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FUZZY GENERAL LINEAR METHODS

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Abstract. This paper concerns with the developing the most general schemes so-called Fuzzy General Linear Methods (FGLM) for solving fuzzy differential equations. The general linear methods (GLM) for ordinary differential equations are the middle state of two extreme extensions (linear multistep and Runge-Kutta methods) of the one step Euler method. In this paper we develop the FGLM framework of the Adams schemes for solving fuzzy differential equations under the strongly generalized differentiability. The stability, consistency and convergent results will be addressed. The numerical results and the order of accuracy is illustrated to show the efficiency and accuracy of the novel scheme.

General linear methods, Adams methods, Strongly generalized differentiability, Generalized fuzzy derivative, Fuzzy differential equations.
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1. Introduction

Many problems in science and engineering have some uncertainty in their nature and fuzzy differential equations are appropriate tools for modeling of such problems [23]. The interpretation of a fuzzy differential equation in the sense of generalized differentiability allows to fuzzifying the appropriate numerical methods of ordinary differential equations to fuzzy differential equations. The Hukuhara derivative with the extension principle or differential inclusions have some disadvantages. The main drawback is that the solutions obtained in this setting have increasing length of their supports [8, 3]. Many authors have been generalized the traditional methods such as Euler's method, Adams-Bashforth methods, predictor-corrector method, Runge-Kutta method,...[5, 10, 12, 14, 15, 16, 18, 19] to fuzzy differential and fuzzy initial value problems. However, they use Hukuhara differentiability and fuzify the numerical method using extension principle or other methods [23]. Under the concept of strongly generalized differentiability there exist fuzzy derivative for a large class of fuzzy-number-valued functions [2, 3]. Another advantage is that there exist two local solutions, so-called, (i)-differentiable and (ii)-differentiable solutions. According to the nature of the initial value problem we can choose the best meaningful practical solution. In this paper we develop the GLM schemes based on strongly generalized differentiability concept. Notion of a fuzzy derivative first introduced by Chang and Zadeh [7] and Dubo and Prade [9] introduced its extension. Stefanini [20, 21] introduced the fuzzy gH-difference and Bede and Stefanini[?] defined and studied new generalization of the differentiability for fuzzy-number-valued functions. The aim of this paper is to develop the GLMs for fuzzy differential equations and study their consistency, stability and convergence. In this paper, under the strongly generalized differentiability we develop a well-known Adams-Bashforth methods in the framework of a general linear method. This starting step will motivate us to develop the arbitrary classes of GLMs with demanded properties in forthcoming research.

Let us denote by $\mathbb{R}_{\mathcal{F}}$ the class of fuzzy numbers, i.e. normal, convex, upper semicontinuous and compactly supported fuzzy subsets of the real numbers. The fuzzy initial value problem is defined as follow:

$$\begin{aligned} y'(t) &= f(t, y(t)), \quad t \in [t_0, T], \\ y(t_0) &= y_0, \end{aligned} \tag{1.1}$$

where, $f : [t_0, T] \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ and $y_0 \in \mathbb{R}_{\mathcal{F}}$.

Here, we explain the GLM for ordinary IVP (1.1) and in next sections we will discuss on the development of GLM for FIVP. Burage and Butcher [6] have presented a standard representation of a GLM in terms of four matrices. These methods were formulated as follows:

$$\begin{aligned} Y &= hAf(Y) + Uy^{[n-1]}, \\ y^{[n]} &= hBf(Y) + Vy^{[n-1]}. \end{aligned} \tag{1.2}$$

where $y^{[n-1]}$ and $y^{[n]}$ are input and output approximations, respectively, and

$$A \in \mathbb{R}^{s\times s}, \quad U \in \mathbb{R}^{s\times r}, \quad B \in \mathbb{R}^{r\times s}, \quad V \in \mathbb{R}^{r\times r}.$$

In this paper we use the fuzzy interpolation for constructing Adams-Bashforth schemes in the general linear methods framework. The organization of this paper is as follow: In section 2 we present the preliminaries from GLM and fuzzy calculus. In section 3 we apply the GLM form of linear multistep methods to solve the fuzzy differential equations and in section 5 numerical results are given.

2. Preliminaries

In this section we present the required concepts from general linear methods and also we shortly review the required definitions form fuzzy calculus, as given in [1]. We will give the main idea of the paper for an important subclass of LMMs, the so-called Adams methods, in GLM framework.

