

PHASE TRANSITION FOR THE ONCE-EXCITED RANDOM WALK ON GENERAL TREES

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ABSTRACT. The phase transition of M -digging random on a general tree was studied by Collecchio, Huynh and Kious [4]. In this paper, we study particularly the critical M -digging random walk on a superperiodic tree that is proved to be recurrent.

We keep using the techniques introduced by Collecchio, Kious and Sidoravicius [5] with the aim of investigating the phase transition of Once-excited random walk on general trees.

In addition, we prove if \mathcal{T} is a tree whose branching number is larger than 1, any multi-excited random walk on \mathcal{T} moving, after excitation, like a simple random walk is transient.

1. INTRODUCTION

In this paper, we study a particular case of multi-excited random walks on trees, introduced by Volkov [11], called the once-excited random walk.

Let $M \in \mathbb{N}$, $(\lambda_1, \dots, \lambda_M) \in (\mathbb{R}_+)^M$ and $\lambda > 0$. Let \mathcal{T} be an infinite, locally-finite, tree rooted at ϱ . The $(\lambda_1, \dots, \lambda_M, \lambda)$ -ERW on \mathcal{T} , is a nearest-neighbor random walk (X_n) started at ϱ such that if X_n is on a site for the i -th time for $i \leq M$, then the walker takes a random step of a biased random walk with bias λ_i (i.e. it jumps on its parent with probability proportional to 1, or jumps on a particular offspring of ν with probability proportional to λ_i); and if $i > M$, then X_n takes a random step of a biased random walk with bias λ . In the case $M = 1$, it is called the once-excited random walk with parameters (λ_1, λ) . We write (λ_1, λ) -OERW for (λ_1, λ) -ERW. The definition of the model and the vocabulary will be made clear in Section 2.3.

Unlike the case of once-reinforced random walk in [5] or digging-random walk in [4], the phase transition of OERW does not depend only on the branching-ruin number and the branching number of tree (see Section 4 for more details). In the case \mathcal{T} is a *spherically symmetric* tree, we give a sharp phase transition recurrence/transience in terms of their *branching number* and *branching-ruin number* and others.

In the following, we denote $br(\mathcal{T})$ the branching number of a tree \mathcal{T} and $br_r(\mathcal{T})$ the branching-ruin number of a tree \mathcal{T} , see (2.1) and (2.2) for their definitions. Let us simply emphasize that, for any tree \mathcal{T} , its branching number is at least one, i.e. $br(\mathcal{T}) \geq 1$, whereas the branching-ruin number is nonnegative, i.e. $br_r(\mathcal{T}) \geq 0$.

A tree \mathcal{T} is said to be spherically symmetric if for every vertex ν , $\deg \nu$ depends only on $|\nu|$, where $|\nu|$ denote its distance from the root and $\deg \nu$ is its number of neighbors. Let \mathcal{T} be a spherically symmetric tree. For any $n \geq 0$, let x_n be the number of children of a vertex at level n . For any $\lambda_1 \geq 0$

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and $\lambda > 0$, we define the following quantities:

$$(1.1) \quad \alpha(\mathcal{T}, \lambda_1, \lambda) = \liminf_{n \rightarrow \infty} \left(\prod_{i=1}^n \frac{\lambda^2 + (x_i - 1)\lambda_1\lambda + \lambda_1}{1 + x_i\lambda_1} \right)^{1/n}.$$

$$(1.2) \quad \beta(\mathcal{T}, \lambda_1, \lambda) = \limsup_{n \rightarrow \infty} \left(\prod_{i=1}^n \frac{\lambda^2 + (x_i - 1)\lambda_1\lambda + \lambda_1}{1 + x_i\lambda_1} \right)^{1/n}.$$

$$(1.3) \quad \gamma(\mathcal{T}, \lambda_1) = \liminf_{n \rightarrow \infty} \frac{-\sum_{i=1}^n \ln \left[1 - \frac{(x_i - 1)\lambda_1 + 2}{(1 + x_i\lambda_1)^i} \right]}{\ln n}.$$

$$(1.4) \quad \eta(\mathcal{T}, \lambda_1) = \limsup_{n \rightarrow \infty} \frac{-\sum_{i=1}^n \ln \left[1 - \frac{(x_i - 1)\lambda_1 + 2}{(1 + x_i\lambda_1)^i} \right]}{\ln n}.$$

Theorem 1. *Let \mathcal{T} be a spherically symmetric tree, and let $\lambda_1 \geq 0$, $\lambda > 0$. Denote \mathbf{X} the (λ_1, λ) -OERW on \mathcal{T} . Assume that there exists a constant $M > 0$ such that $\sup_{\nu \in V} \deg \nu \leq M$, then we have*

- (1) *in the case $\lambda = 1$, if $\eta(\mathcal{T}, \lambda_1) < br_r(\mathcal{T})$ then \mathbf{X} is transient and if $\gamma(\mathcal{T}, \lambda_1) > br_r(\mathcal{T})$ then \mathbf{X} is recurrent;*
- (2) *assume that $\lambda_1 \geq 0$, $\lambda \neq 1$ and $br(\mathcal{T}) > 1$, if $\beta(\mathcal{T}, \lambda_1, \lambda) < \frac{1}{br(\mathcal{T})}$ then \mathbf{X} is recurrent and if $\alpha(\mathcal{T}, \lambda_1, \lambda) > \frac{1}{br(\mathcal{T})}$ then \mathbf{X} is transient.*

Note that, for a b -ary tree, we have $br(\mathcal{T}) = b$ and

$$(1.5) \quad \alpha(\mathcal{T}, \lambda_1, \lambda) = \beta(\mathcal{T}, \lambda_1, \lambda) = \frac{\lambda^2 + (b - 1)\lambda\lambda_1 + \lambda_1}{1 + b\lambda_1}$$

and our result therefore agrees with Corollary 1.6 of [1]. In [1], the authors prove that the walk is recurrent at criticality on regular trees, but this is not expected to be true on any tree). For instance, if $\lambda_1 = \lambda$, the (λ, λ) -OERW \mathbf{X} is the biased random walk with parameter λ . Therefore \mathbf{X} may be recurrent or transient at criticality (see [2], proposition 22).

Volkov [11] conjectured that, any cookie random walk which moves, after excitation, like a simple random walk (i.e. $\lambda = 1$) is transient on any tree containing the binary tree. This conjecture was proved by Basdevant and Singh [1]. Here, we extend this conjecture to any tree \mathcal{T} whose branching number is larger than 1:

Theorem 2. *Let $(\lambda_1, \dots, \lambda_M) \in (\mathbb{R}_+)^M$ and consider $(\lambda_1, \dots, \lambda_M, 1)$ -ERW \mathbf{X} on an infinite, locally finite, rooted tree \mathcal{T} . If $br(\mathcal{T}) > 1$, then \mathbf{X} is transient.*

The techniques used our paper rely on the strategy adopted in [5] or [4]. In particular, for the proof of transience, we here too view the set of edges crossed by \mathbf{X} before returning to ϱ as the cluster of the root in a particular correlated percolation.

There are two key ingredients that allow us to use the rest of the strategy from [5]. First, we need to define *extensions* of \mathbf{X} , which are a family of coupled continuous-time versions of \mathbf{X} defined on subtrees of \mathcal{T} . As in [5], we do this through Rubin's construction in Section 7. But we will see in Section 7, this

construction is actually very different to a once-reinforced random walk in [5] or M -digging random walk in [4].

Second, we need to prove that the correlated percolation mentioned above is in fact a *quasi-independent* percolation, see Lemma 17. From there, the problem boils down to proving that a certain quasi-independent percolation is supercritical.

We refer to Theorem 4 for the more general result on a general tree.

2. THE MODEL

First, we review some basic definitions of graph theory and then we define the model of multi-excited random walk on trees which was introduced by Volkov[11] and then made general by Basdevant and Singh[1].

2.1. Notation. Let $\mathcal{T} = (V, E)$ be an infinite, locally finite, rooted tree with the root ϱ .

Given two vertices ν, μ of \mathcal{T} , we say that ν and μ are *neighbors*, denoted $\nu \sim \mu$, if $\{\nu, \mu\}$ is an edge of \mathcal{T} .

Let $\nu, \mu \in V \setminus \{\varrho\}$, the *distance* between ν and μ , denoted by $d(\nu, \mu)$, is the minimum number of edges of the unique self-avoiding paths joining x and y . The distance between ν and ϱ is called *height* of ν , denoted by $|\nu|$. The *parent* of ν is the vertex ν^{-1} such that $\nu^{-1} \sim \nu$ and $|\nu^{-1}| = |\nu| - 1$. We also call ν is a *child* of ν^{-1} .

For any $\nu \in V$, denote by $\partial(\nu)$ the number of children of ν and $\{\nu_1, \dots, \nu_{\partial\nu}\}$ is the set of children of ν . We define an order on \mathcal{T} by the following way. For all ν and μ , we say that $\nu \leq \mu$ if the unique self-avoiding path joining ϱ and μ contains ν , and we say that $\nu < \mu$ if moreover $\nu \neq \mu$.

Denote by \mathcal{T}_n the set of vertices of \mathcal{T} at height n . For any $\nu \in \mathcal{T}$, denote by \mathcal{T}^ν the biggest sub-tree of \mathcal{T} rooted at ν , i.e. $\mathcal{T}^u = \mathcal{T}[V^u]$, where

$$V^u := \{v \in V(\mathcal{T}) : u \leq v\}.$$

For any edge e of \mathcal{T} , denote by e^+ and e^- its endpoints with $|e^+| = |e^-| + 1$, and we define the *height* of e as $|e| = |e^+|$.

For two edges e and g of \mathcal{T} , we write $g \leq e$ if $g^+ \leq e^+$ and $g < e$ if moreover $g^+ \neq e^+$. For two vertices ν and μ of \mathcal{T} such that $\nu < \mu$, we denote by $[\nu, \mu]$ the unique self-avoiding path joining ν to μ . For two neighboring vertices ν and μ , we use the slight abuse of notation $[\nu, \mu]$ to denote the edge with endpoints ν and μ (note that we allow $\mu < \nu$).

For two edges e_1 and e_2 of E , denote by $e_1 \wedge e_2$ the vertex with maximal distance from the root such that $e_1 \wedge e_2 \leq e_1^+$ and $e_1 \wedge e_2 \leq e_2^+$.

Finally, we define a particular class of trees, which is called *superperiodic tree*. Let $\mathcal{T}_1 = (V_1, E_1)$ and $\mathcal{T}_2 = (V_2, E_2)$ be two trees. A *morphism* of \mathcal{T}_1 to \mathcal{T}_2 is a map $f : \mathcal{T}_1 \rightarrow \mathcal{T}_2$ such that whenever ν and μ are incident in \mathcal{T}_1 , then so are $f(\nu)$ and $f(\mu)$ in \mathcal{T}_2 .

Let $N \geq 0$. An infinite, locally finite and rooted tree \mathcal{T} with the root ϱ , is said to be N -*superperiodic* if for every $\nu \in V(\mathcal{T})$, there exists an injective morphism $f : \mathcal{T} \rightarrow \mathcal{T}^{f(o)}$ with $f(o) \in \mathcal{T}^\nu$ and $|f(o)| - |\nu| \leq N$. A tree \mathcal{T} is called *superperiodic tree* if there exists $N \geq 0$ such that it is N -superperiodic.

2.2. Some quantities on trees. In this section, we review the definitions of branching number, growth rate and branching-ruin number. We refer the reader to ([6] , [8]) for more details on the branching number and growth rate and [5] for more details on the branching-ruin number.

