

A WEIL-ÉTALE VERSION OF THE BIRCH AND SWINNERTON-DYER FORMULA OVER FUNCTION FIELDS

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ABSTRACT. We give a reformulation of the Birch and Swinnerton-Dyer conjecture over global function fields in terms of Weil-étale cohomology of the curve with coefficients in the Néron model, and show that it holds under the assumption of finiteness of the Tate-Shafarevich group.

1. INTRODUCTION

The conjecture of Birch and Swinnerton-Dyer is one of the most important problems in arithmetic geometry. It states that the rank of the rational points $A(K)$ of an abelian variety A over a global field K is equal to the order of vanishing of the L -function $L(A, s)$ associated with A at $s = 1$, and relates the leading term of the L -function to various invariants associated with A . If K has characteristic p and under the assumption of finiteness of the Tate-Shafarevich group $\text{III}(A)$, Schneider [Sch82] proved a formula for the prime to p -part of the leading coefficient, Bauer [Bau92] gave a formula in case A has good reduction at every place, and Kato-Trihan [KT03] proved a formula in general.

In this paper we give a formula for the leading coefficient in terms of Weil-étale cohomology $H_W^*(S, \mathcal{A})$ of the regular complete curve S with function field K with coefficients in the Néron model \mathcal{A} of A . More precisely, let \mathbb{F}_q be the field of constants of S and $e \in H_W^1(\mathbb{F}_q, \mathbb{Z})$ the element corresponding to the q -th power arithmetic Frobenius morphism.

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The cup product with e defines a complex

$$\cdots \xrightarrow{e} H_W^i(S, \mathcal{A}) \xrightarrow{e} H_W^{i+1}(S, \mathcal{A}) \xrightarrow{e} H_W^{i+2}(S, \mathcal{A}) \xrightarrow{e} \cdots,$$

whose cohomology groups are finite if the groups $H_W^*(S, \mathcal{A})$ are finitely generated. In this case, we denote the alternating product of their orders by $\chi(H_W^*(S, \mathcal{A}), e)$. Let $\text{Lie}(\mathcal{A})$ be the vector bundle on S defined by the pullback of the dual of $\Omega_{\mathcal{A}/S}^1$ by the zero section $S \hookrightarrow \mathcal{A}$, and $\chi(S, \text{Lie}(\mathcal{A}))$ the alternating sum of dimensions of the coherent cohomology $H^*(S, \text{Lie}(\mathcal{A}))$ over \mathbb{F}_q .

Theorem 1.1. *Let A be an abelian variety over a global field K of characteristic p and assume that $\text{III}(A)$ is finite. Then the rank r of $A(K)$ equals the order of vanishing of $L(A, s)$ at $s = 1$, the groups $H_W^*(S, \mathcal{A})$ are finitely generated, and*

$$\lim_{s \rightarrow 1} \frac{L(A, s)}{(s-1)^r} = \chi(H_W^*(S, \mathcal{A}), e)^{-1} \cdot q^{\chi(S, \text{Lie}(\mathcal{A}))} \cdot (\log q)^r.$$

The same statements holds if we replace \mathcal{A} by \mathcal{A}^0 , the subgroup scheme with all fibers connected. Our result fits into the general philosophy that important conjectures in arithmetic geometry are equivalent to finite generation statements of Weil-étale cohomology groups, and special values of zeta and L -functions can be expressed as Euler characteristics of Weil-étale cohomology ([Lic05], [Gei04]).

The proof proceeds by showing that finiteness of $\text{III}(A)$ is equivalent to finite generation of the groups $H_W^i(S, \mathcal{A})$, and implies an identity

$$\chi(H_W^*(S, \mathcal{A}), e)^{-1} = \frac{\#\text{III}(A)}{\#A(K)_{\text{tor}} \cdot \#B(K)_{\text{tor}}} \cdot \frac{\text{Disc}(h)}{(\log q)^r} \cdot c(A),$$

where B is the abelian variety dual to A , h the height pairing, and $c(A)$ the product of the orders of the groups of $k(v)$ -rational connected components of the fibers of the Néron model over all places v . Key ingredients are results of the second named author [Suz19]. Theorem 1.1 follows then by applying the result of Kato-Trihan [KT03, Chap. I, Thm.].

At the end of the paper we show that Weil-étale cohomology is an integral model of l -adic cohomology:

Theorem 1.2. *Assume that $\text{III}(A)$ is finite. Let l be a prime and ${}_{\iota}(\mathcal{A}^0)$ the l^i -torsion part of \mathcal{A}^0 . If $l \neq p$, then the canonical homomorphism*

$$H_W^n(S, \mathcal{A}^0) \otimes \mathbb{Z}_l \rightarrow \varprojlim_i H_{\text{et}}^{n+1}(S, {}_{\iota}(\mathcal{A}^0))$$

is an isomorphism. If $l = p$ and A has semistable reduction everywhere, then the canonical homomorphism

$$H_W^n(S, \mathcal{A}^0) \otimes \mathbb{Z}_p \rightarrow \varprojlim_i H_{\text{fppf}}^{n+1}(S, p^i(\mathcal{A}^0))$$

is an isomorphism.

For more complete results (without the semistability assumption for the case $l = p$), see Prop. 9.1 and 9.2.

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Notation. Throughout this paper, we fix a finite field \mathbb{F}_q with q elements of characteristic p , an algebraic closure $\overline{\mathbb{F}_q}$ of \mathbb{F}_q , a proper smooth geometrically connected curve S over \mathbb{F}_q with function field K , and an abelian variety A over K of dimension d . We denote the rank of the Mordell-Weil group $A(K)$ by r . Thus

$$d = \dim(A), \quad r = \text{rank}(A(K)).$$

For a place v of K (i.e. a closed point of S), we denote the residue field of S at v by $k(v)$, its cardinality by $N(v)$, the completed local ring by \mathcal{O}_v , its maximal ideal by m_v and its fraction field by K_v . The adèle ring of K is denoted by \mathbb{A}_K and the integral adèle ring by $\mathcal{O}_{\mathbb{A}_K}$. For a group scheme G over \mathcal{O}_v and $n \geq 0$, we denote the kernel of the reduction map $G(\mathcal{O}_v) \rightarrow G(\mathcal{O}_v/m_v^n)$ by $G(m_v^n)$.

The Néron model over S of A/K is denoted by \mathcal{A} . The open subgroup scheme of \mathcal{A} with connected fibers is denoted by \mathcal{A}^0 and the fiber of \mathcal{A} at a closed point $v \in S$ is denoted by \mathcal{A}_v . Let B be the dual abelian variety of A/K . We have objects $\mathcal{B}, \mathcal{B}^0, \mathcal{B}_v$ correspondingly.

For an abelian group G , its torsion part is denoted by G_{tor} and torsion-free quotient by G/tor . The n -torsion part (kernel of multiplication by n) of G for an integer n is denoted by ${}_nG$. For a pairing φ between finitely generated abelian groups G and H with values in \mathbb{Z}, \mathbb{Q} or \mathbb{R} , we denote by $\text{Disc}(\varphi)$ the absolute value of the determinant of the matrix presentation of φ with respect to some (or equivalently, any) \mathbb{Z} -bases of G/tor and H/tor .

If \mathcal{F} is a coherent sheaf on S , the Euler characteristic of \mathcal{F} is

$$\chi(S, \mathcal{F}) = \sum_i (-1)^i \dim_{\mathbb{F}_q} H^i(S, \mathcal{F}).$$

If C is a complex of abelian groups with finitely many finite cohomology groups, then we denote

$$\chi(C) = \prod_i (\#H^i(C))^{(-1)^i}.$$

If C is a graded object of finite abelian groups with finitely many terms, then we denote

$$\chi(C) = \prod_i (\#C^i)^{(-1)^i}.$$

These two pieces of notation are compatible by viewing a graded object as a complex with zero differentials.

2. THE BIRCH AND SWINNERTON-DYER FORMULAS BY TATE AND BY KATO-TRIHAN

We recall the Birch and Swinnerton-Dyer conjecture for A/K as formulated by Tate [Tat68] and by Kato-Trihan [KT03].