Definition 2.1. *Let $u, v \in \mathbb{R}_{\mathcal{F}}$, the Hukuhara difference (H-difference $\ominus_H$) of u and v is defined by*

$$u \ominus v = w \Longleftrightarrow u = v + w.$$

Where $w \in \mathbb{R}_{\mathcal{F}}$ is called the H-difference of u and v. If H-difference $u \ominus v$ exists, then $[u \ominus v]_r = [u_r^- - v_r^-, u_r^+ - v_r^+]$. The Hukuhara derivative for a fuzzy function was introduced by Puri and Relescu [17]. From Kaleva [13] and Diamond [8], it follows that a Hukuhara differentiable function has increasing length of its support interval. So the Hukuhara difference rarely exists and to overcome this situation strongly generalized differentiability of fuzzy-number-valued functions was introduced and studied by Bede-Gal [3]. Thus, in this case a differentiable function may have the property that the support has increasing or decreasing length.

Definition 2.2. *Let $f : (a, b) \to \mathbb{R}_{\mathcal{F}}$ and $x_0 \in (a, b)$. We say that f is strongly generalized differentiable at x_0, if there exists an element $f'(x_0) \in \mathbb{R}_{\mathcal{F}}$, such that*

(i) *for each* $h > 0$ *sufficiently close to 0, the H-differences* $f(x_0 + h) \ominus f(x_0)$ *and* $f(x_0) \ominus f(x_0 - h)$ *exist and*

$$\lim_{h \to 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \to 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0),$$

or

(ii) *for each* $h > 0$ *sufficiently close to 0, the H-differences* $f(x_0) \ominus f(x_0 + h)$ *and* $f(x_0 - h) \ominus f(x_0)$ *exist and*

$$\lim_{h \to 0} \frac{f(x_0) \ominus f(x_0 + h)}{(-h)} = \lim_{h \to 0} \frac{f(x_0 - h) \ominus f(x_0)}{(-h)} = f'(x_0).$$

Let $f : (a, b) \to \mathbb{R}_{\mathcal{F}}$, we say that f is (i)-differentiable and (ii)-differentiable on (a, b) if f is differentiable in the sense (i) and (ii) of Definition 2.2, respectively. There is also two other differentiability cases - (iii) and (iv) - differentiability - that in these cases there is no existence theorems and we do not discuss them here.

Bede in [5] proved that under certain conditions the fuzzy initial value problem (1.1) has a unique solution and is equivalent to the system of ODEs

$$\begin{cases} (y_r^-)' = f_r^-(t, y_r^-, y_r^+) \\ (y_r^+)' = f_r^+(t, y_r^-, y_r^+) \end{cases}, r \in [0, 1]$$

with respect to H-differentiability.

In this interpretation solutions of a fuzzy differential equation have always an increasing length of its support interval. So a fuzzy dynamical system will have more uncertain behavior in time and it does not allow to have a periodic solutions. Thus, for solve FDEs the different ideas and methods have been investigated. The second interpretation was based on Zadeh's extension principle defined in [22]. Consider the classical ODE $x' = f(t, x, a)$, $x(t_0) = x_0 \in \mathbb{R}$ where $a \in \mathbb{R}$ is a parameter. By using Zadeh's extension principle on the classical solution, we obtain a solution of the FIVP. The third interpretation have been developed based on generalized fuzzy derivative. In this work we will work with interpretation based on strongly generalized differentiability. Fuzzy differential equations based on generalized H-differentiability were investigated by Bede-Gal in [3] and more general results were proposed in Bede-Gal [4]. According to the assumptions of the Theorem 9.11 in [1], the fuzzy initial value problem (1.2) is equivalent to the union of the ODEs:

$$\begin{cases} (y_\alpha^-)'(t) = f_\alpha^-(t, y_\alpha^-(t), y_\alpha^+(t)) \\ (y_\alpha^+)'(t) = f_\alpha^+(t, y_\alpha^-(t), y_\alpha^+(t)), \\ (y_\alpha^-)(t_0) = (y_0)_\alpha^-, \quad (y_\alpha^+)(t_0) = (y_0)_\alpha^+. \end{cases} \qquad \alpha \in [0, 1] \tag{2.1}$$

and

$$\begin{cases} (y_\alpha^-)'(t) = f_\alpha^+(t, y_\alpha^-(t), y_\alpha^+(t)) \\ (y_\alpha^+)'(t) = f_\alpha^-(t, y_\alpha^-(t), y_\alpha^+(t)), \\ (y_\alpha^-)(t_0) = (y_0)_\alpha^-, \quad (y_\alpha^+)(t_0) = (y_0)_\alpha^+. \end{cases} \qquad \alpha \in [0, 1] \tag{2.2}$$

For triangular input data we have the same systems (2.1) and (2.2) with an extra equation $(y_\alpha^1)'(t) = f_\alpha^1(t, y_\alpha^-(t), y_\alpha^1(t), y_\alpha^+(t))$ where $f = (f^-, f^1, f^+)$ (see Theorem 9.12 in [1]).

