$μ_p$ - AND $α_p$ -ACTIONS ON K3 SURFACES IN CHARACTERISTIC p

YUYA MATSUMOTO

ABSTRACT. We consider μ_{p^-} and α_p -actions on RDP K3 surfaces (K3 surfaces with rational double point singularities allowed) in characteristic p > 0. We study possible characteristics, quotient surfaces, and quotient singularities. It turns out that these properties of μ_{p^-} and α_{p^-} actions are analogous to those of $\mathbb{Z}/l\mathbb{Z}$ -actions (for primes $l \neq p$) and $\mathbb{Z}/p\mathbb{Z}$ -quotients respectively. We also show that conversely an RDP K3 surface with a certain configuration of singularities admits a μ_{p^-} or α_p - or $\mathbb{Z}/p\mathbb{Z}$ -covering by a "K3-like" surface, which is often an RDP K3 surface but not always, as in the case of the canonical coverings of Enriques surfaces in characteristic 2.

1. INTRODUCTION

K3 surfaces are proper smooth surfaces X with $\Omega_X^2 \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$. The first condition implies that X has a global non-vanishing 2-form and it is unique up to scalar.

Actions of (finite or infinite) groups on K3 surfaces have been vastly studied. For example, the quotient of a K3 surface by an action of a finite group of order prime to the characteristic is birational to a K3 surface if and only if the action preserves the global 2-form, and moreover the list of possible such finite groups is determined in characteristic 0. Much less studied are infinitesimal actions, or *derivations*, on K3 surfaces in positive characteristic (with the exception of those with Enriques quotients in characteristic 2). Perhaps this is because it is known that smooth K3 surfaces admit no nontrivial global derivations. However we find many examples of nontrivial global derivations when we will look at *RDP K3 surfaces*, by which we mean we allow rational double point singularities (RDPs), the simplest 2-dimensional singularities.

In this paper we consider derivations that correspond to actions of group schemes μ_p and α_p . We study possible characteristic, quotient surfaces, and quotient singularities. It turns out that these properties of μ_p - and α_p actions are quite similar to those of $\mathbb{Z}/p\mathbb{Z}$ -actions in characteristic $\neq p$ and characteristic p respectively.

The actions of μ_p , and more generally of μ_{p^e} and μ_n , on K3 surfaces are also discussed in our previous paper [Mat20a].

The content and the main results of this paper are as follows.

Date: 2021/01/31.

²⁰¹⁰ Mathematics Subject Classification. 14J28 (Primary) 14L15, 14L30 (Secondary). This work was supported by JSPS KAKENHI Grant Numbers 15H05738, 16K17560, and 20K14296.

In Section 2 we introduce fundamental notions of derivations, such as pclosedness and fixed loci, and give their properties. Then in Section 3 we describe local behaviors of derivations related to RDPs. We classify p-closed derivations on RDPs without fixed points (Theorem 3.3) and RDPs arising as p-closed derivation quotients of regular local rings (Lemma 3.6(2)).

We show that a μ_p - or α_p -quotient Y of an RDP K3 surface X in characteristic p is either an RDP K3 surface, an RDP Enriques surface, or a rational surface (Proposition 4.1). For μ_p -actions the author proved in [Mat20a] that the quotient is an RDP K3 surface if and only if the induced action on the global 2-forms is trivial (this is parallel to the case of the actions of finite groups of order not divisible by p). For α_p -actions we could not find a similar criterion, since in this case the action on the 2-form is always trivial (this is parallel to $\mathbb{Z}/p\mathbb{Z}$ -actions).

In [Mat20a] we proved that μ_p -actions on RDP K3 surfaces in characteristic p occurs precisely if $p \leq 19$. In this paper we prove that the corresponding bound for α_p -actions is $p \leq 11$ (Theorem 8.1).

Suppose both X and the quotient Y are RDP K3 surfaces. We determine the possible characteristic p for both μ_p and α_p , and we moreover determine the possible singularities of Y (Theorem 4.6). Again the results are parallel to $\mathbb{Z}/l\mathbb{Z}$ (for a prime $l \neq p$) and $\mathbb{Z}/p\mathbb{Z}$ respectively. We also determine the possible singularities of X when the quotient Y is a supersingular Enriques surface (Theorem 9.1).

We also consider the inverse problem: whether an RDP K3 surface Y with a suitable configuration of singularities (and certain additional properties) can be written as the G-quotient of an RDP K3 surface X. It is known (at least to experts) that the answer is affirmative if $G = \mathbb{Z}/l\mathbb{Z}$. We show a similar result (Theorem 7.3) when G is $\mathbb{Z}/p\mathbb{Z}$, μ_p , or α_p , although if $G = \mu_p$ or $G = \alpha_p$ then X is only "K3-like" (Definition 7.2) in general and it may fail to be an RDP K3 surface. This behavior is analogous to that of the canonical μ_2 - and α_2 -coverings of Enriques surfaces in characteristic 2.

Now suppose $\pi: X \to Y$ is a finite purely inseparable morphism of degree p between RDP K3 surfaces. It is not necessarily the quotient morphism by a (regular) action of μ_p or α_p . We show (Theorem 5.2) that π admits a finite "covering" $\bar{\pi}: \bar{X} \to \bar{Y}$ that is a μ_p - or α_p -quotient morphism between either RDP K3 surfaces or abelian surfaces. We determine the possible covering degree and the characteristic for each case.

In Sections 9–10 we give explicit examples of RDP K3 surfaces and derivations.

Throughout the paper we work over an algebraically closed field k of characteristic $p \ge 0$, and whenever we refer to μ_p , α_p , or p-closed derivations we assume p > 0.

2. Preliminary on derivations

We recall basic facts on derivations, and relate differential forms on X to those on the derivation quotient X^D .

2.1. General properties of derivations. Let X be a scheme over k. A (regular) derivation on X is a k-linear endomorphism D of \mathcal{O}_X satisfying the Leibniz rule D(fg) = fD(g) + D(f)g.

Suppose for simplicity that X is integral. Then a rational derivation on X is a global section of $\text{Der}(\mathcal{O}_X) \otimes_{\mathcal{O}_X} k(X)$, where $\text{Der}(\mathcal{O}_X)$ is the sheaf of derivations on X. Thus, a rational derivation is locally of the form $f^{-1}D$ with f a regular function and D a regular derivation.

Lemma 2.1. If A is a local RDP and D is a derivation on $(\operatorname{Spec} A)^{\operatorname{sm}}$ (the complement of the closed point), then D extends to a derivation on $\operatorname{Spec} A$.

Proof. Indeed, for each $f \in A$ we have $D(f) \in H^0((\operatorname{Spec} A)^{\operatorname{sm}}, \mathcal{O}_A) = H^0(\operatorname{Spec} A, \mathcal{O}_A) = A$ since A is normal.

Lemma 2.2. Suppose A is the localization of a finitely generated k-algebra at a maximal ideal \mathfrak{m} , and D is a derivation on A. Then D extends to a derivation on the completion $\hat{A} = \varprojlim_n A/\mathfrak{m}^n$, and the completion $\widehat{A^D}$ of A^D at $\mathfrak{n} := \mathfrak{m} \cap A^D$ is equal to $(\hat{A})^D$.

Proof. Any derivation D satisfies $D(\mathfrak{m}^n) \subset \mathfrak{m}^{n-1}$, hence D induces a morphism $\lim_{n \to \infty} A/\mathfrak{m}^n \to \lim_{n \to \infty} A/\mathfrak{m}^{n-1}$.

There is a canonical injection $\widehat{A^D} \to (\widehat{A})^D$. Let us show the surjectivity of this map. Suppose $([a_n])_n$ is an element of \widehat{A} (i.e. $a_n \in A$ and $a_{n+l} \equiv a_n$ $(\text{mod } \mathfrak{m}^n)$) that belong to $(\widehat{A})^D$ (i.e. $D(a_n) \in \mathfrak{m}^{n-1}$). It suffices to find an element $b_n \in \mathfrak{m}^n$ with $D(b_n) = D(a_n)$, since then $([a_n]) = ([a_n - b_n]) \in \widehat{A^D}$. Since $D(a_n) = D(a_{n+l}) - D(a_{n+l} - a_n) \in \mathfrak{m}^{n+l-1} + D(\mathfrak{m}^n)$, it suffices to show $D(\mathfrak{m}^n) = \bigcap_{l \geq 0} (D(\mathfrak{m}^n) + \mathfrak{m}^{n+l})$. Suppose \mathfrak{m} is generated by N elements. This follows from Krull's intersection theorem, since $A^{(p)}$ is a Noetherian local ring, A and hence $D(\mathfrak{m}^l)$ are finitely generated $A^{(p)}$ -modules, and $\mathfrak{m}^{n+l} \subset \mathfrak{m}^l \subset (\mathfrak{m}^{(p)})^{\lceil (l-N(p-1))/p \rceil} A$.

Definition 2.3. Suppose D is a derivation on a scheme X. The fixed locus $\operatorname{Fix}(D)$ is the closed subscheme of X corresponding to the sheaf $(\operatorname{Im}(D))$ of ideals generated by $\operatorname{Im}(D) = \{D(a) \mid a \in \mathcal{O}_X\}$. Equivalently, this sheaf is $\operatorname{Im}(\overline{D})$, where $\overline{D} \colon \Omega^1_X \to \mathcal{O}_X$ is the morphism defined below in Definition 2.5. A fixed point of D is a point of $\operatorname{Fix}(D)$.

Assume X is a smooth irreducible variety and $D \neq 0$. Then Fix(D) consists of its divisorial part (D) and non-divisorial part $\langle D \rangle$. If we write $D = f \sum_i g_i \frac{\partial}{\partial x_i}$ for some local coordinate x_i with g_i having no common factor, then (D) and $\langle D \rangle$ corresponds to the ideal (f) and (g_i) respectively.

Assume X is a smooth irreducible variety and suppose $D \neq 0$ is now a *rational* derivation, locally of the form $f^{-1}D'$ for some regular function f and (regular) derivation D'. Then we define $(D) = (D') - \operatorname{div}(f)$ and $\langle D \rangle = \langle D' \rangle$.

If X is only normal, then we can still define (D) as a Weil divisor.

Rudakov–Shafarevich [RS76] uses the term *singularity* for the fixed locus. We do not use this, as we want to distinguish them from the singularities of the varieties.

The next theorem is proved by Rudakov–Shafarevich [RS76, Theorem 3] for regular derivations D satisfying some assumptions, and by Katsura–Takeda [KT89, Proposition 2.1] for general rational derivations.

Theorem 2.4. Let D be a rational derivation on a smooth proper surface X. Then

$$\deg c_2(X) = \deg \langle D \rangle - K_X \cdot (D) - (D)^2.$$

A derivation D on X acts naturally on the sheaves Ω_X^q , as follows.

Definition 2.5. Let D be a derivation on X. Decompose $D: \mathcal{O} \to \mathcal{O}$ as $\overline{D} \circ d: \mathcal{O} \xrightarrow{d} \Omega^1 \xrightarrow{\overline{D}} \mathcal{O}$. Then \overline{D} is \mathcal{O} -linear. Let $\overline{D}_q: \Omega^q \to \Omega^{q-1} \ (q \ge 1)$ be the (\mathcal{O} -linear) homomorphism defined by

$$\bar{D}_q(\beta_1 \wedge \dots \wedge \beta_q) = \sum_{j=1}^q (-1)^{j-1} \bar{D}(\beta_j) \cdot \beta_1 \wedge \dots \wedge \beta_{j-1} \wedge \beta_{j+1} \wedge \dots \wedge \beta_q,$$

for 1-forms β_j , and for q = 0 let \overline{D}_0 be the zero map. We have $\overline{D}_{q_1+q_2}(\beta_1 \wedge \beta_2) = \overline{D}_{q_1}(\beta_1) \wedge \beta_2 + (-1)^{q_1}\beta_1 \wedge \overline{D}_{q_2}(\beta_2)$ for a q_1 -form β_1 and a q_2 -form β_2 . We define $D_q := d \circ \overline{D}_q + \overline{D}_{q+1} \circ d$: $\Omega^q \to \Omega^q \ (q \ge 0)$.

Proposition 2.6. Then we have the following properties.

- $D_0 = D$.
- $D_1(df) = d(D_0(f)).$
- $D_{q_1+q_2}(\beta_1 \wedge \beta_2) = D_{q_1}(\beta) \wedge \beta_2 + \beta_1 \wedge D_{q_2}(\beta_2)$ for a q_1 -form β_1 and a q_2 -form β_2 . Hence, $D_{q_1+\dots+q_l}(\beta_1 \wedge \dots \wedge \beta_l) = \sum_{i=1}^l \beta_1 \wedge \dots \wedge \beta_{i-1} \wedge D_{q_i}(\beta_i) \wedge \beta_{i+1} \wedge \dots \wedge \beta_l$ for q_i -forms β_i .
- $[D, D']_q = [D_q, D'_q]$ and $(D^p)_q = (D_q)^p$.
- $(hD)_q = h \cdot D_q + dh \wedge \overline{D}_q$, which is equal to $h \cdot D_q$ if for example $h \in k(X)^{(p)}$.
- Hence, If $D^p = hD$ for $h \in k(X)^{(p)}$, then $(D_q)^p = hD_q$. (This is not true for general $h \in k(X)$.)

Proof. Straightforward.

We will write simply D in place of D_q .

2.2. General properties of *p*-closed derivations. We say that a derivation D on an integral scheme X is *p*-closed if there exists $h \in k(X)$ with $D^p = hD$. Quotients by such derivations will be studied in the next subsection.

The next formula is well-known.

Lemma 2.7 (Hochschild's formula). Let A be a k-algebra in characteristic p > 0, a an element of A, and D a derivation on A. Then

$$(aD)^p = a^p D^p + (aD)^{p-1}(a)D$$

In particular, if D is p-closed then so is aD.

The following lemmas are useful when analyzing local properties.

Lemma 2.8. Suppose B is a local domain equipped with a p-closed derivation $D \neq 0$ such that Fix(D) is principal. Then the maximal ideal \mathfrak{m} of B is generated by elements x_j $(j \in J)$ and y, satisfying $D(x_j) = 0$. If \mathfrak{m} is generated by n elements then we can take |J| = n - 1.

If B is smooth, then this is proved in [Ses60, Proposition 6] (see also [RS76, Theorem 1 and Corollary]).

Proof. Take $f \in B$ with $(D) = \operatorname{div}(f)$. By replacing D with the (regular) derivation $f^{-1}D$, which is also *p*-closed by Hochschild's formula (Lemma 2.7), we may assume (D) = 0, hence $\operatorname{Fix}(D) = \emptyset$.

Take $h \in B$ such that $D^p = hD$. Note that then D(h) = 0.

Take an element $y \in B$ with $D(y) \notin \mathfrak{m}$ (which exists since $\mathfrak{m} \notin \operatorname{Fix}(D)$). We may assume $y \in \mathfrak{m}$. Let $w = y^{p-1}$. Then $D^k(w) \in yB \subset \mathfrak{m}$ for $0 \leq k \leq p-2$ and $D^{p-1}(w) \in B^*$. We have $u := D^{p-1}(w) - hw \in B^* \cap B^D$. Take elements $(x'_i)_{j \in J'}$ generating \mathfrak{m} . Let

$$x_j = ux'_j + \sum_{k=0}^{p-2} (-1)^k D^k(w) D^{p-1-k}(x'_j).$$

Then we have $D(x_j) = 0$ and, since $x_j \equiv ux'_j \pmod{yB}$, it follows that $x_j \ (j \in J')$ and y generate \mathfrak{m} . If $|J'| < \infty$ then we can remove one of the elements, and the remaining elements still generate \mathfrak{m} . The removed one cannot be y since $(D(x_j)) \subset \mathfrak{m}$, hence the removed one is x_{j_0} for some $j_0 \in J'$, hence $x_j \ (j \in J' \setminus \{j_0\})$ and y generate \mathfrak{m} .

Lemma 2.9. Suppose B is a local domain equipped with a p-closed derivation $D \neq 0$ of additive type such that $Fix(D) = \emptyset$. Then there exists $x \in B$ with D(x) = 1.

Proof. As in the previous lemma, since $\operatorname{Fix}(D) = \emptyset$, there exists $y \in \mathfrak{m}$ with $D(y) \notin \mathfrak{m}$, and then $u := D^{p-1}(y^{p-1}) \in B^* \cap B^D$. Then $D^{p-1}(u^{-1}y^{p-1}) = 1$.

2.3. *p*-closed derivation quotients and differential forms. If D is *p*closed, then X^D is the scheme with underlying topological space homeomorphic to (and often identified with) X, and with structure sheaf $\mathcal{O}_{X^D} = \mathcal{O}_X^D = \{a \in \mathcal{O}_X \mid D(a) = 0\}$ consisting of the *D*-invariant sections of \mathcal{O}_X . The natural morphism $X \to X^D$ is finite of degree p (unless D = 0). If Xis normal then so is X^D .

In this subsection we compare top differential forms on X and the quotient X^D (Propositions 2.12 and 2.14).

Special cases of *p*-closed derivations correspond to (non-reduced) group schemes, as follows, which are the main subject of this paper.

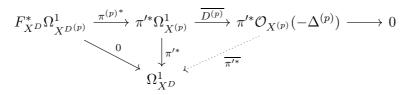
Proposition 2.10. Let $G = \mu_p$ (resp. $G = \alpha_p$). Then the G-actions on a scheme X correspond bijectively to the derivations D on \mathcal{O}_X of multiplicative type (resp. of additive type), that is, $D^p = D$ (resp. $D^p = 0$). The quotient scheme X/G always exists, and coincides with X^D .

Proof. Well-known.

Lemma 2.11. Let X be a smooth variety of dimension m (not necessarily proper) equipped with a p-closed rational derivation D such that $\Delta :=$ Fix(D) is divisorial. Let $\pi: X \to X^D$ be the quotient map. The morphism $\pi^* \colon \pi^* \Omega^1_{X^D} \to \Omega^1_X$ induced by the pullback of 1-forms fits into a canonical exact sequence

$$0 \to \mathcal{O}_X(-p\Delta) \xrightarrow{\overline{\pi'^*}} \pi^* \Omega^1_{X^D} \xrightarrow{\pi^*} \Omega^1_X \xrightarrow{\bar{D}} \mathcal{O}_X(-\Delta) \to 0,$$

where $F_X = \pi' \circ \pi \colon X \to X^D \to X^{(p)}$ is the Frobenius, \overline{D} is defined as in Definition 2.5 (i.e. $\overline{D} \circ d = D$), and $\overline{\pi'^*}$ is the morphism defined in the diagram



and the equality $F_X^*(\mathcal{O}_{X^{(p)}}(-\Delta^{(p)})) = \mathcal{O}_X(-p\Delta)$. Let η (resp. ξ) be the image (resp. preimage) of 1 by the induced isomorphism $\mathcal{O}_X \xrightarrow{\sim} \operatorname{Ker}(\pi^* \otimes \mathcal{O}_X(p\Delta))$ (resp. $\operatorname{Coker}(\pi^* \otimes \mathcal{O}_X(\Delta)) \xrightarrow{\sim} \mathcal{O}_X$). Then $\eta = \frac{d(f^p)}{D(f)^p}$ and $\xi = \frac{df}{D(f)}$ for any local section $f \in \mathcal{O}_X$ satisfying $\operatorname{div}(D(f)) = \Delta$. Moreover $d\eta = 0$.

Proof. By the result of Seshadri (Lemma 2.8), we can take a local coordinate x_0, \ldots, x_{m-1} of X such that $x_0^p, x_1, \ldots, x_{m-1}$ is a local coordinate of X^D . Then $D = \phi \frac{\partial}{\partial x_0}$ for some meromorphic function ϕ on X, and then $\Delta =$ $\operatorname{div}(\phi)$. Then the sequence is

$$0 \to \langle \phi^p \rangle \to \langle d(x_0^p), dx_1, \cdots dx_{m-1} \rangle \to \langle dx_0, dx_1, \cdots dx_{m-1} \rangle \to \langle \phi \rangle \to 0$$

with $\phi^p \mapsto d(x_0^p)$ and $dx_0 \mapsto \phi$, which is clearly exact. The formulas of η and ξ are clear from the construction. $d\eta = 0$ follows either by computation using the formula or from the observation that $d\eta \in \operatorname{Im}(\bigwedge^2 \pi'^* \colon F_X^*\Omega^2_{X^{(p)}} \to \mathbb{I}^2$ $\pi^* \Omega_{XD}^2 = 0$ (since rank $\pi'^* = 1$).

Proposition 2.12. Let D and $\pi: X \to X^D$ as in Lemma 2.11. Then there is an isomorphism

$$\Omega^m_{X/k}(\Delta) \cong \pi^*(\Omega^m_{X^D/k}(\pi_*(\Delta))) = \pi^*\Omega^m_{X^D/k} \otimes \mathcal{O}_X(p\Delta)$$

of \mathcal{O}_X -modules, preserving the zero loci of forms, and sending

 $f_0 \cdot df_1 \wedge \cdots \wedge df_{m-1} \wedge D(g)^{-1} dg \mapsto f_0 \cdot df_1 \wedge \cdots \wedge df_{m-1} \wedge D(g)^{-p} d(g^p)$

for local sections f_i, g of \mathcal{O}_X if $D(f_i) = 0$ for $1 \leq i < m$ and $D(g)^{-1} \in$ $\mathcal{O}_X(\Delta).$

Taking powers and then the D-invariant parts, we also obtain an isomorphism

$$((\pi_*\Omega^m_{X/k}(\Delta))^{\otimes n})^D \cong (\Omega^m_{X^D/k}(\pi_*(\Delta)))^{\otimes n}$$

of \mathcal{O}_{XD} -modules, satisfying the same property when n = 1 if $D(f_0) = 0$.

In particular, if D is regular and fixed-point-free, then we have isomorphisms

$$(\pi_*(\Omega^m_{X/k})^{\otimes n})^D \cong (\Omega^m_{X^D/k})^{\otimes n} \quad and$$
$$H^0(X, (\Omega^m_{X/k})^{\otimes n})^D \cong H^0(X^D, (\Omega^m_{X^D/k})^{\otimes n})$$

with the same properties.

This refines the Rudakov–Shafarevich formula [RS76, Corollary 1 to Proposition 3] $K_X \sim \pi^* K_{X^D} + (p-1)(D)$ (linear equivalence). We note that, by Lemma 2.8, there indeed exist local sections $f_0, f_1, \ldots, f_{m-1}, g$ for which the *m*-forms in the statement are generators.

Proof. This follows immediately from the exact sequence in Lemma 2.11 and the description of the elements η and ξ .

Lemma 2.13. Suppose $V_n \xrightarrow{G_{n-1}} V_{n-1} \xrightarrow{G_{n-2}} \dots \xrightarrow{G_0} V_0$ is a sequence of morphisms between locally-free sheaves of equal finite rank m on an irreducible scheme such that Coker G_i are also locally-free and $\sum_{i \in [0,n[} \operatorname{rank} \operatorname{Coker} G_i$ is equal to the rank of Coker $G_{[0,n[}$ at the generic point, where $G_{[0,i[} :=$ $G_0 \circ \dots \circ G_{i-1}$. Then Coker $G_{[0,n[}$ is also locally-free and there is a unique isomorphism $\bigotimes_i (\det \operatorname{Coker} G_i) \xrightarrow{\sim} \det \operatorname{Coker} G_{[0,n[}$ taking $(v_i)_i$ to $\bigwedge_i G_{[0,i[}(v_i)$ for local sections v_i of V_i .

Proof. For $0 \le p \le q \le n$, let $G_{[p,q]} := G_p \circ G_{p+1} \circ \cdots \circ G_{q-1}$. The assumption on the rank implies that $\operatorname{Coker} G_{[p,q]}$ has rank equal to $\sum_{i \in [p,q]} \operatorname{rank} \operatorname{Coker} G_i$ at the generic point. We show the following.