Let $l \neq p$ be a prime number. The rational l -adic Tate module $V_l(A) = T_l(A) \otimes \mathbb{Q}$ is dual to the l -adic cohomology $H_{\text{cont}}^1(A \times_K K^{\text{sep}}, \mathbb{Q}_l)$ as Galois representations over K , where K^{sep} is a separable closure of K . It is also dual to the negative Tate twist $V_l(B)(-1)$ of $V_l(B)$ via the Weil pairing, where B is the dual abelian variety of A as in Notation at the end of Introduction. Since B is (non-canonically) isogenous to A , $V_l(B)$ is (non-canonically) isomorphic to $V_l(A)$. Summarizing,

$$(1) \quad \begin{aligned} H_{\text{cont}}^1(A \times_K K^{\text{sep}}, \mathbb{Q}_l) &= \text{Hom}(V_l(A), \mathbb{Q}_l) \\ &= V_l(B)(-1) \cong V_l(A)(-1). \end{aligned}$$

For a place v of K where A has good reduction, we define a polynomial in t by

$$P_v(t) = \det(1 - \varphi_{vt} \mid H_{\text{cont}}^1(A \times_K K^{\text{sep}}, \mathbb{Q}_l)),$$

where φ_v is the geometric Frobenius at v . This has \mathbb{Z} -coefficients and does not depend on l . Let $U \subset S$ be an open dense subscheme where A has good reduction. Denote $\Sigma = S \setminus U$. As in [KT03, §1.3], for a complex number s with $\text{Re}(s) > 3/2$, we define the L -function $L(U, A, s)$ without Euler factors outside U by

$$L(U, A, s) = \prod_{v \in U} P_v(N(v)^{-s})^{-1}.$$

This is a rational function in q^{-s} and regular at $s = 1$. In [Tat68, (1.3)], this is denoted by $L_{\Sigma}(s)$.

Let $\text{Lie}(\mathcal{A})/S$ be the Lie algebra (with zero Lie bracket) of the Néron model \mathcal{A}/S ([DG70, Exp. II, 4.11]). It is a group scheme represented by a locally free sheaf on S of rank $d = \dim(A)$ that is given by the pullback along the zero section $S \hookrightarrow \mathcal{A}$ of the $\mathcal{O}_{\mathcal{A}}$ -linear dual of $\Omega_{\mathcal{A}/S}^1$. We similarly have the Lie algebra $\text{Lie}(A)/K$ of A/K .

For each place v , we give K_v the normalized Haar measure μ_v , so that $\mu_v(\mathcal{O}_v) = 1$. (The measure μ_v in [Tat68, §1] is not necessarily normalized for all v . Our normalization is just for simplicity.) The product $\mu = \prod_v \mu_v$ gives a Haar measure on the adèle ring \mathbb{A}_K . In [Tat68, (1.5)], the measure $\mu(\mathbb{A}_K/K)$ of the compact quotient \mathbb{A}_K/K is denoted by $|\mu|$.

We fix a non-zero invariant top degree differential form ω on A/K . As in [Tat68, §1], ω and μ_v on K_v together determine a Haar measure on $\text{Lie}(A)(K_v)$ (still denoted by μ_v) for each v . By definition [Wei82, §2.2.1], this measure is characterized by

$$(2) \quad \mu_v(L) = [\omega(\det(L)) : \mathcal{O}_v]$$

for any full rank \mathcal{O}_v -lattice L of $\text{Lie}(A)(K_v)$, where $\det(L)$ is the top exterior power of L over \mathcal{O}_v , $\omega(\det(L)) \subset K_v$ is the image of $\det(L)$ by ω viewed as a K_v -linear isomorphism $\det(\text{Lie}(A)(K_v)) \xrightarrow{\sim} K_v$, and the index $[\omega(\det(L)) : \mathcal{O}_v]$ means $[\mathcal{O}_v : \omega(\det(L))]^{-1}$ in case $\omega(\det(L))$ does not contain \mathcal{O}_v . One may take this formula as the definition of μ_v . If ω is replaced by its multiple by a rational function $f \in K^\times$, then μ_v is multiplied by the normalized absolute value $|f|_v = N(v)^{-v(f)}$.

We have $\mu_v(\text{Lie}(\mathcal{A})(\mathcal{O}_v)) = 1$ for almost all v . Hence the product $\mu = \prod_v \mu_v$ defines a Haar measure on the adelic points $\text{Lie}(A)(\mathbb{A}_K)$. The measure μ_v on $\text{Lie}(A)(K_v)$ in turn determines a Haar measure on $A(K_v)$ such that

$$(3) \quad \mu_v(\text{Lie}(\mathcal{A})(m_v^n)) = \mu_v(\mathcal{A}(m_v^n))$$

for all $n \geq 1$. The measure $\mu_v(A(K_v))$ is denoted by $\int_{A(K_v)} |\omega|_v \mu_v^d$ in [Tat68, §1].

Assume that $U \subset S$ is large enough so that ω gives a nowhere vanishing section of the dual of the line bundle $\det(\text{Lie}(\mathcal{A}))$ over U , in addition that A has good reduction over U . Following [Tat68, (1.5)], we define

$$L_\Sigma^*(s) = L_\Sigma^*(A, s) = \frac{\mu(\mathbb{A}_K/K)^d}{\prod_{v \notin U} \mu_v(A(K_v))} \cdot L(U, A, s).$$

This is independent of the choice of ω by the product formula. As shown after loc. cit., the asymptotic behavior of $L_\Sigma^*(A, s)$ as $s \rightarrow 1$ does not depend on U (or Σ). Also, following [KT03, §1.7], we define

$$\text{vol}\left(\prod_{v \notin U} A(K_v)\right) = \frac{\prod_{v \notin U} \mu_v(A(K_v))}{\mu(\text{Lie}(A)(\mathbb{A}_K)/\text{Lie}(A)(K))},$$

which is independent of the choice of ω .

Let $\text{III}(A)$ be the Tate-Shafarevich group of A and

$$h: A(K) \times B(K) \rightarrow \mathbb{R}$$

the Néron-Tate height pairing. We have $\text{Disc}(h) \neq 0$.

Now Tate's formulation of the Birch and Swinnerton-Dyer conjecture is the following.

Conjecture 2.1 ([Tat68, §1, (A), (B)]). *The order of zero of $L(U, A, s)$ at $s = 1$ is the Mordell-Weil rank r . The group $\text{III}(A)$ is finite. We have*

$$\lim_{s \rightarrow 1} \frac{L_{\Sigma}^*(A, s)}{(s-1)^r} = \frac{\#\text{III}(A) \cdot \text{Disc}(h)}{\#A(K)_{\text{tor}} \cdot \#B(K)_{\text{tor}}}.$$

On the other hand, Kato-Trihan's formulation is the following.

Conjecture 2.2 ([KT03, 1.3.1, 1.4.1, 1.8.1]). *The order of zero of $L(U, A, s)$ at $s = 1$ is r . The group $\text{III}(A)$ is finite. We have*

$$\lim_{s \rightarrow 1} \frac{L(U, A, s)}{(s-1)^r} = \frac{\#\text{III}(A) \cdot \text{Disc}(h)}{\#A(K)_{\text{tor}} \cdot \#B(K)_{\text{tor}}} \cdot \text{vol}\left(\prod_{v \notin U} A(K_v)\right).$$

3. COMPARISON OF THE TWO FORMULAS

In this section, we show, for the convenience of the reader, that Conj. 2.1 and 2.2 are equivalent. It suffices to show (without hypothesis on the order of zero of $L(U, A, s)$ or finiteness of $\text{III}(A)$) the following.

Proposition 3.1.

$$\mu\left(\frac{\text{Lie}(A)(\mathbb{A}_K)}{\text{Lie}(A)(K)}\right) = \mu(\mathbb{A}_K/K)^d.$$

We are going to reduce this to the Riemann-Roch theorem for the vector bundle $\text{Lie}(\mathcal{A})$ over S . First we relate the left-hand side to the Euler characteristic $\chi(S, \text{Lie}(\mathcal{A}))$. This step will also be used in the next section as one of the keys for the proof of Thm. 1.1.