A linear multistep method is defined by the first characteristic polynomial $\rho(r) = \sum_{j=0}^{k} \alpha_j r^j$ and the second characteristic polynomial $\sigma(r) = \sum_{j=0}^{k} \beta_j r^j$ as follow

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}, \tag{2.3}$$

where $a = t_n \leq t_{n+1} \leq \cdots \leq t_N = b$, $h = \frac{b-a}{N} = t_{n+k} - t_{n+k-1}$, $f_{n+j} = f(t_{n+j}, y_{n+j})$ and α_j and $\beta_j, j = 0, 1, \cdots, k$ are constants. In this scheme we can evaluate an approximate solution y_{n+k} for the exact value $y(x_{n+k})$ using the starting values $y_0, y_1, \ldots, y_{n+k-1}$. The Adams schemes are characterized by their first characteristic polynomial as $\rho(r) = r^k - r^{k-1}$. Therefor, we have

$$y_{n+k} = y_{n+k-1} + h \sum_{j=0}^{k} \beta_j f_{n+j}, \tag{2.4}$$

In (2.4) the case $\beta_k = 0$ means that the method is explicit and otherwise the method is implicit. The stability issue of LMMs are characterized by the root condition for the first characteristic polynomial $\rho(r)$, that means the roots $r_s, s = 1, 2, \ldots, k$ of $\rho(r)$ satisfy $|r_s| \leq 1$ and the roots with $|r_s| = 1$ are simple [11]. The zero-stability of an LMM and correspondingly the its GLM form depends on that the first characteristic polynomial $\rho(r)$ or the minimal polynomial of the matrix V satisfies the root condition.

3. A GLM scheme with strongly generalized differentiability

In this section we present the derivation of a GLM based on linear $k-$step Adams schemes for solving fuzzy initial value problem under strongly generalized differentiability. Assume that for an equally spaced points $0 = t_0 < t_1 < \cdot < t_N = T$ at t_n the exact solutions are indicated by $\mathbf{Y}_1(t_n; r) = [\mathbf{Y}_1^-(t_n; r), \mathbf{Y}_1^+(t_n; r)]$ and $\mathbf{Y}_2(t_n; r) = [\mathbf{Y}_2^-(t_n; r), \mathbf{Y}_2^+(t_n; r)]$ under (i) and (ii)-differentiability, respectively. Also assume that $y_1(t_n; r) = [y_1^-(t_n; r), y_1^+(t_n; r)]$ and $y_2(t_n; r) = [y_2^-(t_n; r), y_2^+(t_n; r)]$ are approximate solutions at t_n under (i) and (ii)-differentiability, respectively.

The k-step Adams methods under Hukuhara or (i)-differentiability can be written as:

$$\begin{aligned} y_{1r}^-(t_{n+k}; r) &= y_{1r}^-(t_{n+k-1}; r) + h \textstyle\sum_{j=0}^{k} \beta_j f^-(t_{n+j}, y_{1r}(t_{n+j}; r)), \\ y_{1r}^+(t_{n+k}; r) &= y_{1r}^+(t_{n+k-1}; r) + h \textstyle\sum_{j=0}^{k} \beta_j f^+(t_{n+j}, y_{1r}(t_{n+j}; r)), \end{aligned} \tag{3.1}$$

and under (ii)-differentiability can be written as:

$$\begin{aligned} y_{2r}^-(t_{n+k}; r) &= y_{2r}^-(t_{n+k-1}; r) + h \textstyle\sum_{j=0}^{k} \beta_j f^+(t_{n+j}, y_{2r}(t_{n+j}; r)), \\ y_{2r}^+(t_{n+k}; r) &= y_{2r}^+(t_{n+k-1}; r) + h \textstyle\sum_{j=0}^{k} \beta_j f^-(t_{n+j}, y_{2r}(t_{n+j}; r)), \end{aligned} \tag{3.2}$$

The Adams schemes are k-step methods (2.4) with $\rho(r) = r^k - r^{k-1}$. In this setting we can find their corresponding general linear method framework. In GLM representation we should first determine the input and output vectors and then find the corresponding matrices. For this end we consider the input and output approximation of general linear methods as follow

$$y^{[n-1]} = \begin{pmatrix} y_{n+k-1} \\ hf_{n+k-1} \\ hf_{n+k-2} \\ \vdots \\ hf_{n+1} \\ hf_n \end{pmatrix}, \qquad y^{[n]} = \begin{pmatrix} y_{n+k} \\ hf_{n+k} \\ hf_{n+k-1} \\ \vdots \\ hf_{n+2} \\ hf_{n+1} \end{pmatrix}.$$