- (1) For $p \leq r$, Coker $G_{[p,r]}$ is locally-free.
- (2) For $p \leq q \leq r$, the sequence $0 \to \operatorname{Coker} G_{[q,r[} \xrightarrow{\beta} \operatorname{Coker} G_{[p,r[} \to \operatorname{Coker} G_{[p,q]} \to 0 \text{ is exact.}$

(1) is clear if $r - p \leq 1$. (2) is clear if p = q or q = r. It suffices to show that if p < q < r and (1) holds for (p,q) and (q,r) then (1) holds for (p,r) and (2) holds for (p,q,r). The exactness at the middle and the right is clear. Since Ker β is a subsheaf of a locally-free sheaf (by the assumption) and its rank at the generic point is 0, we have Ker $\beta = 0$. Thus (2) is true by the assumptions, and this together with the induction hypothesis imply (1).

Now, from (2) we obtain isomorphisms det Coker $G_{[p,q]} \otimes \det \operatorname{Coker} G_{[q,r]} \xrightarrow{\sim} \det \operatorname{Coker} G_{[p,r]} : v \otimes w \mapsto v \wedge G_{[p,q]}(w)$. Composing these isomorphisms inductively, we obtain the desired isomorphism.

Proposition 2.14. Suppose $X_0 \xrightarrow{\pi_0} X_1 \xrightarrow{\pi_1} \dots \xrightarrow{\pi_{m-1}} X_m = X_0^{(p)}$ is a sequence of purely inseparable morphisms of degree p between m-dimensional integral normal varieties, with each π_i given by a p-closed rational derivation D_i on X_i . Then $K_{X_0} \sim -\sum_{i=0}^{m-1} (\pi_{i-1} \circ \cdots \circ \pi_0)^* (D_i)$.

Proof. As the conclusion does not depend on closed subschemes of codimension ≥ 2 , we may assume that $\operatorname{Sing}(X_i) = \emptyset$ and $\langle D_i \rangle = \emptyset$ by restricting to the complement.

We write $\pi_{[0,i]} := \pi_{i-1} \circ \cdots \circ \pi_1 \circ \pi_0 \colon X_0 \to X_i$ and let $G_i \colon \pi^*_{[0,i+1]} \Omega^1_{X_{i+1}} \to \pi^*_{[0,i]} \Omega^1_{X_i}$ be the pullback of $\pi^*_i \colon \pi^*_i \Omega^1_{X_{i+1}} \to \Omega^1_{X_i}$ to X_0 . Then $\operatorname{Coker}(G_i \otimes I_i) \to I_i$

 $\mathcal{O}_{X_0}(\pi^*_{[0,i]}(D_i)))$ is free of rank 1, since it is the pullback of $\operatorname{Coker}(\pi^*_i \otimes$ $\mathcal{O}_{X_i}((D_i))$, which is free of rank 1 by Lemma 2.11. Since $G_0 \circ \cdots \circ G_{m-1} = 0$, we can apply Lemma 2.13 to $G_0 \circ \cdots \circ G_{m-1}$. Then the invertible sheaf

$$\Omega_{X_0}^m \otimes \mathcal{O}_{X_0}\left(\sum_i \pi_{[0,i[}^*(D_i)\right) = (\operatorname{Coker}(G_0 \circ \cdots \circ G_{m-1})) \otimes \bigotimes_i \mathcal{O}_{X_0}(\pi_{[0,i[}^*(D_i)))$$
$$\cong \bigotimes_i \operatorname{Coker}(G_i \otimes \mathcal{O}_{X_0}(\pi_{[0,i[}^*(D_i))))$$
is trivial.

is trivial.

The next proposition, which we will use in Section 7, is a slight generalization of arguments in [BM76, Sections 3 and 5] (where only derivations of multiplicative or additive type are considered).

Proposition 2.15. Let D be a nontrivial p-closed derivation on an integral scheme X, and let $\pi: X \to X^D = Y$ be the quotient map. Suppose $Fix(D) \subset$ $\pi^{-1}(\operatorname{Sing}(Y))$ and $\operatorname{Sing}(X) \subset \pi^{-1}(Y^{\operatorname{sm}})$. Then,

- (1) X is Gorenstein.
- (2) There is a canonical closed 1-form η on $Y^{\rm sm}$ that coincides with the one given in Lemma 2.11 on $Y^{sm} \cap \pi(X^{sm})$. It satisfies Sing(X) = $\pi^{-1}(\operatorname{Zero}(\eta))$. X is normal if and only if $\operatorname{codim} \operatorname{Zero}(\eta) \geq 2$.
- (3) Suppose X and Y are proper, Y admits a dualizing sheaf ω_Y , and it is trivial ($\omega_Y \cong \mathcal{O}_Y$). Then X admits a dualizing sheaf ω_X and it is trivial.
- (4) Suppose X and Y are surfaces. Suppose Y^{sm} admits a global nonvanishing 2-form ω , and fix such a 2-form. Then there is a unique p-closed derivation D_Y on Y satisfying $D_Y(f)\omega = df \wedge \eta$ on Y^{sm} . It moreover satisfies $\operatorname{Zero}(\eta) = \operatorname{Fix}(D_Y|_{Y^{\operatorname{sm}}}), Y^{D_Y} = (X^n)^{(p)}, and$ $D_Y(\omega) = 0.$

Proof. First note that Y is normal. Indeed, for each point $y \in \text{Sing}(Y)$, the point $\pi^{-1}(y) \in X$ is smooth by assumption, in particular normal, and normality inherits to derivation quotients.

(2) Let $Y' = Y^{\text{sm}}$ and $X' = \pi^{-1}(Y')$. Since $\text{Fix}(D) \cap X' = \emptyset$ there exists locally a section $s \in \mathcal{O}_{X'}$ with $D(s) \in \mathcal{O}_{X'}^*$. Consider the 1-form $\eta = d(s^p)/D(s)^p$ on Y'. By Lemma 2.11, the restriction of η to $\pi(X^{sm}) \cap Y'$ (which is dense since X is integral) does not depend on the choice of s, is defined globally, and is killed by d, hence η itself satisfies the same properties.

Two special cases are the following. If D is of multiplicative type then we can take s satisfying D(s) = s ([Mat20a, Lemma 2.13]), and then $\eta =$ $d\log(s^p)$. If D is of additive type then we can take s satisfying D(s) = 1(Lemma 2.9), and then $\eta = d(s^p)$.

By assumption X is regular above $\operatorname{Sing}(Y)$. Locally on Y', we have $\mathcal{O}_{X'}$ = $\mathcal{O}_{Y'}[S]/(S^p-b)$, where $b=s^p$. Hence X is complete intersection, in particular Gorenstein, and we have $\operatorname{Sing}(X) = \pi^{-1}(\operatorname{Zero}(db)) = \pi^{-1}(\operatorname{Zero}(\eta)).$

Since X is regular at the generic point, η is not identically 0. X is normal if and only if $\operatorname{Sing}(X)$ or equivalently $\operatorname{Zero}(\eta)$ is of codimension > 1.

(1) is proved above.

(3) Since X is proper and $\operatorname{codim} \operatorname{Fix}(D) \geq 2$, we have $h \in k$, where $D^p = hD$. We may assume $h \in \{0, 1\}$. We follow [BM76, Proposition 9]. It suffices to give an \mathcal{O}_Y -linear isomorphism $\phi \colon \pi_*\mathcal{O}_X \to \mathcal{H}om(\pi_*\mathcal{O}_X, \mathcal{O}_Y)$. Let ϕ be the morphism $x \mapsto t(x \cdot -)$, where $t = \operatorname{pr}_0 \colon \pi_*\mathcal{O}_X \to (\pi_*\mathcal{O}_X)^{D=0} = \mathcal{O}_Y$ if D is of multiplicative type (i.e. h = 1) and $t = D^{p-1} \colon \pi_*\mathcal{O}_X \to \mathcal{O}_Y$ if D is of additive type (i.e. h = 0). Since $\operatorname{Fix}(D) \cap X' = \emptyset$, $\phi|_{Y'}$ is an isomorphism, and then ϕ itself is an isomorphism since $\pi_*\mathcal{O}_X$ and \mathcal{O}_Y are normal at $\operatorname{Sing}(Y)$.

(4) We define a derivation $D_{Y'}$ on $Y' = Y^{\text{sm}}$ by $D_{Y'}: \mathcal{O}_{Y'} \xrightarrow{d} \Omega_{Y'}^{1} \xrightarrow{\wedge \eta} \Omega_{Y'}^{2} \xleftarrow{\otimes \omega} \mathcal{O}_{Y'}$, hence $D_{Y'}(f)\omega = df \wedge \eta$. Then $\operatorname{Fix}(D_{Y'}) = \operatorname{Zero}(\eta)$. Write $\mathcal{O}_{X'} = \mathcal{O}_{Y'}[S]/(S^p - b)$ locally on Y' as in the proof of (2) and then $\eta = u \cdot db$ for a unit $u \in \mathcal{O}_{Y'}^{*}$. Then it is clear that $b \in \mathcal{O}_{Y'}^{D_{Y'}}$ and $(\mathcal{O}_{Y'})^{(p)} \subset \mathcal{O}_{Y'}^{D_{Y'}}$, hence $(\mathcal{O}_{X'})^{(p)} \subset \mathcal{O}_{Y'}^{D_{Y'}}$. Since $\mathcal{O}_{Y'}^{D_{Y'}}$ is normal (since \mathcal{O}_Y is normal) we obtain $((\mathcal{O}_{X'})^n)^{(p)} \subset \mathcal{O}_{Y'}^{D_{Y'}}$. Since Y is normal, $D_{Y'}$ extends to a derivation D_Y on Y by Lemma 2.1, and we have $((\mathcal{O}_X)^n)^{(p)} \subset \mathcal{O}_Y^{D_Y}$. Comparing the degree with respect to k(X) $(p^2 = [k(X) : k(X^{(p)})] \ge [k(X) : k(Y^{D_Y})] = [k(X) : k(Y^{D_Y})] \ge p^2)$ we observe that this is equality at the generic point, and then since both sides are normal we obtain the equality. We also obtain $[k(Y) : k(Y^{D_Y})] = p$ and hence D_Y is p-closed.

We have $D_Y(\eta) = 0$ since η is the pullback of a 1-form on $\mathcal{O}_X^{(p)} \subset \mathcal{O}_Y^{D_Y}$. Comparing $D_Y(D_Y(f)\omega) = D_Y(df \wedge \eta)$ and $D_Y(D_Y(f))\omega = d(D_Y(f)) \wedge \eta$ (both of which follow from $D_Y(f)\omega = df \wedge \eta$), we obtain $D_Y(\omega) = 0$. \Box

3. Local properties of derivations on smooth points and RDPs

In this section we will recall basic properties of RDPs and then consider derivations on RDPs.

Definition 3.1 (RDPs). *Rational double point* singularities in dimension 2, RDPs for short, are the 2-dimensional canonical singularities.

The exceptional curves of the resolution of singularity and their intersection numbers form a Dynkin diagram of type A_n , D_n , or E_n . We say that the RDP is of type A_n , D_n , or E_n . For D_n and E_n in characteristic 2, E_n in characteristic 3, and E_8 in characteristic 5, and in no other cases, there are more than one, finitely many, isomorphism classes of singularity sharing the same Dynkin diagram. They are classified and named as D_n^r and E_n^r by Artin [Art77], where the range of r is a certain finite set of non-negative integers depending on the characteristic and the Dynkin diagram. In these cases, and also in the cases of A_n with $p \mid (n + 1)$ and D_n with $p \mid (n - 2)$, and in no other cases, the fundamental groups are different from those of the corresponding RDPs in characteristic 0, again see [Art77].

We refer to n and r as the *index* and *coindex* of the RDP.

If A is the localization of a surface at an RDP, or the completion of such an algebra, then we call Spec A a *local RDP* for short.

If Spec A is a local RDP or a 2-dimensional regular local ring, then we denote $\operatorname{Pic}(A) = \operatorname{Pic}((\operatorname{Spec} A)^{\operatorname{sm}})$ and call this the local Picard group of A. If A is Henselian (e.g. if it is complete) then by [Lip69, Proposition 17.1], this group is determined from the Dynkin diagram as in Table 1 and is independent of the characteristic and the coindex.

TABLE 1. Local Picard groups of Henselian RDPs (in any characteristic)

smooth	A_n	D_{2m}	D_{2m+1}	E_6	E_7	E_8
0	$\mathbb{Z}/(n+1)\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0

Definition 3.2 (RDP surfaces). *RDP surfaces* are surfaces that have only RDPs as singularities (if any). In particular, any smooth surface is an RDP surface.

RDP K3 surfaces are proper RDP surfaces whose minimal resolutions are (smooth) K3 surfaces. We similarly define *RDP Enriques surfaces*.

Since abelian surfaces and (quasi-)hyperelliptic surfaces do not admit smooth rational curves, any RDP abelian or RDP (quasi-)hyperelliptic surface is smooth.

Theorem 3.3. Let X be a surface equipped with a nontrivial p-closed derivation D, and $w \in X$ a closed point. Let $\pi: X \to Y = X^D$ be the quotient morphism.

- Assume w ∉ Fix(D). If w is a smooth point then π(w) is also a smooth point. If w is an RDP then π(w) is either a smooth point or an RDP, and more precisely (Ô_{X,w}, D) is isomorphic to (k[[x, y, z]]/(F), u · ∂/∂z) where u is a unit and F is a power series ∈ k[[x, y, z^p]] that is one in Table 2. In either case X ×_Y Ỹ → X is crepant, where Ỹ → Y is the minimal resolution at π(w).
- (2) If $w \in Fix(D)$, then D uniquely extends to a derivation D_1 on $X_1 = Bl_w X$. Suppose moreover that (D) = 0, that w is an RDP, and that $\pi(w)$ is either a smooth point or an RDP. Then $\pi(w)$ is an RDP, $(D_1) = 0$, the image of each point above w is either a smooth point or an RDP, $g: Y_1 = (X_1)^{D_1} \to Y$ is crepant, and $Fix(D_1) \neq \emptyset$.

Proof. (1) Assume w is a smooth point (this case is already proved in [Ses60, Proposition 6]). Taking a coordinate x, y as in Lemma 2.8 (i.e. D(x) = 0 and $D(y) \in \mathcal{O}^*_{X,w}$), we have $\hat{\mathcal{O}}_{Y,\pi(w)} \cong k[[x, y^p]]$, hence $\mathcal{O}_{Y,\pi(w)}$ is smooth.

Assume w is an RDP. By Lemma 2.8 we have a coordinate x, y, z satisfying D(x) = D(y) = 0 and $D(z) \in \mathcal{O}^*_{X,w}$.

We recall the classification [Mat20a, Proposition 4.8] of all formal power series $F \in k[[x, y, z^p]]$ such that k[[x, y, z]]/(F) defines an RDP at the origin, up to multiples by units, and up to coordinate change preserving the invariant subalgebra $k[[x, y, z^p]] \subset k[[x, y, z]]$. The result is displayed in Table 2. We observed that in each case $\pi(w)$ is either a smooth point or an RDP and that $X \times_Y \tilde{Y}$ is an RDP surface crepant over X, where $\tilde{Y} \to Y$ is the resolution at $\pi(w)$. (The entries of the singularities of $X \times_Y \tilde{Y}$ is omitted if Y is already smooth.)

(2) Take a 2-form χ on Y, nonzero on a neighborhood of $\pi(w)$. Let ω be the *D*-invariant 2-form on X corresponding to χ under the isomorphism in Proposition 2.12. Let $\omega_1 = q^* \omega$, where $q: X_1 \to X$ is the blow-up. Let χ_1 be the 2-form on Y_1 corresponding to ω_1 . Then we have

 $\operatorname{div}(\omega) = \pi^*(\operatorname{div}(\chi)) + (p-1)(D), \quad \operatorname{div}(\omega_1) = \pi_1^*(\operatorname{div}(\chi_1)) + (p-1)(D_1),$

p	equation		X	$Y = X^D$	$X \times_Y \tilde{Y}$
any any	$\begin{array}{l} xy + z^{mp} \\ xy + z^p \end{array}$	$(m \ge 2)$	$\begin{array}{c} A_{mp-1} \\ A_{p-1} \end{array}$	A_{m-1} smooth	$\underline{mA_{p-1}}$
5	$x^2 + y^3 + z^5$		E_{8}^{0}	smooth	_
3 3 3	$ \begin{array}{c} x^2 + z^3 + y^4 \\ x^2 + y^3 + yz^3 \\ x^2 + z^3 + y^5 \end{array} $		$E_6^0 \\ E_7^0 \\ E_8^0$	$\begin{array}{c} \text{smooth} \\ A_1 \\ \text{smooth} \end{array}$	E_{6}^{0}
$\begin{array}{c}2\\2\\2\\2\\2\\2\end{array}$	$ \begin{array}{c} z^2 + x^2 y + x y^m \\ x^2 + y z^2 + x y^m \\ x^2 + x z^2 + y^3 \\ z^2 + x^3 + x y^3 \\ z^2 + x^3 + y^5 \end{array} $	$\begin{array}{l} (m \ge 2) \\ (m \ge 2) \end{array}$	$\begin{array}{c} D^0_{2m} \\ D^0_{2m+1} \\ E^0_6 \\ E^0_7 \\ E^0_8 \end{array}$	$ \begin{array}{c} \text{smooth} \\ A_1 \\ A_2 \\ \text{smooth} \\ \text{smooth} \end{array} $	$\begin{array}{c} & & & & & & & & & & & & & & & & & & &$

TABLE 2. Non-fixed p-closed derivations on RDPs

and we have

 $\operatorname{div}(\omega_1) = q^*(\operatorname{div}(\omega)) + K_{X_1/X}, \quad \operatorname{div}(\chi_1) = g^*(\operatorname{div}(\chi)) + K_{Y_1/Y}.$

By assumption we have (D) = 0 and $K_{X_1/X} = 0$. Hence we have

$$\pi_1^* K_{Y_1/Y} + (p-1)(D_1) = 0.$$

Since both terms are effective (since $\pi(w)$ is an RDP) we have $\pi^* K_{Y_1/Y} = (p-1)(D_1) = 0$, and since π is a homeomorphism we have $K_{Y_1/Y} = 0$. In particular $\pi(w)$ is not smooth, and there are no non-RDP singularities above $\pi(w)$. Finally, Fix $(D_1) \neq \emptyset$ is proved in the same way as the corresponding assertion in [Mat20a, Lemma 4.9(2)].

Definition 3.4 (cf. [Mat20a, Definition 4.6]). We say that an RDP surface X equipped with a p-closed derivation D is maximal at a closed point $w \in X$ (not necessarily fixed) if either $w \in X$ is a smooth point or $\pi(w) \in X^D$ is a smooth point.

We say that X, or the quotient morphism $\pi: X \to Y = X^D$, is maximal with respect to the derivation if it is maximal at every closed point. We define the maximality of μ_p - and α_p -actions similarly.

Corollary 3.5. Let $\pi: X \to Y = X^D$ as in the previous theorem. Assume that (D) = 0 and that X and Y are RDP surfaces. Then there exists an RDP surface X' and a derivation D' on X', whose quotient morphism denoted $\pi': X' \to Y'$, fitting into a diagram

$$\begin{array}{cccc} X' & \stackrel{\pi'}{\longrightarrow} & Y' & = & X'^{D'} \\ \downarrow^{g} & & \downarrow \\ X & \stackrel{\pi}{\longrightarrow} & Y & = & X^{D} \end{array}$$

with $X' \to X$ and $Y' \to Y$ surjective birational and crepant, D' = D on the isomorphic locus of $X' \to X$, $\operatorname{Fix}(D')$ isolated, $g(\operatorname{Fix}(D')) = \operatorname{Fix}(D)$, and π' maximal.

X' is characterized as the maximal partial resolution of X to which the derivation extends. If D is of multiplicative type (resp. of additive type, resp. fixed-point-free), then so is D'.

Proof. If D has a fixed RDP w (which is an isolated fixed point by assumption) then consider $X_1 = \operatorname{Bl}_w X \to X$ and $\pi_1 \colon X_1 \to X_1^{D_1} = Y_1$. where D_1 is the induced derivation on X_1 . By Theorem 3.3(2), D_1 on X_1 satisfies the same condition, and $X_1 \to X$ and $Y_1 \to Y$ are crepant. Repeating this finitely many times, we may assume X_1 has no fixed RDP.

If D_1 has a non-fixed RDP w whose image $\pi(w)$ is an RDP, then consider $X_2 = X_1 \times_{Y_1} \tilde{Y}_1$ and the induced derivation D_2 , where $\tilde{Y}_1 \to Y_1$ is the minimal resolution at $\pi(w)$. Since $w \notin \operatorname{Fix}(D_1)$ is equivalent to the existence of $f \in \mathcal{O}_{X_1,w}$ with $D_1(f) \in \mathcal{O}_{X_1,w}$, and since this property inherits to points above w, $\operatorname{Fix}(D_2)$ does not meet the fiber above w. Comparing 2-forms as in the proof of Theorem 3.3(2), we obtain $K_{X_2/X_1} = (p-1)(D_2) = 0$. Therefore $X_2 \to X_1$ and $Y_2 \to Y_1$ are crepant and D_2 on X_2 satisfies the same condition. Repeating this for the (finitely many) points w, we obtain X' with the desired properties.

The characterization follows from Lemma 3.11, which states that each exceptional curve above the remaining singularities appears in (D') with nonzero coefficient.

The final assertion is obvious for multiplicative and additive type, and for fixed-point-freeness this follows from $g(\operatorname{Fix}(D')) = \operatorname{Fix}(D)$.

Next, we classify RDPs that can be written as derivation quotients of smooth points, and give a lower bound for $\deg\langle D\rangle$ of derivations D with non-RDP quotients. The classification, as in (2), of such RDPs in characteristic 2 is also proved by Tziolas [Tzi17, Proposition 3.6].

Lemma 3.6. Let D be a nonzero p-closed derivation on B = k[[x, y]] in characteristic p. Suppose that Supp Fix(D) consists precisely of the closed point. Let $s = \deg\langle D \rangle = \dim_k B/(D(x), D(y))$.

- (1) If D is of additive type then $s \geq 2$.
- (2) Assume B^D is an RDP.
 - (a) Then (p, s, B^D) is one of those listed in Table 3. In particular, we have s = n/(p-1) in every case, where n is the index of the RDP. The table also shows an example of D (satisfying $D^p = hD$) realizing each case.
 - (b) If D is of multiplicative type, then B^D is of type A_{p-1} .
 - (c) If D is of additive type, then (p, B^D) is one of $(5, E_8^0)$, $(3, E_6^0)$, $(2, D_{4m}^0)$, or $(2, E_8^0)$.
 - (d) If D is of additive type and $(p, B^D) = (5, E_8^0), (3, E_6^0)$, then $\operatorname{Im}(D^j|_{\operatorname{Ker} D^{j+1}})$ is equal to the maximal ideal \mathfrak{n} of B^D for each $1 \leq j \leq p-1$.
- (3) Assume D is of additive type and B^D is a non-RDP. If p = 2 then $s \ge 12$. If p = 3 then s > 3. If p = 5 then s > 2.