Proposition 3.2.

$$\mu\left(\frac{\text{Lie}(A)(\mathbb{A}_K)}{\text{Lie}(A)(K)}\right) = \mu(\text{Lie}(\mathcal{A})(\mathcal{O}_{\mathbb{A}_K})) \cdot q^{-\chi(S, \text{Lie}(\mathcal{A}))}.$$

Proof. Let $\mathcal{O}_{(v)}$ be the (Zariski) local ring of S at v . The excision and localization sequences for Zariski cohomology give long exact sequences

$$\begin{aligned} \cdots \rightarrow \bigoplus_v H_v^n(\mathcal{O}_{(v)}, \text{Lie}(\mathcal{A})) \rightarrow H^n(S, \text{Lie}(\mathcal{A})) \rightarrow H^n(K, \text{Lie}(A)) \rightarrow \cdots, \\ \cdots \rightarrow H_v^n(\mathcal{O}_{(v)}, \text{Lie}(\mathcal{A})) \rightarrow H^n(\mathcal{O}_{(v)}, \text{Lie}(\mathcal{A})) \rightarrow H^n(K, \text{Lie}(A)) \rightarrow \cdots \end{aligned}$$

(for each place v of K in the latter sequence), where H_v^n denotes cohomology with closed support. The (Zariski) cohomology groups $H^n(K, \text{Lie}(A))$ and $H^n(\mathcal{O}_{(v)}, \text{Lie}(\mathcal{A}))$ are zero for $n \geq 1$. Hence the finite groups $H^0(S, \text{Lie}(\mathcal{A}))$ and $H^1(S, \text{Lie}(\mathcal{A}))$ are given by the kernel and the cokernel, respectively, of the natural homomorphism

$$\text{Lie}(A)(K) \rightarrow \bigoplus_v \frac{\text{Lie}(A)(K)}{\text{Lie}(\mathcal{A})(\mathcal{O}_{(v)})}.$$

Each summand of the right-hand side is isomorphic to $\text{Lie}(A)(K_v)/\text{Lie}(\mathcal{A})(\mathcal{O}_v)$ by approximation. Therefore the right-hand side (the whole direct sum) is isomorphic to $\text{Lie}(A)(\mathbb{A}_K)/\text{Lie}(\mathcal{A})(\mathcal{O}_{\mathbb{A}_K})$. In other words, we have a natural exact sequence

$$0 \rightarrow \frac{\text{Lie}(\mathcal{A})(\mathcal{O}_{\mathbb{A}_K})}{H^0(S, \text{Lie}(\mathcal{A}))} \rightarrow \frac{\text{Lie}(A)(\mathbb{A}_K)}{\text{Lie}(A)(K)} \rightarrow H^1(S, \text{Lie}(\mathcal{A})) \rightarrow 0.$$

Thus

$$\begin{aligned} \mu \left(\frac{\text{Lie}(A)(\mathbb{A}_K)}{\text{Lie}(A)(K)} \right) &= \mu \left(\frac{\text{Lie}(\mathcal{A})(\mathcal{O}_{\mathbb{A}_K})}{H^0(S, \text{Lie}(\mathcal{A}))} \right) \cdot \#H^1(S, \text{Lie}(\mathcal{A})) \\ &= \mu(\text{Lie}(\mathcal{A})(\mathcal{O}_{\mathbb{A}_K})) \cdot \frac{\#H^1(S, \text{Lie}(\mathcal{A}))}{\#H^0(S, \text{Lie}(\mathcal{A}))} \\ &= \mu(\text{Lie}(\mathcal{A})(\mathcal{O}_{\mathbb{A}_K})) \cdot q^{-\chi(S, \text{Lie}(\mathcal{A}))}. \end{aligned}$$

□

The above proposition is an intermediate step, as the term $\mu(\text{Lie}(\mathcal{A})(\mathcal{O}_{\mathbb{A}_K}))$ can be more explicitly calculated as follows.

Proposition 3.3.

$$\mu(\text{Lie}(\mathcal{A})(\mathcal{O}_{\mathbb{A}_K})) = q^{\deg(\det(\text{Lie}(\mathcal{A})))}.$$

Proof. For any v , let $v(\omega) \in \mathbb{Z}$ be the order of zero at v of the rational section ω of the dual of the line bundle $\det(\text{Lie}(\mathcal{A}))$ over S . Then $\mu_v(\text{Lie}(\mathcal{A})(\mathcal{O}_v)) = N(v)^{-v(\omega)}$ by (2). We have $-\sum_v [k(v) : \mathbb{F}_q] v(\omega) = \deg(\det(\text{Lie}(\mathcal{A})))$. This gives the result. □

Proof of Prop. 3.1. By the previous two propositions, we have

$$\mu \left(\frac{\text{Lie}(A)(\mathbb{A}_K)}{\text{Lie}(A)(K)} \right) = q^{\deg(\det(\text{Lie}(\mathcal{A}))) - \chi(S, \text{Lie}(\mathcal{A}))}.$$

The same calculations, applied to the structure sheaf \mathcal{O}_S instead of $\text{Lie}(\mathcal{A})$, show that

$$\mu(\mathbb{A}_K/K) = \mu(\mathcal{O}_{\mathbb{A}_K}) \cdot q^{-\chi(S, \mathcal{O}_S)} = q^{-\chi(S, \mathcal{O}_S)}.$$

Therefore the result follows from the Riemann-Roch theorem

$$(4) \quad \chi(S, \text{Lie}(\mathcal{A})) = d \cdot \chi(S, \mathcal{O}_S) + \deg(\det(\text{Lie}(\mathcal{A}))).$$

□

4. BAD EULER FACTORS

Using Prop. 3.2 above, we will rewrite the Birch and Swinnerton-Dyer formula in a form including bad Euler factors and $\chi(S, \text{Lie}(\mathcal{A}))$ without terms defined by Haar measures.

As in the previous section, let $l \neq p$ be a prime number. For any place v of K where A may have good or bad reduction, we define a polynomial in t by

$$P_v(t) = \det(1 - \varphi_v t \mid H_{\text{cont}}^1(A \times_K K^{\text{sep}}, \mathbb{Q}_l)^{I_v}),$$

where I_v is the inertial group at v . We define the completed L -function by

$$L(A, s) = L(U, A, s) \cdot \prod_{v \notin U} P_v(N(v)^{-s})^{-1} = \prod_v P_v(N(v)^{-s})^{-1},$$

where the latter product is over all places v . Recall that $d = \dim(A)$.

Proposition 4.1. *For any place v , the polynomial $P_v(t)$ has \mathbb{Z} -coefficients and does not depend on $l \neq p$. We have*

$$P_v(N(v)^{-1}) = \frac{\#\mathcal{A}^0(k(v))}{N(v)^d}.$$

Proof. This is well-known. We recall its proof. By (1), we have

$$P_v(t) = \det(1 - \varphi_v t \mid V_l(A)^{I_v}(-1)) = \det(1 - N(v)\varphi_v t \mid V_l(A)^{I_v}).$$

Let K_v^{ur} be the maximal unramified extension of K_v with ring of integers $\mathcal{O}_v^{\text{ur}}$. Then

$$V_l(A)^{I_v} = V_l(A)(K^{\text{ur}}) = V_l(A(K^{\text{ur}})) = V_l(\mathcal{A}(\mathcal{O}_v^{\text{ur}})).$$

By the smoothness of \mathcal{A} , the reduction map $\mathcal{A}(\mathcal{O}_v^{\text{ur}}) \rightarrow \mathcal{A}(\overline{k(v)})$ is surjective. Its kernel is uniquely l -divisible. With the finiteness of the component group $\pi_0(\mathcal{A}_v) = \mathcal{A}_v/\mathcal{A}_v^0$ of the fiber \mathcal{A}_v , we have $V_l(A)^{I_v} \cong V_l(\mathcal{A}_v^0)$ as l -adic representations over $k(v)$. By the Chevalley structure theorem, the algebraic group \mathcal{A}_v^0 over $k(v)$ has a canonical filtration whose graded pieces are a torus T , a smooth connected unipotent group U and an abelian variety A' . Since U is p -power-torsion, we have an exact sequence $0 \rightarrow V_l(T) \rightarrow V_l(\mathcal{A}_v^0) \rightarrow V_l(A') \rightarrow 0$ and an equality

$$P_v(t) = \det(1 - N(v)\varphi_v t \mid V_l(T)) \cdot \det(1 - N(v)\varphi_v t \mid V_l(A')).$$

On the other hand, a short exact sequence of connected algebraic groups over a finite field induces a short exact sequence of their groups of rational points by Lang's theorem. We have $d = \dim(T) + \dim(U) + \dim(A')$. Hence

$$\frac{\#\mathcal{A}^0(k(v))}{N(v)^d} = \frac{\#T(k(v))}{N(v)^{\dim(T)}} \cdot \frac{\#U(k(v))}{N(v)^{\dim(U)}} \cdot \frac{\#A'(k(v))}{N(v)^{\dim(A')}}.$$

The group U is a finite successive extension of copies of \mathbf{G}_a . Hence the middle factor in the right-hand side is 1.