Similarly, a linear k-steps methods under strongly generalized differentiability (3.1) and (3.2) can be representation in the form of general linear methods. For this representation the input vectors for the GLM form of (3.1) and (3.2) are indicated by $y_{1r}^{[n-1]} = \left[y_{1r}^{-[n-1]}, y_{1r}^{+[n-1]}\right]$ and $y_{2r}^{[n-1]} = \left[y_{2r}^{-[n-1]}, y_{2r}^{+[n-1]}\right]$ under (i) and (ii)-differentiability, respectively. Corresponding to the input vectors, the output vectors are indicated by $y_{1r}^{[n]} = \left[y_{1r}^{-[n]}, y_{1r}^{+[n]}\right]$ and $y_{2r}^{[n]} = \left[y_{2r}^{-[n]}, y_{2r}^{+[n]}\right]$ under (i) and (ii)-differentiability, respectively. Now, we consider the input approximation of general linear methods in terms of (i)-differentiability as:

$$y_{1r}^{-[n-1]} = \begin{pmatrix} y^-_{n+k-1_{1r}} \\ hf^-_{n+k-1_{1r}} \\ hf^-_{n+k-2_{1r}} \\ \vdots \\ hf^-_{n+1_{1r}} \\ hf^-_{n_{1r}} \end{pmatrix}, \qquad y_{1r}^{+[n-1]} = \begin{pmatrix} y^+_{n+k-1_{1r}} \\ hf^+_{n+k-1_{1r}} \\ hf^+_{n+k-2_{1r}} \\ \vdots \\ hf^+_{n+1_{1r}} \\ hf^+_{n_{1r}} \end{pmatrix}, \tag{3.3}$$

and under the (ii)-differentiability we obtain the following input vectors:

$$y_{2r}^{-[n-1]} = \begin{pmatrix} y^-_{n+k-1_{2r}} \\ hf^+_{n+k-1_{2r}} \\ hf^+_{n+k-2_{2r}} \\ \vdots \\ hf^+_{n+1_{2r}} \\ hf^+_{n_{2r}} \end{pmatrix}, \qquad y_{1r}^{+[n-1]} = \begin{pmatrix} y^+_{n+k-1_{2r}} \\ hf^-_{n+k-1_{2r}} \\ hf^-_{n+k-2_{2r}} \\ \vdots \\ hf^-_{n+1_{2r}} \\ hf^-_{n_{2r}} \end{pmatrix}, \tag{3.4}$$

By considering the above input vectors, the fuzzy general linear methods form of (3.1) and (3.2) can be formulated in case of (i)-differentiability as:

$$\left(\frac{Y_{1r}}{y_{1r}^{[n]}}\right) = \left(\begin{array}{c|c} \mathrm{A} & \mathrm{U} \\ \hline \mathrm{B} & \mathrm{V} \end{array}\right) \left(\frac{hf_{1r}(Y_{1r})}{y_{1r}^{[n-1]}}\right), \tag{3.5}$$

and in case of (ii)-differentiability we have:

$$\left(\frac{Y_{2r}}{y_{2r}^{[n]}}\right) = \left(\begin{array}{c|c} \mathrm{A} & \mathrm{U} \\ \hline \mathrm{B} & \mathrm{V} \end{array}\right) \left(\frac{hf_{2r}(Y_{2r})}{y_{2r}^{[n-1]}}\right), \tag{3.6}$$

where $Y_{1r} = [Y_{1r}^{-}, Y_{1r}^{+}]$ and $Y_{2r} = [Y_{2r}^{-}, Y_{2r}^{+}]$ are internal stages under (i) and (ii)-differentiability, respectively. Also

$$\left(\begin{array}{c|c} \mathrm{A} & \mathrm{U} \\ \hline \mathrm{B} & \mathrm{V} \end{array}\right) = \left(\begin{array}{c|cccccc} 0 & 1 & \beta_{k-1} & \cdots & \beta_1 & \beta_0 \\ \hline 0 & 1 & \beta_{k-1} & \cdots & \beta_1 & \beta_0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{array}\right).$$