In order to define the branching number and the branching-ruin number of a tree, we will need the notion of *cutsets*.

Let \mathcal{T} be an infinite, locally finite and rooted tree. A cutset in \mathcal{T} is a set π of edges such that every infinite simple path from a must include an edge in π . The set of cutsets is denoted by Π .

The branching number of \mathcal{T} is defined as

$$(2.1) \quad br(\mathcal{T}) = \sup \left\{ \gamma > 0 : \inf_{\pi \in \Pi} \sum_{e \in \pi} \gamma^{-|e|} > 0 \right\} \in [1, \infty].$$

The branching-ruin number of \mathcal{T} is defined as

$$(2.2) \quad br_r(\mathcal{T}) = \sup \left\{ \gamma > 0 : \inf_{\pi \in \Pi} \sum_{e \in \pi} |e|^{-\gamma} > 0 \right\} \in [0, \infty].$$

These quantities depend on the structure of the tree. If \mathcal{T} is spherically symmetric, then there is really no information in the tree than that contained in the sequence $(|\mathcal{T}_n|, n \geq 0)$. Therefore, a tree which is spherically symmetric and whose n generation grows like b^n (resp. n^b), for $b \geq 1$, has a branching number (resp. branching-ruin number) equal to b . For more general trees, this becomes more complicated. In the other word, there exists a tree whose n generation grows like b^n (resp. n^b), for $b \geq 1$, but its branching number (resp. branching-ruin number) is not equal to b . For instance, the tree 1-3 in ([8], page 4) is an example.

Finally, we review the definition of *growth rate* of an infinite, locally finite and rooted tree \mathcal{T} . Define the *lower growth rate* of \mathcal{T} by

$$(2.3) \quad \underline{gr}(\mathcal{T}) = \liminf |\mathcal{T}_n|^{\frac{1}{n}}.$$

Similarly, we can define *upper growth rate* of \mathcal{T} by

$$(2.4) \quad \overline{gr}(\mathcal{T}) = \limsup |\mathcal{T}_n|^{\frac{1}{n}}.$$

In the case $\overline{gr}(\mathcal{T}) = \underline{gr}(\mathcal{T})$, we define the *growth rate* of \mathcal{T} , denoted by $gr(\mathcal{T})$, by taking the common value of $\overline{gr}(\mathcal{T})$ and $\underline{gr}(\mathcal{T})$.

Now, we state a relationship between the branching number and growth rate of a superperiodic tree.

Theorem 3 (see [8]). *Let \mathcal{T} be a N -superperiodic tree with $\overline{gr}(\mathcal{T}) < \infty$. Then the growth rate of \mathcal{T} exists and $gr(\mathcal{T}) = br(\mathcal{T})$. Moreover, we have $|\mathcal{T}_n| \leq gr(\mathcal{T})^{n+N}$.*

2.3. Definition of the model. Now, we define the model of multi-excited random walk on trees. Let $\mathcal{C} = (\lambda_1, \dots, \lambda_M; \lambda) \in (\mathbb{R}_+)^M \times \mathbb{R}_+^*$ and $\mathcal{T} = (V, E)$ be an infinite, locally finite and rooted tree with the root ϱ . A \mathcal{C} multi-excited random walk is a stochastic process $\mathbf{X} := (X_n)_{n \geq 0}$ defined on some probability space, taking the values in \mathcal{T} with the transition probability defined by:

$$\mathbb{P}(X_0 = \varrho) = 1,$$

$$\mathbb{P}(X_{n+1} = (X_n)_i | X_0, \dots, X_n) = \begin{cases} \frac{\lambda_j}{1 + \partial(X_n)\lambda_j} & \text{if } j \leq M \\ \frac{\lambda}{1 + \partial(X_n)\lambda} & \text{if } j > M \end{cases}$$

$$\mathbb{P}(X_{n+1} = X_n^{-1} | X_0, \dots, X_n) = \begin{cases} \frac{1}{1 + \partial(X_n)\lambda_j} & \text{if } j \leq M \\ \frac{1}{1 + \partial(X_n)\lambda} & \text{if } j > M \end{cases}$$

where $i \in \{1, \dots, k\}$ and $j = |\{0 \leq k \leq n : X_k = X_n\}|$.

We have some particular cases:

- If $\mathcal{C} = (0, \dots, 0; \lambda)$, then \mathcal{C} multi-excited random walk is M -digging random walk with parameter λ (M -DRW $_\lambda$), which was studied in [4].
- If $M = 0$, then \mathcal{C} multi-excited random walk is the biased random walk with parameter λ , which was studied in [7].
- If $\mathcal{C} = (\lambda_1; \lambda)$, then \mathcal{C} multi-excited random walk is (λ_1, λ) -OERW.

The *return time* of \mathbf{X} to a vertex ν is defined by:

$$(2.5) \quad T(\nu) := \inf\{n \geq 1 : X_n = \nu\}.$$

We say that \mathbf{X} is *transient* if

$$(2.6) \quad \mathbb{P}(T(\varrho) = \infty) > 0.$$

Otherwise, we say that \mathbf{X} is *recurrent*.

3. MAIN RESULTS

3.1. Main results about Once-excited random walk. Let $\lambda_1 \geq 0$ and $\lambda > 0$ and we consider the model (λ_1, λ) -OERW on an infinite, locally finite and rooted tree \mathcal{T} . First, we define the following functions. For any $e \in E$, we set $\psi(e, \lambda) = 1$ and $\phi(e, \lambda_1, \lambda) = 1$ if $|e| = 1$ and, for any $e \in E$ with $|e| > 1$, we set

$$(3.1) \quad \psi(e, \lambda) = \frac{\lambda^{|e|-1} - 1}{\lambda^{|e|} - 1} \text{ if } \lambda \neq 1,$$

$$\psi(e, \lambda) = \frac{|e| - 1}{|e|} \text{ if } \lambda = 1.$$

$$(3.2) \quad \phi(e, \lambda_1, \lambda) = \frac{\lambda_1}{1 + \partial(e^-)\lambda_1} + \frac{1}{1 + \partial(e^-)\lambda_1} \psi(e, \lambda) \psi(e^{-1}, \lambda) + \frac{(\partial(e^-) - 1)\lambda_1}{1 + \partial(e^-)\lambda_1} \psi(e, \lambda)$$

Finally, for any $e \in E$, we define:

$$(3.3) \quad \Psi(e, \lambda_1, \lambda) = \prod_{g \leq e} \phi(g, \lambda_1, \lambda).$$

We refer the reader to Lemma 14 for the probabilistic interpretation of these functions.

In the following, we assume that

$$(3.4) \quad \exists M \in \mathbb{N} \text{ such that } \sup\{\deg \nu : \nu \in V\} \leq M.$$

Let us define the quantity $RT(\mathcal{T}, \mathbf{X})$ which was introduced in [5]:

$$(3.5) \quad RT(\mathcal{T}, \mathbf{X}) = \sup\{\gamma > 0 : \inf_{\pi \in \Pi} \sum_{e \in \pi} (\Psi(e))^\gamma > 0\}.$$

Theorem 4. *Consider an (λ_1, λ) -OERW on an infinite, locally finite, rooted tree \mathcal{T} , with parameters $\lambda_1 \geq 0$ and $\lambda > 0$. If $RT(\mathcal{T}, \mathbf{X}) < 1$ then \mathbf{X} is recurrent. If $RT(\mathcal{T}, \mathbf{X}) > 1$ and if (3.4) holds, then \mathbf{X} is transient.*

In the following, we consider the case \mathcal{T} is spherically symmetric.

Lemma 5. *Consider a (λ_1, λ) -OERW \mathbf{X} on a spherically symmetric \mathcal{T} , with parameters $\lambda_1 \geq 0$ and $\lambda > 0$. Assume that there exists a constant $M > 0$ such that $\sup_{\nu \in V} \deg \nu \leq M$. We have that*

- (1) *in the case $\lambda = 1$, if $\eta(\mathcal{T}, \lambda_1) < br_r(\mathcal{T})$ then $RT(\mathcal{T}, \mathbf{X}) > 1$ and if $\gamma(\mathcal{T}, \lambda_1) > br_r(\mathcal{T})$ then $RT(\mathcal{T}, \mathbf{X}) < 1$;*
- (2) *assume that $\lambda_1 \geq 0$, $\lambda \neq 1$ and $br(\mathcal{T}) > 1$, if $\beta(\mathcal{T}, \lambda_1, \lambda) < \frac{1}{br(\mathcal{T})}$ then $RT(\mathcal{T}, \mathbf{X}) < 1$ and if $\alpha(\mathcal{T}, \lambda_1, \lambda) > \frac{1}{br(\mathcal{T})}$ then $RT(\mathcal{T}, \mathbf{X}) > 1$.*

Note that Theorem 1 is a consequence of Theorem 4 and Lemma 5.

3.2. Main results about critical M -Digging random walk. Let $M \in \mathbb{N}^*$, $\lambda > 0$ and we consider the model M -DRW $_\lambda$ on an infinite, locally finite and rooted tree \mathcal{T} . In [4], Collevecchio-Huynh-Kious was proved that there is a phase transition with respect to the parameter λ , i.e there exists a critical parameter λ_c . A natural question that arises: what happens if $\lambda = \lambda_c$? As we said in the introduction, there is no a good answer for this question.

In [1], Basdevant-Singh proved the critical M -digging random walk is recurrent on the regular trees. In this paper, we prove the critical M -digging random walk is still recurrent on a particular class of trees which contains the regular trees.

Theorem 6. *Let $M \in \mathbb{N}^*$ and \mathcal{T} be a superperiodic tree whose upper-growth rate is finite. Then the critical M -digging random walk on \mathcal{T} is recurrent.*

4. AN EXAMPLE

In this section, we give an example to prove that the phase transition of once-excited random walk (λ_1, λ) -OERW on a tree \mathcal{T} does not depend only on the branching-ruin number and the branching number of \mathcal{T} .

If \mathcal{T} is a spherically symmetric tree, recall that $x_n(\mathcal{T})$ is the number of children of a vertex at level n .