Therefore we may treat T and A' separately. The A' -factor is classical and treated by Weil (use [Tat68, (1.1), (1.2)]), resulting that the polynomial $\det(1 - N(v)\varphi_v t \mid V_l(A'))$ has \mathbb{Z} -coefficients, does not depend on l and

$$\det(1 - \varphi_v \mid V_l(A')) = \frac{\#A'(k(v))}{N(v)^{\dim(A')}}.$$

About the T -factor, we have $V_l(T) \cong \text{Hom}(X^*(T), \mathbb{Q}_l(1))$ as l -adic representations, where $X^*(T)$ is the character group of T . Hence

$$\det(1 - N(v)\varphi_v t \mid V_l(T)) = \det(1 - \varphi_v^{-1} t \mid X^*(T)).$$

This has \mathbb{Z} -coefficients and does not depend on l . Its value at $t = N(v)^{-1}$ is

$$\frac{\det(N(v) - \varphi_v^{-1} \mid X^*(T))}{N(v)^{\dim(T)}} = \frac{\#T(k(v))}{N(v)^{\dim(T)}},$$

where the last equality is [Oes84, I, 1.5]. \square

We define the (global) Tamagawa factor of A by

$$c(A) = \prod_v \#\pi_0(\mathcal{A}_v)(k(v)).$$

Let U and ω be as in Conj. 2.1.

Proposition 4.2.

$$\prod_{v \notin U} \mu_v(A(K_v)) = c(A) \cdot \mu(\text{Lie}(\mathcal{A})(\mathcal{O}_{\mathbb{A}_K})) \cdot \prod_{v \notin U} P_v(N(v)^{-1}).$$

Proof. For any place v , the reduction map $A(K_v) = \mathcal{A}(\mathcal{O}_v) \rightarrow \mathcal{A}(k(v))$ is surjective since \mathcal{A} is smooth and \mathcal{O}_v is henselian. Therefore we have an exact sequence

$$0 \rightarrow \mathcal{A}(m_v) \rightarrow A(K_v) \rightarrow \mathcal{A}(k(v)) \rightarrow 0$$

and an equality

$$\mu_v(A(K_v)) = \#\mathcal{A}(k(v)) \cdot \mu_v(\mathcal{A}(m_v)).$$

By Lang's theorem, we have an exact sequence

$$0 \rightarrow \mathcal{A}^0(k(v)) \rightarrow \mathcal{A}(k(v)) \rightarrow \pi_0(\mathcal{A}_v)(k(v)) \rightarrow 0$$

and hence an equality

$$\#\mathcal{A}(k(v)) = \#\pi_0(\mathcal{A}_v)(k(v)) \cdot \#\mathcal{A}^0(k(v)).$$

By (3), we have

$$\mu_v(\mathcal{A}(m_v)) = \mu_v(\text{Lie}(\mathcal{A})(m_v)) = \frac{\mu_v(\text{Lie}(\mathcal{A})(\mathcal{O}_v))}{\#\text{Lie}(\mathcal{A})(k(v))} = \frac{\mu_v(\text{Lie}(\mathcal{A})(\mathcal{O}_v))}{N(v)^d}.$$

Combining all the above, we get

$$\mu_v(A(K_v)) = \#\pi_0(\mathcal{A}_v)(k(v)) \cdot \mu_v(\text{Lie}(\mathcal{A})(\mathcal{O}_v)) \cdot \frac{\#\mathcal{A}^0(k(v))}{N(v)^d}.$$

The third factor in the right-hand side is $P_v(N(v)^{-1})$ by Prop. 4.1. Taking the product over $v \notin U$, we get the result. \square

Proposition 4.3.

$$\text{vol}\left(\prod_{v \notin U} A(K_v)\right) = c(A) \cdot q^{\chi(S, \text{Lie}(\mathcal{A}))} \cdot \prod_{v \notin U} P_v(N(v)^{-1}).$$

Proof. This follows from Prop. 3.2 and 4.2 \square

Proposition 4.4.

$$\lim_{s \rightarrow 1} \frac{L(A, s)}{L_\Sigma^*(A, s)} = c(A) \cdot q^{\chi(S, \text{Lie}(\mathcal{A}))}.$$

Proof. We have

$$\begin{aligned} \frac{L(U, A, s)}{L_\Sigma^*(A, s)} &= \frac{\prod_{v \notin U} \mu_v(A(K_v))}{\mu(\mathbb{A}_K/K)^d} \\ &= \frac{\prod_{v \notin U} \mu_v(A(K_v))}{\mu(\text{Lie}(A)(\mathbb{A}_K)/\text{Lie}(A)(K))} \\ &= \text{vol}\left(\prod_{v \notin U} A(K_v)\right) \\ &= c(A) \cdot q^{\chi(S, \text{Lie}(\mathcal{A}))} \cdot \prod_{v \notin U} P_v(N(v)^{-1}), \end{aligned}$$

where the first and third equalities are by definition, the second by Prop. 3.1 and the fourth by Prop. 4.3. On the other hand,

$$\frac{L(A, s)}{L(U, A, s)} = \prod_{v \notin U} P_v(N(v)^{-s})^{-1} \rightarrow \prod_{v \notin U} P_v(N(v)^{-1})^{-1}$$

as $s \rightarrow 1$. Multiplying these two, we get the result. \square

Corollary 4.5. *The formula in Conj. 2.1 or 2.2 is equivalent to the formula*

$$\lim_{s \rightarrow 1} \frac{L(A, s)}{(1 - q^{1-s})^r} = \frac{\#\text{III}(A)}{\#A(K)_{\text{tor}} \cdot \#B(K)_{\text{tor}}} \cdot \frac{\text{Disc}(h)}{(\log q)^r} \cdot c(A) \cdot q^{\chi(S, \text{Lie}(\mathcal{A}))}.$$

By (4), we have

$$\chi(S, \text{Lie}(\mathcal{A})) = d \cdot \chi(S, \mathcal{O}_S) + \deg(\det(\text{Lie}(\mathcal{A}))) = d(1 - g) - \deg(\omega),$$

where g is the genus of the curve S and $\deg(\omega) = \sum_v [k(v) : \mathbb{F}_q] v(\omega)$. Hence we can also write the above conjectural formula as

$$\lim_{s \rightarrow 1} \frac{L(A, s)}{(s - 1)^r} = \frac{\#\text{III}(A) \cdot \text{Disc}(h)}{\#A(K)_{\text{tor}} \cdot \#B(K)_{\text{tor}}} \cdot q^{-\deg(\omega) + d(1-g)} \cdot c(A).$$

If A has good reduction everywhere, this is the formula in [Bau92, Thm. 4.7], which omits the factor $c(A)$ since $c(A) = 1$ for such A .

In the rest of this paper, we will rewrite the right-hand side of the formula in Cor. 4.5 using Weil-étale cohomology of S with coefficients in \mathcal{A} . We begin with the definition of Weil-étale cohomology and need some preparations.

5. REVIEW OF WEIL-ÉTALE COHOMOLOGY

We recall the definition of Weil-étale cohomology following [Gei04]. For a scheme X over \mathbb{F}_q , its base change to $\overline{\mathbb{F}_q}$ is denoted by \overline{X} . For a sheaf \mathcal{F} on S_{et} , its pullback to \overline{S}_{et} is denoted by $\overline{\mathcal{F}}$. If \mathcal{F} is representable by a scheme locally of finite type over S , then these two pieces of notation are compatible by a limit argument ([Mil80, II, Lem. 3.3, also Rmk. 3.4]).

Let $G \cong \mathbb{Z}$ be the Weil group of \mathbb{F}_q and $\phi \in G$ the q -th power arithmetic Frobenius. We denote the category of abelian groups (resp. G -modules) by Ab (resp. Mod_G) and the category of sheaves of abelian groups on S_{et} by $\text{Ab}(S_{\text{et}})$. Consider the left exact functor $\text{Ab}(S_{\text{et}}) \rightarrow \text{Mod}_G$ sending a sheaf \mathcal{F} to the abelian group $\Gamma(\overline{S}, \overline{\mathcal{F}})$ with its natural G -action. If \mathcal{F} is an injective sheaf, then $H_{\text{et}}^i(\overline{S}, \overline{\mathcal{F}}) = 0$ for $i > 0$ by a limit argument ([Mil80, III, Lem. 1.16]). Therefore the i -th right derived functor of $\mathcal{F} \mapsto \Gamma(\overline{S}, \overline{\mathcal{F}})$ is $\mathcal{F} \mapsto H_{\text{et}}^i(\overline{S}, \overline{\mathcal{F}})$ with the natural G -action. Hence this derived functor agrees with what is denoted by $R^i \Gamma_{\overline{S}}(\gamma^* \mathcal{F})$ in the notation of [Gei04, §6] by [Gei04, Lem. 6.1]. Let

$$D^+(S_{\text{et}}) \rightarrow D^+(\text{Mod}_G), \quad \mathcal{F} \mapsto R\Gamma_{\text{et}}(\overline{S}, \overline{\mathcal{F}})$$

be the total right derived functor on the bounded below derived categories. By composing it with the group cohomology functor $R\Gamma(G, \cdot)$, we have a triangulated functor to $D^+(\text{Ab})$, which agrees with what is

denoted by $R\Gamma_W(S, \gamma^* \cdot)$ in the notation of [Gei04, §6]. Omitting γ^* from the notation, we denote the resulting functor by

$$D^+(S_{\text{et}}) \rightarrow D^+(\text{Ab}), \quad \mathcal{F} \mapsto R\Gamma_W(S, \mathcal{F}) := R\Gamma(G, R\Gamma_{\text{et}}(\overline{S}, \overline{\mathcal{F}})).$$

One may take this as the definition of Weil-étale cohomology of étale sheaves, but see [Gei04] for the full details.