The following corollary is an immediate consequence of this lemma and will be used in Section 4.

p	$\mathrm{deg} \langle D \rangle$	RDP	example of $D(x), D(y)$	h
any	1	A_{p-1}	x, -y	1
5	2	E_{8}^{0}	y, x^2	0
$\frac{3}{3}$	$\frac{3}{4}$	E_{6}^{0} E_{8}^{0}	$egin{array}{c} y^3,x\ y^4,x \end{array}$	$\begin{array}{c} 0 \\ y^3 \end{array}$
2 2 2 2	$ \begin{array}{c} 4m \\ 4m+2 \\ 7 \\ 8 \end{array} $	$\begin{array}{c} D^0_{4m} \\ D^0_{4m+2} \\ E^0_{7} \\ E^0_{8} \end{array}$	x^2, y^{2m} $x^2 + xy^{2m}, y^{2m+1}$ $xy^2, x^2 + y^3$ y^4, x^2	$\begin{array}{c} 0\\ y^{2m}\\ y^2\\ 0 \end{array}$

TABLE 3. RDPs arising as quotients of smooth points by *p*-closed derivations, and examples of derivations

Corollary 3.7. Suppose $A_i = k[[x, y]], 1 \le i \le N$, are respectively equipped with derivations D_i of additive type and suppose Supp Fix (D_i) consists precisely of the closed point for each *i*. Let $s_i = \deg \langle D_i \rangle = \dim_k A_i / (D_i(x), D_i(y))$. Assume $\sum s_i = 24/(p+1)$. Then either

- N = 1 and $A_1^{D_1}$ is a non-RDP and $p \ge 3$, or each $A_i^{D_i}$ is an RDP, and more precisely $(p, \{A_i^{D_i}\})$ is $(2, 2D_4^0)$, $(2, 1D_8^0)$, $(2, 1E_8^0)$, $(3, 2E_6^0)$, or $(5, 2E_8^0)$.

Proof of Lemma 3.6. (1) Since $D^p = 0$, it follows that $D|_{\mathfrak{m}/\mathfrak{m}^2}$ is nilpotent, hence for some coordinate $x, y \in \mathfrak{m}$ we have $D(x) \in \mathfrak{m}^2$.

(2a-2c) We observe that the derivation D described in Table 3 satisfies (D) = 0 and $D^p = hD$, and it realizes the RDP.

Suppose B^D is an RDP. Since the composite $\operatorname{Pic}(B^D) \to \operatorname{Pic}(B) \to$ $\operatorname{Pic}((B^D)^{(1/p)}) \cong \operatorname{Pic}(B^D)$ is equal to the *p*-th power map, and since $\operatorname{Pic}(B)$ is trivial, $Pic(B^D)$ is a p-torsion group and has no nontrivial prime-to-p torsion. The natural morphism $\operatorname{Spec} B^D \to \operatorname{Spec} B^{(p)}$ is the quotient morphism with respect to some rational p-closed derivation D' on B^{D} . Then by the Rudakov–Shafarevich formula we have

$$K_{BD} \sim \pi^* K_{B(p)} + (p-1)(D'),$$

but since both canonical divisors are trivial, we have $(p-1)(D') \sim 0$, and by above we have in fact $(D') \sim 0$. Replacing D' with $g^{-1}D'$ where (D') = $\operatorname{div}(g)$, we may assume D' is regular with (D') = 0. Then, by Theorem 3.3, the closed point is not an isolated fixed point either, and (p, B^D, D') is one of (p, X, D) listed in Table 2 with X^D smooth. Hence, after a coordinate change, (p, D) is one of those listed in Table 3 up to replacing D by a unit multiple. We obtain (2a).

It remains to check the impossibility for the derivation to be of multiplicative or additive type. Suppose D_1 is a derivation on B satisfying $(D_1) = 0$ and realizing the RDP. Then $D_1 = fD$ for some $f \in B^*$, where D is the derivation given in Table 3. By Hochschild's formula (Lemma 2.7) we have $D_1^p = (f^{p-1}h + D(g))D_1$ where $g = (fD)^{p-2}(f)$. If $h \in \mathfrak{m}$ and $(\operatorname{Im}(D)) \subset \mathfrak{m}$ then $f^{p-1}h + D(g) \neq 1$ for any $f \in B^*$. Thus we obtain (2b). If $h \notin (\text{Im}(D))$ then $f^{p-1}h + D(g) \neq 0$ for any $f \in B^*$. Thus we obtain (2c).

(3) If p > 5 then there is nothing to prove. We will check that if $p \leq 5$ and D is of additive type with s less than the bound then B^D is an RDP.

Suppose p = 5 and s = 2. We have $D|_{\mathfrak{m}/\mathfrak{m}^2} \neq 0$ and $(D|_{\mathfrak{m}/\mathfrak{m}^2})^2 = 0$. We may assume D(y) = x, $D(x) = f = y^2 + g$, $g \in (x^2, xy, y^3)$. we say that the monomial $x^i y^j$ has degree 3i + 2j and let I_n be the ideal generated by the monomials of degree $\geq n$. We have $D(I_n) \subset I_{n+1}$, $f \equiv y^2 \pmod{I_5}$, and $D^2(f) - 2(x^2 + y^3) =: h \in I_7$. Let $B' = k[[X, Y, Z]]/(-Z^5 + 2(X^2 + Y^3) + h^5) \subset B^D$ where $X = x^5$, $Y = y^5$, $Z = D^2(f) = D^4(y)$. Since $h^5 \in I_7^{(5)} = (X^3, X^2Y, XY^2, Y^4)_{k[[X,Y]]}$, B' is normal and hence $B' = B^D$, and it is an RDP of type E_8^0 .

Suppose p = 3 and s = 2, 3. We have $D|_{\mathfrak{m}/\mathfrak{m}^2} \neq 0$ and $(D|_{\mathfrak{m}/\mathfrak{m}^2})^2 = 0$. We may assume D(y) = x, D(x) = f, D(f) = 0, $f = y^s + g$, $g \in (x^2, xy, y^{s+1})$. Then since D(f) = 0 it follows that $s \neq 2$, hence s = 3, and that $g \in (x^2, xy^2, y^4)$. We say that the monomial $x^i y^j$ has degree 2i + j and let I_n be the ideal generated by the monomials of degree $\geq n$. We have $D(I_n) \subset I_{n+1}$ and $g \in I_4 = (x^2, xy^2, y^4)$. Let $B' = k[[X, Y, Z]]/(-Z^3 + X^2 + Yf^3) \subset B^D$ where $X = x^3$, $Y = y^3$, $Z = x^2 + yf$. Since $f^3 = Y^3 + g^3$ with $g^3 \in I_4^{(3)} = (X^2, XY^2, Y^4)_{k[[X,Y]]}$, B' is normal and hence $B' = B^D$, and it is an RDP of type E_6^6 .

Suppose p = 2. By Theorem 3.8 there exists $h \in k[[x, y]]^*$ and $R, S, T \in k[[x, y]]$ such that $D' = h^{-1}D$ satisfies $D'(x) = S^2 + T^2x$, $D'(y) = R^2 + T^2y$, and $D'^2 = T^2D'$. (This derivation D' is *p*-closed but not necessarily of additive type.) Suppose s < 12 and that $B^D = B^{D'}$ is not an RDP. Then by Corollary 3.9 we have $R, S \in \mathfrak{m}^2, T \in \mathfrak{m}$, and $T \notin \mathfrak{m}^2$. Since D = hD' is of additive type we have $D'(h) + hT^2 = 0$, but this is impossible since $\operatorname{Im}(D') \subset \mathfrak{m}^3$ and $hT^2 \notin \mathfrak{m}^3$.

(2d) We use the description given in the proof of (3). Suppose D is additive and $(p, B^D) = (3, E_6^0), (5, E_8^0)$. Since $\operatorname{Im}(D^{p-1}) \subset \operatorname{Im}(D^j|_{\operatorname{Ker} D_{j+1}}) \subset \operatorname{Im}(D|_{\operatorname{Ker} D^2}) \subset \mathfrak{m} \cap B^D = \mathfrak{n}$, it suffices to show $\mathfrak{n} \subset \operatorname{Im}(D^{p-1})$. If (p, s) = (5, 2), a straightforward calculation yields $D^4(y) = Z$, $D^4(x^2) \equiv y^5 = Y$ (mod $I_{11} \cap B^D$), $D^4(x^3y) \equiv x^5 = X$ (mod $I_{16} \cap B^D$). Since the initial terms of the elements Z, Z^2, Y, X have different degrees 6, 12, 10, 15, these elements are linearly independent modulo I_{15+1} , hence $D^4(y), D^4(x^2), D^4(x^3y)$ generate $\mathfrak{n}/\mathfrak{n}^2$. If (p, s) = (3, 3), a straightforward calculation yields $D^2(y) = Y + g$ $(g \in I_4 \cap B^D), D^2(y^2) = 2Z, D^2(xy^2) = 2X$. Clearly these elements generate $\mathfrak{n}/\mathfrak{n}^2$.

Theorem 3.8. Let k be an algebraically closed field of characteristic 2. Let D be a nonzero p-closed derivation on B = k[[x, y]]. Then there exist $h, R, S, T \in k[[x, y]]$, such that D = hD' where D' is the p-closed derivation defined by $D'(x) = S^2 + T^2x$ and $D'(y) = R^2 + T^2y$. It follows that

$$B^{D} = B^{D'} = k[[x^{2}, y^{2}, R^{2}x + S^{2}y + T^{2}xy]]$$

$$\cong k[[X, Y, Z]]/(Z^{2} + (R^{(2)})^{2}X + (S^{(2)})^{2}Y + (T^{(2)})^{2}XY),$$

where $X = x^2$, $Y = y^2$, $Z = R^2x + S^2y + T^2xy$. We have $D'^2 = T^2D'$ and $D^2 = (D'(h) + hT^2)D$.

Here $R^{(2)} = R^{(2)}(X, Y) \in k[[X, Y]]$ is the power series satisfying $R^{(2)}(x^2, y^2) = R(x, y)^2$, and $S^{(2)}$ and $T^{(2)}$ are defined in the same way.

We can give a classification of quotient singularities with small deg $\langle D \rangle$, using which we can complete the proof of Lemma 3.6.

Corollary 3.9. Let D, h, R, S, T, and D' be as in the previous theorem. Assume (D) = 0.

- (1) If R or S is a unit, then B^D is smooth and $\deg\langle D\rangle = 0$. Hereafter we assume this is not the case, and we implicitly make similar assumptions cumulatively.
- (2) If T is a unit, then B^D is an RDP of type A_1 .
- (3) If R and S generate \mathfrak{m} , then B^D is an RDP of type D_4^0 .
- (4) Suppose R and S generate a 1-dimensional subspace of $\mathfrak{m}/\mathfrak{m}^2$. We may assume $R \notin \mathfrak{m}^2$ and $S \in \mathfrak{m}^2$. Suppose moreover that x and R generate \mathfrak{m} . Let $m = \dim_k B/(R,S)$ and $n = \dim_k B/(R,T)$ (so $2 \leq m \leq \infty$ and $1 \leq n \leq \infty$). Since (D') = 0, at least one of m and n is finite. (e.g. $(R,S,T) = (y,x^m,0), (y,0,x^n)$.) Then B^D is an RDP of type $D^0_{\min\{4m,4n+2\}}$.
- (5) Suppose $R \notin \mathfrak{m}^2$, $S \in \mathfrak{m}^2$, and that x and R do not generate \mathfrak{m} .
 - If $\dim_k B/(R,T) = 1$ (e.g. (R,S,T) = (x,0,y)), then B^D is an RDP of type E_7^0 .
 - If $\dim_k B/(R,T) > 1$ and $\dim_k B/(R,S) = 2$ (e.g. $(R,S,T) = (x,y^2,0)$), then B^D is an RDP of type E_8^0 .
 - If $\dim_k B/(R,T) > 1$ and $\dim_k B/(R,S) = 3$ (e.g. $(R,S,T) = (x, y^3, 0)$), then B^D is an elliptic double point of the form $Z^2 + X^3 + Y^7 + \varepsilon = 0$, where $\varepsilon \in (X^5, X^3Y, X^2Y^3, XY^4, Y^9)$, and $\deg\langle D \rangle = 12$.
- (6) Suppose $R, S \in \mathfrak{m}^2$, $T \notin \mathfrak{m}^2$. We may assume $T \equiv x \pmod{\mathfrak{m}^2}$.
 - If dim_k B/(T,S) = 2 (e.g. $(R,S,T) = (0, y^2, x)$), then B^D is an elliptic double point of the form $Z^2 + X^3Y + Y^5 + \varepsilon = 0$, where $\varepsilon \in (X^5, X^4Y, X^3Y^2, X^2Y^3, XY^4, Y^7)$, and deg $\langle D \rangle = 11$.
 - If dim_k B/(T,S) > 2 and dim_k B/(T,R) = 2 (e.g. $(R,S,T) = (y^2,0,x)$), then B^D is an elliptic double point of the form $Z^2 + X^3Y + XY^4 + \varepsilon = 0$, where $\varepsilon \in (X^5, X^4Y, X^3Y^2, X^2Y^3, XY^5, Y^7)$, and deg $\langle D \rangle = 12$.
- (7) In all other cases, $\deg\langle D\rangle > 12$ and B^D is not an RDP.

If B^D is A_n , D_n , or E_n , then we have $\deg\langle D \rangle = n$.

Proof of Theorem 3.8. B^D satisfies $k[[x^2, y^2]] \subset B^D \subset k[[x, y]]$ and hence there exists $f \in k[[x, y]]$ such that $B^D = k[[x^2, y^2, f]]$. Write $f = Q^2 + R^2x + S^2y + T^2xy$ with $Q, R, S, T \in k[[x^2, y^2]]$. We have gcd(Q, R, S, T) = 1. We may assume Q = 0.

Since D(f) = 0 we have $(R^2 + T^2y)D(x) + (S^2 + T^2x)D(y) = 0$. There exists $h \in \text{Frac } B$ such that $D(x) = (S^2 + T^2x)h$ and $D(y) = (R^2 + T^2y)h$. It remains to show $h \in B$.

It suffices to show that $R^2 + T^2y$ and $S^2 + T^2x$ have no nontrivial common factor. Suppose there exists an irreducible non-unit power series $P \in k[[x, y]]$ dividing both $S^2 + T^2x$ and $R^2 + T^2y$. Since P does not divide T (since gcd(R, S, T) = 1), we have $x = S^2/T^2$ and $y = R^2/T^2$ in the quotient ring B/P, hence $B/P = (B/P)^{(2)}$, contradiction.

Proof of Corollary 3.9. Straightforward.

Convention 3.10. We use the following numbering for the exceptional curves of the resolutions of RDPs.

- $A_n: e_1, \ldots, e_n$, where $e_i \cdot e_{i+1} = 1$.
- $D_n: e_1, \ldots, e_n$, where $\{(i, j) \mid i < j, e_i \cdot e_j = 1\} = \{(1, 2), \ldots, (n 2, n 1)\} \cup \{(n 2, n)\}.$
- $E_6: e_1, e_{2\pm}, e_{3\pm}, e_4$, where $e_1 \cdot e_4 = e_{2+} \cdot e_{3+} = e_{2-} \cdot e_{3-} = e_{3\pm} \cdot e_4 = 1$.
- $E_7: e_1, \ldots, e_7$, where $\{(i, j) \mid i < j, e_i \cdot e_j = 1\} = \{(1, 2), \ldots, (5, 6)\} \cup \{(4, 7)\}.$
- E_8 : e_1, \ldots, e_8 , where $\{(i, j) \mid i < j, e_i \cdot e_j = 1\} = \{(1, 2), \ldots, (6, 7)\} \cup \{(5, 8)\}.$

Lemma 3.11. Let $X = \operatorname{Spec} B$ be a local RDP of index n in characteristic p, equipped with a p-closed derivation D, with $\operatorname{Fix}(D) = \emptyset$ and $X^D = \operatorname{Spec} B^D$ smooth. Let \tilde{X} be the resolution of X and \tilde{D} the rational derivation on \tilde{X} induced by D. Then $(\tilde{D})^2 = -2n/(p-1)$ and $\operatorname{deg}\langle \tilde{D} \rangle = n(p-2)/(p-1)$.

Proof. For each case of (p, Sing(X)), a straightforward computation yields the following description of (\tilde{D}) and $\langle \tilde{D} \rangle$, from which the stated equalities follow. The cases for p = 2 also appear in [EHSB12, Lemma 6.5].

If p = 2, then $\langle D \rangle = 0$. For every case, each closed point in $\operatorname{Supp} \langle D \rangle$ appears with degree 1, so we write only the support. We denote by q_{ij} the intersection of e_i and e_j , and by q'_i a certain point on e_i (not lying on the other components).

$$(p, A_{p-1}): (D) = -\sum e_i, \langle D \rangle = \{q_{i,i+1} \mid 1 \le i \le p-2\}.$$

$$(2, D_{2m}^0): (\tilde{D}) = -(\sum_{i=1}^{m-1} (2ie_{2i-1} + 2ie_{2i}) + me_{2m-1} + me_{2m}).$$

$$(2, E_7^0): (\tilde{D}) = -(3e_1 + 4e_2 + 7e_3 + 8e_4 + 6e_5 + 2e_6 + 5e_7).$$

$$(2, E_8^0): (\tilde{D}) = -(2e_1 + 6e_2 + 8e_3 + 12e_4 + 14e_5 + 10e_6 + 4e_7 + 8e_8).$$

$$(3, E_6^0): (\tilde{D}) = -(2e_1 + 2e_{2+} + 2e_{2-} + 3e_{3+} + 3e_{3-} + 3e_4), \langle \tilde{D} \rangle = \{q_1', q_{2+,3+}, q_{2-,3-}\}.$$

$$(3, E_8^0): (\tilde{D}) = -(2e_1 + 3e_2 + 6e_3 + 8e_4 + 9e_5 + 7e_6 + 4e_7 + 5e_8), \langle \tilde{D} \rangle =$$

$$\{q_1', q_{34}, q_{67}, q_8'\}.$$

$$(5, E_9^0): (\tilde{D}) = -(2e_1 + 3e_2 + 4e_2 + 5e_4 + 5e_5 + 4e_6 + 2e_7 + 3e_8), \langle \tilde{D} \rangle =$$

 $(5, E_8^0): (D) = -(2e_1 + 3e_2 + 4e_3 + 5e_4 + 5e_5 + 4e_6 + 2e_7 + 3e_8), \langle D \rangle = \{q_{12}, q_{23}, q_{34}, q_{67}, q'_7, q'_8\}.$

4. μ_p - and α_p -actions on RDP K3 surfaces

Proposition 4.1. Let $G = \mu_p$ or $G = \alpha_p$. Let X be an RDP K3 surface or an RDP Enriques surface equipped with a nontrivial G-action and let D be the corresponding derivation. If the divisorial part (D) of Fix(D) is zero and each point in $\pi(\text{Fix}(D))$ is either smooth or an RDP, then X/G is an RDP K3 surface or an RDP Enriques surface. Otherwise, X/G is a (possibly singular) rational surface.

If X is an RDP K3 surface, then X/G is an RDP Enriques surface if and only if the G-action is fixed-point-free (Fix $(D) = \emptyset$), and in this case we have p = 2.

Proof. Let Y = X/G. By the Rudakov–Shafarevich formula, $\pi^*K_Y \sim K_X - (p-1)(D)$, hence $K_Y \leq 0$ in $(\operatorname{Pic}(Y) \otimes \mathbb{Q})/\equiv$, and $K_Y \equiv 0$ if and only if (D) = 0. We have $\operatorname{Sing}(Y) \subset \pi(\operatorname{Sing}(X) \cup \operatorname{Fix}(D))$, and each point of $\pi(\operatorname{Sing}(X) \setminus \operatorname{Fix}(D))$ is either a smooth point or an RDP by Theorem 3.3(1). Let $\rho: \tilde{Y} \to Y$ be the resolution. Then $K_{\tilde{Y}} \leq \rho^* K_Y$ and the equality holds if and only if $\operatorname{Sing}(Y)$ consists only of RDPs. We deduce that $K_{\tilde{Y}} \equiv 0$ if and only if (D) = 0 and each point in $\pi(\operatorname{Fix}(D))$ is either smooth or an RDP. In this case Y is a proper RDP surface with $\kappa(\tilde{Y}) = 0$. Otherwise we have $\kappa(\tilde{Y}) = -\infty$.

Next we will show that Y is not birational to abelian, (quasi-)hyperelliptic, or non-rational ruled surface. Since π is purely inseparable we have $b_1(\tilde{X}) = b_1(X) = b_1(\tilde{Y}) = b_1(\tilde{Y})$, where $b_i = \dim_{\mathbb{Q}_l} H^i_{\text{ét}}(-, \mathbb{Q}_l)$ are the *l*-adic Betti numbers for an auxiliary prime $l \neq p$. Since \tilde{X} is a K3 surface or an Enriques surface we have $b_1(\tilde{X}) = 0$. Hence \tilde{Y} is not abelian, (quasi-)hyperelliptic, nor non-rational ruled, since such surfaces have $b_1 > 0$. Thus the first assertion follows.

Suppose X is an RDP K3 surface. To show the equivalence of freeness and Enriques quotient, we may assume that Fix(D) is isolated and that, by Corollary 3.5, π is maximal. By the equality s = n/(p-1) of Lemma 3.6(2a), [Mat20a, Proposition 6.10] (which is stated for μ_p -actions) holds also for α_p -actions, from which the equivalence and p = 2 follows. \Box

Remark 4.2. Suppose X is an RDP K3 surface. If $G = \mu_p$, the author showed [Mat20a, Theorems 6.1 and 6.2] that X/μ_p is an RDP K3 surface if and only if the action is *symplectic* ([Mat20a, Definition 2.6]) in the sense that the nonzero global 2-form ω on X^{sm} , which is unique up to scalar, is *D*-invariant (i.e. $D(\omega) = 0$). Note that since $D^p = D$ we always have $D(\omega) = i\omega$ for some $i \in \mathbb{F}_p$. If $G = \alpha_p$, then this criterion fails since, in fact, any action is symplectic in this sense, since $D^p = 0$. This difference is parallel to that of actions of tame and wild finite groups (i.e. of order not divisible or divisible by p).

Theorem 4.3. Let X and Y be RDP surfaces with K_X numerically trivial and K_Y trivial. If $\pi: X \to Y$ is the quotient morphism by either a μ_p action or an α_p -action, then so is the induced morphism $\pi': Y \to X^{(p)}$ (not necessarily by the same group).

Proof. Let D be the derivation on X corresponding to the action. By the Rudakov–Shafarevich formula $K_X \sim \pi^* K_Y + (p-1)(D)$, we have $(p-1)(D) \equiv 0$. Since (D) is effective and numerically trivial, it follows that $(D) \sim 0$.

Let D' be a rational *p*-closed derivation on Y inducing π' , i.e. $Y^{D'} = X^{(p)}$. (To find one, take a generator h of $k(Y)/k(X^{(p)})$ (so $h^p \in k(X^{(p)})$), and define D' by $D'|_{k(X^{(p)})} = 0$ and D'(h) = 1. Then $D'^p = 0$, in particular D' is *p*-closed.) By Proposition 2.14, we have $K_Y \sim -(D') - \pi'^*((D^{(p)})$. Since $K_Y \sim 0$ and $(D^{(p)}) \sim 0$, we have $(D') = \operatorname{div}(g)$ for some rational function $g \in k(Y)^*$. Then $D'' := g^{-1}D'$ is a regular derivation on Y with $Y^{D''} = Y^{D'} = X^{(p)}$ and (D'') = 0. By Hochschild's formula D'' is also *p*-closed, hence $D''^p = \lambda D''$ for some everywhere regular function λ on Y, hence $\lambda \in k$, and by replacing D'' with a scalar multiple we may assume $\lambda = 0$ or $\lambda = 1$, and then D'' gives either an α_p - or μ_p -action respectively.

Remark 4.4. There exist finite inseparable morphisms of degree p between RDP K3 surfaces that are not μ_p - nor α_p -quotients. Classification of such morphisms will be given in Section 5.

Theorem 4.3 fails also if π is a μ_2 -quotient with Y an Enriques surface (so that K_Y is nontrivial), as in the next proposition, proved by the same way as Theorem 4.3.

Proposition 4.5 (cf. [CD89, Section 1.2]). Let X be an RDP K3 surface in characteristic p = 2 and $\pi: X \to Y$ a μ_2 -quotient morphism with Y an RDP classical Enriques surface. Then $\pi': Y \to X^{(2)}$ is not the quotient morphism by a p-closed (regular) derivation. Instead π' is the quotient morphism by a p-closed rational derivation D' on Y with $(D') \sim K_Y$.