Let \mathcal{T} (resp. $\tilde{\mathcal{T}}$) be a spherically symmetric such that for any $n \geq 0$, we have $x_n(\mathcal{T}) = 2$ (resp. $x_n(\tilde{\mathcal{T}}) = 1$ if n is odd and $x_n(\tilde{\mathcal{T}}) = 4$ if not). Then we obtain :

$$(4.1) \quad br(\mathcal{T}) = br(\tilde{\mathcal{T}}) = 2.$$

$$(4.2) \quad br_r(\mathcal{T}) = br_r(\tilde{\mathcal{T}}) = \infty.$$

Lemma 7. *Consider a $(1, (\sqrt{3} - 1)/2)$ -OERW \mathbf{X} (resp. $\tilde{\mathbf{X}}$) on \mathcal{T} (resp. $\tilde{\mathcal{T}}$). Then \mathbf{X} is recurrent, but $\tilde{\mathbf{X}}$ is transient.*

Proof. Note that \mathcal{T} is a binary tree, then we can apply Corollary 1.6 of [1] to imply that \mathbf{X} is recurrent. On the other hand, by a simple computation we have

$$(4.3) \quad \alpha \left(\tilde{\mathcal{T}}, 1, \frac{\sqrt{3}-1}{2} \right) = \beta \left(\tilde{\mathcal{T}}, 1, \frac{\sqrt{3}-1}{2} \right) > \frac{1}{2}.$$

By Theorem 1 and 4.3, we obtain $\tilde{\mathbf{X}}$ is transient. \square

5. PROOF OF THEOREM 2

Lemma 8. *Let \mathcal{T} be an infinite, locally finite and rooted tree. If $br(\mathcal{T}) > 1$ then $br_r(\mathcal{T}) = +\infty$.*

Proof. See ([4], proof of Lemma 8, Case V). \square

Lemma 9. *Let $(\lambda_1, \dots, \lambda_M) \in (\mathbb{R}_+)^M$ and \mathcal{T} be an infinite, locally finite and rooted tree. If M -DRW₁ is transient, then $(\lambda_1, \dots, \lambda_M, 1)$ -ERW is transient.*

Proof. See ([1], Section 3). \square

Remark 10. *Let $T(\varrho)$ (resp. $S(\varrho)$) the return of of M -DRW₁ (resp. $(\lambda_1, \dots, \lambda_M, 1)$ -ERW) to the root ϱ of \mathcal{T} . It is simple to see that*

$$(5.1) \quad \mathbb{P}(T(\varrho) < \infty) \leq \mathbb{P}(S(\varrho) < \infty).$$

Proposition 11. *Let $(\lambda_1, \dots, \lambda_M) \in (\mathbb{R}_+)^M$ and consider $(\lambda_1, \dots, \lambda_M, 1)$ -ERW \mathbf{X} on an infinite, locally finite, rooted tree \mathcal{T} . If $br(\mathcal{T}) > 1$, then \mathbf{X} is transient.*

Proof. Note that if $\lambda_i = 0$ for all $1 \leq i \leq M$ and $\lambda = 1$, then \mathbf{X} is a M -digging random walk with parameter 1 (M -DRW₁). On the other hand, we have $(\lambda_1, \dots, \lambda_M, 1)$ -ERW is more transient than M -DRW₁, i.e if M -DRW₁ is transient then $(\lambda_1, \dots, \lambda_M, 1)$ -ERW is transient. We complete the proof by using Lemma (8) and Theorem 2 in [4]. \square

6. PROOF OF LEMMA 5 AND THEOREM 1

In this section, we prove Lemma 5. Theorem 1 then trivially follows from Theorem 4.

Lemma 12. *Recall the definition of $\Psi(e, \lambda_1, \lambda)$ as in 3.3. We have that, if $\lambda \neq 1$, for any $|e| > 1$,*

$$(6.1) \quad \Psi(e, \lambda_1, \lambda) = \left(\prod_{g \leq e, |g| > 1} \frac{\lambda^2 + (\partial(g^-) - 1)\lambda_1\lambda + \lambda_1}{1 + \partial(g^-)\lambda_1} \right) \prod_{g \leq e, |g| > 1} \left(\frac{1 - \lambda^{|g|} \left(\frac{1 + \partial(g^-)\lambda_1}{\lambda^2 + (\partial(g^-) - 1)\lambda_1\lambda + \lambda_1} \right)}{1 - \lambda^{|g|}} \right).$$

and if $\lambda = 1$, for any $|e| > 1$,

$$(6.2) \quad \Psi(e, \lambda_1, \lambda) = \prod_{g \leq e, |g| > 1} \left(1 - \frac{(\partial(g^-) - 1)\lambda_1 + 2}{|g|(1 + \partial(g^-)\lambda_1)} \right).$$

Proof. We compute the quantity $\Psi(e, \lambda, \lambda_1)$ by using (3.1), 3.2 and (3.3). We will proceed by distinguishing two cases.

Case I: $\lambda \neq 1$.

By (3.1), 3.2 and (3.3), we have

$$\Psi(e, \lambda_1, \lambda) = \prod_{g \leq e, |g| > 1} \phi(g, \lambda_1, \lambda)$$

$$\begin{aligned}
&= \prod_{g \leq e, |g| > 1} \left(\frac{\lambda_1}{1 + \partial(g^-)\lambda_1} + \frac{1}{1 + \partial(g^-)\lambda_1} \psi(e, \lambda) \psi(e^{-1}, \lambda) + \frac{(\partial(g^-) - 1)\lambda_1}{1 + \partial(g^-)\lambda_1} \psi(e, \lambda) \right) \\
&= \left(\prod_{g \leq e, |g| > 1} \frac{1}{1 + \partial(g^-)\lambda_1} \right) \prod_{g \leq e, |g| > 1} (\lambda_1 + \psi(e, \lambda) \psi(e^{-1}, \lambda) + (\partial(g^-) - 1)\lambda_1 \psi(e, \lambda))
\end{aligned}$$

By 3.1, we have:

$$\begin{aligned}
&\lambda_1 + \psi(g, \lambda) \psi(g^{-1}, \lambda) + (\partial(g^-) - 1)\lambda_1 \psi(g, \lambda) \\
&= \lambda_1 + \left(\frac{1 - (1/\lambda)^{|g|-2}}{1 - (1/\lambda)^{|g|}} \right) + \left((\partial(g^-) - 1)\lambda_1 \frac{1 - (1/\lambda)^{|g|-1}}{1 - (1/\lambda)^{|g|}} \right) \\
(6.3) \quad &= \lambda_1 + \left(\frac{\lambda^{|g|} - \lambda^2}{\lambda^{|g|} - 1} \right) + (\partial(g^-) - 1)\lambda_1 \left(\frac{\lambda^{|g|} - \lambda}{\lambda^{|g|} - 1} \right) \\
&= \frac{\lambda^2 + (\partial(g^-) - 1)\lambda_1 \lambda + \lambda_1 - \lambda^{|g|} (1 + \partial(g^-)\lambda_1)}{1 - \lambda^{|g|}} \\
&= (\lambda^2 + (\partial(g^-) - 1)\lambda_1 \lambda + \lambda_1) \left(\frac{1 - \lambda^{|g|} \left(\frac{1 + \partial(g^-)\lambda_1}{\lambda^2 + (\partial(g^-) - 1)\lambda_1 \lambda + \lambda_1} \right)}{1 - \lambda^{|g|}} \right).
\end{aligned}$$

Therefore we obtain 6.1.

Case II: $\lambda = 1$.

By (3.1), 3.2 and (3.3), we have

$$\begin{aligned}
\Psi(e, \lambda_1, \lambda) &= \prod_{g \leq e, |g| > 1} \phi(g, \lambda_1, \lambda) \\
&= \prod_{g \leq e, |g| > 1} \left(\frac{\lambda_1}{1 + \partial(g^-)\lambda_1} + \frac{1}{1 + \partial(g^-)\lambda_1} \psi(e, \lambda) \psi(e^{-1}, \lambda) + \frac{(\partial(g^-) - 1)\lambda_1}{1 + \partial(g^-)\lambda_1} \psi(e, \lambda) \right) \\
&= \left(\prod_{g \leq e, |g| > 1} \frac{1}{1 + \partial(g^-)\lambda_1} \right) \prod_{g \leq e, |g| > 1} (\lambda_1 + \psi(e, \lambda) \psi(e^{-1}, \lambda) + (\partial(g^-) - 1)\lambda_1 \psi(e, \lambda))
\end{aligned}$$

By 3.1, we have:

$$\begin{aligned}
&\lambda_1 + \psi(g, \lambda) \psi(g^{-1}, \lambda) + (\partial(g^-) - 1)\lambda_1 \psi(g, \lambda) \\
(6.4) \quad &= \lambda_1 + \frac{|g| - 2}{|g|} + (\partial(g^-) - 1)\lambda_1 \frac{|g| - 1}{|g|} \\
&= \frac{\lambda_1 |g| + |g| - 2 + (\partial(g^-) - 1)\lambda_1 (|g| - 1)}{|g|} \\
&= 1 + \partial(g^-)\lambda_1 - \frac{(\partial(g^-) - 1)\lambda_1 + 2}{|g|}
\end{aligned}$$

Therefore we obtain 6.2. □

Proof of Lemma 5. We will proceed by distinguishing a few cases.

Case I: $\lambda \neq 1$, $br(\mathcal{T}) > 1$ and $\beta(\mathcal{T}, \lambda_1, \lambda) < \frac{1}{br(\mathcal{T})}$.

By (2.1), there exists $\delta \in (0, 1)$ such that

$$(6.5) \quad \inf_{\pi \in \Pi} \sum_{e \in \Pi} \beta^{(1-\delta)^2|e|} = 0.$$

As $\beta < \beta^{(1-\delta)}$, there exists $c > 0$, for any $n > 0$,

$$(6.6) \quad \prod_{i=1}^n \frac{\lambda^2 + (x_i - 1)\lambda_1\lambda + \lambda_1}{1 + x_i\lambda_1} \leq c\beta^{(1-\delta)n}.$$

By 6.1 and 6.6, there exists $C > 0$ such that for any $\pi \in \Pi$,

$$(6.7) \quad \sum_{e \in \pi} \Psi(e)^{1-\delta} \leq C \sum_{e \in \Pi} \beta^{(1-\delta)^2|e|}.$$

Therefore, by (6.5),

$$(6.8) \quad \inf_{\pi \in \Pi} \sum_{e \in \pi} \Psi(e)^{1-\delta} = 0,$$

which implies that $RT(\mathcal{T}, \mathbf{X}) < 1$.

Case II: $\lambda \neq 1$, $br(\mathcal{T}) > 1$ and $\alpha(\mathcal{T}, \lambda_1, \lambda) > \frac{1}{br(\mathcal{T})}$.

First, note that if $\lambda > 1$ and $br(\mathcal{T}) > 1$ then \mathbf{X} is transient. Now, assume that $\lambda < 1$, $br(\mathcal{T}) > 1$ and $\alpha(\mathcal{T}, \lambda_1, \lambda) > \frac{1}{br(\mathcal{T})}$. We have that there exists $\delta > 0$ and $\varepsilon > 0$ such that

$$(6.9) \quad \inf_{\pi \in \Pi} \sum_{e \in \Pi} \alpha^{(1+\delta)^2|e|} > \varepsilon.$$

By 1.1 and $\lambda < 1$, we obtain $\alpha < 1$, therefore $\alpha^{1+\delta} < \alpha$. We have that there exists $c > 0$, for any $n > 0$,

$$(6.10) \quad \prod_{i=1}^n \frac{\lambda^2 + (x_i - 1)\lambda_1\lambda + \lambda_1}{1 + x_i\lambda_1} \geq c\alpha^{(1+\delta)n}.$$

By 6.1 and 6.10, there exists $C > 0$ such that for any $\pi \in \Pi$,

$$(6.11) \quad \sum_{e \in \pi} \Psi(e)^{1+\delta} \geq C \sum_{e \in \Pi} \alpha^{(1+\delta)^2|e|}.$$

Therefore, by (6.9),

$$(6.12) \quad \inf_{\pi \in \Pi} \sum_{e \in \pi} \Psi(e)^{1+\delta} > 0,$$

which implies that $RT(\mathcal{T}, \mathbf{X}) > 1$.

Case III: $\lambda = 1$ and $\eta(\mathcal{T}, \lambda_1) < br_r(\mathcal{T})$.