For $\mathcal{F} \in \text{Ab}(S_{\text{et}})$, we have $H_W^0(S, \mathcal{F}) = \Gamma(S, \mathcal{F})$. Since $G \cong \mathbb{Z}$ is generated by ϕ , we have a long exact sequence

$$\cdots \rightarrow H_W^i(S, \mathcal{F}) \rightarrow H_{\text{et}}^i(\overline{S}, \overline{\mathcal{F}}) \xrightarrow{\phi^{-1}} H_{\text{et}}^i(\overline{S}, \overline{\mathcal{F}}) \rightarrow H_W^{i+1}(S, \mathcal{F}) \rightarrow \cdots$$

and a short exact sequence

$$(5) \quad 0 \rightarrow H_{\text{et}}^{i-1}(\overline{S}, \overline{\mathcal{F}})_G \rightarrow H_W^i(S, \mathcal{F}) \rightarrow H_{\text{et}}^i(\overline{S}, \overline{\mathcal{F}})^G \rightarrow 0$$

for $\mathcal{F} \in \text{Ab}(S_{\text{et}})$, where $(\cdot)_G$ and $(\cdot)^G$ denote the G -coinvariants and G -invariants, respectively. By [Gei04, Cor. 5.2], there exists a canonical long exact sequence

$$(6) \quad \cdots \rightarrow H_{\text{et}}^i(S, \mathcal{F}) \rightarrow H_W^i(S, \mathcal{F}) \rightarrow H_{\text{et}}^{i-1}(S, \mathcal{F}) \otimes \mathbb{Q} \rightarrow H_{\text{et}}^{i+1}(S, \mathcal{F}) \rightarrow \cdots$$

Let $e \in H^1(G, \mathbb{Z}) = \text{Hom}(G, \mathbb{Z})$ be the homomorphism sending ϕ to 1. The cup product with e gives a canonical homomorphism $e: H_W^i(S, \mathcal{F}) \rightarrow H_W^{i+1}(S, \mathcal{F})$. This agrees with the composite

$$(7) \quad H_W^i(S, \mathcal{F}) \rightarrow H_{\text{et}}^i(\overline{S}, \overline{\mathcal{F}})^G \xrightarrow{\text{can}} H_{\text{et}}^i(\overline{S}, \overline{\mathcal{F}})_G \rightarrow H_W^{i+1}(S, \mathcal{F})$$

by [Gei04, Lem. 6.2 b)]. Since $e \cup e = 0$, we obtain a complex

$$(H_W^*(S, \mathcal{F}), e) = [\cdots \xrightarrow{e} H_W^i(S, \mathcal{F}) \xrightarrow{e} H_W^{i+1}(S, \mathcal{F}) \xrightarrow{e} H_W^{i+2}(S, \mathcal{F}) \xrightarrow{e} \cdots]$$

of abelian groups.

6. FINITE GENERATION FOR NÉRON MODEL COEFFICIENTS

The Néron model \mathcal{A} and its subgroup scheme \mathcal{A}^0 represent sheaves on S_{et} , so that their Weil-étale cohomology groups $H_W^*(S, \mathcal{A})$ and $H_W^*(S, \mathcal{A}^0)$ make sense. In this section, we study finiteness properties of $H_W^*(S, \mathcal{A})$ and $H_W^*(S, \mathcal{A}^0)$. This is a continuation of what is studied in [Suz19, Prop. 4.2.10] and the paragraph after.

First recall from [Suz19] the commutative group schemes $\mathbf{H}^n(S, \mathcal{A})$ over \mathbb{F}_q for each n , a canonical subgroup scheme $\mathbf{H}^1(S, \mathcal{A})_{\text{div}}$ of $\mathbf{H}^1(S, \mathcal{A})$ and similar objects $\mathbf{H}^n(S, \mathcal{A}^0)$, $\mathbf{H}^1(S, \mathcal{A}^0)_{\text{div}}$. We use the following results.

Proposition 6.1 ([Suz19]).

- (a) *The group of $\overline{\mathbb{F}_q}$ -points of $\mathbf{H}^n(S, \mathcal{A})$ is given by $H_{\text{et}}^n(\overline{S}, \overline{\mathcal{A}})$ including the G -actions ([Suz19, Prop. 2.7.8]).*

- (b) $\mathbf{H}^n(S, \mathcal{A})$ is the perfection (inverse limit along Frobenius morphisms) of a smooth group scheme over \mathbb{F}_q for any n and $\mathbf{H}^n(S, \mathcal{A}) = 0$ for $n \neq 0, 1, 2$ ([Suz19, Thm. 3.4.1 (1)]).
- (c) The identity component of $\mathbf{H}^0(S, \mathcal{A})$ is the perfection of an abelian variety and the component group of $\mathbf{H}^0(S, \mathcal{A})$ is an étale group with finitely generated group of geometric points ([Suz19, Thm. 3.4.1 (2)]).
- (d) $\mathbf{H}^2(S, \mathcal{A})$ is a torsion étale group whose Pontryagin dual is the profinite Tate module of an abelian variety ([Suz19, Thm. 3.4.1 (2), (6a)]).
- (e) The quotient $\mathbf{H}^1(S, \mathcal{A})/\text{div}$ of $\mathbf{H}^1(S, \mathcal{A})$ by $\mathbf{H}^1(S, \mathcal{A})_{\text{div}}$ is the perfection of a commutative algebraic group with unipotent identity component ([Suz19, Thm. 3.4.1 (2)]).
- (f) $\mathbf{H}^1(S, \mathcal{A})_{\text{div}}$ is a divisible torsion étale group scheme with finite n -torsion part for any $n \geq 1$ ([Suz19, Thm. 3.4.1 (2)]).
- (g) Let $T(\mathbf{H}^1(S, \mathcal{A})_{\text{div}})$ be the profinite Tate module of $\mathbf{H}^1(S, \mathcal{A})_{\text{div}}$. Let $V(\mathbf{H}^1(S, \mathcal{A})_{\text{div}})$ be $T(\mathbf{H}^1(S, \mathcal{A})_{\text{div}}) \otimes \mathbb{Q}$. Then

$$(V(\mathbf{H}^1(S, \mathcal{A})_{\text{div}}))^G = (V(\mathbf{H}^1(S, \mathcal{A})_{\text{div}}))_G = 0$$

if and only if $\text{III}(A)$ is finite ([Suz19, Prop. 4.2.5]).

- (h) The statements above also hold with \mathcal{A} replaced by \mathcal{A}^0 ([Suz19, Thm. 3.4.1 (3), Prop. 3.2.4]).

In [Suz19, §4], the G -coinvariants $(V(\mathbf{H}^1(S, \mathcal{A})_{\text{div}}))_G$ is taken in (a category containing) the ind-category of profinite abelian groups (see also the proof of Prop. 6.4 below). The object $(V(\mathbf{H}^1(S, \mathcal{A})_{\text{div}}))_G$ is zero as an ind-object of profinite abelian groups if and only if it is zero as an (abstract) abelian group, since the l -adic Tate module $T_l(\mathbf{H}^1(S, \mathcal{A})_{\text{div}})$ is a finite free \mathbb{Z}_l -module for any prime l by Assertion (f). Hence one may equivalently take the G -coinvariants $(V(\mathbf{H}^1(S, \mathcal{A})_{\text{div}}))_G$ in the category of abelian groups in Assertion (g). A priori, $(V(\mathbf{H}^1(S, \mathcal{A})_{\text{div}}))_G$ might contain a subgroup isomorphic to $(\prod_l \mathbb{Z}/l\mathbb{Z})/(\bigoplus_l \mathbb{Z}/l\mathbb{Z})$ for example.

We denote the groups of $\overline{\mathbb{F}_q}$ -points of $\mathbf{H}^1(S, \mathcal{A})_{\text{div}}$ and $\mathbf{H}^1(S, \mathcal{A})/\text{div}$ by $H_{\text{et}}^1(\overline{S}, \overline{\mathcal{A}})_{\text{div}}$ and $H_{\text{et}}^1(\overline{S}, \overline{\mathcal{A}})/\text{div}$, respectively. We use the same notation with \mathcal{A} replaced by \mathcal{A}^0 .