Suppose X and Y are RDP K3 surfaces. We will determine possible characteristics and singularities.

Theorem 4.6. Let $\pi: X \to Y$ be a *G*-quotient morphism between RDP K3 surfaces in characteristic p, where $G \in \{\mu_p, \alpha_p\}$. If $G = \mu_p$ then $p \leq 7$. If $G = \alpha_p$ then $p \leq 5$.

If moreover π is maximal, then $\operatorname{Sing}(Y)$ are as follows.

- $\frac{24}{p+1}A_{p-1}$ if $G = \mu_p$. $2D_4^0$, $1D_8^0$, or $1E_8^0$ if $G = \alpha_2$. $2E_6^0$ if $G = \alpha_3$. $2E_8^0$ if $G = \alpha_5$.

By Theorem 4.3, X is a G'-quotient of $Y^{(1/p)}$ for $G' \in \{\mu_p, \alpha_p\}$, and hence Sing(X) is also as described above. In particular, the total index of RDPs of X and that of Y are both equal to 24(p-1)/(p+1).

Remark 4.7. Suppose X is a smooth K3 surface and $G \subset Aut(X)$ a cyclic subgroup of prime order p. Assume Y = X/G is an RDP K3 surface. If $\operatorname{char}(k) \neq p$ then it is well-known that $\operatorname{Sing}(Y)$ is $\frac{24}{p+1}A_{p-1}$, and in particular the total index of RDPs of Y is equal to 24(p-1)/(p+1). We will see below (Theorem 7.3) that this value is equal to 24(p-1)/(p+1) even in characteristic p. Consequently, this value 24(p-1)/(p+1) appears for actions of any group scheme G of order p in any characteristic!

Proof of Theorem 4.6. We may assume π is maximal. First we prove the assertion for the total indices of $\operatorname{Sing}(X)$ and $\operatorname{Sing}(Y)$. Let $\{w_i\} \subset X$ and $\{v_i\} \subset Y$ be the RDPs, of indices m_i and n_j respectively. Let X be the resolution of X and \tilde{D} the induced rational derivation on \tilde{X} . Using Lemma 3.6(2) and Lemma 3.11 we obtain

$$(\tilde{D})^2 = -\sum_i \frac{2}{p-1} m_i, \qquad \deg \langle \tilde{D} \rangle = \sum_i \frac{p-2}{p-1} m_i + \sum_j \frac{1}{p-1} n_j.$$

By Theorem 2.4 we have $24 = \deg \langle \tilde{D} \rangle - (\tilde{D})^2 = \sum_i \frac{p}{p-1} m_i + \sum_j \frac{1}{p-1} n_j$.

We can apply the same argument to $\pi' \colon Y \to X^{(p)}$ to obtain another equality. Also, since π is purely inseparable we have dim $H^2_{\text{ét}}(X, \mathbb{Q}_l) =$

dim $H^2_{\text{ét}}(Y, \mathbb{Q}_l)$ and hence $\sum_i m_i = \sum_j n_j$. By either way, we obtain $\sum_i m_i = \sum_j n_j = 24(p-1)/(p+1)$.

Each v_j is one of those appearing in Table 3. If $G = \alpha_p$ then we have $p \leq 5$ and then $\operatorname{Sing}(Y)$ is as stated. If $G = \mu_p$ then $\operatorname{Sing}(Y)$ is as stated, and hence $(p+1) \mid 24$ and 24(p-1)/(p+1) < 22. This implies $p \leq 11$. We refer to [Mat20a, Theorem 7.1] for a proof of $p \neq 11$.

5. Inseparable morphisms of degree p between RDP K3 surfaces

Suppose $\pi: X \to Y$ is a finite inseparable morphism of degree p between RDP K3 surfaces. It is not always a quotient morphism by a global regular derivation. However it can be covered by such a quotient morphism, and we have a classification as in Theorem 5.2.

Lemma 5.1. Let r > 1 be an integer prime to $p = \operatorname{char} k$. Suppose either $M = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$ $(\lambda \in k^*)$, or r is even and $M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then there is no $g \in \operatorname{SL}_2(k)$ of order r such that $g^{-1}Mg = \zeta M$ with a primitive r-th root ζ of 1.

Proof. If 2 | r, then $g^{r/2} \in SL_2(k)$ is of order 2, hence $g^{r/2} = -I_2$, which is central. If r > 2 in the former case, then M and ζM have different eigenvalues.

Theorem 5.2. Suppose $\pi: X \to Y$ is a finite inseparable morphism of degree p between RDP K3 surfaces. Then for some $r \ge 1$ and some $G \in \{\mu_p, \alpha_p\}$, there exists a $\mathbb{Z}/r\mathbb{Z}$ -equivariant G-quotient morphism $\overline{\pi}: \overline{X} \to \overline{Y}$ between proper RDP surfaces equipped with $\mathbb{Z}/r\mathbb{Z}$ -actions, fitting into a commutative diagram

$$\begin{array}{ccc} \bar{X} & \xrightarrow{\pi} & \bar{Y} \\ \downarrow \phi_X & & \downarrow \phi_Y \\ X & \xrightarrow{\pi} & Y \end{array}$$

such that $\phi_X \colon \overline{X} \to X$ and $\phi_Y \colon \overline{Y} \to Y$ are the $\mathbb{Z}/r\mathbb{Z}$ -quotient morphisms.

Among such "coverings" $\bar{\pi}$, there exists a minimal one (i.e. any other such covering admits $\bar{\pi}$ as a subcovering). If $\bar{\pi}$ is minimal, then $r \in \{1, 2, 3, 4, 6\}$ and $r \mid p - 1$, the $\mathbb{Z}/r\mathbb{Z}$ -actions on \bar{X} and \bar{Y} are symplectic (in the usual sense on abelian and K3 surfaces), and moreover exactly one the following holds:

- (1) \bar{X} and \bar{Y} are (smooth) abelian surfaces, and $r \neq 1$;
- (2) \bar{X} and \bar{Y} are RDP K3 surfaces, $G = \mu_p, p \leq 7$, and $(p,r) \neq (7,2), (7,6); or$
- (3) \overline{X} and \overline{Y} are RDP K3 surfaces, $G = \alpha_p, p \leq 5$, and $(p, r) \neq (5, 4)$.

Every case and every remaining (p, r) occurs.

If $\bar{\pi}$ is minimal and moreover maximal (in the sense of Definition 3.4), then Sing(Y) is as described in Table 4.

Proof. As in the proof of Theorem 4.3, take a rational derivation D with $Y = X^D$. Then we have (p-1)(D) = 0 in $\operatorname{Pic}(X^{\operatorname{sm}})$. Let $\phi: \overline{X^{\operatorname{sm}}} \to X^{\operatorname{sm}}$

covering	p	r	$\operatorname{Sing}(Y)$
abelian	$\equiv 1 \pmod{6}$	6	$A_5 + 4A_2 + 5A_1$
abelian	$\equiv 1 \pmod{4}$	4	$4A_3 + 6A_1$
abelian	$\equiv 1 \pmod{3}$	3	$9A_2$
abelian	$\equiv 1 \pmod{2}$	2	$16A_1$
K3, μ_7	7	3	$A_6 + 6A_2$
K3, μ_7	7	1	$3A_6$
K3, μ_5	5	4	$A_4 + 4A_3 + 2A_1$
K3, μ_5	5	2	$2A_4 + 8A_1$
K3, μ_5	5	1	$4A_4$
K3, μ_3	3	2	$3A_2 + 8A_1$
K3, μ_3	3	1	$6A_2$
K3, μ_2	2	1	$8A_1$
K3, α_5	5	2	$E_8^0 + 8A_1$
K3, α_5	5	1	$2\tilde{E}_{8}^{0}$
K3, α_3	3	2	$E_{6}^{0} + 8A_{1}$
K3, α_3	3	1	$2E_{6}^{0}$
K3, α_2	2	1	$2D_4^0, 1D_8^0, \text{ or } 1E_8^0$

TABLE 4. Structure of purely inseparable morphisms of degree p between RDP K3 surfaces

be the étale covering trivializing (D) (so $r = \deg \phi$ divides p - 1). Then the normalization \bar{X} of X in $k(\overline{X^{\text{sm}}})$ is an RDP surface.

We claim that \bar{X} is an RDP K3 surface or an abelian surface. This is trivial if r = 1. Assume $r \ge 2$, hence $p \ge 3$. By construction \bar{X} has trivial canonical divisor. If \bar{X} is not RDP K3 nor abelian, then it is (quasi-)hyperelliptic surface in characteristic p = 3. Hence r = 2. Comparing the *l*-adic Euler– Poincaré characteristic (which is 0 and 24 for (quasi-)hyperelliptic and K3 surfaces), we observe that the involution g on the resolution \tilde{X} has 16 fixed points, but then we have

$$22 - 16 = \dim H^2_{\text{\'et}}(\tilde{\bar{X}}/\langle g \rangle, \mathbb{Q}_l) = \dim H^2_{\text{\'et}}(\tilde{\bar{X}}, \mathbb{Q}_l)^{\langle g \rangle} \le \dim H^2_{\text{\'et}}(\tilde{\bar{X}}, \mathbb{Q}_l) = 2,$$

a contradiction.

We have $\phi^{-1}((D)) = \operatorname{div}(h)$ for some $h \in k(\bar{X})$, and then $\bar{D} := h^{-1} \cdot \phi^*(D)$ is a regular derivation. Write $\bar{X}^{\bar{D}} = \bar{Y}$. Take a generator g_X of the $\mathbb{Z}/r\mathbb{Z}$ action on \bar{X} . Then g_X acts on \bar{D} by multiplication by a *r*-th root λ of unity. This λ is in fact a primitive *r*-th root of unity since, if $\lambda^s = 1$, then \bar{D} descends to $\bar{X}/\langle g_X^s \rangle$, hence (D) is trivialized on $\bar{X}/\langle g_X^s \rangle$, hence $g_X^s = 1$, hence $r \mid s$. Hence g_X induces an automorphism g_Y on \bar{Y} of order r with $\bar{Y}/\langle g_Y \rangle = Y$.

We show the minimality. Let $\psi: \bar{X}' \to X$ with \bar{D}' be another covering of π with the required properties. Then the pullback $\psi^*(D)$ of D to \bar{X}' coincide with \bar{D}' up to $k(X)^*$, in particular $(\psi^*(D)) \sim 0$ on $\operatorname{Pic}(\psi^{-1}(X^{\operatorname{sm}}))$. Hence $\psi|_{\psi^{-1}(X^{\operatorname{sm}})}$ factors through $\phi|_{\phi^{-1}(X^{\operatorname{sm}})}$, and ψ factors through ϕ . We show $r \in \{2, 3, 4, 6\}$ and the description of the singularities in the case \bar{X} is an abelian surface. It is proved by Katsura [Kat87, Theorem 3.7 and Table in page 17] that, if \bar{X} is an abelian surface and g is a nontrivial symplectic automorphism (fixing the origin) of order r prime to $p = \operatorname{char} k$, then $r \in \{2, 3, 4, 5, 6, 8, 10, 12\}, \bar{X}/\langle g \rangle$ is an RDP K3 surface, and $\operatorname{Sing}(\bar{X}/\langle g \rangle)$ are as in Table 5 (in [Kat87] the coefficient of A_7 in order 8 is written as 1, but this is a misprint and actually it is 2). In particular, if $r \in \{5, 8, 10, 12\}$ then (since the exceptional curves of the resolution of $\bar{X}/\langle g \rangle$ generate a rank 20 negative-definite lattice) $\bar{X}/\langle g \rangle$ is a supersingular RDP K3 surface and \bar{X} is a supersingular abelian surface. It is showed [Kat87, Lemma 6.3] that supersingular abelian surfaces in characteristic p do not have symplectic automorphisms of order r = 5 if $p \equiv 1 \pmod{5}$. One observes that the proof of this lemma relies only on the fact that $[\mathbb{Q}(\zeta_5): \mathbb{Q}] = 4$, therefore it remains valid if we replace 5 with 8, 10, or 12. Hence we obtain $r \in \{2, 3, 4, 6\}$ in our case.

Suppose X is an RDP K3 surface and $\bar{\pi}$ is a μ_p -quotient or an α_p -quotient. Then respectively $p \leq 7$ or $p \leq 5$ by Theorem 4.6. We show that if r > 1 then g_X does not fix any point of $\operatorname{Fix}(\bar{D})$. If $G = \mu_p$, then the action of \bar{D} on the tangent space of a point of $\operatorname{Fix}(\bar{D})$ is diagonalizable with eigenvalues $\pm i$ $(i \in \mathbb{F}_p^*)$. If $G = \alpha_p$, then $p \in \{3, 5\}$ and hence $r \in \{2, 4\}$, and the action of \bar{D} on the tangent space is nilpotent and nontrivial (otherwise $s \geq 3$ in Lemma 3.6). Hence in either case it is impossible by Lemma 5.1.

Using this, we show that $(G,r) = (\mu_7, 2), (\mu_7, 6), (\alpha_5, 4)$ cannot happen. If $(G,r) = (\mu_7, 2)$, then Fix (\overline{D}) consists of 3 points $w_1, w_2, w_4 \in \overline{X}$, on whose tangent spaces \overline{D} acts by eigenvalues $\pm 1, \pm 2, \pm 4$ respectively. Since $g_X^*\overline{D} = -\overline{D}$, we have $g_X(w_1) \neq w_2, w_4$, and we have $g_X(w_1) \neq w_1$ by above. Contradiction. The case $(G,r) = (\mu_7, 6)$ is reduced to the previous case. If $(G,r) = (\alpha_5, 4)$, then Fix (\overline{D}) consists of 2 points, hence g_X^2 fixes each point. Contradiction.

The assertion on $\operatorname{Sing}(Y)$ follows from the description of $\operatorname{Sing}(\bar{Y})$ (Theorem 4.6), the description of the fixed locus and the quotient singularities of a symplectic automorphism of finite order prime to the characteristic (Nikulin [Nik79, Section 5] (p = 0) and Dolgachev–Keum [DK09, Theorem 3.3] (p > 0)), and the observation above that $\langle g_X \rangle$ acts freely on Fix(\bar{D}).

We will see in Examples 10.2–10.8 (r = 1), 10.12 $(r > 1, \bar{X} \text{ abelian})$, 10.14 $(r > 1, \bar{X} \text{ K3})$ that all cases indeed occur.

6. $\mathbb{Z}/p\mathbb{Z}$ -, μ_p -, α_p -coverings of RDPs

In this section we describe $\mathbb{Z}/p\mathbb{Z}$ -, μ_p -, and α_p -coverings of certain RDPs that are related to $\mathbb{Z}/p\mathbb{Z}$ -, μ_p -, and α_p -coverings of RDP K3 surfaces discussed in Section 7.

6.1. μ_p -coverings. Let $Z = \operatorname{Spec} A$ be a local ring that is an RDP of type A_{n-1} , in characteristic $p \geq 0$ (possibly dividing n). Let $\tilde{Z} \to Z$ be the minimal resolution and let e_j $(1 \leq j \leq n-1)$ be the exceptional curves numbered as in Convention 3.10 (i.e. $e_j \cdot e_{j'} = 1$ if and only if |j - j'| = 1).

Lemma 6.1.

r	$\operatorname{Sing}(X)$
2	$16A_1$
3	$9A_2$
4	$4A_3 + 6A_1$
5	$5A_4$
6	$A_5 + 4A_2 + 5A_1$
8	$2A_7 + A_3 + 3A_1$
10	$A_9 + 2A_4 + 3A_1$
12	$A_{11} + A_3 + 2A_2 + 2A_1$

TABLE 5. RDP K3 surfaces arising as symplectic cyclic quotients of abelian surfaces [Kat87, Table in page 17]

(1) There is a canonical injection from $\operatorname{Pic}(Z^{\operatorname{sm}})$ to a cyclic group of order n. It is compatible with étale extensions of A and it is an isomorphism if A is Henselian.

In the following assertions, we assume that the injection in (1) is an isomorphism.

(2) For each 0 < h < n, let L_h be a line bundle on Z^{sm} belonging to the class $h \in \mathbb{Z}/n\mathbb{Z} \cong \text{Pic}(Z^{\text{sm}})$. Let $L_0 = \mathcal{O}_{Z^{\text{sm}}}$. Let $\phi_0 = \text{id}_{L_0}$ and $\phi_1 = \text{id}_{L_1}$. Take isomorphisms $\phi_h \colon L_h \xrightarrow{\sim} L_1^{\otimes h}$ $(2 \le h < n)$ and $\psi \colon L_0 \xrightarrow{\sim} L_1^{\otimes n}$. Then the morphisms

$$\phi_{h+h'}^{-1} \circ (\phi_h \otimes \phi_{h'}) \colon L_h \otimes L_{h'} \to L_{h+h'} \quad (h+h' < n),$$
$$(\phi_{h+h'-n}^{-1} \otimes \psi^{-1}) \circ (\phi_h \otimes \phi_{h'}) \colon L_h \otimes L_{h'} \to L_{h+h'-n} \quad (h+h' \ge n)$$

define an $\mathcal{O}_{Z^{\mathrm{sm}}}$ -algebra structure on $V := \bigoplus_{h=0}^{n-1} L_h$.

- (3) Let $\bar{L}_h := \iota_* L_h$ and $\bar{V} = \iota_* V = \bigoplus_{h=0}^{n-1} \bar{L}_h$, where $\iota: Z^{\mathrm{sm}} \to Z$ is the inclusion. Then the $\mathcal{O}_{Z^{\mathrm{sm}}}$ -algebra structure on V extends to an \mathcal{O}_Z -algebra structure on \bar{V} , and \bar{V} is regular. $U := \operatorname{Spec} \bar{V} \to Z$ is a μ_n -covering.
- (4) Let $\tilde{L}_h = \tilde{\iota}_* L_h$, where $\tilde{\iota}: Z^{\mathrm{sm}} \to \tilde{Z}$ is the inclusion. Then $I_h := \operatorname{Im}((\tilde{L}_h)^{\otimes n} \to \mathcal{O}_{\tilde{Z}})$ is an invertible sheaf and, writing $I_h = \mathcal{O}_{\tilde{Z}}(-\sum b_{h,j}e_j)$, there exists $a \in (\mathbb{Z}/n\mathbb{Z})^*$ such that $b_{h,j} \equiv ahj \pmod{n}$. More precisely, we have

$$I_h = \mathcal{O}(-\sum_j f_n((ah \mod n), j)e_j).$$

Here $m \mod n$ denotes the remainder modulo n, i.e., the unique integer $\in \{0, \ldots, n-1\}$ congruent to m modulo n, and the function $f_n: \{1, 2, \ldots, n-1\}^2 \to \mathbb{Z}$ is defined by

$$f_n(h,j) = \begin{cases} hj & (j \le n-h) \\ (n-h)(n-j) & (j \ge n-h). \end{cases}$$

Proof. (1) This is [Lip69, Proposition 17.1].

(2) Straightforward.

(3) We may assume that A is complete. By changing the isomorphism $\mathbb{Z}/n\mathbb{Z} \cong \operatorname{Pic}(Z^{\operatorname{sm}})$, we may assume that $A = k[[x^n, y^n, xy]] \subset B = k[[x, y]]$

and identify \overline{L}_h with $x^h A + y^{n-h} A \subset B$ for 0 < h < n, and ϕ_h^{-1} with the multiplication in B. We have $\psi^{-1}(x^{\otimes n}) = ax^n$ with $a \in A^*$. Replacing B = k[[x, y]] with k[[x', y']] $(x' = a^{1/n}x, y' = a^{-1/n}y)$, and identifying $x^hA + y^{n-h}A \xrightarrow{\sim} x'^hA + y'^{n-h}A$ by the multiplication by $(a^{1/n})^h$, we may assume a = 1. Then $\overline{V} = B$ and is regular.

(4) Straightforward (cf. [Mat20a, Lemma 4.16]).

Remark 6.2. Suppose A is Henselian. If $p \nmid n$, then $U \rightarrow Z$ is independent of the choices (since, under the notation in the proof, $a^{1/n}$ exists in A^*) and $U|_{Z^{\rm sm}} \to Z^{\rm sm}$ is the fundamental covering. To the contrary, if $p \mid n$, then $U \to Z$ does depend on the choice of the isomorphisms ϕ_h and ψ , and is not unique.

6.2. $\mathbb{Z}/p\mathbb{Z}$ - and α_p -coverings.

Lemma 6.3. Let A be a Noetherian Gorenstein 2-dimensional local kalgebra, and $I \subset A$ an ideal with $\operatorname{Supp}(A/I) \subset \{\mathfrak{m}_A\}$ (equivalently $I \supset \mathfrak{m}_A^n$) for some n). Then $\dim_k \operatorname{Ext}^1(I, A) = \dim_k A/I$. For any other such ideal I' with $I' \subset I$, the induced map $\operatorname{Ext}^1(I, A) \to \operatorname{Ext}^1(I', A)$ is injective. The map $\operatorname{Ext}^1(I,A) \xrightarrow{\sim} \operatorname{Ext}^2(A/I,A) \to H^2_{\mathfrak{m}_A}(A)$ is injective and its image is the *I*-torsion part $H^2_{\mathfrak{m}_A}(A)[I]$.

If x, y is a regular sequence in \mathfrak{m}_A , then we have an isomorphism $H^2_{\mathfrak{m}_A}(A) \xrightarrow{\sim} H^2_{\mathfrak{m}_A}(A)$ $\operatorname{Coker} \left(A[x^{-1}] \oplus A[y^{-1}] \to A[(xy)^{-1}] \right).$

Proof. See [Mat20b, Lemma 3.1]. The assertion on the dimension follows from dim $\operatorname{Ext}^{1}(\mathfrak{m}, A) = \operatorname{dim} \operatorname{Ext}^{2}(k, A) = 1$, which follows from Gorenstein.

Lemma 6.4. Let A and I be as in the previous Lemma. Then there are canonical semilinear maps $F \colon \operatorname{Ext}^1(I, A) \to \operatorname{Ext}^1(I^{(p)}, A)$ and $F \colon \operatorname{Ext}^2(A/I, A) \to$ $\operatorname{Ext}^2(A/I^{(p)}, A)$, which we call the Frobenius, satisfying the following properties.

- F commute with the boundary maps and the pullbacks by inclusions $I' \hookrightarrow I$ of ideals.
- $h_{I^{(p)}}(F(e)) = (h_I(e))^p$, where h_I is the map $\operatorname{Ext}^1(I, A) \xrightarrow{\sim} \operatorname{Ext}^2(A/I, A) \rightarrow H^2_{\mathfrak{m}_A}(A) \xrightarrow{\sim} \operatorname{Coker}\left(A[x^{-1}] \oplus A[y^{-1}] \rightarrow A[(xy)^{-1}]\right)$ defined in Lemma 6.3.

Proof. We define the maps on the local cohomology groups $H^2_{\mathfrak{m}_A}(A)$, and use the identification of Lemma 6.3.

Now let $Z = \operatorname{Spec} A$ be a local RDP in characteristic p and suppose $(p, \operatorname{Sing}(Z))$ is one of the following, and define an integer $m \geq 1$ accordingly.

- $(p, \operatorname{Sing}(Z)) = (2, D_{4m}^r), m \ge 1, r \in \{0, \dots, m\}.$
- $(p, \operatorname{Sing}(Z)) = (2, E_8^r), r \in \{0, 1, 2\}, \text{ let } m = 2.$
- $(p, \operatorname{Sing}(Z)) = (3, E_6^r), r \in \{0, 1\}, \text{ let } m = 1.$
- $(p, \operatorname{Sing}(Z)) = (5, E_8^r), r \in \{0, 1\}, \text{ let } m = 1.$

Thus, in each case, the range of r is $\{0, \ldots, m\}$.