We have that there exists $\delta > 0$ and $\varepsilon > 0$ such that

$$(6.13) \quad \inf_{\pi \in \Pi} \sum_{e \in \pi} |e|^{-(1+\delta)^2\eta} > \varepsilon.$$

As $\eta < (1 + \delta)\eta$, by 1.4 there exists $c > 0$, for any $n > 0$,

$$(6.14) \quad \prod_{i=1}^n \left[1 - \frac{(x_i - 1)\lambda_1 + 2}{(1 + x_i\lambda_1)i} \right] \geq c n^{-(1+\delta)\eta}.$$

By 6.2 and 6.14, there exists $C > 0$ such that for any $\pi \in \Pi$,

$$(6.15) \quad \sum_{e \in \pi} \Psi(e)^{1+\delta} \geq C \sum_{e \in \Pi} |e|^{-(1+\delta)^2\eta}.$$

Therefore, by (6.13),

$$(6.16) \quad \inf_{\pi \in \Pi} \sum_{e \in \pi} \Psi(e)^{1+\delta} > 0,$$

which implies that $RT(\mathcal{T}, \mathbf{X}) > 1$.

Case IV: $\lambda = 1$ and $\gamma(\mathcal{T}, \lambda_1) > br_r(\mathcal{T})$

We have that there exists $\delta > 0$ such that

$$(6.17) \quad \inf_{\pi \in \Pi} \sum_{e \in \pi} |e|^{-(1-\delta)^2\eta} = 0.$$

As $\eta > (1 - \delta)\eta$, by 1.4 there exists $c > 0$, for any $n > 0$,

$$(6.18) \quad \prod_{i=1}^n \left[1 - \frac{(x_i - 1)\lambda_1 + 2}{(1 + x_i\lambda_1)i} \right] \leq c n^{-(1-\delta)\eta}.$$

By 6.2 and 6.18, there exists $C > 0$ such that for any $\pi \in \Pi$,

$$(6.19) \quad \sum_{e \in \pi} \Psi(e)^{1-\delta} \leq C \sum_{e \in \Pi} |e|^{-(1-\delta)^2\eta}.$$

Therefore, by (6.17),

$$(6.20) \quad \inf_{\pi \in \Pi} \sum_{e \in \pi} \Psi(e)^{1-\delta} > 0,$$

which implies that $RT(\mathcal{T}, \mathbf{X}) < 1$. □

7. EXTENSIONS

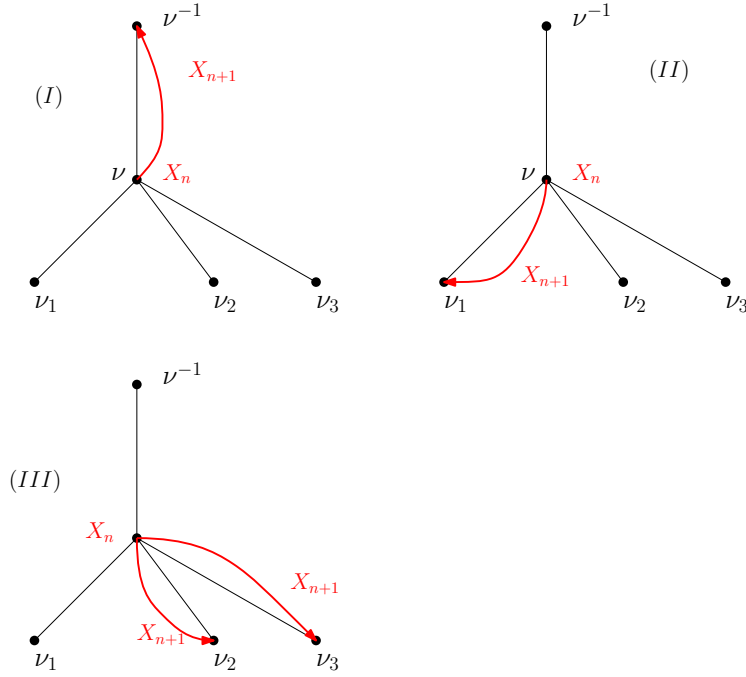
First of all, let us describe the dynamic of this model. If \mathbf{X} visits a vertex ν for the first time, three cases can occur for visiting ν_1 (see Figure 1):

- It eats the cookie at ν and returns to the parent of ν (i.e. ν^{-1}) with probability $\frac{1}{1+\partial(\nu)\lambda_1}$. It then visits ν for the second time, and goes to ν_1 with probability $\frac{\lambda}{1+\partial(\nu)\lambda}$.
- It goes directly to ν_1 with probability $\frac{\lambda_1}{1+\partial(\nu)\lambda_1}$.
- It goes to one of the children of ν except for ν_1 , with probability $\frac{(\partial\nu-1)\lambda_1}{1+\partial(\nu)\lambda_1}$. It then visits ν for the second time, and goes to ν_1 with probability $\frac{\lambda}{1+\partial(\nu)\lambda}$.

Now, we introduce a construction of once-excited random walk by using the Rubin's construction. Let $(\Omega, \mathcal{F}, \mathbf{P})$ denote a probability space on which

$$(7.1) \quad \mathbf{Y} = (Y(\nu, \mu, k) : (\nu, \mu) \in V^2, \text{ with } \nu \sim \mu, \text{ and } k \in \mathbb{N})$$

$$(7.2) \quad \mathbf{Z} = (Z(\nu, \mu) : (\nu, \mu) \in V^2, \text{ with } \nu \sim \mu)$$

FIGURE 1. The movement of \mathbf{X} to ν_1 after visiting ν .

are two families of independent mean 1 exponential random variables, where (ν, μ) denotes an *ordered* pair of vertices. Let

$$(7.3) \quad \mathbf{U} = (U_\nu : \nu \in V)$$

is a family of independent uniformly random variables on $[0, 1]$ which is independent to \mathbf{Y} and \mathbf{Z} .

For any pair vertices $\nu, \mu \in V$ with $\nu \sim \mu$, we define the following quantities

$$(7.4) \quad r(\nu, \mu) = \begin{cases} \lambda^{|\nu|-1}, & \text{if } \mu < \nu, \\ \lambda^{|\mu|-1}, & \text{if } \nu < \mu. \end{cases}$$

Let \mathcal{T}' be a sub-tree of \mathcal{T} , we define the *extension* $\mathbf{X}^{(\mathcal{T}')} = (V', E')$ on \mathcal{T}' in the following way. Denote by ϱ' the root of \mathcal{T}' which be defined as the vertex of V' with smallest distance to the root of \mathcal{T} . For any family of nonnegative integers $\bar{k} = (k_\mu)_{\mu: [\nu, \mu] \in E'}$, we let

$$(7.5) \quad A_{\bar{k}, n, \nu}^{(\mathcal{T}')} := \{X_n^{(\mathcal{T}')} = \nu\} \cap \bigcap_{\mu: [\nu, \mu] \in E'} \{\#\{1 \leq j \leq n : (X_{j-1}^{(\mathcal{T}'), X_j^{(\mathcal{T}')}) = (\nu, \mu)\} = k_\mu\}.$$

$$(7.6) \quad t_\nu(n) := \#\{1 \leq j \leq n : X_j^{(\mathcal{T}')} = \nu\}.$$

$$(7.7) \quad h_\nu := \inf\{i \geq 1 : t_\nu(i) = 2\}.$$

$$(7.8) \quad \tilde{A}_{\bar{k}, n, \nu}^{(\mathcal{T}')} := \{X_n^{(\mathcal{T}')} = \nu\} \cap \bigcap_{\mu: [\nu, \mu] \in E'} \{\#\{h_\nu \leq j \leq n : (X_{j-1}^{(\mathcal{T}'), X_j^{(\mathcal{T}')}) = (\nu, \mu)\} = k_\mu\}.$$

$$(7.9) \quad \mathcal{I}^\mathcal{T}(\nu) := \#\{i \in \{1, 2, \dots, \partial(\nu)\} : \nu_i \in V(\mathcal{T}')\}.$$

Set $X_0^{(\mathcal{T}')} = \varrho'$ and on the event $A_{\bar{k}, n, \nu}^{(\mathcal{T}')} \cap \{t_\nu(n) \leq 1\}$:

- If $U_\nu < \frac{1}{1+\partial(\nu)\lambda_1}$, then we set $X_{n+1}^{(\mathcal{T}')} = \nu^{-1}$.
- If $U_\nu \in \left[\frac{1+(j-1)\lambda_1}{1+\partial(\nu)\lambda_1}, \frac{1+j\lambda_1}{1+\partial(\nu)\lambda_1} \right]$ and $j \in \mathcal{I}^\mathcal{T}(\nu)$, then we set $X_{n+1}^{(\mathcal{T}')} = v_j$.
- If $U_\nu \in \left[\frac{1+(j-1)\lambda_1}{1+\partial(\nu)\lambda_1}, \frac{1+j\lambda_1}{1+\partial(\nu)\lambda_1} \right]$ for some $j \notin \mathcal{I}^\mathcal{T}(\nu)$ and

$$\left\{ \nu' = \arg \min_{\mu: [\nu, \mu] \in E'} \left\{ \frac{Z(\nu, \mu)}{r(\nu, \mu)} \right\} \right\},$$

we set $X_{n+1}^{(\mathcal{T}')} = \nu'$.

On the event

$$(7.10) \quad \tilde{A}_{k, n, \nu}^{(\mathcal{T}')} \cap \{t_\nu(n) \geq 2\} \cap \left\{ \nu' = \arg \min_{\mu: [\nu, \mu] \in E'} \left\{ \sum_{i=0}^{k_\mu} \frac{Y(\nu, \mu, i)}{r(\nu, \mu)} \right\} \right\},$$

we set $X_{n+1}^{(\mathcal{T}')} = \nu'$, where the function r is defined in (7.4) and the clocks Y 's are from the same collection \mathbf{Y} fixed in (7.1).

Thus, this defines $\mathbf{X}^{(\mathcal{T})}$ as the extension on the whole tree. By using the properties of independent exponential random variables, it is easy to check that this construction is a construction of (λ_1, λ) -OERW on the tree \mathcal{T} . We refer the reader to ([4], section 7) for more discussions on this construction.

In the case $\mathcal{T}' = [\varrho, \nu]$ for some vertex ν of \mathcal{T} , we write $\mathbf{X}^{(\nu)}$ instead of $\mathbf{X}^{([\varrho, \nu])}$, and we denote $T^{(\nu)}(\cdot)$ the return times associated to $\mathbf{X}^{(\nu)}$. For simplicity, we will also write $\mathbf{X}^{(e)}$ and $T^{(e)}(\cdot)$ instead of $\mathbf{X}^{(e^+)}$ and $T^{(e^+)}(\cdot)$ for $e \in E$.

Remark 13. Let \mathcal{T}' be a proper subtree of \mathcal{T} . Note that $\mathbf{X}^{(\mathcal{T}')}$ is not (λ_1, λ) -OERW on \mathcal{T}' , that is different with M -digging random walk (see [4], section 7) and once-reinforced random walk (see [5], section 5).

Finally, we give a probabilistic interpretation of the functions ϕ and Ψ :

Lemma 14. For any $e \in E$ and any $g \leq e$, we have

$$(7.11) \quad \phi(g, \lambda_1, \lambda) = \mathbb{P} \left(T^{(e)}(g^+) \circ \theta_{T^{(e)}(g^-)} < T^{(e)}(\varrho) \circ \theta_{T^{(e)}(g^-)} \right),$$

$$(7.12) \quad \Psi(e, \lambda_1, \lambda) = \mathbb{P} \left(T^{(e)}(e^+) < T^{(e)}(\varrho) \right),$$

where θ is the canonical shift on the trajectories.