Proposition 6.2.

- (a) We have $H_{\text{et}}^n(\overline{S}, \overline{\mathcal{A}}) = 0$ for $n \neq 0, 1, 2$.
- (b) The groups $\overline{\mathcal{A}}(\overline{S})^G$ and $\overline{\mathcal{A}}(\overline{S})_G$ are finitely generated.
- (c) The group $H_{\text{et}}^2(\overline{S}, \overline{\mathcal{A}})^G$ is finite, and $H_{\text{et}}^2(\overline{S}, \overline{\mathcal{A}})_G$ is trivial.
- (d) The groups $(H_{\text{et}}^1(\overline{S}, \overline{\mathcal{A}})/\text{div})^G$ and $(H_{\text{et}}^1(\overline{S}, \overline{\mathcal{A}})/\text{div})_G$ are finite.

- (e) The group $H_{\text{et}}^1(\overline{S}, \overline{\mathcal{A}})_{\text{div}}$ is divisible torsion with finite n -torsion part for any $n \geq 1$.
- (f) Let $T(H_{\text{et}}^1(\overline{S}, \overline{\mathcal{A}})_{\text{div}})$ be the profinite Tate module of $H_{\text{et}}^1(\overline{S}, \overline{\mathcal{A}})_{\text{div}}$. Let $V(H_{\text{et}}^1(\overline{S}, \overline{\mathcal{A}})_{\text{div}})$ be $T(H_{\text{et}}^1(\overline{S}, \overline{\mathcal{A}})_{\text{div}}) \otimes \mathbb{Q}$. Then we have

$$(V(H_{\text{et}}^1(\overline{S}, \overline{\mathcal{A}})_{\text{div}}))^G = (V(H_{\text{et}}^1(\overline{S}, \overline{\mathcal{A}})_{\text{div}}))_G = 0$$

if and only if $\text{III}(A)$ is finite.

- (g) The statements above also hold with \mathcal{A} replaced by \mathcal{A}^0 .

Proof. (a) This follows from Prop. 6.1 (a), (b).

(b) First, the endomorphism $\phi - 1$ on any commutative connected algebraic group over \mathbb{F}_q is surjective with finite kernel by Lang's theorem. The same is true with "commutative connected algebraic group" replaced by the perfection of such a group. Hence Prop. 6.1 (c) implies the result.

(c), (d) The same argument as the proof of the previous assertion applies by Prop. 6.1 (d), (e), respectively.

(e), (f), (g) These follow from Prop. 6.1 (f), (g), (h), respectively. \square

Proposition 6.3.

- (a) We have $H_W^n(S, \mathcal{A}) = 0$ for $n \neq 0, 1, 2$.
- (b) The group $H_W^0(S, \mathcal{A})$ is finitely generated.
- (c) The group $H_W^2(S, \mathcal{A})$ is torsion.
- (d) The group $H_W^1(S, \mathcal{A})$ is finitely generated if and only if the torsion group $(H_{\text{et}}^1(\overline{S}, \overline{\mathcal{A}})_{\text{div}})^G$ is finite.
- (e) The group $H_W^2(S, \mathcal{A})$ is finite if and only if the divisible group $(H_{\text{et}}^1(\overline{S}, \overline{\mathcal{A}})_{\text{div}})_G$ is trivial.
- (f) The statements above also hold with \mathcal{A} replaced by \mathcal{A}^0 .

Proof. (a) follows from Prop. 6.2 (a) and (c) and the exact sequence (5). The rest of the statements follow from the exact sequences

$$\begin{aligned} 0 &\rightarrow \overline{\mathcal{A}}(\overline{S})_G \rightarrow H_W^1(S, \mathcal{A}) \rightarrow H_{\text{et}}^1(\overline{S}, \overline{\mathcal{A}})^G \rightarrow 0, \\ 0 &\rightarrow H_{\text{et}}^1(\overline{S}, \overline{\mathcal{A}})_G \rightarrow H_W^2(S, \mathcal{A}) \rightarrow H_{\text{et}}^2(\overline{S}, \overline{\mathcal{A}})^G \rightarrow 0, \end{aligned}$$

and

$$\begin{aligned} 0 &\rightarrow (H_{\text{et}}^1(\overline{S}, \overline{\mathcal{A}})_{\text{div}})^G \rightarrow H_{\text{et}}^1(\overline{S}, \overline{\mathcal{A}})^G \rightarrow (H_{\text{et}}^1(\overline{S}, \overline{\mathcal{A}})/\text{div})^G \\ &\rightarrow (H_{\text{et}}^1(\overline{S}, \overline{\mathcal{A}})_{\text{div}})_G \rightarrow H_{\text{et}}^1(\overline{S}, \overline{\mathcal{A}})_G \rightarrow (H_{\text{et}}^1(\overline{S}, \overline{\mathcal{A}})/\text{div})_G \rightarrow 0, \end{aligned}$$

and Prop. 6.2. \square

Of course $H_W^0(S, \mathcal{A}) \cong A(K)$ is finitely generated also by the Mordell-Weil theorem. The group $H_W^0(S, \mathcal{A}^0) \cong \mathcal{A}^0(S)$ is a finite index subgroup of $A(K)$.

Proposition 6.4. *The group $H_W^1(S, \mathcal{A})$ is finitely generated if and only if all the groups $H_W^*(S, \mathcal{A})$ are finitely generated if and only if $\text{III}(A)$ is finite. The same is true with \mathcal{A} replaced by \mathcal{A}^0 .*

Proof. Let \mathcal{C} be the category of finite abelian groups. Let $\text{Pro}(\mathcal{C})$ be the pro-category of \mathcal{C} and $\text{Ind}(\text{Pro}(\mathcal{C}))$ the ind-category of $\text{Pro}(\mathcal{C})$ ([KS06, Def. 6.1.1]). They are abelian categories by [KS06, Thm. 8.6.5 (i)]. The category $\text{Pro}(\mathcal{C})$ is just the category of profinite abelian groups. Since the natural functor $\mathcal{C} \rightarrow \text{Pro}(\mathcal{C})$ is fully faithful, the induced functor $\text{Ind}(\mathcal{C}) \rightarrow \text{Ind}(\text{Pro}(\mathcal{C}))$ is also fully faithful ([KS06, Prop. 6.1.10]), where the ind-category $\text{Ind}(\mathcal{C})$ of \mathcal{C} is just the category of torsion abelian groups.

Consider the short exact sequence

$$0 \rightarrow T(H_{\text{et}}^1(\overline{S}, \overline{A})_{\text{div}}) \rightarrow V(H_{\text{et}}^1(\overline{S}, \overline{A})_{\text{div}}) \rightarrow H_{\text{et}}^1(\overline{S}, \overline{A})_{\text{div}} \rightarrow 0$$

of G -modules. We view $H_{\text{et}}^1(\overline{S}, \overline{A})_{\text{div}} \in \text{Ind}(\mathcal{C})$, $T(H_{\text{et}}^1(\overline{S}, \overline{A})_{\text{div}}) \in \text{Pro}(\mathcal{C})$ and $V(H_{\text{et}}^1(\overline{S}, \overline{A})_{\text{div}}) \in \text{Ind}(\text{Pro}(\mathcal{C}))$. (The object $V(H_{\text{et}}^1(\overline{S}, \overline{A})_{\text{div}})$ is of course a locally compact group.) Consequently, we may view the above sequence as a short exact sequence of G -module objects in the abelian category $\text{Ind}(\text{Pro}(\mathcal{C}))$. We have the induced long exact sequence

$$\begin{aligned} 0 \rightarrow (T(H_{\text{et}}^1(\overline{S}, \overline{A})_{\text{div}}))^G &\rightarrow (V(H_{\text{et}}^1(\overline{S}, \overline{A})_{\text{div}}))^G \rightarrow (H_{\text{et}}^1(\overline{S}, \overline{A})_{\text{div}})^G \\ &\rightarrow (T(H_{\text{et}}^1(\overline{S}, \overline{A})_{\text{div}}))_G \rightarrow (V(H_{\text{et}}^1(\overline{S}, \overline{A})_{\text{div}}))_G \rightarrow (H_{\text{et}}^1(\overline{S}, \overline{A})_{\text{div}})_G \rightarrow 0 \end{aligned}$$

in $\text{Ind}(\text{Pro}(\mathcal{C}))$.

