdots \wedge dz_n}{f_1 \cdots f_n}$ up to a sign, see formula (4.6). Thus we recover the local duality theorem, see [6, Page 659].

Corollary 3.7. *Let $V = \Omega_M$ be the holomorphic cotangent bundle of a small open ball $M = \{z \in \mathbb{C}^n \mid |z| < \epsilon\}$, and $F = \mathcal{O}_M$. Let $s = df$ be a holomorphic section of V, where f is a holomorphic function on M. Let $\psi = dz_1 \wedge \cdots \wedge dz_n \otimes e_1 \wedge \cdots \wedge e_n$, where $\{z_i\}$ is the coordinate of $\mathbb{C}^n$ and $\{e_i\}$ is the holomorphic frame of Ω_M. Assume that $s = df = f_1 e_1 + \cdots + f_n e_n$ and $Z = s^{-1}(0) = 0$. Then*

$$\operatorname{res}_s : \Gamma(M, \mathcal{O}_M)/(f_1, \cdots, f_n) \times \Gamma(M, \mathcal{O}_M)/(f_1, \cdots, f_n) \to \mathbb{C}$$

is non-degenerate.

4. Appendix

In this appendix we recall the construction of the virtual residue given by Chang and the author in [3], and prove the relation between the virtual residue and the Grothendieck residue when the zero loci is zero-dimensional.

Let V be a holomorphic bundle over a compact complex manifold M with $\operatorname{rank} V = \dim M = n$. Let s be a holomorphic section of V, and $Z = s^{-1}(0)$ be the compact zero loci.

Let $U := M \setminus Z$, and let V_U be the restriction of V over U. Since s is nowhere zero over U, the following Koszul sequence is exact over U

$$0 \longrightarrow K_U \xrightarrow{s} K_U \otimes V_U \xrightarrow{s\wedge} \cdots \xrightarrow{s\wedge} K_U \otimes \wedge^{n-1} V_U \xrightarrow{s\wedge} K_U \otimes \wedge^n V_U \longrightarrow 0.$$

The exact Koszul sequence induces a homomorphism

$$\mathrm{H}^0(U, K_U \otimes \wedge^n V_U) \longrightarrow \mathrm{H}^{n-1}(U, K_U). \tag{4.1}$$

One also has a canonical Dolbeault isomorphism

$$\mathrm{H}^{n-1}(U, K_U) \cong \mathrm{H}^{n,n-1}_{\overline{\partial}}(U). \tag{4.2}$$

Applying (4.1) and (4.2) to the holomorphic section $\psi \in \Gamma(M, K_M \otimes \det V)$, and using that every $(n, n-1)$ form is ∂-closed, one obtains a (unique) De-Rham cohomology class

$$\eta_\psi \in \mathrm{H}^{2n-1}(U, \mathbb{C}). \tag{4.3}$$

Then the virtual residue is defined as

$$\mathrm{Res}_Z \frac{\psi}{s} := \left(\frac{1}{2\pi\sqrt{-1}}\right)^n \int_N \eta_\psi \in \mathbb{C}, \tag{4.4}$$

where N is a real $(2n-1)$-dimensional piecewise smooth compact subset of M that surrounds Z, in the sense that $N = \partial T$ for some compact domain $T \subset M$, which contains Z and is homotopically equivalent to Z.

When $M = \{z \in \mathbb{C}^n \| z| < \epsilon\}$ is a small open ball, and $V = \Omega_M$ with the standard Hermitian metric h_V. Let $F = \mathcal{O}_M$ and $s = df$, where f is a holomorphic function on M. Let $\{z_i\}$ be the coordinate of $\mathbb{C}^n$, and $\{e_i\}$ be the holomorphic frame of Ω_M. Assume that $s = df = f_1 e_1 + \cdots + f_n e_n$ and $Z = s^{-1}(0) = 0$. Let $\bar{s} = \langle s, s\rangle_{h_V}^{-1} \sum_{i=1}^n \bar{f}_i e_i^*$, where e_i^* is the dual basis of $V^\vee$. Then we have the following equalities on U