Proof. Let $e \in E$ and $g \leq e$. For simplicity, we set

$$\mathcal{A} := \{T^{(e)}(g^+) \circ \theta_{T^{(e)}(g^-)} < T^{(e)}(\varrho) \circ \theta_{T^{(e)}(g^-)}\},$$

$$\mathcal{I}_1 := \left[\frac{1+(j-1)\lambda_1}{1+\partial(g^-)\lambda_1}, \frac{1+j\lambda_1}{1+\partial(g^-)\lambda_1} \right],$$

$$\mathcal{I}_2 := [0, 1] \setminus \left(\left[\frac{1+(j-1)\lambda_1}{1+\partial(g^-)\lambda_1}, \frac{1+j\lambda_1}{1+\partial(g^-)\lambda_1} \right] \cup \left[0, \frac{1}{1+\partial(g^-)\lambda_1} \right] \right),$$

where $j \in \{1, \dots, \partial(g^-)\}$ such that $(g^-)_j = g^+$. We have that

$$(7.13) \quad \begin{aligned} \mathbb{P}(\mathcal{A}) &= \mathbb{P} \left(A \mid U_{g^-} < \frac{1}{1+\partial(g^-)\lambda_1} \right) \times \mathbb{P} \left(U_{g^-} < \frac{1}{1+\partial(g^-)\lambda_1} \right) \\ &+ \mathbb{P}(A \mid \mathcal{I}_1) \times \mathbb{P}(U_{g^-} \in \mathcal{I}_1) + \mathbb{P}(A \mid \mathcal{I}_2) \times \mathbb{P}(U_{g^-} \in \mathcal{I}_2). \end{aligned}$$

On the other hand, we have the following equalities:

$$(7.14) \quad \mathbb{P}\left(A \mid U_{g^-} < \frac{1}{1 + \partial(g^-)\lambda_1}\right) \times \mathbb{P}\left(U_{g^-} < \frac{1}{1 + \partial(g^-)\lambda_1}\right) = \frac{1}{1 + \partial(g^-)\lambda_1} \psi(g, \lambda) \psi(g^{-1}, \lambda)$$

$$(7.15) \quad \mathbb{P}(A \mid \mathcal{I}_1) \times \mathbb{P}(U_{g^-} \in \mathcal{I}_1) = \frac{\lambda_1}{1 + \partial(g^-)\lambda_1}.$$

$$(7.16) \quad \mathbb{P}(A \mid \mathcal{I}_2) \times \mathbb{P}(U_{g^-} \in \mathcal{I}_2) = \frac{(\partial(g^-) - 1)\lambda_1}{1 + \partial(g^-)\lambda_1} \psi(g, \lambda).$$

We use (7.13), (7.14), (7.15) and (7.16) to obtain the results. \square

8. RECURRENCE IN THEOREM 4: THE CASE $RT(\mathcal{T}, \mathbf{X}) < 1$

Proposition 15. *If*

$$(8.1) \quad \inf_{\pi \in \Pi} \sum_{e \in \pi} \Psi(e) = 0,$$

then \mathbf{X} is recurrent.

Proof. The proof is identical to the proof of Proposition 10 of [5]. \square

9. TRANSIENCE IN THEOREM 4: THE CASE $RT(\mathcal{T}, \mathbf{X}) > 1$

In order to prove transience, we use the relationship between the walk \mathbf{X} and its associated percolation.

9.1. Link with percolation. Denote by $\mathcal{C}(\varrho)$ the set of edges which are crossed by \mathbf{X} before returning to ϱ , that is:

$$(9.1) \quad \mathcal{C}(\varrho) = \{e \in E : T(e^+) < T(\varrho)\}.$$

We define an other percolation which will be more easy to study. In order to do this, we use the Rubin's construction and the extensions introduced in Section 7. We define

$$(9.2) \quad \mathcal{C}_{CP}(\varrho) = \{e \in E : T^{(e)}(e^+) < T^{(e)}(\varrho)\}.$$

We say that an edge $e \in E$ is open if and only if $e \in \mathcal{C}_{CP}(\varrho)$.

Lemma 16. *We have that*

$$(9.3) \quad \mathbb{P}(T(\varrho) = \infty) = \mathbb{P}(|\mathcal{C}(\varrho)| = \infty) = \mathbb{P}(|\mathcal{C}_{CP}(\varrho)| = \infty).$$

Proof. We can follow line by line the proof of Lemma 11 in [5]. \square

For simplicity, for a vertex $v \in V$, we write $v \in \mathcal{C}_{CP}(\varrho)$ if one of the edges incident to v is in $\mathcal{C}_{CP}(\varrho)$. Besides, recall that for two edges e_1 and e_2 , their common ancestor with highest generation is the vertex denoted $e_1 \wedge e_2$.

Lemma 17. *Let $\lambda_1 \geq 0$, $\lambda > 0$ and \mathcal{T} be an infinite, locally finite and rooted tree with the root ϱ . Assume that the condition (3.4) holds with some constant M . Then the correlated percolation induced by \mathcal{C}_{CP} is quasi-independent, i.e. there exists a constant $C_Q \in (0, +\infty)$ such that, for any two edges e_1, e_2 , we have that*

$$(9.4) \quad \mathbb{P}(e_1, e_2 \in \mathcal{C}_{CP}(\varrho) \mid e_1 \wedge e_2 \in \mathcal{C}_{CP}(\varrho)) \leq C_Q \mathbb{P}(e_1 \in \mathcal{C}_{CP}(\varrho) \mid e_1 \wedge e_2 \in \mathcal{C}_{CP}(\varrho)) \\ \times \mathbb{P}(e_2 \in \mathcal{C}_{CP}(\varrho) \mid e_1 \wedge e_2 \in \mathcal{C}_{CP}(\varrho)).$$

Proof. Recall the construction of Section 7. Note that if $e_1 \wedge e_2 = \varrho$, then the extensions on $[\varrho, e_1]$ and $[\varrho, e_2]$ are independent, then the conclusion of Lemma holds with $C = 1$. Assume that $e_1 \wedge e_2 \neq \varrho$, and note that the extensions on $[\varrho, e_1]$ and $[\varrho, e_2]$ are dependent since they use the same clocks on $[\varrho, e_1 \wedge e_2]$. Denote by e the unique edge of \mathcal{T} such that $e^+ = e_1 \wedge e_2$. For $i \in \{1, 2\}$, let v_i be the vertex which is the offspring of e^+ lying the path from ϱ to e_i . Note that v_i could be equal to e_i^+ . Let i_1 (resp. i_2) be an element of $\{1, \dots, \partial(e^+)\}$ such that $(e^+)_{i_1} = v_1$ (resp. $(e^+)_{i_2} = v_2$). As the events $\{e \in \mathcal{C}_{CP}\}$ and $U_{e_1 \wedge e_2}$ are independent, therefore:

$$\mathbb{P}(e_1, e_2 \in \mathcal{C}_{CP}(\varrho) | e \in \mathcal{C}_{CP}(\varrho)) = A + B + C + D,$$

where

$$(9.5) \quad A = \mathbb{P}\left(e_1, e_2 \in \mathcal{C}_{CP}(\varrho) | e \in \mathcal{C}_{CP}(\varrho), U_{e^+} < \frac{1}{1 + \partial(e^+)\lambda_1}\right) \mathbb{P}\left(U_{e^+} < \frac{1}{1 + \partial(e^+)\lambda_1}\right)$$

$$(9.6) \quad B = \mathbb{P}\left(e_1, e_2 \in \mathcal{C}_{CP}(\varrho) | e \in \mathcal{C}_{CP}(\varrho), U_{e^+} \in \left[\frac{1 + (i_1 - 1)\lambda_1}{1 + \partial(e^+)\lambda_1}, \frac{1 + i_1\lambda_1}{1 + \partial(e^+)\lambda_1}\right]\right) \\ \times \mathbb{P}\left(U_{e^+} \in \left[\frac{1 + (i_1 - 1)\lambda_1}{1 + \partial(e^+)\lambda_1}, \frac{1 + i_1\lambda_1}{1 + \partial(e^+)\lambda_1}\right]\right).$$

$$(9.7) \quad C = \mathbb{P}\left(e_1, e_2 \in \mathcal{C}_{CP}(\varrho) | e \in \mathcal{C}_{CP}(\varrho), U_{e^+} \in \left[\frac{1 + (i_2 - 1)\lambda_1}{1 + \partial(e^+)\lambda_1}, \frac{1 + i_2\lambda_1}{1 + \partial(e^+)\lambda_1}\right]\right) \\ \times \mathbb{P}\left(U_{e^+} \in \left[\frac{1 + (i_2 - 1)\lambda_1}{1 + \partial(e^+)\lambda_1}, \frac{1 + i_2\lambda_1}{1 + \partial(e^+)\lambda_1}\right]\right).$$

$$(9.8) \quad D = \mathbb{P}\left(e_1, e_2 \in \mathcal{C}_{CP}(\varrho) | e \in \mathcal{C}_{CP}(\varrho), U_{e^+} \in \bigcup_{i \in \{1, \dots, \partial(e^+)\} \setminus \{i_1, i_2\}} \left[\frac{1 + (i - 1)\lambda_1}{1 + \partial(e^+)\lambda_1}, \frac{1 + i\lambda_1}{1 + \partial(e^+)\lambda_1}\right]\right) \\ \times \mathbb{P}\left(U_{e^+} \in \bigcup_{i \in \{1, \dots, \partial(e^+)\} \setminus \{i_1, i_2\}} \left[\frac{1 + (i - 1)\lambda_1}{1 + \partial(e^+)\lambda_1}, \frac{1 + i\lambda_1}{1 + \partial(e^+)\lambda_1}\right]\right).$$

In the same way, for any $j \in \{1, 2\}$, we have:

$$\mathbb{P}(e_j \in \mathcal{C}_{CP}(\varrho) | e \in \mathcal{C}_{CP}(\varrho)) = E_j + F_j + G_j,$$

where

$$(9.9) \quad E_j = \mathbb{P}\left(e_j \in \mathcal{C}_{CP}(\varrho) | e \in \mathcal{C}_{CP}(\varrho), U_{e^+} < \frac{1}{1 + \partial(e^+)\lambda_1}\right) \mathbb{P}\left(U_{e^+} < \frac{1}{1 + \partial(e^+)\lambda_1}\right)$$

$$(9.10) \quad F_j = \mathbb{P}\left(e_j \in \mathcal{C}_{CP}(\varrho) | e \in \mathcal{C}_{CP}(\varrho), U_{e^+} \in \left[\frac{1 + (i_j - 1)\lambda_1}{1 + \partial(e^+)\lambda_1}, \frac{1 + i_j\lambda_1}{1 + \partial(e^+)\lambda_1}\right]\right) \\ \times \mathbb{P}\left(U_{e^+} \in \left[\frac{1 + (i_j - 1)\lambda_1}{1 + \partial(e^+)\lambda_1}, \frac{1 + i_j\lambda_1}{1 + \partial(e^+)\lambda_1}\right]\right)$$

$$(9.11) \quad G_j = \mathbb{P}\left(e_j \in \mathcal{C}_{CP}(\varrho) | e \in \mathcal{C}_{CP}(\varrho), U_{e^+} \in \bigcup_{i \in \{1, \dots, \partial(e^+)\} \setminus \{i_j\}} \left[\frac{1 + (i - 1)\lambda_1}{1 + \partial(e^+)\lambda_1}, \frac{1 + i\lambda_1}{1 + \partial(e^+)\lambda_1}\right]\right) \\ \times \mathbb{P}\left(U_{e^+} \in \bigcup_{i \in \{1, \dots, \partial(e^+)\} \setminus \{i_j\}} \left[\frac{1 + (i - 1)\lambda_1}{1 + \partial(e^+)\lambda_1}, \frac{1 + i\lambda_1}{1 + \partial(e^+)\lambda_1}\right]\right).$$

Lemma 18. *There exists four constants $(\alpha_1, \alpha_2, \alpha_3, \alpha)$ depend on \mathcal{T} , λ and λ_1 such that:*

$$(9.12) \quad A \leq \alpha_1 E_1 E_2.$$

$$(9.13) \quad B \leq \alpha_2 F_1 E_2.$$

$$(9.14) \quad C \leq \alpha_3 F_2 E_1.$$

$$(9.15) \quad D \leq \alpha_4 G_1 G_2.$$

We deduce from Lemma 18 that

$$A + B + C + D \leq \alpha(E_1 + F_1 + G_1)(E_2 + F_2 + G_2),$$

where $\alpha = \max_{i \in \{1, 2, 3, 4\}} \alpha_i$. The latter inequality concludes the proof of Proposition. \square

Proof of Lemma 18. Now, we will adapt the argument from the proof of Lemma 12 in [5]. We prove that there exists α_1 such that $A \leq \alpha_1 E_1 E_2$ and we use the same argument for the other inequalities.