Therefore if $\text{III}(A)$ is finite, then by Prop. 6.2 (f), the groups $(T(H_{\text{et}}^1(\overline{S}, \overline{A})_{\text{div}}))^G$ and $(H_{\text{et}}^1(\overline{S}, \overline{A})_{\text{div}})_G$ are trivial, and we have an isomorphism from $(H_{\text{et}}^1(\overline{S}, \overline{A})_{\text{div}})^G \in \text{Ind}(\mathcal{C})$ to $(T(H_{\text{et}}^1(\overline{S}, \overline{A})_{\text{div}}))_G \in \text{Pro}(\mathcal{C})$ in $\text{Ind}(\text{Pro}(\mathcal{C}))$. This implies that these isomorphic groups are in \mathcal{C} , i.e. finite. Therefore all of $H_W^*(S, \mathcal{A})$ are finitely generated by Prop. 6.2.

Conversely, if $H_W^1(S, \mathcal{A})$ is finitely generated, then $(H_{\text{et}}^1(\overline{S}, \overline{A})_{\text{div}})^G$ is finite. Therefore by Prop. 6.2 (e), the endomorphism $\phi - 1$ on the l -primary part of $H_{\text{et}}^1(\overline{S}, \overline{A})_{\text{div}}$ is surjective for any prime number l (possibly equal to p) and invertible for almost all l . Hence $\phi - 1$ on $T(H_{\text{et}}^1(\overline{S}, \overline{A})_{\text{div}})$ is injective with finite cokernel. Hence $\phi - 1$ on $V(H_{\text{et}}^1(\overline{S}, \overline{A})_{\text{div}})$ is an isomorphism. Thus by Prop. 6.2 (f), we know that $\text{III}(A)$ is finite.

The case of \mathcal{A}^0 can be treated similarly (or reduced to the case of \mathcal{A}). \square

7. DUALITY

We recall the duality result [Suz19, Prop. 4.2.10] on $H_W^*(S, \mathcal{A})$. The Poincaré bundle P on $A \times_K B$ canonically extends to a line bundle \mathcal{P}

on $\mathcal{A} \times_S \mathcal{B}^0$ and defines a morphism $\mathcal{A} \otimes^L \mathcal{B}^0 \rightarrow \mathbf{G}_m[1]$ in $D(S_{\text{et}})$ by [Gro72, IX, 1.4.3], where \otimes^L denotes the derived tensor product. See [Mil06, III, Appendix C] for a good review of this theory. Applying $R\Gamma_W(S, \cdot)$ and cup product, we have a morphism

$$(8) \quad R\Gamma_W(S, \mathcal{A}) \otimes^L R\Gamma_W(S, \mathcal{B}^0) \rightarrow R\Gamma_W(S, \mathbf{G}_m)[1]$$

in $D(\text{Ab})$. By [Gei04, Prop. 7.4], we have canonical isomorphisms

$$(9) \quad H_W^n(S, \mathbf{G}_m) \cong \begin{cases} \mathbb{F}_q^\times & \text{if } n = 0, \\ \text{Pic}(S) & \text{if } n = 1, \\ \mathbb{Z} & \text{if } n = 2, \\ 0 & \text{if } n \geq 3. \end{cases}$$

In particular, we have a canonical morphism $R\Gamma_W(S, \mathbf{G}_m) \rightarrow \mathbb{Z}[-2]$. This induces a morphism

$$(10) \quad R\Gamma_W(S, \mathcal{A}) \otimes^L R\Gamma_W(S, \mathcal{B}^0) \rightarrow \mathbb{Z}[-1].$$

Proposition 7.1. *Assume that $\text{III}(A)$ is finite (which implies finite generation of $H_W^*(S, \mathcal{A})$ and $H_W^*(S, \mathcal{B}^0)$ by Prop. 6.4). Then the morphism*

$$R\Gamma_W(S, \mathcal{A}) \rightarrow R\text{Hom}(R\Gamma_W(S, \mathcal{B}^0), \mathbb{Z})[-1]$$

induced by (10) is an isomorphism in $D(\text{Ab})$. In particular, for any n , we have a perfect pairing

$$H_W^n(S, \mathcal{A})/\text{tor} \times H_W^{1-n}(S, \mathcal{B}^0)/\text{tor} \rightarrow \mathbb{Z}$$

of finite free abelian groups and a perfect pairing

$$H_W^n(S, \mathcal{A})_{\text{tor}} \times H_W^{2-n}(S, \mathcal{B}^0)_{\text{tor}} \rightarrow \mathbb{Q}/\mathbb{Z}$$

of finite abelian groups.

Proof. This follows from [Suz19, Prop. 4.2.10]. □

8. EULER CHARACTERISTICS FOR NÉRON MODELS

In this section, we assume that

$$\#\text{III}(A) < \infty,$$

so that the Weil-étale cohomology groups $H_W^*(S, \mathcal{A})$ are finitely generated by Prop. 6.4. We relate $\chi(H_W^*(S, \mathcal{A}, e))$ to the product

$$\frac{\#\text{III}(A)}{\#A(K)_{\text{tor}} \cdot \#B(K)_{\text{tor}}} \cdot \frac{\text{Disc}(h)}{(\log q)^r} \cdot c(A)$$

that appears in Cor. 4.5, thereby finishing the proof of Thm. 1.1.

As in §5, the cup product with $e \in H^1(G, \mathbb{Z})$ turns the groups $H_W^*(S, \mathcal{A})$ into a complex $(H_W^*(S, \mathcal{A}), e)$. The rationalized complex

$(H_W^*(S, \mathcal{A}) \otimes \mathbb{Q}, e)$ is exact by the general result on uniquely divisible sheaves [Gei04, Cor. 5.2]. Hence the cohomology groups of the complex $(H_W^*(S, \mathcal{A}), e)$ are finite. Its Euler characteristic $\chi(H_W^*(S, \mathcal{A}), e)$ is thus well-defined. On the other hand, the groups $H_W^*(S, \mathcal{A})_{\text{tor}}$ are finite, so that $\chi(H_W^*(S, \mathcal{A})_{\text{tor}})$ also is well-defined.

Proposition 8.1.

$$\chi(H_W^*(S, \mathcal{A})_{\text{tor}})^{-1} = \frac{\#\text{III}(A)}{\#A(K)_{\text{tor}} \cdot \#B(K)_{\text{tor}}} \cdot \frac{c(A)}{[B(K)/\text{tor} : \mathcal{B}^0(S)/\text{tor}]}.$$

Proof. We have $H_W^0(S, \mathcal{A}) = A(K)$. Also the finite group $H_W^2(S, \mathcal{A})$ is Pontryagin dual to $\mathcal{B}^0(S)_{\text{tor}}$ by Prop. 7.1.

We treat $H_W^1(S, \mathcal{A})$. By Prop. 7.1, we have $\#H_W^1(S, \mathcal{A})_{\text{tor}} = \#H_W^1(S, \mathcal{B}^0)_{\text{tor}}$. By (6), we have a natural exact sequence

$$0 \rightarrow H_{\text{et}}^1(S, \mathcal{B}^0) \rightarrow H_W^1(S, \mathcal{B}^0) \rightarrow \mathcal{B}^0(S) \otimes \mathbb{Q}.$$

Hence $H_{\text{et}}^1(S, \mathcal{B}^0)_{\text{tor}} \xrightarrow{\sim} H_W^1(S, \mathcal{B}^0)_{\text{tor}}$. By [Mil06, III, Prop. 9.2], we have a natural exact sequence

$$0 \rightarrow \mathcal{B}^0(S) \rightarrow B(K) \rightarrow \bigoplus_v \pi_0(\mathcal{B}_v)(k(v)) \rightarrow H_{\text{et}}^1(S, \mathcal{B}^0) \rightarrow \text{III}(B) \rightarrow 0.$$

Hence $H_{\text{et}}^1(S, \mathcal{B}^0)$ is finite and

$$\#H_{\text{et}}^1(S, \mathcal{B}^0) = \frac{\#\text{III}(B) \cdot c(B)}{[B(K) : \mathcal{B}^0(S)]}.$$

We have $\#\text{III}(B) = \#\text{III}(A)$ by the perfectness of the Cassels-Tate pairing [Mil06, III, Cor. 9.5] and $c(B) = c(A)$ by the perfectness of the Grothendieck pairing [Mil06, III, Thm. 7.11]. Thus

$$\#H_W^1(S, \mathcal{A})_{\text{tor}} = \frac{\#\text{III}(A) \cdot c(A)}{[B(K) : \mathcal{B}^0(S)]}.$$

Therefore

$$\begin{aligned} \chi(H_W^*(S, \mathcal{A})_{\text{tor}})^{-1} &= \frac{\#\text{III}(A)}{\#A(K)_{\text{tor}} \cdot \#\mathcal{B}^0(S)_{\text{tor}}} \cdot \frac{c(A)}{[B(K) : \mathcal{B}^0(S)]} \\ &= \frac{\#\text{III}(A)}{\#A(K)_{\text{tor}} \cdot \#B(K)_{\text{tor}}} \cdot \frac{c(A)}{[B(K)/\text{tor} : \mathcal{B}^0(S)/\text{tor}]}. \end{aligned}$$

□

As in §7, let P be the Poincaré bundle on $A \times_K B$ and \mathcal{P} its canonical extension to $\mathcal{A} \times_S \mathcal{B}^0$. By pullback, we have a pairing on $\mathcal{A}(S) \times \mathcal{B}^0(S) = A(K) \times \mathcal{B}^0(S)$ with values in $\text{Pic}(S)$. Let $\langle \cdot, \cdot \rangle$ be the pairing defined by the composite of the maps

$$A(K) \times \mathcal{B}^0(S) \rightarrow \text{Pic}(S) \rightarrow \mathbb{Z},$$

where the last map is the degree map. This pairing $\langle \cdot, \cdot \rangle$ is non-degenerate modulo torsion subgroups by [Sch82, Lem. 9 iii), Satz 11].