(The RDPs of type D_{4m}^r $(r \in \{m + 1, \dots, 2m - 1\})$ and E_8^r $(r \in \{3, 4\})$ in characteristic 2 will not be discussed in this paper.)

We assume A is complete, and we fix the presentation A = k[[x, y, z]]/(F) as follows, for each case of (p, Sing(Z)).

$$\begin{split} &(2,D_{4m}^r)\colon F=z^2+x^2y+xy^{2m}+\lambda zxy^m,\qquad \lambda=0,y^{m-r}\quad (r=0,r>0),\\ &(2,E_8^r)\colon F=z^2+x^3+y^5+\lambda zxy^2,\qquad \lambda=0,y,1\quad (r=0,1,2),\\ &(3,E_6^r)\colon F=-z^2+x^3+y^4+\lambda x^2y^2,\qquad \lambda=0,1\quad (r=0,1),\\ &(5,E_8^r)\colon F=z^2+x^3+y^5+(\lambda/2)xy^4,\qquad \lambda=0,2\quad (r=0,1). \end{split}$$

Write $x_1 = x$ and $x_2 = y$. Let $Z_i = \operatorname{Spec} A[x_i^{-1}]$. Define $\bar{q}_i \in A[x_i^{-1}]$ as below and let $\bar{\varepsilon} := z/(xy^m)$. Then we have $\bar{\varepsilon}^p - \lambda \bar{\varepsilon} = \bar{q}_1 - \bar{q}_2$.

$$\bar{q}_1 := \begin{cases} x^{-1}, & & \\ x^{-2}y, & & \\ x^{-3}yz, & & \\ x^{-5}(y^5 + \lambda xy^4 + (\lambda^2/4)x^2y^3 + 2x^3)z, & & \\ \end{array} \quad \bar{q}_2 := \begin{cases} y^{-(2m-1)}, & \\ y^{-4}x, & \\ -y^{-3}z, & \\ -y^{-3}z, & \\ -y^{-5}xz. \end{cases}$$

Note that $\bar{\varepsilon}$ itself cannot be written as $\bar{\varepsilon} = q'_1 - q'_2$ with $q'_i \in A[x_i^{-1}]$.

Define an ideal $I \subset A$ to be $I = (x, y^m, z)$ according to the convention on m and the presentation given above. In fact, this ideal can be defined intrinsically (without assuming completeness) as follows:

- If $(p, \operatorname{Sing}(Z))$ is $(2, D_4^r)$, $(3, E_6^r)$, or $(5, E_8^r)$, then I is the maximal ideal \mathfrak{m} .
- If $(p, \operatorname{Sing}(Z))$ is $(2, D_{4m}^r)$ (resp. $(2, E_8^r)$), then *I* consists of the elements that vanish on the component e_{2m} (resp. e_4) with order $\geq 2m$ (resp. ≥ 8), where the components are numbered as in Convention 3.10.

Lemma 6.5. $\operatorname{Ext}_{A}^{1}(I, A)$ is m-dimensional as a k-vector space, and generated by the class \overline{e} of $\overline{\varepsilon}$ as an A-module (under the identification of Lemma 6.3).

Proof. By Lemma 6.3, we have dim $\operatorname{Ext}_A^1(I, A) = \dim_k(A/I) = m$. We also have $I \subset \operatorname{Ann}(\bar{e})$. It remains to show that $\operatorname{Ann}(\bar{e}) \subset I$. It suffices to show $y^{m-1} \notin \operatorname{Ann}(\bar{e})$. Using the isomorphism $A = k[[x,y]] \oplus k[[x,y]]z$ of k[[x,y]]-modules, we see that the class of $y^{m-1}\bar{\varepsilon} = zy^{m-1}/(xy^m) = z/(xy)$ in $\operatorname{Coker}\left(A[x^{-1}] \oplus A[y^{-1}] \to A[(xy)^{-1}]\right)$ is nontrivial. \Box

Lemma 6.6. Let $q_i \in A[x_i^{-1}]$ and $\varepsilon \in A[(xy)^{-1}]$. Suppose $\varepsilon^p - \lambda \varepsilon = q_1 - q_2$, and the class $[\varepsilon]$ is a generator of $\operatorname{Ext}^1_A(I, A)$. Let $U_i \to Z_i$ be the coverings given by $\mathcal{O}_{U_i} = \mathcal{O}_{Z_i}[t_i]/(t_i^p - \lambda t_i - q_i)$, glue them on $Z_1 \cap Z_2$ by $t_1 - t_2 = \varepsilon$, and let $U = \operatorname{Spec} B \to Z = \operatorname{Spec} A$ be the normalization of $U_1 \cup U_2 \to Z_1 \cup Z_2 = Z^{\operatorname{sm}} \subset Z$. Then the following assertions hold.

- (1) Let e be the class of ε in $\operatorname{Ext}_{A}^{1}(I, A)$ Then $e = h \cdot \overline{e}$ for some $h \in (k[y]/y^{m})^{*}$. We have $(\iota^{*}(e) \neq 0 \text{ and}) F(e) = \lambda \cdot \iota^{*}(e)$, where $F \colon \operatorname{Ext}^{1}(I, A) \to \operatorname{Ext}^{1}(I^{(p)}, A)$ is the Frobenius (Lemma 6.4), and $\iota^{*} \colon \operatorname{Ext}^{1}(I, A) \to \operatorname{Ext}^{1}(I^{(p)}, A)$ is the morphism induced from the inclusion $\iota \colon I^{(p)} \to I$. If r = m then $h \in \mu_{p-1} \subset k^{*}$.
- (2) $U_1 \cup U_2$ is regular, and U is regular.

24

(3) There is a unique endomorphism $\delta \in \text{End}(B)$ (of the A-module B) satisfying $\delta|_A = 0$, $\delta(t_i) = 1$, $\delta(bc) = \delta(b)c + b\delta(c) + \lambda^{1/(p-1)}\delta(b)\delta(c)$, and $\delta^p = 0$. Here we fix a (p-1)-th root $\lambda^{1/(p-1)}$ of λ .

If r = m (resp. 0 < r < m), then $g := \mathrm{id} + \lambda^{1/(p-1)}\delta$ is an automorphism of order p generating $\mathrm{Aut}_Z(U)$, and π is a $\mathbb{Z}/p\mathbb{Z}$ -covering with $\mathrm{Supp}\operatorname{Fix}(g)$ consisting precisely of the closed point (resp. $\mathrm{dim}\operatorname{Supp}\operatorname{Fix}(g) = 1$). If r = m, this means that $U \times_Z Z^{\mathrm{sm}} \to Z^{\mathrm{sm}}$ is the fundamental covering.

If r = 0, then δ is a derivation of additive type, and π is an α_p -covering with Supp Fix(δ) consisting precisely of the closed point. (4) We have $\operatorname{Im}(\delta^j|_{\operatorname{Ker}\delta^{j+1}}) = I$ for all $1 \leq j \leq p-1$.

(5) Let $V = \operatorname{Ker} \delta^2 \subset B$. The extension

$$0 \to A \to V \xrightarrow{o} I \to 0$$

is non-split. The corresponding class in $\text{Ext}^1(I, A)$ is e.

These descriptions of the coverings for the cases r > 0 are essentially the ones given in [Art77, Sections 4–5]. (We note that the equations for p = 3, 5 given there should be fixed as $-\alpha^3 - \alpha$ for p = 3 and $\alpha^5 - 2\alpha$ for p = 5.)

Proof. (1) By Lemma 6.5, the first assertion is clear. The assumption on ε yields the equality $F(e) - \lambda \cdot \iota^*(e) = 0$. Suppose r = m. Since \bar{e} satisfies the same equality and since $\lambda \in k^*$, we have $h^p = h$ in $k[y]/(y^m)$, hence $h \in \mu_{p-1}$.

(2) For the cases r = m, this is proved by Artin [Art77, Sections 4–5].

Suppose $(p, \operatorname{Sing}(Z)) = (2, D_{4m}^r)$ (resp. $(2, E_8^r)$). Let $u = xt_1$ and $v = y^m t_2$. Then we have $u^2 + \lambda xu - x^2 f = x$ (resp. = y), $v^2 + \lambda y^m v - y^{2m} f = y$ (resp. = x), and $y^m u - xv = z$. Let B' = A[u, v]. Then by above B' is integral over A, satisfies $A \subsetneq B' \subset \operatorname{Frac} B$, and its maximal ideal is generated by u and v, hence B' is regular, in particular normal, hence B' = B.

Suppose r = 0. Let $Z' := Z^{sm} = Z_1 \cup Z_2$ and $U' := U \times_Z Z'$. Let ω be the 2-form on Z' satisfying $F_{x_i}\omega = dx_{i+1} \wedge dx_{i+2}$, where we write $(x_1, x_2, x_3) = (x, y, z)$ and consider the indices modulo 3. Applying Proposition 2.15 to $U' \to Z'$, we obtain a 1-form η satisfying $\eta = dq_i = d\bar{q}_i + df$ on Z_i and a derivation D' satisfying $D'(g)\omega = dg \wedge \eta$, $\pi(\operatorname{Sing}(U')) = \operatorname{Zero}(\eta) = \operatorname{Fix}(D')$, and $Z'^{D'} = ((U')^n)^{(p)}$. Since Z is normal, D' extends to a derivation D on Z, and since U is normal we have $Z^D = U^{(p)}$. It remains to show $\operatorname{Fix}(D) = \emptyset$, since then by Theorem 3.3(1) it follows that $U^{(p)}$ and hence U are regular. Let c = 1, 1, 3 and i = 3, 1, 2 for p = 2, 3, 5 respectively (hence $F_{x_i} = 0$). A straightforward calculation yields $\eta = cF_{x_{i+1}}^{-1}dx_{i+2} + df$ $(= -cF_{x_{i+2}}^{-1}dx_{i+1}+df)$. Hence we have $D(x_i) = -c+F_{x_{i+2}}f_{x_{i+1}}-F_{x_{i+1}}f_{x_{i+2}} \in \mathcal{O}_Z^*$ (where $df = \sum_h f_{x_h} dx_h$), hence $\operatorname{Fix}(D) = \emptyset$.

(3) On each U_i there exists a unique $\delta \in \operatorname{End}(\mathcal{O}_{U_i})$ with the required properties. They glue to an endomorphism δ on $\mathcal{O}_{U_1 \cup U_2}$. Since U is normal and $U_1 \cup U_2$ is the complement in U of a codimension 2 subscheme, this δ extends to U.

If r = m (resp. 0 < r < m), then $g := \mathrm{id} + \lambda^{1/(p-1)}\delta \in \mathrm{End}(B)$ preserves products and satisfies $g^p = \mathrm{id}$, hence is an automorphism. It is nontrivial since $\lambda \neq 0$ and $\delta \neq 0$. Since the ideal of $\mathcal{O}_{U_1 \cup U_2}$ generated by $\mathrm{Im}(g - \mathrm{id})$ is (y^{m-r}) , we have $\operatorname{Supp}\operatorname{Fix}(g)|_{U_1\cup U_2} = \emptyset$ (resp. $\operatorname{Supp}\operatorname{Fix}(g)|_{U_1\cup U_2} = (y=0)$). Since the image of the closed point of U is singular, the closed point belongs to $\operatorname{Supp}\operatorname{Fix}(g)$.

If r = 0, then δ is a derivation (since $\lambda = 0$) and is of additive type, we have $\operatorname{Supp} \operatorname{Fix}(\delta)|_{U_1 \cup U_2} = \emptyset$, and similarly the closed point belongs to $\operatorname{Supp} \operatorname{Fix}(\delta)$.

(4) If $(p, \text{Sing}(Z)) = (3, E_6^0), (5, E_8^0)$, this is proved in Lemma 3.6(2d).

Let $I_j := \operatorname{Im}(\delta^j|_{\operatorname{Ker}\delta^{j+1}})$ for each $1 \leq j \leq p-1$. We have $I_{p-1} \subset I_j \subset I_1$. By assumption we have $\varepsilon \notin A[x^{-1}] + A[y^{-1}]$, hence $I_1 \subsetneq A$, hence $I_1 \subset \mathfrak{m}$.

Suppose p = 2. Let u, v be as in the proof of (2). We have $\delta(1) = 0$, $\delta(u) = x, \ \delta(v) = y^m, \ \delta(uv) = xv + uy^m + \lambda xy^m = z + \lambda xy^m$. Since the *A*-module B = k[[u, v]] is generated by 1, u, v, uv, we obtain $I_1 = \text{Im}(\delta) = (x, y^m, z) = I$.

Suppose $(p, \operatorname{Sing}(Z)) = (3, E_6^1), (5, E_8^1)$. Let $a_j = \dim_k(A/I_j)$ for each $1 \leq j \leq p-1$. Since $I_j \subset I_1 = I = \mathfrak{m}$ we have $a_j \geq 1$. It suffices to show $\sum_j (a_j - 1) = 0$. Suppose there is an action of $G = \mathbb{Z}/p\mathbb{Z} = \langle g \rangle$ on a K3 surface X such that the quotient Y = X/G is an RDP K3 surface with $(p, \operatorname{Sing}(Y)) = (3, nE_6^1), (5, nE_8^1)$ and that $\operatorname{Supp}\operatorname{Fix}(G) = \pi^{-1}(\operatorname{Sing}(Y)),$ where $\pi \colon X \to Y$ is the quotient morphism. At each singular point w of Y, the morphism $\hat{\mathcal{O}}_{X,\pi^{-1}(w)} \to \hat{\mathcal{O}}_{Y,w}$ is as above (since it is the fundamental covering of $(\operatorname{Spec} \hat{\mathcal{O}}_{Y,w})^{\operatorname{sm}}$). Let $\delta := g - \operatorname{id} \in \operatorname{End}(\pi_*\mathcal{O}_X)$ and $\mathcal{I}_j := \operatorname{Im}(\delta^j|_{\operatorname{Ker}\delta^{j+1}}) \subset \mathcal{O}_Y$ for each $1 \leq j \leq p-1$. We have $\chi_Y(\mathcal{I}_j) = \chi_Y(\mathcal{O}_Y) - \chi_Y(\mathcal{O}_Y/\mathcal{I}_j) = 2 - na_j$ and $2 = \chi_X(\mathcal{O}_X) = \chi_Y(\pi_*\mathcal{O}_X) = \chi_Y(\mathcal{O}_Y) + \sum_{j=1}^{p-1} \chi_Y(\mathcal{I}_j) = 2 + \sum_j (2 - na_j)$. Since there indeed exist examples with n = 2 (Examples 10.10 and 10.11), we obtain $\sum_j (a_j - 1) = 0$.

Lemma 6.7. Suppose Y is an RDP K3 surface and let $Z_i = \operatorname{Spec} \hat{\mathcal{O}}_{Y,w_i}$ for $w_i \in \operatorname{Sing}(Y)$.

- (1) Suppose $\operatorname{Sing}(Y) = \{w_1, w_2\}$. Let \mathcal{I} be the ideal $\mathcal{I} = \operatorname{Ker}(\mathcal{O}_Y \to \bigoplus_{i=1,2} \mathcal{O}_{Y,w_i}/\mathfrak{m}_{w_i})$, where \mathfrak{m}_{w_i} are the maximal ideals. Then the restriction $\operatorname{Ext}^1_Y(\mathcal{I}, \mathcal{O}_Y) \to \operatorname{Ext}^1_{Z_1}(\mathfrak{m}_{w_1}, \mathcal{O}_{Z_1})$ is an isomorphism.
- (2) Suppose Sing(Y) = {w₁} and (p, w₁) is either (2, D^r₈) or (2, E^r₈) with r ∈ {0,1,2}. Let I ⊂ O_{Y,w1} be the ideal defined above (just before Lemma 6.5) and I = Ker(O_Y → O_{Y,w1}/I). Then Ext¹_Y(I, O_Y) → Ext¹_{Z1}(I, O_{Z1}) is injective and its image is a 1-dimensional k-vector space generated by a · ē for some a ∈ A^{*}, where ē is an element as in Lemma 6.5.

Proof. Let $\mathcal{I} \subsetneq \mathcal{O}_Y$ be any ideal on an RDP K3 surface Y with dim $\operatorname{Supp}(\mathcal{O}_Y/\mathcal{I}) = 0$ (hence $\operatorname{Supp}(\mathcal{O}_Y/\mathcal{I}) \neq \emptyset$). By Serre duality (and the equalities $h^1(\mathcal{O}_Y) = 0$ and $h^2(\mathcal{O}_Y) = 1$), we obtain dim $\operatorname{Ext}^1(\mathcal{I}, \mathcal{O}_Y) = h^0(\mathcal{O}/\mathcal{I}) - 1$ and dim $\operatorname{Ext}^2(\mathcal{I}, \mathcal{O}_Y) = 0$.

26

Comparing the long exact sequences for $0 \to \mathcal{I} \to \mathcal{O} \to \mathcal{O}/\mathcal{I} \to 0$ on Y and $\prod_i Z_i$, we have (since $H^1(Y, \mathcal{O}) = H^1(Z_i, \mathcal{O}) = H^2(Z_i, \mathcal{O}) = 0$)

hence we obtain an exact sequence

$$0 \longrightarrow \operatorname{Ext}^{1}_{Y}(\mathcal{I}, \mathcal{O}) \longrightarrow \bigoplus_{i} \operatorname{Ext}^{1}_{Z_{i}}(\mathcal{I}, \mathcal{O}) \longrightarrow H^{2}(Y, \mathcal{O}) \longrightarrow 0$$

compatible with the Frobenius and the pullbacks by inclusions of ideals. Here, the Frobenius on $\operatorname{Ext}^1_Y(\mathcal{I}, \mathcal{O})$ is induced by the one on $H^2_{\operatorname{Supp}(\mathcal{O}/\mathcal{I}))}(Y, \mathcal{O})$. In particular, for any inclusion $\mathcal{I} \hookrightarrow \mathcal{J} \subsetneq \mathcal{O}_Y$, the diagram

is a cartesian diagram.

(1) Apply this to $\mathcal{J} = \operatorname{Ker}(\mathcal{O}_Y \to \mathcal{O}_{Y,w_2}/\mathfrak{m}_{w_2}).$

(2) Let $\mathcal{J} = \operatorname{Ker}(\mathcal{O}_Y \to \mathcal{O}_{Y,w_1}/\mathfrak{m}_{w_1})$ and consider the diagram above. Write $A = \hat{\mathcal{O}}_{Y,w_1}, M := \operatorname{Ext}_{Z_1}^1(I,\mathcal{O}), M_J := \operatorname{Im}(\operatorname{Ext}_{Z_1}^1(J,\mathcal{O}) \to \operatorname{Ext}_{Z_1}^1(I,\mathcal{O})),$ and $M_Y := \operatorname{Im}(\operatorname{Ext}_Y^1(\mathcal{I},\mathcal{O}) \to \operatorname{Ext}_{Z_1}^1(I,\mathcal{O})).$ We know that M is generated by an element \bar{e} with $\operatorname{Ann}(\bar{e}) = I = (x, y^2, z)$, and that $M_J = M[J] = yM$ by Lemma 6.3. Now $M_Y \subset M$ is a 1-dimensional k-vector subspace with $M_Y \cap M_J = \operatorname{Ext}_Y^1(\mathcal{J},\mathcal{O}) = 0$ by the above cartesian diagram. This shows that M_Y has a basis $a \cdot \bar{e}$ for some $a \in A^*$.

7. $\mathbb{Z}/p\mathbb{Z}$ -, μ_p -, α_p -coverings of K3 surfaces by K3-like surfaces

Let G be one of $\mathbb{Z}/l\mathbb{Z}$, $\mathbb{Z}/p\mathbb{Z}$, μ_p , or α_p (l is a prime $\neq p$). Suppose $\pi \colon X \to Y$ is a G-quotient morphism between RDP K3 surfaces in characteristic p, and suppose moreover that π is maximal (Definition 3.4) if $G = \mu_p$ or $G = \alpha_p$ and that X is smooth if $G = \mathbb{Z}/l\mathbb{Z}$ or $G = \mathbb{Z}/p\mathbb{Z}$. Let $\rho \colon \tilde{Y} \to Y$ be the minimal resolution.

Quotient singularities on Y and some additional properties on $\operatorname{Pic}(Y^{\operatorname{sm}})$ are known for $G = \mathbb{Z}/l\mathbb{Z}$ (Theorem 7.1(1)). We prove its analogue for $G = \mu_p, \mathbb{Z}/p\mathbb{Z}, \alpha_p$ (Theorem 7.3(1)). For $G = \mathbb{Z}/l\mathbb{Z}$, conversely, such properties on Y recovers a $\mathbb{Z}/l\mathbb{Z}$ -covering $X \to Y$ (Theorem 7.1(2)). We state and prove its analogue for $G = \mu_p, \mathbb{Z}/p\mathbb{Z}, \alpha_p$ (Theorem 7.3(2)). However, in the converse statement for μ_p and α_p , the covering is a K3-like surface (Definition 7.2) but not necessarily birational to a K3 surface. This situation is similar to the canonical μ_2 - or α_2 -coverings of classical or supersingular Enriques surfaces in characteristic 2, where the covering is K3-like ([BM76, Proposition 9]) but not necessarily birational to a K3 surface.

Theorem 7.1.

- (1) Let π be as above and suppose G = Z/lZ. Then l ≤ 7, Sing(Y) = ²⁴/_{l+1}A_{l-1}, and |Pic(Ysm)_{tors}| = l. The l-torsion is given by a divisor on Ŷ whose multiple by l is linearly equivalent to ∑_{i,j} ja_ie_{i,j} for a suitable numbering e_{i,j} (1 ≤ i ≤ ²⁴/_{l+1}, 1 ≤ j ≤ l − 1, e_{i,j} · e_{i,j+1} = 1) of exceptional curves of Ŷ. Here (a₁,..., a_{24/(l+1)}) is given by (1,...,1), (1,...,1), (1,1,2,2), (1,2,4) for l = 2,3,5,7 respectively. Every prime l ≤ 7 occur in every characteristic ≠ l.
- (2) Conversely, let Y be an RDP K3 surface in characteristic $\neq l$ with $\operatorname{Sing}(Y) = \frac{24}{l+1}A_{l-1}$ and $\operatorname{Pic}(Y^{\operatorname{sm}})_{\operatorname{tors}} \neq 0$. Then there exists a smooth K3 surface X and a $\mathbb{Z}/l\mathbb{Z}$ -quotient morphism $\pi \colon X \to Y$ with $\operatorname{Supp}\operatorname{Fix}(\mathbb{Z}/l\mathbb{Z}) = \pi^{-1}(\operatorname{Sing}(Y))$.

Proof. (1) The assertions $l \leq 7$ and $\operatorname{Sing}(Y) = \frac{24}{l+1}A_{l-1}$ are proved by Nikulin [Nik79, Section 5] (p = 0) and Dolgachev–Keum [DK09, Theorem 3.3] (p > 0). Then the eigenspace of $\pi_* \mathcal{O}_X$ for a nontrivial eigenvalue gives an invertible sheaf whose *l*-th power is isomorphic to $\mathcal{O}_{\tilde{Y}}(-\sum_{i,j} f_l(a_i, j)e_{i,j})$ for a suitable numbering, where f_l is the function defined in Lemma 6.1. See [Mat20a, Theorem 7.1] for details. See the proof of (2) to show that Pic(Y^{sm}) has no more torsion.

Examples for each l are well-known.