First, by using condition 3.4, note that,

$$\mathbb{P}\left(U_{e^+} < \frac{1}{1 + \partial(e^+) \lambda_1}\right) = \frac{1}{1 + \partial(e^+) \lambda_1} \geq \frac{1}{1 + M \lambda_1}, \text{ we then obtain:}$$

$$(9.16) \quad \mathbb{P}\left(U_{e^+} < \frac{1}{1 + \partial(e^+) \lambda_1}\right) \leq (1 + M \lambda_1) \left[\mathbb{P}\left(U_{e^+} < \frac{1}{1 + \partial(e^+) \lambda_1}\right)\right]^2.$$

On the event $\left\{e \in \mathcal{C}_{CP}(\varrho), U_{e^+} < \frac{1}{1 + \partial(e^+) \lambda_1}\right\}$ we have $X_{T^{(e)}(e^+) + 1}^{(e)} = e^-$. We then define $\tilde{T}^{(e)}(e^+) := \inf \left\{n \geq T^{(e)}(e^+) + 1 : X_n^{(e)} = e^+\right\}$. We define the following quantities:

$$(9.17) \quad \begin{aligned} N(e) &= \left| \left\{ \tilde{T}^{(e)}(e^+) \leq n \leq T^{(e)}(\varrho) \circ \theta_{\tilde{T}^{(e)}(e^+)} : (X_n^{(e)}, X_{n+1}^{(e)}) = (e^+, e^-) \right\} \right|, \\ L(e) &= \sum_{j=0}^{N(e)-1} \frac{Y(e^+, e^-, j)}{r(e^+, e^-)}, \end{aligned}$$

where $|A|$ denotes the cardinality of a set A and θ is the canonical shift on trajectories. Note that $L(e)$ is the time consumed by the clocks attached to the oriented edge (e^+, e^-) before $\mathbf{X}^{(e)}$, $X^{(e_1)}$ or $X^{(e_2)}$ goes back to ϱ once it has returned e^+ after the time $T^{(e)}(e^+)$. Recall that these three extensions are coupled and thus the time $L(e)$ is the same for the three of them.

For $i \in \{1, 2\}$, recall that v_i is the vertex which is the offspring of e^+ lying the path from ϱ to e_i . Note that v_i could be equal to e_i^+ . We define for $i \in \{1, 2\}$:

$$(9.18) \quad \begin{aligned} N^*(e_i) &= \left| \left\{ \tilde{T}^{(e)}(e^+) \leq n \leq T^{(e_i)}(e_i^+) : (X_n^{[e^+, e_i^+]}, X_{n+1}^{[e^+, e_i^+]}) = (e^+, v_i) \right\} \right|, \\ L^*(e_i) &= \sum_{j=0}^{N^*(e_i)-1} \frac{Y(e^+, e^-, j)}{r(e^+, e^-)}. \end{aligned}$$

Here, $L^*(e_i)$, $i \in \{1, 2\}$, is the time consumed by the clocks attached to the oriented edge (e^+, v_i) before $\mathbf{X}^{(e_i)}$, or $\mathbf{X}^{[e^+, e_i^+]}$, hits e_i^+ .

Notice that the three quantities $L(e)$, $L^*(e_1)$ and $L^*(e_2)$ are independent, and we also have:

$$(9.19) \quad \begin{aligned} \mathbb{P} \left(e_1, e_2 \in \mathcal{C}_{CP}(\varrho) \mid e \in \mathcal{C}_{CP}(\varrho), U_{e^+} < \frac{1}{1 + \partial(e^+) \lambda_1} \right) \\ = \psi(e, \lambda) \mathbb{P} (L(e) > L^*(e_1) \vee L^*(e_2)). \end{aligned}$$

$$(9.20) \quad \mathbb{P} \left(e_1 \in \mathcal{C}_{CP}(\varrho) \mid e \in \mathcal{C}_{CP}(\varrho), U_{e^+} < \frac{1}{1 + \partial(e^+) \lambda_1} \right) = \psi(e, \lambda) \mathbb{P} (L(e) > L^*(e_1)).$$

$$(9.21) \quad \mathbb{P} \left(e_2 \in \mathcal{C}_{CP}(\varrho) \mid e \in \mathcal{C}_{CP}(\varrho), U_{e^+} < \frac{1}{1 + \partial(e^+) \lambda_1} \right) = \psi(e, \lambda) \mathbb{P} (L(e) > L^*(e_2)).$$

Now, the random variable $N(e)$ is simply a geometric random variable (counting the number of trials) with success probability $\lambda^{1-|e|} / \sum_{g \leq e} \lambda^{1-|g|}$. The random variable $N(e)$ is independent of the family $Y(e^+, e^-, \cdot)$. As $Y(e^+, e^-, j)$ are independent exponential random variable for $j \geq 0$, we then have that $L(e)$ is an exponential random variables with parameter

$$(9.22) \quad p := \frac{\lambda^{1-|e|}}{\sum_{g \leq e} \lambda^{1-|g|}} \times \lambda^{|e|-1} = \frac{1}{\sum_{g \leq e} \lambda^{1-|g|}}.$$

A priori, $L^*(e_1)$ and $L^*(e_2)$ are not exponential random variable, but they have a continuous distribution. Denote f_1 and f_2 respectively the densities of $L^*(e_1)$ and $L^*(e_2)$. Then, we have that

$$(9.23) \quad \begin{aligned} \mathbb{P} (L(e) > L^*(e_1) \vee L^*(e_2)) &= \int_0^{+\infty} \int_0^{+\infty} \int_{x_1 \vee x_2}^{+\infty} p e^{-pt} f_1(x_1) f_2(x_2) dt dx_1 dx_2 \\ &= \int_0^{+\infty} \int_0^{+\infty} e^{-p(x_1 \vee x_2)} f_1(x_1) f_2(x_2) dx_1 dx_2. \\ &\leq \int_0^{+\infty} \int_0^{+\infty} e^{-\frac{p}{2}(x_1+x_2)} f_1(x_1) f_2(x_2) dx_1 dx_2. \end{aligned}$$

Thus, one can write

$$(9.24) \quad \begin{aligned} \mathbb{P} (L(e) > L^*(e_1) \vee L^*(e_2)) \\ \leq \left(\int_0^{+\infty} e^{-px_1/2} f_1(x_1) dx_1 \right) \cdot \left(\int_0^{+\infty} e^{-px_2/2} f_2(x_2) dx_2 \right). \end{aligned}$$

Note that:

$$(9.25) \quad \int_0^{+\infty} e^{-px_1/2} f_1(x_1) dx_1 = \mathbb{P} \left(\tilde{L}(e) > L^*(e_1) \right),$$

where $\tilde{L}(e)$ is an exponential variable with parameter $p/2$. Note that, in view of (9.22), $\tilde{L}(e)$ has the same law as $L(e)$ when we replace the weight of an edge g' by $\lambda^{-|g'|+1}/2$ for $g' \leq e$ only, and keep the other weights the same.

For simplicity, for any $g \in E$, we set $w(g) = \lambda^{|g|-1}$. For $g \in E$ such that $e < g$, define the functions $\tilde{\psi}$ and $\tilde{\phi}$ in a similar way as ψ and ϕ , except that we replace the weight of an edge g' by $\lambda^{-|g'|+1}/2$ for $g' \leq e$ only, and keep the other weights the same, that is, for $g \in E$, $e < g$,

$$(9.26) \quad \tilde{\psi}(g, \lambda) = \frac{\sum_{g' < g} w(g')^{-1} + \sum_{g' \leq e} w(g')^{-1}}{\sum_{g' \leq g} w(g')^{-1} + \sum_{g' \leq e} w(g')^{-1}}.$$

$$(9.27) \quad \tilde{\phi}(g, \lambda_1, \lambda) = \frac{\lambda_1}{1 + \partial(g^-)\lambda_1} + \frac{1}{1 + \partial(g^-)\lambda_1} \tilde{\psi}(g, \lambda) \tilde{\psi}(g^{-1}, \lambda) + \frac{(\partial(g^-) - 1)\lambda_1}{1 + \partial(g^-)\lambda_1} \tilde{\psi}(g, \lambda).$$

We obtain:

$$(9.28) \quad \begin{aligned} \mathbb{P}(\tilde{L}(e) > L^*(e_1)) &= \prod_{e < g \leq e_1} \tilde{\phi}(g, \lambda_1, \lambda) = \prod_{e < g \leq e_1} \phi(g, \lambda_1, \lambda) \prod_{e < g \leq e_1} \left(\frac{\tilde{\phi}(g, \lambda_1, \lambda)}{\phi(g, \lambda_1, \lambda)} \right) \\ &= \mathbb{P}(L(e) > L^*(e_1)) \times \prod_{e < g \leq e_1} \left(\frac{\lambda_1 + \tilde{\psi}(g, \lambda) \tilde{\psi}(g^{-1}, \lambda) + (\partial(g^{-1}) - 1)\lambda_1 \tilde{\psi}(g, \lambda)}{\lambda_1 + \psi(g, \lambda) \psi(g^{-1}, \lambda) + (\partial(g^{-1}) - 1)\lambda_1 \psi(g, \lambda)} \right) \\ &= \mathbb{P}(L(e) > L^*(e_1)) \\ &\quad \times \prod_{e < g \leq e_1} \left(1 + \frac{\tilde{\psi}(g, \lambda) \tilde{\psi}(g^{-1}, \lambda) - \psi(g, \lambda) \psi(g^{-1}, \lambda) + (\partial(g^{-1}) - 1)\lambda_1 (\tilde{\psi}(g, \lambda) - \psi(g, \lambda))}{\lambda_1 + \psi(g, \lambda) \psi(g^{-1}, \lambda) + (\partial(g^{-1}) - 1)\lambda_1 \psi(g, \lambda)} \right). \end{aligned}$$

Now, we compute the product:

$$\begin{aligned} &\prod_{e < g \leq e_1} \left(1 + \frac{\tilde{\psi}(g, \lambda) \tilde{\psi}(g^{-1}, \lambda) - \psi(g, \lambda) \psi(g^{-1}, \lambda) + (\partial(g^{-1}) - 1)\lambda_1 (\tilde{\psi}(g, \lambda) - \psi(g, \lambda))}{\lambda_1 + \psi(g, \lambda) \psi(g^{-1}, \lambda) + (\partial(g^{-1}) - 1)\lambda_1 \psi(g, \lambda)} \right) \\ &\leq \prod_{e < g \leq e_1} \left(1 + \frac{\tilde{\psi}(g, \lambda) \tilde{\psi}(g^{-1}, \lambda) - \psi(g, \lambda) \psi(g^{-1}, \lambda) + (\partial(g^{-1}) - 1)\lambda_1 (\tilde{\psi}(g, \lambda) - \psi(g, \lambda))}{\lambda_1} \right) \\ &\leq \exp \left(\frac{1}{\lambda_1} \sum_{e < g \leq e_1} \left(\tilde{\psi}(g, \lambda) \tilde{\psi}(g^{-1}, \lambda) - \psi(g, \lambda) \psi(g^{-1}, \lambda) + (\partial(g^{-1}) - 1)\lambda_1 (\tilde{\psi}(g, \lambda) - \psi(g, \lambda)) \right) \right) \end{aligned}$$