Now, we consider two example of Fuzzy GLMs form of k-step methods under strongly generalized differentiability for $k = 4, 5$. First, Consider $k = 4$. The input vectors for $k = 4$ under (i) and (ii)-differentiability are as follow, respectively:

$$y_{1r}^{\mp[n-1]} = \begin{pmatrix} y_{1r}^{\mp}(t_{n+3}) \\ hf_{1r}^{\mp}(t_{n+3}, y_{1r}(t_{n+3})) \\ hf_{1r}^{\mp}(t_{n+2}, y_{1r}(t_{n+2})) \\ hf_{1r}^{\mp}(t_{n+1}, y_{1r}(t_{n+1})) \\ hf_{1r}^{\mp}(t_{n}, y_{1r}(t_{n})) \end{pmatrix}, \quad y_{2r}^{\mp[n-1]} = \begin{pmatrix} y_{2r}^{\mp}(t_{n+3}) \\ hf_{2r}^{\pm}(t_{n+3}, y_{2r}(t_{n+3})) \\ hf_{2r}^{\pm}(t_{n+2}, y_{2r}(t_{n+2})) \\ hf_{2r}^{\pm}(t_{n+1}, y_{2r}(t_{n+1})) \\ hf_{2r}^{\pm}(t_{n}, y_{2r}(t_{n})) \end{pmatrix},$$

and

$$\left(\begin{array}{c|cccccc} 0 & 1 & \frac{55}{24} & \frac{-59}{24} & \frac{37}{24} & \frac{-9}{24} \\ \hline 0 & 1 & \frac{55}{24} & \frac{-59}{24} & \frac{37}{24} & \frac{-9}{24} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right).$$

similarly, for $k = 5$ we obtain

$$y_{1r}^{\mp[n-1]} = \begin{pmatrix} y_{1r}^{\mp}(t_{n+4}) \\ hf_{1r}^{\mp}(t_{n+4}, y_{1r}(t_{n+4})) \\ hf_{1r}^{\mp}(t_{n+3}, y_{1r}(t_{n+3})) \\ hf_{1r}^{\mp}(t_{n+2}, y_{1r}(t_{n+2})) \\ hf_{1r}^{\mp}(t_{n+1}, y_{1r}(t_{n+1})) \\ hf_{1r}^{\mp}(t_{n}, y_{1r}(t_{n})) \end{pmatrix}, \quad y_{2r}^{\mp[n-1]} = \begin{pmatrix} y_{2r}^{\mp}(t_{n+4}) \\ hf_{2r}^{\pm}(t_{n+4}, y_{2r}(t_{n+4})) \\ hf_{2r}^{\pm}(t_{n+3}, y_{2r}(t_{n+3})) \\ hf_{2r}^{\pm}(t_{n+2}, y_{2r}(t_{n+2})) \\ hf_{2r}^{\pm}(t_{n+1}, y_{2r}(t_{n+1})) \\ hf_{2r}^{\pm}(t_{n}, y_{2r}(t_{n})) \end{pmatrix},$$

and

$$\left(\begin{array}{c|ccccccc} 0 & 1 & \frac{1901}{720} & \frac{-2774}{720} & \frac{2616}{720} & \frac{-1274}{720} & \frac{251}{720} \\ \hline 0 & 1 & \frac{1901}{720} & \frac{-2774}{720} & \frac{2616}{720} & \frac{-1274}{720} & \frac{251}{720} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right).$$

4. Convergence, consistency and stability

To address the convergence of the presented FGLMs we consider the numerical solutions $y_1(t_{n+j}; r) = [y_1^{-}(t_{n+j}; r), y_1^{+}(t_{n+j}; r)]$ and $y_2(t_{n+j}; r) = [y_2^{-}(t_{n+j}; r), y_2^{+}(t_{n+j}; r)]$ and the corresponding exact solutions $\mathbf{Y}_1(t_{n+j}; r) = [\mathbf{Y}_1^{-}(t_{n+j}; r), \mathbf{Y}_1^{+}(t_{n+j}; r)]$ and

$\mathbf{Y}_2(t_{n+j};r) = [\mathbf{Y}_2^-(t_{n+j};r), \mathbf{Y}_2^+(t_{n+j};r)]$ under (i) and (ii)-differentiability, respectively. The local truncation errors (LTEs) of the FGLMs under strongly generalized differentiability are defined by

$$\begin{array}{l} \Psi_1(t_{n+k};r) = \sum_{j=0}^{k} r_j y_1(t_{n+j};r) - h\psi_{f_1}\big(y_1(t_{n+k};r), \cdots, y_1(t_n;r)\big), \\ \Psi_2(t_{n+k};r) = \sum_{j=0}^{k} r_j y_2(t_{n+j};r) - h\psi_{f_2}\big(y_2(t_{n+k};r), \cdots, y_2(t_n;r)\big), \end{array} \tag{4.1}$$

where $r_k = -r_{k-1} = 1$ and $r_j = 0$ for $j = 0, 1, \ldots, k-2$ and

$$\begin{aligned} \psi_{f_1}\big(y_1(t_{n+k};r), \cdots, y_1(t_n;r) &= \sum_{j=0}^{k-1} \beta_j f_1(t_{n+j}, y_1(t_{n+j};r)) \\ \psi_{f_2}\big(y_2(t_{n+k};r), \cdots, y_2(t_n;r) &= \sum_{j=0}^{k-1} \beta_j f_2(t_{n+j}, y_2(t_{n+j};r)) \end{aligned}$$

Consistency and stability are two essential conditions for convergent.