$$\overline{\partial}\iota_{\bar{s}} = \sum \Big(\frac{\overline{\partial}\bar{f}_i}{\langle s, s\rangle_{h_V}} - \frac{\bar{f}_i \overline{\partial}\langle s, s\rangle_{h_V}}{\langle s, s\rangle_{h_V}^2}\Big)\iota_{e_i^*},$$

and

$$(\langle s, s\rangle_{h_V}^{-1} \sum_{i=1}^n \bar{f}_i \iota_{e_i^*})(\sum \frac{\bar{f}_i \overline{\partial}\langle s, s\rangle_{h_V}}{\langle s, s\rangle_{h_V}^2}\iota_{e_i^*}) = -\frac{\overline{\partial}\langle s, s\rangle_{h_V}}{\langle s, s\rangle_{h_V}}(\langle s, s\rangle_{h_V}^{-1} \sum_{i=1}^n \bar{f}_i \iota_{e_i^*})^2 = 0.$$

Let $g,\ h$ be holomorphic functions on M. Then $\psi = gh dz_1 \wedge \cdots \wedge dz_n \otimes\ e_1 \wedge \cdots \wedge e_n$ is a holomorphic section of $\Gamma(M, K_M \otimes \det V)$. Therefore

$$\begin{aligned}
\eta_\psi &= \langle s, s\rangle_{h_V}^{-1}(\sum \bar{f}_i \iota_{e_i^*})(\overline{\partial}\iota_{\bar{s}})^{n-1}\psi \\
&= \langle s, s\rangle_{h_V}^{-1}(\sum \bar{f}_i \iota_{e_i^*})\Big(\sum \frac{\overline{\partial}\bar{f}_i}{\langle s, s\rangle_{h_V}}\iota_{e_i^*}\Big)^{n-1}\psi \\
&\quad -(n-1)\langle s, s\rangle_{h_V}^{-1}(\sum \bar{f}_i \iota_{e_i^*})\Big(\sum \frac{\overline{\partial}\bar{f}_i}{\langle s, s\rangle_{h_V}}\iota_{e_i^*}\Big)^{n-2}\Big(\sum \frac{\bar{f}_i \overline{\partial}\langle s, s\rangle_{h_V}}{\langle s, s\rangle_{h_V}^2}\Big)\iota_{e_i^*}\psi \\
&= (\langle s, s\rangle_{h_V}^{-1} \sum \bar{f}_i \iota_{e_i^*})\Big(\sum \frac{\overline{\partial}\bar{f}_i}{\langle s, s\rangle_{h_V}}\iota_{e_i^*}\Big)^{n-1}\psi \\
&= (-1)^{\frac{n(n-1)}{2} + \frac{n(n+1)}{2}}(n-1)!gh \sum_{i=1}^n (-1)^{i-1} \\
&\quad \frac{\bar{f}_i}{\langle s, s\rangle_{h_V}^n}\overline{\partial}\bar{f}_1 \wedge \cdots \wedge \widehat{\overline{\partial}\bar{f}_i} \wedge \cdots \wedge \overline{\partial}\bar{f}_n \wedge dz_1 \wedge \cdots \wedge dz_n.
\end{aligned}$$

Let N be a small sphere around 0, the virtual residue

$$\operatorname{Res}_Z \frac{\psi}{s} = \left(\frac{1}{2\pi\sqrt{-1}}\right)^n \int_N \eta_\psi$$
$$= (-1)^{\frac{n(n+1)}{2}+\frac{n(n-1)}{2}}(n-1)!\left(\frac{1}{2\pi\sqrt{-1}}\right)^n \int_N gh\sum_{i=1}^n(-1)^{i-1}$$
$$\frac{\bar{f}_i}{\langle s,s\rangle_{h_V}^n}\overline{\partial}\bar{f}_1\wedge\cdots\wedge\widehat{\overline{\partial}\bar{f}_i}\wedge\cdots\wedge\overline{\partial}\bar{f}_n\wedge dz_1\wedge\cdots\wedge dz_n. \tag{4.5}$$

By Lemma in [6, Page 651] and the definition of $\operatorname{res}_s(g,h)$ in [6, Page 659], we have

$$\operatorname{Res}_Z \frac{\psi}{s} = (-1)^{\frac{n(n+1)}{2}}\operatorname{res}_s(g,h). \tag{4.6}$$

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