Lemma 19. *There exists a constant $c = c(\lambda_1, \lambda)$ which do not depend on e , e_1 and e_2 , such that:*

$$(9.29) \quad \sum_{e < g \leq e_1} \left(\tilde{\psi}(g, \lambda) - \psi(g, \lambda) \right) \leq c.$$

On the other hand, by using Lemma 19, for any e and e_1 we have that

$$(9.30) \quad \sum_{e < g \leq e_1} \left(\tilde{\psi}(g, \lambda) \tilde{\psi}(g^{-1}, \lambda) - \psi(g, \lambda) \psi(g^{-1}, \lambda) \right) \leq 2c.$$

By using 9.30, Lemma 19 and condition (3.4), we obtain:

$$(9.31) \quad \begin{aligned} &\prod_{e < g \leq e_1} \left(1 + \frac{\tilde{\psi}(g, \lambda) \tilde{\psi}(g^{-1}, \lambda) - \psi(g, \lambda) \psi(g^{-1}, \lambda) + (\partial(g^{-1}) - 1)\lambda_1 (\tilde{\psi}(g, \lambda) - \psi(g, \lambda))}{\lambda_1 + \psi(g, \lambda) \psi(g^{-1}, \lambda) + (\partial(g^{-1}) - 1)\lambda_1 \psi(g, \lambda)} \right) \\ &\leq \exp \left(Mc + \frac{2c}{\lambda_1} \right). \end{aligned}$$

We have just proved that

$$(9.32) \quad \int_0^{+\infty} e^{-px_1/2} f_1(x_1) dx_1 \leq \exp\left(Mc + \frac{2c}{\lambda_1}\right) \times \mathbb{P}(e_1 \in \mathcal{C}_{CP}(\varrho) | e_1 \wedge e_2 \in \mathcal{C}_{CP}(\varrho)).$$

By doing a very similar computation, one can prove that

$$(9.33) \quad \int_0^{+\infty} e^{-px_2/2} f_1(x_2) dx_2 \leq \exp\left(Mc + \frac{2c}{\lambda_1}\right) \times \mathbb{P}(e_2 \in \mathcal{C}_{CP}(\varrho) | e_1 \wedge e_2 \in \mathcal{C}_{CP}(\varrho)).$$

Moreover, we have

$$(9.34) \quad \psi(e, \lambda) \geq \frac{\lambda}{1 + \lambda}.$$

The conclusion (9.4) follows by using (9.16), (9.24), (9.34), together with (9.32) and (9.33). \square

It remains to prove Lemma 19.

Proof of Lemma 19. By a simple computation, for any $e < g \leq e_1$,

$$(9.35) \quad \tilde{\psi}(g, \lambda) - \psi(g, \lambda) = \frac{\left(\sum_{g' \leq e} w(g')^{-1}\right) w(g)^{-1}}{\left(\sum_{g' \leq g} w(g')^{-1} + \sum_{g' \leq e} w(g')^{-1}\right) \left(\sum_{g' \leq g} w(g')^{-1}\right)}.$$

We will proceed by distinguishing three cases.

Case I: $\lambda < 1$.

By (9.35), we have that

$$(9.36) \quad \tilde{\psi}(g, \lambda) - \psi(g, \lambda) = \frac{\left(1 - \frac{1}{\lambda^{|e|}}\right) \frac{1}{\lambda^{|g|-1}}}{\left(1 - \frac{1}{\lambda^{|g|}} + 1 - \frac{1}{\lambda^{|e|}}\right) \left(1 - \frac{1}{\lambda^{|g|}}\right)} \times \left(1 - \frac{1}{\lambda}\right).$$

Hence, there exists a constants c_1 such that

$$(9.37) \quad 0 \leq \tilde{\psi}(g, \lambda) - \psi(g, \lambda) \leq c_1 \lambda^{|g|-|e|}.$$

Therefore we obtain

$$(9.38) \quad \sum_{e < g \leq e_1} \left(\tilde{\psi}(g, \lambda) - \psi(g, \lambda)\right) \leq c_1 \sum_{e < g \leq e_1} \lambda^{|g|-|e|} \leq c_1 \sum_{i \geq 0} \lambda^i < \infty.$$

Case II: $\lambda = 1$.

By (9.35), we have that

$$(9.39) \quad \tilde{\psi}(g, \lambda) - \psi(g, \lambda) = \frac{|e|}{|g|(|g| + |e|)}.$$

Therefore we obtain

$$(9.40) \quad \begin{aligned} \sum_{e < g \leq e_1} \left(\tilde{\psi}(g, \lambda) - \psi(g, \lambda)\right) &\leq \sum_{n \geq |e|} \left(\frac{|e|}{n(n + |e|)}\right) \leq \sum_{n \geq |e|} \left(\frac{1}{n} - \frac{1}{n + |e|}\right) \\ &\leq \sum_{n=|e|}^{2|e|-1} \frac{1}{n}. \end{aligned}$$

On the other hand, we have:

$$(9.41) \quad \lim_{n \rightarrow \infty} \left(\sum_{k=n}^{2n-1} \frac{1}{k} \right) = \lim_{k \rightarrow \infty} \left(\sum_{k=0}^{n-1} \frac{1}{n+k} \right) = \lim_{k \rightarrow \infty} \left(\frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{1+k/n} \right) = \int_0^1 \frac{dx}{1+x}.$$

We use (9.40) and (9.41) to obtain the result.

Case III: $\lambda > 1$.

By (9.35), we have that

$$(9.42) \quad \tilde{\psi}(g, \lambda) - \psi(g, \lambda) = \frac{\left(1 - \frac{1}{\lambda^{|e|}}\right) \frac{1}{\lambda^{|g|-1}}}{\left(1 - \frac{1}{\lambda^{|g|}} + 1 - \frac{1}{\lambda^{|e|}}\right) \left(1 - \frac{1}{\lambda^{|g|}}\right)} \times \left(1 - \frac{1}{\lambda}\right).$$

Hence, there exists a constants c_2 such that

$$(9.43) \quad 0 \leq \tilde{\psi}(g, \lambda) - \psi(g, \lambda) \leq \frac{c_2}{\lambda^{|g|}}.$$

Therefore we obtain

$$(9.44) \quad \sum_{e < g \leq e_1} \left(\tilde{\psi}(g, \lambda) - \psi(g, \lambda) \right) \leq c_2 \sum_{e < g \leq e_1} \frac{1}{\lambda^{|g|}} \leq c_2 \sum_{i \geq 0} \left(\frac{1}{\lambda} \right)^i < \infty.$$

□

9.2. Transience in Theorem 4: The case $RT(\mathcal{T}, \mathbf{X}) > 1$.

Proposition 20. *If $RT(\mathcal{T}, \mathbf{X}) > 1$ and if (3.4) is satisfied then \mathbf{X} is transient.*

Proof. The proof is now easy, we can follow line by line the Appendix A.2 of [4].

□

10. PROOF OF THEOREM 6

This section is independent with the previous sections. In this section, we prove a criterion which can apply to the critical M -digging random walk on superperiodic trees. We will use the Rubin's construction (resp. the definition of $\mathcal{C}(\varrho)$, $\mathcal{C}_{CP}(\varrho)$) from section 7 (resp. section 8.1) of [4]. We will allow ourselves to omit these definitions and refer the readers to [4] for more details.

The main idea for the proof of Theorem 6 is that the number of surviving rays of the percolation $\mathcal{C}_{CP}(\varrho)$ almost surely is either zero or infinite. This property was proved in the case of *Bernoulli percolation* (see [8] proposition 5.27) or *target percolation* (see [10], lemma 4.2). The main difficulty that we have to face is that the FKG inequality is not true for our percolation.

10.1. Some definitions. Let $\lambda > 0$, $M \in \mathbb{N}$ and \mathcal{T} be an infinite, locally finite and rooted tree. For each $v \in V(\mathcal{T})$, recall the definition of subtree \mathcal{T}^v of \mathcal{T} from Section 2.1. Let $\mathbf{X}^{v, \lambda}$ be the M -digging random walk on \mathcal{T}^v . We say that \mathcal{T} is *uniformly transient* if for any λ such that the M -digging random walk on \mathcal{T} with parameter λ is transient (i.e. $\mathbf{X}^{e, \lambda}$ is transient),

$$(10.1) \quad \exists \alpha_\lambda > 0, \forall v \in V(\mathcal{T}), \mathbb{P}(\forall n > 0, X_n^{v, \lambda} \neq v) \geq \alpha_\lambda.$$

It is called *weakly uniformly transient* if there exists a sequence of finite pairwise disjoint π_n such that

$$(10.2) \quad \exists \alpha_\lambda > 0, \forall v \in \bigcup_n V(\pi_n), \mathbb{P}(\forall n > 0, X_n^{v, \lambda} \neq v) \geq \alpha_\lambda$$

where $V(\pi_n) = \{e^- : e \in \pi_n\}$.

Remark 21. • If \mathcal{T} is uniformly transient, then \mathcal{T} is also weakly uniformly transient, but the reverse is not always true.

- The superperiodic trees are uniformly transient.

An infinite self-avoiding path starting at ϱ is called a *ray*. The set of all rays, denoted by $\partial\mathcal{T}$, is called the *boundary* of \mathcal{T} . Let $\phi : \mathbb{Z}^+ \rightarrow \mathbb{R}$ be a decreasing positive function with $\phi(n) \rightarrow 0$ as $n \rightarrow \infty$. The *Hausdorff measure* of \mathcal{T} in gauge ϕ is

$$\liminf_{\Pi} \sum_{v \in \Pi} \phi(|v|),$$

where the \liminf is taken over Π such that the distance from ϱ to the nearest vertex in Π goes to infinity. We say that \mathcal{T} has σ -finite Hausdorff measure in gauge ϕ if $\partial\mathcal{T}$ is the union of countably many subsets with finite Hausdorff measure in gauge ϕ .

Finally, If λ is such that the M -digging random walk X with parameter λ on \mathcal{T} is transient, on the event $\{T(\varrho) = \infty\}$, its path determines an infinite branch in \mathcal{T} , which can be seen as a random ray ω^∞ , and call it the *limit walk* of X . Equivalently, on the event $\{T(\varrho) = \infty\}$, we define the limit walk as follows: For any $k \geq 1$,

$$(10.3) \quad \omega^\infty(k) = v \iff v \in \mathcal{T}_k \text{ and } \exists n_0, \forall n > n_0 : X_n \in \mathcal{T}^v.$$

Note that $\mathbb{P}(\omega^\infty(0) = \varrho) = 1$. For any $k \geq 1$, we call the k -first steps of ω^∞ is $(\omega^\infty(0), \dots, \omega^\infty(k))$, denoted by $\omega_{|[0, n]}^\infty$.