(2) By the exact sequence

$$0 \to \bigoplus_{i,j} \mathbb{Z}[e_{i,j}] \to \operatorname{Pic}(\tilde{Y}) \to \operatorname{Pic}(Y^{\operatorname{sm}}) \to 0,$$

where $e_{i,j}$ runs through the exceptional curves of $\tilde{Y} \to Y$ over $w_i \in \operatorname{Sing}(Y)$, and the fact that discriminant group of the A_{l-1} lattice is cyclic of order l, we see that a nontrivial element of $\operatorname{Pic}(Y^{\operatorname{sm}})_{\operatorname{tors}}$ is of order l and induces $\Delta \in \operatorname{Pic}(\tilde{Y})$ satisfying $\sum b_{i,j}e_{i,j} = l\Delta \in l\operatorname{Pic}(\tilde{Y})$ for some coefficients $b_{i,j} \in \mathbb{Z}$ not all divisible by l. By Lemma 6.1(4), there exist integers a_i satisfying $b_{i,j} \equiv ja_i \pmod{l}$. We may assume $a_i \in \{0, \ldots, \lfloor l/2 \rfloor\}$ and $b_{i,j} = (ja_i \mod l) \in \{0, 1, \ldots, l-1\}$. Computing the intersection number $(l\Delta)^2$, we obtain $\Delta^2 = -l^{-1}\sum_i a_i(l-a_i) \in 2\mathbb{Z}$. Moreover we have $\Delta^2 \neq -2$ since if $\Delta^2 = -2$ then Δ or $-\Delta$ is effective, which leads to a contradiction. The only solution (a_i) is as in the statement of (1), up to the numbering of the RDPs w_i .

Suppose there are two *l*-torsion elements $\sum (ja_i \mod l)e_{i,j}$ and $\sum (ja'_i \mod l)e_{i,j}$ with (a_i) and (a'_i) linearly independent in $\mathbb{F}_l^{24/(l+1)}$. Then for some $m \in \mathbb{Z}$, the elements $a_i - ma'_i \in \mathbb{F}_l$ are neither all zero nor all nonzero, contradicting the observation above. Hence $\operatorname{Pic}(Y^{\operatorname{sm}})_{\operatorname{tors}}$ is of order *l*.

Now suppose there is a nontrivial *l*-torsion of Y^{sm} . Construct a μ_l -covering $\pi: X \to Y$ as in Lemma 6.1. Then X is regular above Sing(Y).

It is clear from the construction that π is finite étale outside $\operatorname{Sing}(Y)$. Hence X is a smooth proper surface. A non-vanishing 2-form on Y^{sm} pullbacks to a non-vanishing 2-form on $X \setminus \pi^{-1}(\operatorname{Sing}(Y))$, which then extends to X. For each 0 < k < l, we have $(\tilde{L}_k)^2 = -4$ by the calculation of Δ^2 above, hence $\chi(\tilde{Y}, \tilde{L}_k) = 0$, hence $\chi(Y, \bar{L}_k) = \chi(Y, \rho_* \tilde{L}_k) = \chi(\tilde{Y}, \tilde{L}_k) = 0$ since $R^i \rho_* \tilde{L}_k = 0$ for i > 0. Here χ is the Euler–Poincaré characteristic of the

28

sheaf cohomology. Hence $\chi(X, \mathcal{O}) = \chi(Y, \mathcal{O}) + \sum_{0 \le k \le l} \chi(Y, \bar{L}_k) = 2 + 0 = 2.$ Hence X is a K3 surface.

Alternatively, we can conclude that X is a K3 surface from by computing the Euler–Poincaré characteristic χ of the l'-adic cohomology for an auxiliary prime $l' \neq \operatorname{char} k$. Indeed, as π is finite étale outside $\operatorname{Sing}(Y)$, we have $\chi(X \setminus \pi^{-1}(\text{Sing}(Y))) = l \cdot \chi(Y^{\text{sm}}), \text{ hence } \chi(X) - \frac{24}{l+1} = l \cdot (\chi(Y) - l\frac{24}{l+1}),$ therefore $\chi(X) = 24$.

Definition 7.2 (following [BM76, Proposition 9]). A proper reduced Gorenstein (not necessarily normal) surface X is K3-like if $h^i(X, \mathcal{O}_X) = 1, 0, 1$ for i = 0, 1, 2, and the dualizing sheaf ω_X is isomorphic to \mathcal{O}_X .

RDP K3 surfaces are K3-like.

Theorem 7.3.

- (1) Let G be μ_p , $\mathbb{Z}/p\mathbb{Z}$, or α_p . Let π be as in the beginning of this section. Then $(G, \operatorname{Sing}(Y), |\operatorname{Pic}(Y^{\operatorname{sm}})_{\operatorname{tors}}|)$ is one of those listed in Table 6. If $G = \mu_p$, then the p-torsion is given by a divisor on \tilde{Y} whose multiple by p is linearly equivalent to $\sum_{i,j} ja_i e_{i,j}$, with a_i as in Theorem 7.1(1). Every case occur.
- (2) Conversely, suppose Y is an RDP K3 surface in characteristic p with Sing(Y) as in Table 6, let G be the corresponding group scheme, and if $G = \mu_p$ suppose moreover $\operatorname{Pic}(Y^{\operatorname{sm}})_{\operatorname{tors}} \neq 0$. Then there exists a G-quotient morphism $\pi: X \to Y$ from a proper K3-like surface X with $\operatorname{Sing}(X) \cap \pi^{-1}(\operatorname{Sing}(Y)) = \emptyset$ and $\operatorname{Supp}\operatorname{Fix}(G) = \pi^{-1}(\operatorname{Sing}(Y)).$ If $G = \mathbb{Z}/p\mathbb{Z}$ then X is a smooth K3 surface. If $G = \mu_p$ or $G = \alpha_p$, then one of the following holds:
 - X is an RDP K3 surface.
 - X is a normal rational surface with Sing(X) consisting of a single non-RDP singularity, and $p \geq 3$.
 - X is a non-normal rational surface with $\dim \operatorname{Sing}(X) = 1$.

All three cases of (2) occur for all $G \in \{\mu_p \ (p \leq 7), \alpha_p \ (p \leq 5)\}$ unless otherwise stated. See Section 10.4 for examples.

Remark 7.4. Dolgachev–Keum studied $\mathbb{Z}/p\mathbb{Z}$ -actions on K3 surfaces in characteristic p. Their results in the case of K3 quotients are as follows [DK01, Theorem 2.4 and Remark 2.6]: Suppose $G = \mathbb{Z}/p\mathbb{Z}$ acts on a K3 surface X in characteristic p with quotient Y birational to a K3 surface. Then

- Fix(G) is isolated and Sing(Y) = $\pi(Fix(G))$, and each singularity of Y is an RDP.
- $1 \le \#\operatorname{Sing}(Y) \le 2$ and $p \le 5$.
- If p = 2, then Sing(Y) is one of $1D_4^1$, $2D_4^1$, $1D_8^2$, or $1E_8^2$.

(The E_8^2 on the last is misprinted as E_8^4 in [DK01, Remark 2.6].)

Also note that if $G = \mathbb{Z}/p\mathbb{Z}$ then each quotient RDP singularity on Y should be one of those having fundamental group $\mathbb{Z}/p\mathbb{Z}$ and smooth fundamental covering, which, due to Artin [Art77, Sections 4–5], are the following:

- $D_{4r}^r \ (r \ge 1)$ and E_8^2 if p = 2. E_6^1 if p = 3. E_8^1 if p = 5.

char.	G		$\operatorname{Sing}(Y)$	$ \operatorname{Pic}(Y^{\operatorname{sm}})_{\operatorname{tors}} $
$p \ge 0$	$\mathbb{Z}/l\mathbb{Z}$	$l \leq 7$ prime, $l \neq p$	$\frac{24}{l+1}A_{l-1}$	l
p	μ_p	$p \leq 7$	$\frac{24}{p+1}A_{p-1}$	p
5	$\mathbb{Z}/5\mathbb{Z}$		$2E_8^1 \\ 2E_6^1$	1
3	$\mathbb{Z}/3\mathbb{Z}$		$2E_{6}^{1}$	1
2	$\mathbb{Z}/2\mathbb{Z}$		$2D_4^1, 1D_8^2, \text{ or } 1E_8^2$	1
5	α_5		$2E_8^0 \\ 2E_6^0$	1
3	α_3		$2E_{6}^{0}$	1
2	α_2		$2D_4^0, 1D_8^0, \text{ or } 1E_8^0$	1

TABLE 6. Singularities of $\mathbb{Z}/l\mathbb{Z}$ -, μ_p -, $\mathbb{Z}/p\mathbb{Z}$ -, and α_p -quotient K3 surfaces in characteristic p

• There are no such RDP if $p \ge 7$.

Note that these RDPs and their $\mathbb{Z}/p\mathbb{Z}$ -coverings are discussed in Section 6.2. Thus, it was known that $\operatorname{Sing}(Y)$ is $1E_6^1$ or $2E_6^1$ if p = 3, and $1E_8^1$ or $2E_8^1$ if p = 5. Compared to these results, we exclude the possibility of $1D_4^1$ (p = 2), $1E_6^1$ (p = 3), and $1E_8^1$ (p = 5).

Proof of Theorem 7.3. (1) Consider the case $G = \mu_p, \alpha_p$. The assertion on $\operatorname{Sing}(Y)$ is showed in Theorem 4.6. If $G = \mu_p$, the author showed [Mat20a, Theorem 7.1] that the eigenspace of $\pi_*\mathcal{O}_X$ for a nontrivial eigenvalue (of the derivation D of multiplicative type corresponding to the μ_p -action) gives an invertible sheaf whose p-th power is isomorphic to $\mathcal{O}_{\tilde{Y}}(-\sum_{i,j} f_p(a_i, j)e_{i,j})$ for a suitable numbering. Here f_p is the function defined in Lemma 6.1. The same (characteristic-free) argument as in the proof of Theorem 7.1(2) shows $|\operatorname{Pic}(Y^{\operatorname{sm}})_{\operatorname{tors}}| = p$. A similar calculation shows that if $(p, \operatorname{Sing}(Y))$ is $(2, 2D_4^r)$ etc. then $\operatorname{Pic}(Y^{\operatorname{sm}})_{\operatorname{tors}} = 0$.

Consider the case $G = \mathbb{Z}/p\mathbb{Z}$. As in Proposition 4.1 (using the usual ramification formula in place of the Rudakov–Shafarevich formula) we have that Fix(G) is finite and each point in $\pi(\text{Fix}(G))$ is an RDP. Let $\text{Sing}(Y) = \{w_i\}$. Then each w_i is one of the RDPs appearing in Remark 7.4, hence in Section 6.2, and let m_i be the integer defined there. Let $\mathcal{I}_j = \text{Im}(\delta^j|_{\text{Ker}\,\delta^{j+1}})$ for $1 \leq j \leq p-1$, where $\delta = g - \text{id} \in \text{End}(\pi_*\mathcal{O}_X)$. We have $\chi_Y(\mathcal{I}_j) = \chi_Y(\mathcal{O}_Y) - \chi_Y(\mathcal{O}/\mathcal{I}_j) = 2 - \sum_i m_i$ by Lemma 6.6(4). Since $2 = \chi_X(\mathcal{O}_X) = \chi_Y(\mathcal{O}_Y) + \sum_j \chi_Y(\mathcal{I}_j) = 2 + (p-1)(2 - \sum_i m_i)$, we obtain $\sum_i m_i = 2$. This proves the assertion on Sing(Y).

(2) Suppose $G = \mu_p$. As in the case of $\mathbb{Z}/l\mathbb{Z}$ (Theorem 7.1(2)), with l replaced with p, we obtain a μ_p -covering $\pi: X \to Y$. Since in this case π is not étale over Y^{sm} , X may be singular.

By Proposition 2.15, X is Gorenstein with $\omega_X \cong \mathcal{O}_X$, and we have a derivation D on Y satisfying $\operatorname{Fix}(D) = \pi(\operatorname{Sing}(X))$ and $Y^D = (X^n)^{(p)}$. Here $-^n$ is the normalization. Also X is normal if and only if the divisorial part $(D|_{Y^{\operatorname{sm}}})$ of $\operatorname{Fix}(D|_{Y^{\operatorname{sm}}})$ is zero. As in the $\mathbb{Z}/l\mathbb{Z}$ case, we have $\chi(X, \mathcal{O}_X) = 2$. Since X is connected and reduced we have $h^0(X, \mathcal{O}_X) = 1$, and $h^2(X, \mathcal{O}_X) = h^2(X, \omega_X) = h^0(X, \mathcal{O}_X) = 1$. Thus X is K3-like.

Let $D' = D|_{Y^{\text{sm}}}$ and suppose $(D') \neq 0$. Then X is non-normal. By Proposition 4.1, $Y^D = (X^n)^{(p)}$ is rational, and hence X is rational.

Now suppose (D') = 0. Then X is normal and we have $Y^D = X^{(p)}$. As in the proof of Theorem 4.3 we have $D^p = \lambda D$ for some scalar λ , and we may assume $\lambda = 1$ or $\lambda = 0$ (by replacing D by a suitable multiple).

Suppose $\lambda = 1$. Since *D* is a derivation of multiplicative type with $D(\omega) = 0$ (Proposition 2.15(4)), where ω is a global 2-form on Y^{sm} , it follows from [Mat20a, Theorem 6.1] that Y^D is an RDP K3 surface.

Next suppose $\lambda = 0$. By Theorem 2.4 and Lemma 3.11 and the assumption on $\operatorname{Sing}(Y)$, we have $\operatorname{deg}\langle D' \rangle = 24/(p+1)$. Then, by Corollary 3.7, either every singularity of X is an RDP, or X has a single singularity and it is non-RDP and $p \geq 3$. In the latter case X is a rational surface by Proposition 4.1.

Now we consider the cases $G = \mathbb{Z}/p\mathbb{Z}$ and $G = \alpha_p$ simultaneously. Write Sing $(Y) = \{w_i\}_{i=1}^N$. Define an ideal $\mathcal{I} = \mathcal{I}_1 \subset \mathcal{O}_Y$ by $\mathcal{I} = \text{Ker}(\mathcal{O}_Y \to \bigoplus_i (\mathcal{O}_{Y,w_i}/I_{w_i}))$, where $I_{w_i} \subset \mathcal{O}_{Y,w_i}$ is as in Section 6.2. Then we have $h^0(\mathcal{O}_Y/\mathcal{I}) = 2$ and hence dim Ext¹ $(\mathcal{I}, \mathcal{O}) = 1$ (as in the proof of Lemma 6.7). Take a nonzero element $e \in \text{Ext}^1(\mathcal{I}, \mathcal{O})$ corresponding to a non-split extension

$$0 \to \mathcal{O}_Y \to V \xrightarrow{o} \mathcal{I} \to 0$$

(which is unique up to scalar) and let $e_i := e|_{Z_i} \in \operatorname{Ext}_{Z_i}^1(I_{w_i}, \mathcal{O})$ be its restriction to $Z_i = \operatorname{Spec} \mathcal{O}_{Y,w_i}$. By Lemma 6.7, e_i generates this group as an \mathcal{O}_{Y,w_i} -module.

As in the proof of Lemma 6.7, we have a diagram with exact rows

where the double vertical arrows are F and ι^* . By Lemma 6.6(1), we have $\operatorname{Im}(F_{\operatorname{middle}}) \subset \operatorname{Im}(\iota^*_{\operatorname{middle}})$. Hence $\operatorname{Im}(F_{\operatorname{left}}) \subset \beta'^{-1}(\operatorname{Im}(F_{\operatorname{middle}})) \subset \beta'^{-1}(\operatorname{Im}(\iota^*_{\operatorname{middle}})) = \operatorname{Im}(\iota^*_{\operatorname{left}})$, where the last equality follows from snake lemma (applied to the commutative diagram for ι^*) since $\iota^*_{\operatorname{right}} = \operatorname{id}$. Since $\operatorname{Ext}^1_Y(\mathcal{I}, \mathcal{O})$ is 1-dimensional, we obtain $F(e) = \lambda \cdot \iota^*(e)$ for some $\lambda \in k$. Clearly the same equality holds for e_i for each i, and since e_i is a generator, we have the following equivalence: $\lambda \neq 0$ (resp. $\lambda = 0$) if and only if the coindex r of the RDP(s) is $\neq 0$ (resp. r = 0) if and only if $G = \mathbb{Z}/p\mathbb{Z}$ (resp. $G = \alpha_p$).

Since the restriction $e|_{Y^{\text{sm}}} \in H^1(Y^{\text{sm}}, \mathcal{O})$ is annihilated by $F - \lambda$, it induces a *G*-covering $X|_{Y^{\text{sm}}} \to Y^{\text{sm}}$ as follows. Take an open covering $\{O_h\}$ of Y^{sm} fine enough and take local sections $t_h \in V$ with $\delta(t_h) = 1 \in \mathcal{I}|_{Y^{\text{sm}}} = \mathcal{O}_{Y^{\text{sm}}}$. Let $e_{hh'} = t_h - t_{h'} \in \mathcal{O}$ (this 1-cocycle represents $e|_{Y^{\text{sm}}}$). Then since $F(e) = \lambda \cdot \iota^*(e)$ there exists $c_h \in \mathcal{O}$ with $e_{hh'}^p - \lambda e_{hh'} = c_h - c_{h'}$. We equip the

locally-free sheaf $V_{p-1} := \operatorname{Sym}_{\mathcal{O}_{Y^{\mathrm{sm}}}}^{p-1}(V|_{Y^{\mathrm{sm}}})$ on Y^{sm} with an $\mathcal{O}_{Y^{\mathrm{sm}}}$ -algebra structure by $t_h^p := \lambda t_h + c_h$, and let $X|_{Y^{\mathrm{sm}}} = \mathcal{S}pec V_{p-1}$. Since each e_i is a generator, this Y^{sm} -scheme is regular above a neighborhood of w_i by Lemma 6.6(2). By filling the holes above $\operatorname{Sing}(Y)$ by normalization, we obtain a Yscheme X that is isomorphic to $X|_{Y^{\mathrm{sm}}}$ outside $\operatorname{Sing}(Y)$ and regular above a neighborhood of $\operatorname{Sing}(Y)$ (again by Lemma 6.6(2)). Extend $\delta : V|_{Y^{\mathrm{sm}}} \to \mathcal{O}_Y$ to an endomorphism $\delta \in \operatorname{End}_{\mathcal{O}_{Y^{\mathrm{sm}}}}(V_{p-1})$ by $\delta(a \otimes b) = \delta(a) \otimes b + a \otimes$ $\delta(b) + \lambda^{1/(p-1)}\delta(a) \otimes \delta(b)$ (note that this is compatible with the equality $t_h^p = \lambda t_h + c_h$) and then extend it to an endomorphism $\delta \in \operatorname{End}_{\mathcal{O}_Y}(\mathcal{O}_X)$ (by using normality of X above a neighborhood of $\operatorname{Sing}(Y)$). Then δ corresponds to a G-action on X, and $\pi : X \to Y$ is a G-covering.

Let $\mathcal{I}_j = \operatorname{Im}(\delta^j|_{\operatorname{Ker}\delta^{j+1}})$ for $1 \leq j \leq p-1$. Then we have $\mathcal{I}_j = \operatorname{Ker}(\mathcal{O}_Y \to \bigoplus_i (\mathcal{O}_{Y,w_i}/I_{w_i,j}))$ with $I_{w_i,j}$ as in Section 6.2, hence we have $\chi(\mathcal{I}_j) = 2 - \sum_i \dim(\mathcal{O}_{Y,w_i}/I_{w_i,j}) = 0$. Since $\pi_*\mathcal{O}_X$ has $\mathcal{O}_Y, \mathcal{I}_1, \ldots, \mathcal{I}_{p-1}$ as a composition series and since $\chi(\mathcal{O}_Y) = 2$ and $\chi(\mathcal{I}_j) = 0$ for each $1 \leq j \leq p-1$, we have $\chi(\mathcal{O}_X) = 2$.

Suppose $G = \mathbb{Z}/p\mathbb{Z}$. It is clear from the construction that π is finite étale outside $\operatorname{Sing}(Y)$. Hence X is smooth, and a non-vanishing 2-form on Y^{sm} pullbacks to a non-vanishing 2-form on $X \setminus \pi^{-1}(\operatorname{Sing}(Y))$, which then extends to X. Hence X is a K3 surface.

Suppose $G = \alpha_p$. We conclude by using Proposition 2.15 as in the case of $G = \mu_p$.

Remark 7.5. In the proof of the case of $G = \mathbb{Z}/p\mathbb{Z}$, α_p of Theorem 7.3(2), we showed $F(e_i) = \lambda \cdot \iota^*(e_i)$ for each RDP w_i . A similar argument proves an unexpected consequence on non-existence of certain RDP K3 surfaces: If Y is an RDP K3 surface in characteristic p, then $(p, \operatorname{Sing}(Y))$ cannot be $(2, D_4^0 + D_4^1)$, $(3, E_6^0 + E_6^1)$, $(5, E_8^0 + E_8^1)$, $(2, D_{4m}^1)$ $(m \ge 2)$, nor $(2, E_8^1)$. This implies that RDP K3 surfaces in characteristic 2 cannot have D_{2n}^r nor D_{2n+1}^r if 0 < r < n-1 and $2 \nmid (n-r)$, since a partial resolution of such an RDP produces a $D_{2(n-r+1)}^1$ with $2 \mid (n-r+1)$ and n-r+1 > 2.

We do not prove this in this paper, as we give a more general result, relating singularities to the height of the K3 surface, in a subsequent paper [Mat20b, Theorem 1.2].

8. Bound of p for α_p -actions

Theorem 8.1. Let k be an algebraically closed field of characteristic p > 0. Then there exists an RDP K3 surface equipped with a nontrivial action of μ_p (resp. α_p) if and only if $p \le 19$ (resp. $p \le 11$).

Proof. The case of μ_p is given in [Mat20a, Theorem 8.2]. Examples of RDP K3 surfaces with a nontrivial action of α_p in characteristic p for $p \leq 11$ are given in Section 10.2. It remains to show that if $p \geq 13$ then there is no such example.

Suppose $p \ge 13$ and X is an RDP K3 surface equipped with a nontrivial action of α_p . Since smooth K3 surface have no global derivations, X has an RDP x, and since RDPs fixed by the α_p -action can be blown up, we may assume x and all other RDPs are not fixed. Since $p \ge 7$, by Theorem 3.3(1),

x and all other RDPs are of type A_{mp-1} . Since $p \ge 13$, It follows that m = 1 and that x is the only RDP.

We observe that the tangent module T_B of the RDP $B = \hat{\mathcal{O}}_{X,x} = k[[x, y, z]]/(xy - z^p)$ of type A_{p-1} is a free *B*-module with basis $e_1 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, e_2 = \frac{\partial}{\partial z}$, and that $a_1e_1 + a_2e_2$ $(a_1, a_2 \in B)$ fixes the closed point if and only if $a_2 \in \mathfrak{m}_B$.

Let $\Delta := (D)$. The morphism $H^0(X, \mathcal{O}_X(\Delta)) \to H^0(X, T_X) \to T_B \to B/\mathfrak{m}_B = k$ taking f to the coefficient modulo \mathfrak{m}_B of e_2 in $(fD)|_B$ is injective, since if f is in the kernel then fD extends to an element of $H^0(\tilde{X}, T_{\tilde{X}}) = 0$. Hence dim $H^0(X, \mathcal{O}_X(\Delta)) = 1$. It follows that $\operatorname{Supp}(\Delta)$ is a finite disjoint union (possibly empty) of ADE configurations of smooth rational curves.

Let $X' = X \setminus \text{Supp}(\text{Fix}(D)) = X \setminus (\text{Supp}(\Delta) \cup \text{Supp}(\langle D \rangle)), X'' = X' \setminus \{x\}$, and $Y' = X'^D$, $Y'' = X''^D$. Then X'' and Y'' are smooth. Since X'' is the complement in a K3 surface of a finite disjoint union of ADE configurations and closed points, we have $H^0(X'', \mathcal{O}_X) = k$. Moreover we can compute Pic(X'')[p] as in the proof of Theorem 7.1(2), and since there is at most one RDP of type A_{p-1} (since $p \geq 13$) we obtain Pic(X'')[p] = 0.