Definition 4.1. *A Fuzzy GLM form of k-step method under generalized differentiability is said to be consistent if for all fuzzy initial value problems, the residual $\Psi_1(t_{n+k};r)$ and $\Psi_2(t_{n+k};r)$ defined by (4.1) satisfies*

$$\lim_{h\to 0} \frac{1}{h}\Psi_1(t_{n+k};r) = 0,$$
$$\lim_{h\to 0} \frac{1}{h}\Psi_2(t_{n+k};r) = 0.$$

Definition 4.2. *A Fuzzy GLM is stable if the minimal polynomial of coefficient matrix V has no zeros greater than 1 and all zeros equal to 1 are simple, in other words it satisfies the root condition.*

To verify the stability of the given Fuzzy GLMs under generalized differentiability in section 3 we found the minimal polynomial $p_k(w)$ of the coefficient matrix V for $k = 4, 5$:

$$p_k(w) = w^k(w-1), \quad k = 4, 5,$$

which simply satisfies the root condition and the corresponding Fuzzy GLMs are stable.

5. Numerical results

In this section, we report among many test problems an example to show the numerical results of FGLMs for solving fuzzy differential equations under strongly generalized differentiability. We utilize the FGLMs ($k = 4, 5$) presented in section 3. The absolute error numerical results concerning the order of convergence is provided. We can estimate the order of convergence p by evaluation of the fraction $\frac{E(h/2)}{E(h)} = O(\frac{1}{2^p})$.

Test 5.1. *(Bede [1]) Consider the following fuzzy initial value problem*

$$y' = -y + e^{-t}(-1, 0, 1), \qquad y_0 = (-1, 0, 1). \tag{5.1}$$

The system of ODEs corresponding to (i)-differentiability is given by

$$\begin{cases} (y^-)' = -y^+ - e^{-t}, \\ (y^1)' = -y^1, \\ (y^+)' = -y^- + e^{-t}, \\ y_0 = (-1, 0, 1). \end{cases}$$

The analytical solution under (i)-differentiability is

$$\begin{aligned} Y_1^-(t; r) &= (1-r)(\frac{1}{2}e^{-t} - \frac{3}{2}e^t) \\ Y_1^+(t; r) &= (1-r)(\frac{3}{2}e^t - \frac{1}{2}e^{-t}) \end{aligned}$$

Similarly, the system of ODEs corresponding to (ii)-differentiability is given by

$$\begin{cases} (y^-)' = -y^- + e^{-t}, \\ (y^1)' = -y^1, \\ (y^+)' = -y^+ - e^{-t}, \\ y_0 = (-1, 0, 1), \end{cases}$$

and the analytical solution under (ii)-differentiability is

$$\begin{aligned} Y_2^-(t; r) &= (-1+r)(1-t)\exp(-t) \\ Y_2^+(t; r) &= (1-r)(1-t)\exp(-t). \end{aligned}$$

We demonstrate the numerical solution of FIVP (5.1) *in the interval* $[0, 2]$. *The (i) and (ii)-exact and approximate solutions, resulted by FGLMs for* $k = 4$ *and* $k = 5$, *are presented in Tables 1 and 2 at* $t = 2$ *with* $N = 20$ *and* $h = \frac{T - t_0}{N}$. *Moreover, the results for their convergence provided in Tables 3 and 4.*

From Tables 3 and 4, it follows that the Fuzzy GLMs of 4-step methods under strongly generalized differentiability have convergence order 4 and the Fuzzy GLMs form of 5-step methods have convergence order 5.

6. Conclusion

In this paper we have developed the linear multistep methods (Adams-Bashforth methods) in the framework of general linear methods for solving fuzzy differential equations under strongly generalized differentiability. We have shown the consistency, stability, and convergence of the new FGLM formulation. The general framework of FGLMs will be studied in the forthcoming paper.