10.2. Proof of Theorem 6.

We begin with the following proposition:

Proposition 22. *Let \mathbf{X} be a M -digging random walk with parameter λ_c on an uniformly transient tree \mathcal{T} and recall the definition of \mathcal{C}_{CP} from \mathbf{X} as in ([4], Section 7). Consider the percolation induced by \mathcal{C}_{CP} and let $\phi(n) = \mathbb{P}(\varrho \leftrightarrow v)$ for $v \in \mathcal{T}_n$.*

- (1) *Almost surely, the number of surviving rays is either zero or infinite.*
- (2) *If $\partial\mathcal{T}$ has σ -finite Hausdorff measure in the gauge $\{\phi(n)\}$, then $\mathbb{P}(\varrho \leftrightarrow \infty) = 0$. In particular, \mathbf{X} is recurrent.*

The overall strategy for the proof of Proposition 22 is as follows. First, if \mathbf{X} is recurrent, then the percolation induced by \mathcal{C}_{CP} almost surely have no surviving ray. Next, assume that \mathbf{X} is transient. On the event $\{T(\varrho) = \infty\}$, the limit walk ω^∞ is a surviving ray of $\mathcal{C}_{CP}(\varrho)$. Given $n \in \mathbb{N}$ and conditioning on $\omega_{|[0, n]}^\infty$, by using the Rubin's construction and the definition of uniformly transient, we prove that there exists a surviving ray in $\mathcal{T}^{\omega^\infty(i)}$ with probability larger than a constant c which do not depend on i and ω^∞ (see Figure 2). The following basic lemma is necessary:

Lemma 23. *Let $\lambda > 0$ and \mathcal{T} be an infinite, locally finite and rooted tree. Let $\overline{M} := (m_v, v \in V(\mathcal{T}))$ be a family of non-negative integers. Denote by \mathbf{X} the M -digging random walk with parameter λ and \mathbf{Y} the \overline{M} -digging random walk associated with the inhomogeneous initial number of cookies \overline{M} with parameter λ (see [4], section 2.3.2 for more details on the definition of \overline{M} -digging random walk). Denote by $T^{\mathbf{X}}(\varrho)$ (resp. $T^{\mathbf{Y}}(\varrho)$) the return time of \mathbf{X} (resp. \mathbf{Y}) to ϱ . Assume that $m(v) \leq M$ for all $v \in V(\mathcal{T})$, we then have*

$$(10.4) \quad \mathbb{P}(T^{\mathbf{X}}(\varrho) < \infty) \leq \mathbb{P}(T^{\mathbf{Y}}(\varrho) < \infty).$$

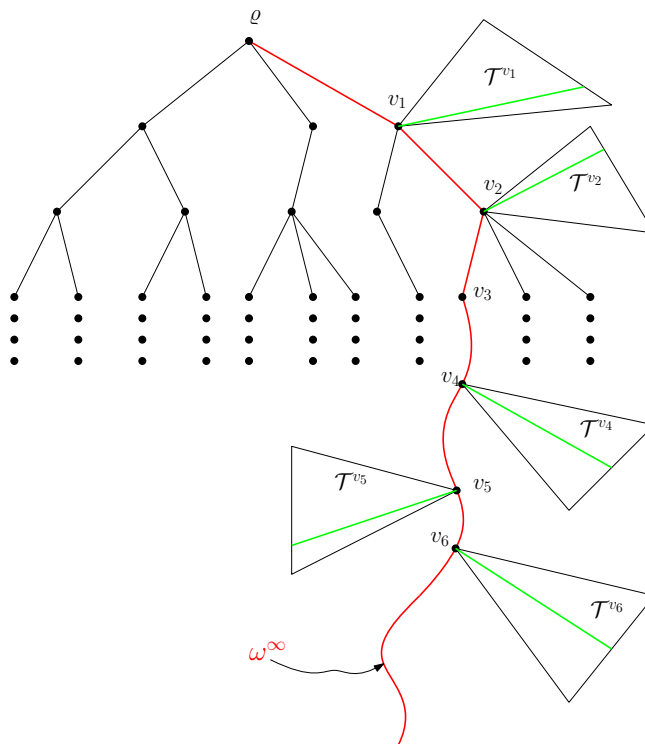


FIGURE 2. The proof’s idea of Proposition 22. The limit walk ω^∞ is in red. Conditioning on the event $\{\omega^\infty(0) = \varrho, \omega^\infty(1) = v_1, \dots, \omega^\infty(6) = v_6\}$ and denote by ℓ the last time the critical M -digging random walk \mathbf{X} on \mathcal{T} visits v_6 . For each $1 \leq i \leq 6$, running the walk $\mathbf{X}^{v_i, \lambda_c}$ on \mathcal{T}^{v_i} . The property of uniformly transient implies that there exists a surviving ray (in green) in \mathcal{T}^{v_i} with probability is larger than a constant which do not depend on i .

Proof. The proof is simple, therefore it is omitted. □

Proof of Proposition 22. Let \mathcal{A}_k denote the event that exactly k rays survive and assume that

$$(10.5) \quad \mathbb{P}(\mathcal{A}_k) > 0,$$

Hence,

$$(10.6) \quad P(|\mathcal{C}_{CP}(\varrho)| = \infty) > 0.$$

By (10.6) and Lemma 22 in [4], we have that:

$$(10.7) \quad \mathbb{P}(T(\varrho) = \infty) > 0,$$

and therefore \mathbf{X} is transient.

On the event $\{T(\varrho) = \infty\}$, the limit walk ω^∞ of \mathbf{X} is well defined and it is a surviving ray. Let n be a positive integer and $\gamma := (\gamma_0 = \varrho, \gamma_1 = v_1, \dots, \gamma_n = v_n)$ be a path of length n of \mathcal{T} . Denote by $\mathcal{B}_{n, \gamma}$ the following event:

$$(10.8) \quad \mathcal{B}_{n, \gamma} := \{\omega|_{[0, n]}^\infty = \gamma\}.$$

For any $1 \leq k \leq n$, define a sub-tree \mathcal{T}^{v_i} of \mathcal{T} in the following way (see Figure 2).

- The root of \mathcal{T}^{v_i} is the vertex v_i .
- If $\partial(v_i) < 2$ then \mathcal{T}^{v_i} is a tree with a single vertex v_i : for example, \mathcal{T}^{v_3} in Figure 2.
- If $\partial(\gamma_i) \geq 2$, choose one of its children which is different to v_{i+1} , denoted by v . We then set:

$$\begin{cases} (\mathcal{T}^{v_i})_1 = \{v\} \\ (\mathcal{T}^{v_i})^v = \mathcal{T}^v \end{cases}$$

Note that for every pair $(i, j) \in [1, n]^2$, we have $V(\mathcal{T}^{v_i}) \cap V(\mathcal{T}^{v_j}) = \emptyset$.

Now, conditioning on the event $\mathcal{B}_{n, \gamma}$. Let ℓ be the last time \mathbf{X} visits v_n , i.e.

$$(10.9) \quad \ell := \sup\{k > 0 : X_k = v_n\}.$$

By the definition of limit walk, ℓ is finite on the event $\mathcal{B}_{n, \gamma}$. For each $i \in [1, n]$ and for all $v \in V(\mathcal{T}^{v_i})$, denote by $m^i(v)$ the remaining number of cookies at v after time ℓ , i.e.

$$(10.10) \quad m^i(v) := M - \#\{k \leq \ell : X_k = v\}.$$

By using the extensions introduced in ([4], Section 7), the next steps on the tree \mathcal{T}^{v_i} are given by the digging random walk associated with the inhomogeneous initial number of cookies $(m^i(v), v \in V(\mathcal{T}^{v_i}))$ and the same parameter λ_c as \mathbf{X} , denoted by $\mathbf{X}^{v_i, m^i, \lambda_c}$ (see [4], section 2.3.2 for more details on the definition of $\mathbf{X}^{v_i, m^i, \lambda_c}$). Denote by T^{v_i, m^i, λ_c} the return time of $\mathbf{X}^{v_i, m^i, \lambda_c}$ to the root v_i of \mathcal{T}^{v_i} . By the definition of uniformly transient and Lemma 23, there exists a constant $c > 0$ which do not depend on n and γ such that for any i ,

$$(10.11) \quad \mathbb{P}\left(T^{v_i, m^i, \lambda_c} < \infty\right) > c.$$

On the event $\{T^{v_i, m^i, \lambda_c} < \infty\}$, note that \mathcal{C}_{CP} contains a surviving ray in \mathcal{T}^{v_i} . By (10.11), we have

$$(10.12) \quad \mathbb{P}(\mathcal{A}_k | \mathcal{B}_{n, \gamma}) \leq \binom{n}{k} (1-c)^{n-k}$$

On the other hand, we have $\mathcal{A}_k \subset \bigcup_{\gamma: |\gamma|=n} \mathcal{B}_{n, \gamma}$, therefore by (10.12) we obtain:

$$(10.13) \quad \mathbb{P}(\mathcal{A}_k) = \sum_{\gamma: |\gamma|=n} \mathbb{P}(\mathcal{A}_k | \mathcal{B}_{n, \gamma}) \times \mathbb{P}(\mathcal{B}_{n, \gamma}) \leq \left(\sum_{i=1}^k \binom{n}{i} \right) (1-c)^n \underbrace{\sum_{\gamma: |\gamma|=n} \mathbb{P}(\mathcal{B}_{n, \gamma})}_{\leq 1} \leq \left(\sum_{i=1}^k \binom{n}{i} \right) (1-c)^n.$$

Since 10.13 holds for any n then we obtain the following contradiction

$$(10.14) \quad \mathbb{P}(\mathcal{A}_k) = 0.$$

For part (2), the proof is similar to part (ii), Lemma 4.2 in [10]. \square

In the same method as in the proof of Proposition 22, we can prove the slightly stronger result (the proof of which we omit):

Proposition 24. *Let \mathbf{X} be a M -digging random walk with parameter λ_c on a weakly uniformly transient tree \mathcal{T} and recall the definition of \mathcal{C}_{CP} from \mathbf{X} as in ([4], Section 7). Consider the percolation induced by \mathcal{C}_{CP} and let $\phi(n) = \mathbb{P}(\varrho \leftrightarrow v)$ for $v \in \mathcal{T}_n$.*

- (1) *With probability one, the number of surviving rays is either zero or infinite.*

- (2) If $\partial\mathcal{T}$ has σ -finite Hausdorff measure in the gauge $\{\phi(n)\}$, then $\mathbb{P}(\varrho \leftrightarrow \infty) = 0$. In particular, \mathbf{X} is recurrent.

The following corollary is an immediate consequence of Proposition 24.

Corollary 25. *Let $M \in \mathbb{N}$ and \mathcal{T} be a weakly uniformly transient tree such that $\partial\mathcal{T}$ has σ -finite Hausdorff measure in the gauge $\{\phi(n)\} = \left(\frac{1}{br(\mathcal{T})}\right)^n$ if $br(\mathcal{T}) > 1$ and $\{\phi(n)\} = \frac{1}{n^{M+1}}$ if $br(\mathcal{T}) = 1$. Then the critical M -digging random walk on \mathcal{T} is recurrent.*

Proposition 26. *Let $M \in \mathbb{N}^*$ and \mathcal{T} be a superperiodic tree whose upper-growth rate is finite. The critical M -digging random walk on \mathcal{T} is recurrent.*

Proof. This is a consequence of Corollary 25 and Theorem 3. □

Remark 27. *If $M = 0$, then M -DRW $_\lambda$ is the biased random walk with parameter λ . The recurrence of critical biased random walk on \mathcal{T} is a consequence of Theorem 3 and Nash-Williams criterion (see [8] or [9]).*

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