Fix a nonzero element ω_X of $H^0(X^{\text{sm}}, \Omega_X^2)$ (which is unique up to k^*). We have $D(\omega_X) = 0$ since D acts nilpotently on this 1-dimensional space. Let ω_Y be the 2-form on Y'' corresponding to $\omega_X|_{X''}$ via the isomorphism $H^0(X'', \Omega_X^2)^D \cong H^0(Y'', \Omega_Y^2)$ of Proposition 2.12. Let $D_{Y''}$ be the derivation on Y'' of Proposition 2.15. Then $\text{Fix}(D_{Y''}) = \emptyset$, $D_{Y''}$ is p-closed, and $Y''^{D_{Y''}} = X''^{(p)}$. Write $D_{Y''}^p = hD_{Y''}$ with $h \in k(Y)$. Then $h \in$ $H^0(Y'', \mathcal{O}_Y) \subset H^0(X'', \mathcal{O}_X) = k$.

As $D_{Y''}$ extends to $D_{Y'}$ on Y' with $\operatorname{Fix}(D_{Y'}) \subset \{\pi(x)\}\)$, we have $h \neq 0$, since x (of type A_{p-1}) is not α_p -quotient by Lemma 3.6. We may assume h = 1 (by replacing $D_{Y'}$ with a multiple by k^*). Then $Y''^{(1/p)} \to X''$ is a μ_p -covering, which corresponds to an element of $\operatorname{Pic}(X'')[p] = 0$. Since $H^0(X'', \mathcal{O}_X) = k$, such a covering is non-reduced, which is absurd. \Box

9. Coverings of supersingular Enriques surfaces in Characteristic 2

In this section we give a restriction on the singularities of the canonical α_2 -covering of a supersingular Enriques surface in characteristic 2, and give some examples. A more detailed study will be given in a subsequent paper [Mat21].

Let X be a classical or supersingular (smooth) Enriques surface in characteristic 2 (i.e. an Enriques surface with $\operatorname{Pic}^{\tau}(X) = \mathbb{Z}/2\mathbb{Z}$ or α_2 respectively). Let $\pi: Y \to X$ be its canonical μ_2 - or α_2 -cover. We recall some known properties of Y.

- ([BM76, Proposition 9]) Y is K3-like (as in Definition 7.2, i.e. $h^i(Y, \mathcal{O}_Y) = 1, 0, 1$ for i = 0, 1, 2, and the dualizing sheaf ω_Y is isomorphic to \mathcal{O}_Y). There exists a global regular 1-form $\eta \neq 0$ on X, unique up to scalar, and it satisfies $\operatorname{Sing}(Y) = \pi^{-1}(\operatorname{Zero}(\eta))$. The zero locus $\operatorname{Zero}(\eta)$ is nonempty (hence Y is singular somewhere), and if it has no divisorial part then it is of degree 12.
- ([CD89, Theorem 1.3.1]) One of the following holds.

- -Y has only RDPs as singularities, and Y is an RDP K3 surface.
- -Y has only isolated singularities, it has exactly one non-RDP singularity and that is an elliptic double point, and Y is a normal rational surface.
- -Y has 1-dimensional singularities, and Y is a non-normal rational surface.
- ([ESB04]) Non-normal examples exist. More detailed properties, for example on the structure of the divisorial part of $\text{Zero}(\eta)$, are proved.
- ([EHSB12, Corollary 6.16]) If Y is an RDP K3 surface, then Sing(Y) is one of $12A_1$, $8A_1 + D_4^0$, $6A_1 + D_6^0$, $5A_1 + E_7^0$, $3D_4^0$, $D_8^0 + D_4^0$, $E_8^0 + D_4^0$, or D_{12}^0 .
- ([Sch19, Sections 13–14]) If Y has an elliptic double point singularity, then there are no other singularities on Y. Such examples exist.

By using similar arguments as in Theorem 7.3(2), we can give some restrictions on the singularities of the canonical α_2 -coverings of supersingular Enriques surfaces in characteristic 2, assuming it is an RDP K3 surface. Since this method depends on the triviality of the canonical divisor of X, it cannot be applied to classical Enriques surfaces.

Theorem 9.1. Let $\pi: Y \to X$ be the canonical α_2 -covering of a supersingular Enriques surface X. If Y is an RDP K3 surface, then $\operatorname{Sing}(Y)$ is one of $12A_1, 3D_4^0, D_8^0 + D_4^0, E_8^0 + D_4^0$, or D_{12}^0 .

Proof. By Theorem 4.3, $X \to Y^{(2)}$ is the quotient by a derivation D of multiplicative or additive type with (D) = 0. Then $\deg\langle D \rangle = 12$ by Theorem 2.4. The assertion follows from by Lemma 3.6.

Remark 9.2. $12A_1$ is the most generic case, and explicit examples are given for example by [KK15, Section 4]. We give examples with Sing(Y) being $3D_4^0$, $D_8^0 + D_4^0$, $E_8^0 + D_4^0$, or single non-RDP, in Example 9.4, and we will give an example of the remaining RDP case (i.e. $\text{Sing}(Y) = D_{12}^0$) in a subsequent paper [Mat21, Section 5]. See also [Sch19, Sections 13–15] for various examples, although classical and supersingular Enriques surfaces are not distinguished explicitly.

Remark 9.3. We note an error of an example of Bombieri–Mumford [BM76, Section 5]. Let X be a supersingular Enriques surface (in characteristic 2). They showed that there exists a regular vector field ϑ (canonical up to scalar) and they gave two examples of X, second of which is claimed to have $\delta_X = 0$, where δ_X is the scalar defined by $\vartheta^2 = \delta_X \vartheta$ (by normalizing ϑ we may assume $\delta_X \in \{0, 1\}$). However their calculation is incorrect and this X actually has $\delta_X = 1$. Note that $\delta_X = 1$ (resp. $\delta_X = 0$) is equivalent to the morphism $X \to (Y^{(2)})^n$ being a μ_2 -quotient (resp. an α_2 -quotient), where $Y \to X$ is the canonical covering of the Enriques surface.

Their construction is as follows. Let $Y \subset \mathbb{P}^5$ be the complete intersection of the three quadrics

$$x_1^2 + x_1x_2 + y_3^2 + y_1x_2 + x_1y_2 = 0,$$

$$x_2^2 + x_2x_3 + y_1^2 + y_2x_3 + x_2y_3 = 0,$$

$$x_3^2 + x_3x_1 + y_2^2 + y_3x_1 + x_3y_1 = 0.$$

This surface Y has exactly 6 isolated singular points:

$$(1, 1, 1, 0, 0, 0);$$

$$(t^{3}, t, 1, t^{3}, t, 1), \quad t^{3} + t^{2} + 1 = 0;$$

$$(t^{2}, t, 1, t^{3}, t^{2}, t), \quad t^{2} + t + 1 = 0.$$

(We corrected the error on the coordinates of the points of the third type.) Let X be the quotient of Y by the α_2 -action $(x_i, y_i) \mapsto (x_i, \varepsilon x_i + y_i)$, that is, $D(x_i) = 0$ and $D(y_i) = x_i$. They claim that X is a smooth supersingular Enriques surface, but actually it has $3A_2$ singularities at the images of the 3 points $(t^3, t, 1, t^3, t, 1), t^3 + t^2 + 1 = 0$, of type A_5 . (The other singularities of Y are all A_1 and their images are smooth points.) Then $\operatorname{Sing}(\tilde{X} \times_X Y)$ is $12A_1$, with three A_1 above each A_5 of Y, where $\tilde{X} \times_X Y$ is the canonical α_2 -covering of the resolution \tilde{X} of X. Consequently \tilde{X} has $\delta_{\tilde{X}} = 1$.

We will construct supersingular Enriques surfaces with $\delta_X = 0$.

Example 9.4. We consider the indices modulo 3. Let $F_i \in k[x_1, x_2, x_3, y_1, y_2, y_3]$ (i = 1, 2, 3) be homogeneous quadratic polynomials belonging to the subring $k[x_j^2, y_j^2, t_j, s_j]_{j=1,2,3}$ (resp. $k[x_j^2, y_j^2, t_j, u_j]_{j=1,2,3}$), where $t_j = x_{j+1}x_{j+2}$, $s_j = y_{j+1}y_{j+2}$, $u_j = x_{j+1}y_{j+2} + x_{j+2}y_{j+1}$, and let $Y = (F_1 = F_2 = F_3 = 0) \subset \mathbb{P}^5$. Endow Y with a derivation D of multiplicative (resp. additive) type with

$$(D(x_j), D(y_j)) = (0, y_j)$$

(resp. $(D(x_j), D(y_j)) = (0, x_j)$)

(see the convention before Example 10.2). If F_i are generic, then Y is an RDP K3 surface and the quotient $X = Y^D$ is a classical (resp. supersingular) Enriques surface. Liedtke [Lie15, Proposition 3.4] showed that any classical (resp. supersingular) Enriques surface is birational to an RDP Enriques surface of this form. (Liedtke's theorem also covers singular Enriques surfaces (i.e. those with $\text{Pic}^{\tau} = \mu_2$), which we do not discuss in this paper.)

As showed in Proposition 4.5, in the classical case there is no (regular) derivation D' on X with $X^{D'} = (Y^{(2)})^n$.

Consider the supersingular case. Write $F_i = A_i + B_i + C_i$, where $A_i \in \langle x_j^2, y_j^2 \rangle_j$, $B_i \in \langle t_j \rangle_j$, $C_i \in \langle u_j \rangle_j$. For simplicity assume C_1, C_2, C_3 are linearly independent, and then we may assume $C_i = u_i$. Write $B_i = \sum_j b_{ij} t_j$. The derivation D' on X defined by

$$D'(B_i + u_i) = 0,$$

$$D'(t_j) = b_{j+1,j+2}x_{j+1}^2 + b_{j+2,j+1}x_{j+2}^2 + et_j + A_j,$$

where $e = \sum_{j} b_{jj}$, satisfies $X^{D'} = (Y^{(2)})^n$ and $D'^2 = eD'$. (To check that this is well-defined, it suffices to observe $D'(t_{j+1})t_{j+2} + t_{j+1}D'(t_{j+2}) = x_j^2 D'(t_j)$, and it is straightforward.) If $e \neq 0$ then $e^{-1}D'$ is of multiplicative type and if e = 0 then D' is of additive type. One can check (e.g. by using the Jacobian criterion) that if F_i is generic with $C_i = u_i$ and e = 0 then $\operatorname{Sing}(Y)$ is $3D_4^0$ at $(G_1 = G_2 = G_3 = H_1 = H_2 = H_3 = 0)$,

$$G_{j} = \sqrt{A_{j}} + \sqrt{b_{j+1,j+2}x_{j+1}} + \sqrt{b_{j+2,j+1}x_{j+2}},$$

$$H_{j} = B_{j} + u_{j} + b_{j+1,j+2}x_{j+1}^{2} + b_{j+2,j+1}x_{j+2}^{2}.$$

Note that the subscheme $(H_1 = H_2 = H_3 = 0) \subset \mathbb{P}^5$ is of codimension 2 and degree 3, since $\sum x_i H_i = 0$.

Now, for simplicity let $F_i = A_i + u_i$ (so $b_{ij} = 0$ and e = 0).

- If $A_1 = x_1^2 + x_3^2$, $A_2 = y_1^2 + y_2^2$, $A_3 = x_3^2 + y_3^2$, then Sing(Y) is $3D_4^0$ at $(x_1, x_2, x_3, y_1, y_2, y_3) = (0, 1, 0, 0, 0, 0)$, (1, 1, 1, 1, 1, 1), (0, 0, 0, 1, 1, 0). If $A_1 = x_1^2 + x_2^2 + y_1^2$, $A_2 = x_3^2$, $A_3 = y_1^2 + y_2^2$, then Sing(Y) is $D_8^0 + D_4^0$
- at (1, 1, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 1).
- If $A_1 = x_1^2 + x_2^2 + y_1^2$, $A_2 = y_1^2 + y_2^2$, $A_3 = x_3^2 + y_3^2$, then Sing(Y) is $E_8^0 + D_4^0$ at (1, 1, 0, 0, 0, 0), (0, 0, 1, 0, 0, 1). If $A_1 = x_1^2 + x_3^2 + y_1^2$, $A_2 = x_2^2 + y_1^2 + y_3^2$, and $A_3 = y_2^2$, then Sing(Y) consists of a single non-RDP singularity at (1, 0, 1, 0, 0, 0).

We will give an example of the remaining RDP case (i.e. Sing(Y) = D_{12}^0 , and also examples in the classical case, in a subsequent paper [Mat21, Section 5].

10. Examples

10.1. Local α_p -actions.

Example 10.1. For p = 2, 3, 5, 7, let D be the derivation on A = k[[x, y]]defined as in the table. Then D is of additive type, with (D) = 0, deg $\langle D \rangle$ is as in the table, and $A^D = k[[X, Y, Z]]/(F)$, where $X = x^p$, $Y = y^p$, Z is as in the table, and F is as in the table, and A^D is a non-RDP. (cf. Lemma 3.6.)

The non-RDPs appearing in Examples 9.4 and 10.3–10.6 are isomorphic to the quotient singularities listed here, at least up to terms of high degree.

p	D(x)	D(y)	$\mathrm{deg}\langle D\rangle$	F	Ζ
2	y^2	x^6	12	$X^7 + Y^3 - Z^2$	$x^7 + y^3$
3	y	x^6	6	$X^7 + Y^2 - Z^3$	$x^7 + y^2$
5	xy	$2(x^2 + y^2)$	4	$2X^3 + XY^2 - Z^5$	$2x^3 + xy^2$
7	y	$-2x^{3}$	3	$X^4 + Y^2 - Z^7$	$x^4 + y^2$

10.2. Actions on RDP K3 surfaces with rational quotients. Examples for $G = \mathbb{Z}/l\mathbb{Z}$, $l \leq 19$ and $l \neq p$, are well-known.

Examples for $G = \mathbb{Z}/p\mathbb{Z}$, $p \leq 11$, are given in [DK01].

Examples for $G = \mu_p$, $p \leq 19$ and $p \neq 5$, are given in [Mat20a, Section 9]. For $G = \mu_5$, the derivation $D = t\partial/\partial t$ on the elliptic RDP K3 surface $(y^{2} + x^{3} + x^{2} + t^{10} = 0)$ gives an example.

Examples for $G = \alpha_p$, $p \leq 7$, are given in Section 10.4. For $G = \alpha_{11}$, the derivation $D = \partial/\partial t$ on the elliptic RDP K3 surface $(y^2 + x^3 + x^2 + t^{11} = 0)$ gives an example.

We do not know whether examples with $G = \alpha_p$, p = 13, 17, 19, exist.

10.3. Actions on RDP K3 surfaces with RDP Enriques quotients. As noted in Proposition 4.1, this is possible only if p = 2. We gave examples in Example 9.4.

10.4. Actions with RDP K3 quotients. In this section, we give the following examples of G-quotient morphisms $\pi: X \to Y$ in the following characteristics p.

36

 $\mu_p\text{-}$ AND $\alpha_p\text{-}\text{ACTIONS}$ ON K3 SURFACES IN CHARACTERISTIC p

- X and Y are RDP K3 surfaces, X is smooth, $G = \mathbb{Z}/p\mathbb{Z}$, $(p, \text{Sing}(Y)) = (2, 2D_4^1), (2, D_8^2), (2, E_8^2), (3, 2E_6^1), (5, 2E_8^1).$
- X and Y are RDP K3 surfaces, and the induced morphism $\pi': Y \to X^{(p)}$ is a G'-quotient morphism, with
 - $-(G,G') = (\mu_p,\mu_p), p \le 7;$
 - $-(G,G') = (\mu_p, \alpha_p), p \le 5;$
 - $(G, G') = (\alpha_p, \alpha_p), p \le 3.$

(We note that if π is an example for $(G, G') = (\mu_p, \alpha_p)$, then π' is an example for $(G, G') = (\alpha_p, \mu_p)$.)

When p = 2, we give examples with all pairs $(\text{Sing}(X), \text{Sing}(Y)) \in \{8A_1, 2D_4^0, 1D_8^0, 1E_8^0\}^2$ except $(1E_8^0, 1E_8^0)$.

• Y is an RDP K3 surface with Sing(Y) and Pic(Y) as in Table 6, X is the corresponding G-covering that is a K3-like rational surface, and

- X has a single singularity, which is a non-RDP, $G = \mu_p$ ($p \le 7, p \ne 2$) and $G = \alpha_p$ ($p \le 5, p \ne 2$).

- X is non-normal, $G = \mu_p \ (p \le 7)$ and $G = \alpha_p \ (p \le 5)$.

In this case $\pi' \colon Y \to (X^{(p)})^n$ is an α_p -quotient morphism with rational quotient.

We prove in a subsequent paper [Mat20b, Corollary 6.8] that if X and Y are RDP K3 surfaces then the following are impossible:

- $(G, G') = (\alpha_5, \alpha_5).$
- $(G, G', \operatorname{Sing}(X), \operatorname{Sing}(Y)) = (\alpha_2, \alpha_2, 1E_8^0, 1E_8^0).$

Below we use the following description of derivations. Suppose X is a projective scheme over k, L is an ample line bundle on it, and $D^* \in$ $\operatorname{End}_k(H^0(X,L))$ is a k-linear endomorphism that extends to a derivation D^* of the k-algebra $\bigoplus_{m\geq 0} H^0(X,mL)$. Then D^* induces a derivation D on X by $D(f/g) = D^*(f)/g - fD^*(g)/g^2$ on $(g \neq 0) \subset X$ for $f, g \in$ $H^0(X,mL)$. This can be applied for example to $X = (F = 0) \subset \mathbb{P}^3$ and $D^* \in \operatorname{End}_k(H^0(\mathcal{O}_{\mathbb{P}^3}(1)))$ satisfying $D^*(F) = cF$ for some $c \in k$. Below we write simply D in place of D^* .

Example 10.2 $(G = \mu_2 \text{ (resp. } G = \alpha_2))$. Let $F \in k[w, x, y, z]$ be a homogeneous quartic polynomial belonging to

$$k[w^2, x^2, y^2, z^2, wx, yz]$$
 (resp. $k[w^2, x^2, y^2, z^2, xz, wz + xy]$)

and let $X = (F = 0) \subset \mathbb{P}^3$. Such F is uniquely written as

$$F = H + wxI + yzJ + wxyzK$$

(resp. $F = H + xzI + (wz + xy)J + xz(wz + xy)K$)

with $H, I, J, K \in k[w^2, x^2, y^2, z^2]$ of respective degree 4, 2, 2, 0. Endow X with a derivation D of multiplicative (resp. additive) type with

$$(D(w), D(x), D(y), D(z)) = (0, 0, y, z)$$

(resp. $(D(w), D(x), D(y), D(z)) = (x, 0, z, 0)$)

If F is generic, then X and the quotient $Y = X^D$ are RDP K3 surfaces. Let L' be the line bundle on Y with $H^0(Y, L') = H^0(X, pL)^D$. The derivation D' on $H^0(Y, L')$ defined by $D'(w^2) = D'(x^2) = D'(y^2) = D'(z^2) = 0$

and

$$D'(wx) = J + wxK, \qquad D'(yz) = I + yzK$$

(resp. $D'(xz) = J + xzK, \qquad D'(wz + xy) = I + (wz + xy)K$)

satisfies $Y^{D'} = X^{(2)}$ and $D'^2 = KD'$. If $K \neq 0$ then $K^{-1}D'$ is of multiplicative type, and if K = 0 then D' is of additive type. This gives an 11- (resp. 10-) dimensional family Y of μ_2 -actions which degenerate to α_2 -actions in codimension 1. One can check that if F is generic then $\operatorname{Sing}(X)$ is $8A_1$, if F is generic with K = 0 then $\operatorname{Sing}(X)$ is $2D_4^0$, and if F is generic with K = 0 and #(H = I = J = 0) = 1 then $\operatorname{Sing}(X)$ is $1D_8^0$. If $G = \mu_2$ and $(H, I, J, K) = (w^4 + y^4, x^2 + y^2, w^2 + x^2 + y^2 + z^2, 0)$ then $\operatorname{Sing}(X)$ is $1E_8^0$ and $\operatorname{Sing}(Y)$ is $2D_4^0$. If $G = \alpha_2$ and $(H, I, J, K) = (x^4 + z^4 + w^2y^2, w^2, y^2, 0)$ then $\operatorname{Sing}(X)$ is $1E_8^0$ and $\operatorname{Sing}(Y)$ is $2D_4^0$. If $G = \alpha_2$ and $(H, I, J, K) = (w^4 + x^4 + z^4, w^2, x^2 + y^2 + z^2, 0)$ then $\operatorname{Sing}(X)$ is $1D_8^0$ and $\operatorname{Sing}(Y)$ is $1D_8^0$. If $G = \alpha_2$ and $(H, I, J, K) = (w^4 + y^2z^2, x^2, y^2, 0)$ then $\operatorname{Sing}(X)$ is $1D_8^0$ and $\operatorname{Sing}(Y)$ is $1D_8^0$. If $G = \alpha_2$ and $(H, I, J, K) = (w^4 + y^2z^2, x^2, y^2, 0)$ then $\operatorname{Sing}(X)$ is $1D_8^0$.

then $\operatorname{Sing}(X)$ is ID_8 and $\operatorname{Sing}(I')$ is ID_8 . If $G = \alpha_2$ and $(II, I, J, K) = (w^4 + y^2 z^2, x^2, y^2, 0)$ then $\operatorname{Sing}(X)$ is $1E_8^0$ and $\operatorname{Sing}(Y)$ is $1D_8^0$. If $G = \mu_2$ and $(H, I, J, K) = (y^2 I + x^2 J, w^2 + y^2, x^2 + \lambda^2 z^2, 0)$ (resp. $G = \alpha_2$ and $(H, I, J, K) = ((z^2 + w^2)I + x^2 J, w^2 + z^2, x^2 + \lambda^2 y^2, 0))$, with $\lambda \in k \setminus \mathbb{F}_2$, then $\operatorname{Sing}(X) = (I = J = 0)$, hence X is non-normal, and $\operatorname{Sing}(Y)$ consists of $\pi(\operatorname{Fix}(D)) = 8A_1$ (resp. $\pi(\operatorname{Fix}(D)) = 1D_8^0$) and $4A_1$ (resp. $1A_1$) contained in $\pi(\operatorname{Sing}(X))$. Let $Y' \to Y$ be the resolution of the latter singularities. Then $X \times_Y Y' \to Y'$ is an example of a non-normal μ_2 -(resp. α_2 -) covering.

Example 10.3 $(G = \mu_3 \text{ (resp. } G = \alpha_3))$. Let $F \in k[x, y, z]$ be a homogeneous sextic polynomial belonging to $k[x, y^3, z^3, A]$, where A = yz (resp. $A = xz + y^2$), and let $X = (w^2 + F = 0) \subset \mathbb{P}(3, 1, 1, 1)$. Such F is uniquely written as

$$F = H + xAI + (xA)^2J$$

with $H, I, J \in k[x^3, y^3, z^3]$ of respective degree 6, 3, 0. Endow X with a derivation D of multiplicative (resp. additive) type with

$$(D(w), D(x), D(y), D(z)) = (0, 0, y, -z)$$

(resp. $(D(w), D(x), D(y), D(z)) = (0, 0, x, y)$).