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r	y_{1r}	Y_{1r}	E_{1r}
0	[-1.101531E1, 1.101531E1]	[-1.101592E1, 1.101592E1]	6.024101E-4
0.1	[-9.913783E0, 9.913783E0]	[-9.914325E0, 9.914325E0]	5.421691E-4
0.2	[-8.812251E0, 8.812251E0]	[-8.812733E0, 8.812733E0]	4.819281E-4
0.3	[-7.710720E0, 7.710720E0]	[-7.711142E0, 7.711142E0]	4.216871E-4
0.4	[-6.609188E0, 6.609188E0]	[-6.609550E0, 6.609550E0]	3.614461E-4
0.5	[-5.507657E0, 5.507657E0]	[-5.507958E0, 5.507958E0]	3.012050E-4
0.6	[-4.406126E0, 4.406126E0]	[-4.406367E0, 4.406367E0]	2.409640E-4
0.7	[-3.304594E0, 3.304594E0]	[-3.304775E0, 3.304775E0]	1.807230E-4
0.8	[-2.203063E0, 2.203063E0]	[-2.203183E0, 2.203183E0]	1.204820E-4
0.9	[-1.101531E0, 1.101531E0]	[-1.101592E0, 1.101592E0]	6.024101E-5
1.0	[0,0]	[0,0]	0

(a)

r	y_{2r}	Y_{2r}	E_{2r}
0	[1.352883E-1, -1.352883E-1]	[1.353353E-1, -1.353353E-1]	4.699417E-5
0.1	[1.217595E-1, -1.217595E-1]	[1.218018E-1, -1.218018E-1]	4.229476E-5
0.2	[1.082306E-1, -1.082306E-1]	[1.082682E-1, -1.082682E-1]	3.759534E-5
0.3	[9.470180E-2, -9.470180E-2]	[9.473470E-2, -9.473470E-2]	3.289592E-5
0.4	[8.117297E-2, -8.117297E-2]	[8.120117E-2, -8.120117E-2]	2.819650E-5
0.5	[6.764414E-2, -6.764414E-2]	[6.766764E-2, -6.766764E-2]	2.349709E-5
0.6	[5.411532E-2, -5.411532E-2]	[5.413411E-2, -5.413411E-2]	1.879767E-5
0.7	[4.058649E-2, -4.058649E-2]	[4.060058E-2, -4.060058E-2]	1.409825E-5
0.8	[2.705766E-2, -2.705766E-2]	[2.706706E-2, -2.706706E-2]	9.398835E-6
0.9	[1.352883E-2, -1.352883E-2]	[1.353353E-2, -1.353353E-2]	4.699417E-6
1.0	[0,0]	[0,0]	0

(b)

TABLE 1. (a)Approximate solution of the FGLM ($k = 4$) $y_{1r} = [y_{1r}^-, y_{1r}^+]$, exact solution $Y_{1r} = [Y_{1r}^-, Y_{1r}^+]$ and absolute error E_{1r} under (i)-differentiability, (b)Approximate solution of the FGLM ($k = 4$) $y_{2r} = [y_{2r}^-, y_{2r}^+]$, exact solution $Y_{2r} = [Y_{2r}^-, Y_{2r}^+]$ and absolute error E_{2r} under (ii)-differentiability, Test 5.1.

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r	y_{1r}	Y_{1r}	E_{1r}
0	[-1.101587E1, 1.101587E1]	[-1.101592E1, 1.101592E1]	4.451187E-5
0.1	[-9.914285E0, 9.914285E0]	[-9.914325E0, 9.914325E0]	4.006069E-5
0.2	[-8.812698E0, 8.812698E0]	[-8.812733E0, 8.812733E0]	3.560950E-5
0.3	[-7.711110E0, 7.711110E0]	[-7.711142E0, 7.711142E0]	3.115831E-5
0.4	[-6.609523E0, 6.609523E0]	[-6.609550E0, 6.609550E0]	2.670712E-5
0.5	[-5.507936E0, 5.507936E0]	[-5.507958E0, 5.507958E0]	2.225594E-5
0.6	[-4.406349E0, 4.406349E0]	[-4.406367E0, 4.406367E0]	1.780475E-5
0.7	[-3.304762E0, 3.304762E0]	[-3.304775E0, 3.304775E0]	1.335356E-5
0.8	[-2.203174E0, 2.203174E0]	[-2.203183E0, 2.203183E0]	8.902375E-6
0.9	[-1.101587E0, 1.101587E0]	[-1.101592E0, 1.101592E0]	4.451187E-6
1.0	[0,0]	[0,0]	0