If F is generic, then X and the quotient $Y = X^D$ are RDP K3 surfaces. The derivation D' on Y defined by

$$D'(y^3) = D'(z^3) = 0, \quad D'(w) = I + 2JxA, \quad D'(xA) = w$$

satisfies $Y^{D'} = X^{(3)}$ and $D'^3 = 2JD'$. If $J \neq 0$ then $(2J)^{-1/2}D'$ is of multiplicative type and if J = 0 then D' is of additive type. This gives a 7-(resp. 6-) dimensional family Y of μ_3 -actions which degenerate to α_3 -actions in codimension 1. One can check that if F is generic then $\operatorname{Sing}(X)$ is $6A_2$, and if F is generic with J = 0 then $\operatorname{Sing}(X)$ is $2E_6^0$.

and if F is generic with J = 0 then $\operatorname{Sing}(X)$ is $2E_6^0$. If $(H, I, J) = ((\lambda^3 x^3 + y^3)^2 + (y^3 - z^3)^2, y^3 - z^3, 0)$ with $\lambda \in k \setminus \mathbb{F}_3$, then X has a single singularity at $(0, 1, \lambda, \lambda)$ (resp. (0, 0, 1, 0)), which is a non-RDP, X is a rational surface, and Y is an RDP K3 surface with $\operatorname{Sing}(Y) = 6A_2$ (resp. $\operatorname{Sing}(Y) = 2E_6^0$).

If $(H, I, J) = ((x^3 + y^3 + z^3)^2, x^3 + y^3 + z^3, 0)$, then X is non-normal rational surface with $\operatorname{Sing}(X) = (w = x + y + z = 0)$, and Y is an RDP K3 surface with $\operatorname{Sing}(Y) = 6A_2$ (resp. $\operatorname{Sing}(Y)$ consists of $\pi(\operatorname{Fix}(D)) = 2E_6^0$

38

and $3A_1$ contained in $\pi(\operatorname{Sing}(X))$, and $X \times_Y Y' \to Y'$, where Y' = Y (resp. $Y' \to Y$ is the resolution of RDPs of other than $2E_6^0$) is an example of a non-normal μ_3 - (resp. α_3 -) covering.

Example 10.4 $(G = \mu_5)$. Let $F \in k[x, y, z]$ be a homogeneous sextic polynomial belonging to $k[x, y^5, z^5, A]$ where A = yz and let $X = (w^2 + F = 0) \subset \mathbb{P}(3, 1, 1, 1)$. Endow X with a derivation D of multiplicative (resp. additive) type with

$$(D(w), D(x), D(y), D(z)) = (0, 0, y, -z)$$

If F is generic, then X and the quotient $Y = X^D$ are RDP K3 surfaces. Write

$$F = a_6 x^6 + a_4 x^4 A + a_2 x^2 A^2 + a_0 A^3 + bxy^5 + cxz^5.$$

Define a derivation D' on Y by

$$D'(x^{5}) = D'(y^{5}) = D'(z^{5}) = 0,$$

$$D'(wx^{2}) = 3x\frac{\partial F}{\partial A}, D'(wA) = \frac{\partial F}{\partial x}, D'(x^{3}A) = -wx^{2}, D'(xA^{2}) = -2wA.$$

Then it satisfies $Y^{D'} = X^{(5)}$ and $D'^5 = eD'$, where $e = a_2^2 - 3a_0a_4$. If $e \neq 0$ then $e^{-1/4}D'$ is of multiplicative type and if e = 0 then D' is of additive type. This gives a 3-dimensional family Y of μ_5 -actions which degenerate to α_5 -actions in codimension 1. One can check that if F is generic then $\operatorname{Sing}(X)$ is $4A_4$, and if F is generic with e = 0 then $\operatorname{Sing}(X)$ is $2E_8^0$.

If $F = (A - x^2)^3 + x(2x^5 + y^5 + z^5)$, then X has a single singularity at (w, x, y, z) = (0, 1, -1, -1), which is a non-RDP, X is a rational surface, and Y is an RDP K3 surface with $\operatorname{Sing}(Y) = 4A_4 + A_2$, where A_2 is the image of the non-RDP. Let $Y' \to Y$ be the resolution of the A_2 point, then $\operatorname{Sing}(X \times_Y Y')$ is a single non-RDP.

Example 10.5 $(G = \mu_7)$. Let $a \in k$, $F = w^2 + x_1^5 x_2 + x_2^5 x_4 + x_4^5 x_1 + ax_1^2 x_2^2 x_4^2 \in k[w, x_1, x_2, x_4]$ and $X = (F = 0) \subset \mathbb{P}(3, 1, 1, 1)$. Let $b = (a^{-3} - 1)^{1/3} \in k \cup \{\infty\}$, hence b = 0 if and only if $a^3 = 1$. Then $\operatorname{Sing}(X)$ consists of the points $(0, x_1, x_2, x_4)$ satisfying

$$(x_1^5x_2: x_2^5x_4: x_4^5x_1: ax_1^2x_2^2x_4^2) = (1+4jb: 1+2jb: 1+jb: 4)$$

for some $j \in \{1, 2, 4\}$, and it is $3A_6$ if $b \neq 0$ and a single non-RDP if b = 0. X admits a derivation D of multiplicative type with

$$D(w) = 0, \ D(x_i) = ix_i.$$

whose quotient $Y = X^D$ is an RDP K3 surface. If $b \neq 0$ then $\operatorname{Sing}(Y) = \pi(\operatorname{Fix}(D))$ is $3A_6$, and if b = 0 then $\operatorname{Sing}(Y) = \pi(\operatorname{Fix}(D)) \cup \pi(\operatorname{Sing}(X))$ is $3A_6 + A_1$. In the latter case, let $Y' \to Y$ be the resolution of the A_1 point, then $\operatorname{Sing}(X \times_Y Y')$ is a single non-RDP whose completion is isomorphic to $k[[X, Y, Z]]/(X^2 + Y^4 + Z^7 + \dots)$.

Y admits a derivation D' defined by

$$D'(x_i^7) = 0,$$

$$D'(x_i x_{2i}^2 x_{4i}^4) = i^2 w x_{2i} x_{4i}^3,$$

$$D'(w x_i x_{2i}^3) = i^2 (-x_{2i}^7 + 2x_i x_{2i}^2 x_{4i}^4 - 2a x_i^2 x_{2i}^4 x_{4i}),$$

i = 1, 2, 4, where the indices are considered modulo 7, satisfying $D'^7 = (1 - a^3)D'$.

Example 10.6 $(G = \alpha_5)$. Let Y be the RDP K3 surface $w^2 + (y^2 - 2xz)^3 + z(x^5 + y^5 + z^5) = 0$, equipped with the derivation D' defined by D'(w) = 0, D'(x) = y, D'(y) = z, D'(z) = 0. Then Sing(Y) is $2E_8^0$ at $w = y^2 - 2xz = x + y + z = 0$. Then $(Y^{D'})^{(1/p)}$ is the α_5 -covering of Y, with a single singularity that is non-RDP.

Example 10.7 ($G = \mu_5$ (resp. $G = \alpha_5$)). Let $a \in k$ and assume $a(a^3-2) \neq 0$ (resp. a = 0). Let S be the elliptic RDP K3 surface $y^2 = x^3 + ax^2 + t^5(t-1)^5$, equipped with the derivation $D' = \partial/\partial t$ having 1-dimensional fixed locus at $t = \infty$. Then Sing(S) is $4A_4$ at t = 0, t = 1, $t^5(t-1)^5 + 2a^3 = 0$ (resp. $2E_8^0$ at t = 0, t = 1). S admits a non-normal μ_5 - (resp. α_5 -) covering, birational to $(S^{D'})^{(1/p)}$. We see that $S^{D'}$ is a certain compactification of $y^2 = x^3 + ax^2 + T(T-1)$, where $T = t^5$.

Example 10.8 $(G = \mu_7)$. Let *S* be the elliptic RDP K3 surface $y^2 = x^3 + t^7x + 1$, equipped with the derivation $D' = \partial/\partial t$ having 1-dimensional fixed locus at $t = \infty$. Then $\operatorname{Sing}(S) = 3A_6$ at $-4(t^7)^3 - 27 = 0$. Similarly to the previous example, *S* admits a non-normal μ_7 -covering birational to $(S^{D'})^{(1/p)}$. We see that $S^{D'}$ is a certain compactification of $y^2 = x^3 + Tx + 1$, where $T = t^7$.

Example 10.9 ($G = \mathbb{Z}/2\mathbb{Z}$; See also [DK01, Examples 2.8]). Let $F \in k[w, x, y, z]$ be a homogeneous quartic polynomial belonging to

$$k[w^{2} + x^{2}, y^{2} + z^{2}, wx, yz, wy + xz, wz + xy]$$

and let $X = (F = 0) \subset \mathbb{P}^3$. Endow X with an automorphism g of order 2 with g(w, x, y, z) = (x, w, z, y). If F is generic, then X is a smooth K3 surface and $Y = X/\langle g \rangle$ is an RDP K3 surface, with

Fix(g) = {(
$$\alpha, \alpha, \beta, \beta$$
) | $\alpha^2 c (w^2 x^2)^{1/2} + \alpha \beta c (w x y z)^{1/2} + \beta^2 c (y^2 z^2)^{1/2} = 0$ },

where c(m) are the coefficients of the monomials m in F. If F is generic (resp. generic with c(wxyz) = 0), then $\operatorname{Sing}(Y) = \pi(\operatorname{Fix}(g))$ is $2D_4^1$ (resp. $1D_8^2$).

Now let $X \subset \mathbb{P}^5 = \operatorname{Proj} k[x_1, x_2, y_1, y_2, y_3, y_4]$ be the K3 surface defined by

$$x_1^2 + x_1y_1 + y_3y_2 = x_2^2 + x_2y_2 + y_1^2 + y_3y_4 = y_1y_3 + y_2y_4 + y_4^2 = 0,$$

with automorphism g defined by $g(x_i) = x_i + y_i$, $g(y_i) = y_i$. Then $\# \operatorname{Fix}(g) = 1$ (at $x_1 = x_2 = y_1 = y_2 = y_4 = 0$), and $Y = X/\langle g \rangle$ is an RDP K3 surface with $\operatorname{Sing}(Y) = \pi(\operatorname{Fix}(g)) = 1E_8^2$.

Example 10.10 ($G = \mathbb{Z}/3\mathbb{Z}$). Let $F \in k[w, x, y, z]$ be a homogeneous quartic polynomial belonging to

k[w, x + y + z, xy + yz + zx, xyz, (x - y)(y - z)(z - x)],

and let $X = (F = 0) \subset \mathbb{P}^3$. Endow X with an automorphism g of order 3 with g(w, x, y, z) = (w, y, z, x). If F is generic (e.g. if $F = w^4 + x^4 + y^4 + z^4 - \lambda^3 wxyz$ with $\lambda \neq 0, 1$), then X is a smooth K3 surface, Fix(g) =

40

 $\{(0, 1, 1, 1), (\lambda, 1, 1, 1)\}$ where $\lambda = (-c(wxyz)/c(w^4))^{1/3}$, and $Y = X/\langle g \rangle$ is an RDP K3 surface with $\text{Sing}(Y) = 2E_6^1$.

Example 10.11 $(G = \mathbb{Z}/5\mathbb{Z}, \text{ and } G = \alpha_5)$. Let $a, b_{-1}, b_0, b_1 \in k$ with $b_{-1}b_1 \neq 0$. Let $b = b(t) = b_{-1}t^{-1} + b_0 + b_1t$ and $c = c(t) = tb(t) = b_{-1} + b_0t + b_1t^2$. Let S and T be two RDP K3 surfaces defined by

$$S: y^{2} = x^{3} + at^{4}x + t^{5}c,$$

$$T: Y^{2} = X^{3} + a^{5}t^{4}X + tc^{5}.$$

Let $\xi = t^{-2}X + ab$. Let $\Delta = -4a^3 - 27b^2$. Let $f: T \dashrightarrow S$ be the rational map defined by f(X, Y) =

$$\bigg(t^2 \frac{\xi^5 - ab\xi^4 - a^2\Delta\xi^3 - a\Delta^3\xi}{(2a\xi^2 + \Delta^2)^2}, Y \frac{\xi^6 + a^2\Delta\xi^4 - 2b\Delta^2\xi^3 - a\Delta^3\xi^2 + 2\Delta^5}{(2a\xi^2 + \Delta^2)^3}\bigg).$$

Over k(t), this defines a separable (resp. inseparable) isogeny of degree 5 between ordinary (resp. supersingular) elliptic curves if $a \neq 0$ (resp. a = 0).

Suppose b is generic and $a \neq 0$. Then T and S are RDP K3 surfaces with $4A_4$ and $2E_8^1$ respectively. Let $\tilde{T} \to T$ be the resolution. Then f induces a finite morphism $\tilde{T} \to S$ that is the quotient morphism of a $\mathbb{Z}/5\mathbb{Z}$ action generated by the translation by a 5-torsion point $(X, Y) = (\frac{2}{e^2}\Delta - ab, 2\Delta(e^3 + \frac{b}{e^3})), e^4 = 2a.$

Suppose a = 0 and $\operatorname{disc}(c) = b_0^2 - 4b_{-1}b_1 \neq 0$ (so c is not a square). Then Tand S are both RDP K3 surfaces with $2E_8^0$. Let $\tilde{T} \to T$ be the resolution, Cbe the unique $4A_4$ configuration contained in the union of the two fibers over t = 0 and $t = \infty$, and $\tilde{T} \to T'$ be the contraction of C. Then T' is an RDP K3 surface with $4A_4$, and f induces a finite morphism $f': T' \to S$ which is an α_5 -quotient morphism. Define a derivation D' on S by D'(x) = 2c'(t)x, D'(y) = 3c'(t)y, D'(t) = c(t). We have $D'^5 = (\operatorname{disc}(c))^2 D'$. This defines a μ_5 -action on S whose quotient is $T'^{(5)}$.

Suppose a = 0 and $\operatorname{disc}(c) = b_0^2 - 4b_{-1}b_1 = 0$ (so c is a square). Then $\operatorname{Sing}(S)$ contains $2E_8^0$, the derivation D' on S defined as above has divisorial fixed locus, and the corresponding α_5 -covering of S is non-normal.

10.5. Inseparable morphisms of degree p between RDP K3 surfaces. We give an example for each case with r > 1 mentioned in Theorem 5.2.

Example 10.12 (Kummer surfaces and generalized Kummer surfaces (cf. [Kat87])). Let $r \in \{2, 3, 4, 6\}$. Let p be a prime with $p \equiv 1 \pmod{r}$. Let $\bar{\pi} \colon A \to B$ be a purely inseparable isogeny of degree p between abelian surfaces in characteristic p, (automatically) induced by a derivation, say D. Suppose we have symplectic automorphisms $g_A \in \operatorname{Aut}_0(A)$ and $g_B \in \operatorname{Aut}_0(B)$ of same order r satisfying $\bar{\pi} \circ g_A = g_B \circ \bar{\pi}$ and $g_A^*(D) = \zeta D$ for a primitive r-th root ζ of unity. Here Aut_0 is the group of automorphisms preserving the origin. Then $\pi \colon A/\langle g_A \rangle \to B/\langle g_B \rangle$ is a purely inseparable morphism of degree p between RDP K3 surfaces, whose covering as in Theorem 5.2 is $\bar{\pi}$.

The singularities of the quotients are as in Table 5 [Kat87, Table in page 17]: $16A_1$, $9A_2$, $4A_3 + 6A_1$, $A_5 + 4A_2 + 5A_1$ for r = 2, 3, 4, 6 respectively.

Examples of such $\bar{\pi}, g_A, g_B$ are given as follows. If r = 2, take $\bar{\pi}$ arbitrarily and let $g_A = [-1]_A, g_B = [-1]_B$. If r = 3, 4, 6, take an elliptic curve Eequipped with an automorphism $h \in \operatorname{Aut}_0(E)$ of order r, and let $\bar{\pi} \colon A = E \times E \to B = E \times E^{(p)}$ and $g_A = h \times h^{-1}, g_B = h \times (h^{(p)})^{-1}$. Then g_B is symplectic since $p \equiv 1 \pmod{r}$.

Remark 10.13. If $\bar{\pi}: A \to B$ be a purely inseparable morphism of degree p between non-supersingular abelian surfaces in characteristic p = 2, then $\pi: A/\{\pm 1\} \to B/\{\pm 1\}$ is a μ_2 - or α_2 -quotient morphism between RDP K3 surfaces. More precisely, if p-rank(A) = 2 (resp. p-rank(A) = 1) then both $\operatorname{Sing}(A/\{\pm 1\})$ and $\operatorname{Sing}(B/\{\pm 1\})$ are $4D_4^1$ (resp. $2D_8^2$) (Katsura [Kat78, Proposition 3]), and both $\bar{\pi}$ and π are μ_2 -quotient (resp. either both are μ_2 -quotient or both are α_2 -quotient).

If A is (and hence B is) supersingular, then $A/\{\pm 1\}$ is not birational to a K3 surface, instead it is a rational surface with a single non-RDP singularity (Katsura [Kat78, Proposition 3]), and so is $B/\{\pm 1\}$.

Example 10.14. For each pair of $G \in \{\mu_p, \alpha_p\}$ and r > 1 appearing in Theorem 5.2(2,3), we give an example of an RDP K3 surface \bar{X} with a derivation D of multiplicative type or additive type and a symplectic automorphism $g \in \operatorname{Aut}(X)$ of order r such that $\bar{Y} = \bar{X}^D$ is an RDP K3 surface and $g^*(D) = \zeta D$ for a primitive r-th root ζ of unity, hence g induces a symplectic automorphism $g' \in \operatorname{Aut}(Y)$ (of order r), and the induced morphism $\pi \colon X = \bar{X}/\langle g \rangle \to Y = \bar{Y}/\langle g' \rangle$ has $\bar{\pi} \colon \bar{X} \to \bar{Y}$ as its minimal covering as in Theorem 5.2.

[$\mu_5, r = 4$] Let $\overline{X} = (x_1^3 x_2 - x_2^3 x_4 + x_4^3 x_3 - x_3^3 x_1 = 0) \subset \mathbb{P}^3$ be the quartic RDP K3 surface (with $4A_4$ at $\{(x_1 : x_2 : x_3 : x_4) = (1 : 2e^3 : e : 3e^2) \mid e^4 = -1\}$), and define a derivation D and an automorphism g of \overline{X} by $D(x_i) = ix_i, g(x_i) = x_{(2i \mod 5)}$. Then both D and g are symplectic, and $g^*D = 2^{-1}D$. Hence $\pi : X = \overline{X}/\langle g \rangle \to Y = \overline{Y}/\langle g \rangle$ is an example with $\overline{\pi}$ a μ_5 -quotient and r = 4.

 $[\mu_7, r = 3]$ Suppose $b \neq 0$ in Example 10.5, and let $g(w, x_1, x_2, x_4) = (w, x_4, x_1, x_2)$. Then g is symplectic and $g^*D = 2D$.

 $[\alpha_5, r=2]$ Suppose e=0 in Example 10.4 and suppose moreover b=c, and let g(w, x, y, z) = (-w, x, z, y). Then $g^*D = -D$ and $g^*D' = -D'$.

 $[\mu_3 \text{ (resp. } \alpha_3), r = 2]$ In Example 10.3 suppose that H and I are invariant under $(x, y, z) \mapsto (x, z, y)$ (resp. $(x, y, z) \mapsto (x, -y, z)$). For example, let $F = x^6 + y^6 + z^6 + xyz(y^3 + z^3)$ (resp. $F = x^6 + y^6 + z^6 + x(xz + y^2)(x^3 - z^3)$). Let g(w, x, y, z) = (-w, x, z, y) (resp. g(w, x, y, z) = (-w, x, -y, z)). Then $g^*(D) = -D$.

Acknowledgments. I thank Simon Brandhorst, Hiroyuki Ito, Tetsushi Ito, Yukari Ito, Shigeyuki Kondo, Hisanori Ohashi, and Takehiko Yasuda for helpful comments and discussions.

References

- [Art77] M. Artin, Coverings of the rational double points in characteristic p, Complex analysis and algebraic geometry, Iwanami Shoten, Tokyo, 1977, pp. 11–22.
- [BM76] E. Bombieri and D. Mumford, Enriques' classification of surfaces in char. p. III, Invent. Math. 35 (1976), 197–232.

42

- [CD89] François R. Cossec and Igor V. Dolgachev, *Enriques surfaces. I*, Progress in Mathematics, vol. 76, Birkhäuser Boston, Inc., Boston, MA, 1989.
- [DK01] Igor V. Dolgachev and JongHae Keum, Wild p-cyclic actions on K3-surfaces, J. Algebraic Geom. 10 (2001), no. 1, 101–131.
- [DK09] _____, Finite groups of symplectic automorphisms of K3 surfaces in positive characteristic, Ann. of Math. (2) **169** (2009), no. 1, 269–313.
- [ESB04] T. Ekedahl and N. I. Shepherd-Barron, On exceptional Enriques surfaces (2004), available at http://arxiv.org/abs/math/0405510.
- [EHSB12] T. Ekedahl, J. M. E. Hyland, and N. I. Shepherd-Barron, Moduli and periods of simply connected Enriques surfaces (2012), available at http://arxiv.org/abs/1210.0342.
 - [Kat78] Toshiyuki Katsura, On Kummer surfaces in characteristic 2, Proceedings of the International Symposium on Algebraic Geometry (Kyoto Univ., Kyoto, 1977), Kinokuniya Book Store, Tokyo, 1978, pp. 525–542.
 - [Kat87] _____, Generalized Kummer surfaces and their unirationality in characteristic p, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 34 (1987), no. 1, 1–41.
 - [KK15] Toshiyuki Katsura and Shigeyuki Kondō, A 1-dimensional family of Enriques surfaces in characteristic 2 covered by the supersingular K3 surface with Artin invariant 1, Pure Appl. Math. Q. 11 (2015), no. 4, 683–709.
 - [KT89] Toshiyuki Katsura and Y. Takeda, Quotients of abelian and hyperelliptic surfaces by rational vector fields, J. Algebra 124 (1989), no. 2, 472–492.
 - [Lie15] Christian Liedtke, Arithmetic moduli and lifting of Enriques surfaces, J. Reine Angew. Math. 706 (2015), 35–65.
 - [Lip69] Joseph Lipman, Rational singularities, with applications to algebraic surfaces and unique factorization, Inst. Hautes Études Sci. Publ. Math. 36 (1969), 195– 279.
- [Mat20a] Yuya Matsumoto, μ_n -actions on K3 surfaces in positive characteristic (2020), available at http://arxiv.org/abs/1710.07158v3.
- [Mat21] _____, Canonical coverings of Enriques surfaces in characteristic 2 (2021), available at http://arxiv.org/abs/1812.06914v3.
- [Mat20b] _____, Inseparable maps on W_n -valued Ext groups of non-taut rational double point singularities and the height of K3 surfaces (2020), available at http://arxiv.org/abs/1907.04686v2.
 - [Nik79] V. V. Nikulin, Finite automorphism groups of Kähler K3 surfaces, Trudy Moskov. Mat. Obshch. 38 (1979), 75–137 (Russian). English translation: Trans. Moscow Math. Soc. 1980, no. 2, 71–135.
 - [RS76] A. N. Rudakov and I. R. Shafarevich, Inseparable morphisms of algebraic surfaces, Izv. Akad. Nauk SSSR Ser. Mat. 40 (1976), no. 6, 1269–1307, 1439 (Russian). English translation: Math. USSR-Izv. 10 (1976), no. 6, 1205–1237.
 - [Sch19] Stefan Schröer, Enriques surfaces with normal K3-like coverings (2019), available at http://arxiv.org/abs/1703.03081v2.
 - [Ses60] Conjeerveram Srirangachari Seshadri, L'opération de Cartier. Applications, Séminaire C. Chevalley, 3ième année: 1958/59. Variétés de Picard, École Normale Supérieure, Paris, 1960, pp. 1–26 (French).
 - [Tzi17] Nikolaos Tziolas, Quotients of schemes by α_p or μ_p actions in characteristic p > 0, Manuscripta Math. **152** (2017), no. 1–2, 247–279.

Department of Mathematics, Faculty of Science and Technology, Tokyo University of Science, 2641 Yamazaki, Noda, Chiba, 278-8510, Japan

Email address: matsumoto.yuya.m@gmail.com

Email address: matsumoto_yuya@ma.noda.tus.ac.jp