(a)

r	y_{2r}	Y_{2r}	E_{2r}
0	[1.353406E-1, -1.353406E-1]	[1.353353E-1, -1.353353E-1]	5.270043E-6
0.1	[1.218065E-1, -1.218065E-1]	[1.218018E-1, -1.218018E-1]	4.743039E-6
0.2	[1.082724E-1, -1.082724E-1]	[1.082682E-1, -1.082682E-1]	4.216035E-6
0.3	[9.473839E-2, -9.473839E-2]	[9.473470E-2, -9.473470E-2]	3.689030E-6
0.4	[8.120433E-2, -8.120433E-2]	[8.120117E-2, -8.120117E-2]	3.162026E-6
0.5	[6.767028E-2, -6.767028E-2]	[6.766764E-2, -6.766764E-2]	2.635022E-6
0.6	[5.413622E-2, -5.413622E-2]	[5.413411E-2, -5.413411E-2]	2.108017E-6
0.7	[4.060217E-2, -4.060217E-2]	[4.060058E-2, -4.060058E-2]	1.581013E-6
0.8	[2.706811E-2, -2.706811E-2]	[2.706706E-2, -2.706706E-2]	1.054009E-6
0.9	[1.353406E-2, -1.353406E-2]	[1.353353E-2, -1.353353E-2]	5.270043E-7
1.0	[0,0]	[0,0]	0

(b)

TABLE 2. (a)Approximate solution of the FGLM ($k = 5$), $y_{1r} = [y^-_{1r}, y^+_{1r}]$, exact solution $Y_{1r} = [Y^-_{1r}, Y^+_{1r}]$ and absolute error E_{1r} under (i)-differentiability, (b)Approximate solution of the FGLM ($k = 5$), $y_{2r} = [y^-_{2r}, y^+_{2r}]$, exact solution $Y_{2r} = [Y^-_{2r}, Y^+_{2r}]$ and absolute error E_{2r} under (ii)-differentiability, Test 5.1.

r	h_i	$E_{2r}(h_i)$	p
0.2	$\frac{1}{10}$	3.759533862475462E-5	
	$\frac{1}{20}$	2.360816361970941E-6	3.993196066118324E0
	$\frac{1}{40}$	1.475860954835984E-7	3.999657112312050E0
	$\frac{1}{80}$	9.220812974275461E-9	4.000519042265663E0
0.4	$\frac{1}{10}$	2.819650396852780E-5	
	$\frac{1}{20}$	1.770612271481675E-6	3.993196066113545E0
	$\frac{1}{40}$	1.106895716057599E-7	3.999657112405316E0
	$\frac{1}{80}$	6.915609765401065E-9	4.000519034937461E0
0.6	$\frac{1}{10}$	1.879766931239119E-5	
	$\frac{1}{20}$	1.180408180992409E-6	3.993196066110909E0
	$\frac{1}{40}$	7.379304775567697E-8	3.999657112049211E0
	$\frac{1}{80}$	4.610406514893306E-9	4.000519033851667E0
0.8	$\frac{1}{10}$	9.398834656195593E-6	
	$\frac{1}{20}$	5.902040904962047E-7	3.993196066110909E0
	$\frac{1}{40}$	3.689652387783848E-8	3.999657112049211E0
	$\frac{1}{80}$	2.305203257446653E-9	4.000519033851667E0

TABLE 3. Convergence of the FGLM ($k = 4$)under (ii)-differentiability

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r	h_i	$E_{2r}(h_i)$	p
0.2	$\frac{1}{10}$	4.216034534335056E-6	
	$\frac{1}{20}$	1.336319765954386E-7	4.979549509841708E0
	$\frac{1}{40}$	4.185707641601866E-9	4.996649911789974E0
	$\frac{1}{80}$	1.308334413030465E-10	4.999668298467584E0
	$\frac{1}{160}$	4.088021587911328E-12	5.000184718796247E0
0.4	$\frac{1}{10}$	3.162025900754761E-6	
	$\frac{1}{20}$	1.002239824188234E-7	4.979549510242824E0
	$\frac{1}{40}$	3.139280738140293E-9	4.996649908201587E0
	$\frac{1}{80}$	9.812499424111110E-11	4.999669576905354E0
	$\frac{1}{160}$	3.065672715685253E-12	5.000345072764134E0
0.6	$\frac{1}{10}$	2.108017267167528E-6	
	$\frac{1}{20}$	6.681598831159707E-8	4.979549509542057E0
	$\frac{1}{40}$	2.092853827739827E-9	4.996649907306344E0
	$\frac{1}{80}$	6.541663044590251E-11	4.999670292639665E0
	$\frac{1}{160}$	2.043996916167856E-12	5.000192524602108E0
0.8	$\frac{1}{10}$	1.054008633583764E-6	
	$\frac{1}{20}$	3.340799415579854E-8	4.979549509542057E0
	$\frac{1}{40}$	1.046426913869913E-9	4.996649907306344E0
	$\frac{1}{80}$	3.270831522295126E-11	4.999670292639665E0
	$\frac{1}{80}$	1.021998458083928E-12	5.000192524602108E0

TABLE 4. Convergence of the FGLM ($k = 5$)under (ii)-differentiability

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