Proposition 8.2.

$$\chi(H_W^*(S, \mathcal{A})/\text{tor}, e)^{-1} = \text{Disc}(\langle \cdot, \cdot \rangle).$$

Proof. Since $H_W^2(S, \mathcal{A})$ is torsion by Prop. 6.3 (e), the only relevant morphism for the left-hand side is $e: H_W^0(S, \mathcal{A})/\text{tor} \rightarrow H_W^1(S, \mathcal{A})/\text{tor}$. By (7), (8) and (9), we have a commutative diagram

$$\begin{array}{ccccc} H_W^0(S, \mathcal{A}) \times H_W^0(S, \mathcal{B}^0) & \longrightarrow & H_W^1(S, \mathbf{G}_m) & \xlongequal{\quad} & \text{Pic}(S) \\ \downarrow e \times \text{id} & & \downarrow e & & \downarrow \text{deg} \\ H_W^1(S, \mathcal{A}) \times H_W^0(S, \mathcal{B}^0) & \longrightarrow & H_W^2(S, \mathbf{G}_m) & \xlongequal{\quad} & \mathbb{Z}. \end{array}$$

The upper pairing gives $\langle \cdot, \cdot \rangle$. The lower pairing modulo torsion subgroups is perfect by Prop. 7.1. Therefore the homomorphism $e: H_W^0(S, \mathcal{A})/\text{tor} \rightarrow H_W^1(S, \mathcal{A})/\text{tor}$ can be identified with the injective homomorphism $A(K)/\text{tor} \hookrightarrow \text{Hom}(\mathcal{B}^0(S)/\text{tor}, \mathbb{Z})$ given by $\langle \cdot, \cdot \rangle$. This implies the result. \square

Proposition 8.3.

$$\chi(H_W^*(S, \mathcal{A}), e)^{-1} = \frac{\#\text{III}(A)}{\#A(K)_{\text{tor}} \cdot \#B(K)_{\text{tor}}} \cdot \frac{\text{Disc}(h)}{(\log q)^r} \cdot c(A).$$

Proof. By the displayed equation right before [Sch82, Theorem], we have

$$\frac{\text{Disc}(h)}{(\log q)^r} = \frac{\text{Disc}(\langle \cdot, \cdot \rangle)}{[B(K)/\text{tor} : \mathcal{B}^0(S)/\text{tor}]}.$$

Also we have

$$\begin{aligned} \chi(H_W^*(S, \mathcal{A}), e) &= \chi(H_W^*(S, \mathcal{A})_{\text{tor}}, e) \cdot \chi(H_W^*(S, \mathcal{A})/\text{tor}, e) \\ &= \chi(H_W^*(S, \mathcal{A})_{\text{tor}}) \cdot \chi(H_W^*(S, \mathcal{A})/\text{tor}, e); \end{aligned}$$

see the proof of [Gei04, Thm. 9.1]. Hence the result follows from the previous two propositions. \square

Proposition 8.4. *The formula in Conj. 2.1 or 2.2 is equivalent to the formula*

$$\lim_{s \rightarrow 1} \frac{L(A, s)}{(1 - q^{1-s})^r} = \chi(H_W^*(S, \mathcal{A}), e)^{-1} \cdot q^{\chi(S, \text{Lie}(\mathcal{A}))}.$$

Proof. This follows from the previous proposition and Cor. 4.5 \square

Now Thm. 1.1 is a consequence of this proposition and the result of Kato-Trihan [KT03, Chap. I, Thm.].

9. INTEGRAL MODELS FOR l -ADIC AND p -ADIC COHOMOLOGY

In this section, we will see that the Weil-étale cohomology $H_W^*(S, \mathcal{A})$ is an integral model for the corresponding l -adic and p -adic cohomology theory if $\mathbf{III}(A)$ is finite. This is a Néron model version of the corresponding result [Gei04, Thm. 8.4] for motivic Tate twists $\mathbb{Z}(n)$. We follow Jannsen's adic formalism [Jan88].

We need some notation about inverse limits. Let l be a prime number that may be equal to p . Let S_{fppf} be the fppf site of S . Let $\text{Ab}(S_{\text{fppf}})^{\mathbb{N}}$ be the category of inverse systems in $\text{Ab}(S_{\text{fppf}})$ indexed by positive integers with the usual ordering. It has enough injectives ([Jan88, (1.1 a)]). As in [Jan88, §3] (adapted to the fppf site), the functor $\text{Ab}(S_{\text{fppf}})^{\mathbb{N}} \rightarrow \text{Ab}$ given by $\{\mathcal{F}_i\} \mapsto \varprojlim_i \Gamma(S, \mathcal{F}_i)$ is denoted by $\Gamma(S, \{\mathcal{F}_i\})$, with right derived functors $H_{\text{fppf}}^n(S, \{\mathcal{F}_i\})$ and total right derived functor $R\Gamma_{\text{fppf}}(S, \{\mathcal{F}_i\})$. A system $\{\mathcal{F}_i\}$ is said to be ML-zero (Mittag-Leffler zero; [Jan88, (1.10)]) if for any $i \geq 1$, there exists an integer $j = j(i) \geq 1$ such that the transition morphism $\mathcal{F}_{i+j} \rightarrow \mathcal{F}_i$ is zero. In this case, we have $R\Gamma_{\text{fppf}}(S, \{\mathcal{F}_i\}) = 0$ (use [Jan88, (1.11), (3.1)]). The ML-zero systems form a Serre subcategory of $\text{Ab}(S_{\text{fppf}})^{\mathbb{N}}$ ([Jan88, (1.12)]). Two objects of $\text{Ab}(S_{\text{fppf}})^{\mathbb{N}}$ are said to be ML-isomorphic if they are isomorphic in the quotient category of $\text{Ab}(S_{\text{fppf}})^{\mathbb{N}}$ by ML-zero systems.

As in [Jan88, (5.1)], define a functor $\underline{T}_l: \text{Ab}(S_{\text{fppf}}) \rightarrow \text{Ab}(S_{\text{fppf}})^{\mathbb{N}}$ by sending a sheaf \mathcal{F} to the inverse system of sheaves

$$\cdots \xrightarrow{l} {}_l^2\mathcal{F} \xrightarrow{l} {}_l\mathcal{F},$$

where ${}_l\mathcal{F}$ means the kernel of multiplication by l^i on \mathcal{F} . By [Jan88, (5.1 a)], for any $\mathcal{F} \in \text{Ab}(S_{\text{fppf}})$, we have a canonical isomorphism

$$(11) \quad R^1\underline{T}_l\mathcal{F} \cong \{\mathcal{F} \otimes \mathbb{Z}/l^i\mathbb{Z}\}_i,$$

where the transition morphisms are the natural reduction morphisms, and $R^n\underline{T}_l\mathcal{F} = 0$ for $n \geq 2$. As before, let $T_l = \text{Hom}(\mathbb{Q}_l/\mathbb{Z}_l, \cdot): \text{Ab} \rightarrow \text{Ab}$ be the l -adic Tate module functor. The natural isomorphism $\Gamma(S, \underline{T}_l(\mathcal{F})) \cong T_l(\Gamma(S, \mathcal{F}))$ induces a canonical isomorphism

$$R\Gamma_{\text{fppf}}(S, R\underline{T}_l(\mathcal{F})) \cong RT_l(R\Gamma_{\text{fppf}}(S, \mathcal{F}))$$

in $D(\text{Ab})$ for $F \in \text{Ab}(S_{\text{fppf}})$ by [Jan88, (5.2), (5.4)]. The same definitions and statements hold for the étale site S_{et} . We use similar notation $\underline{T}_l: \text{Ab}(S_{\text{et}}) \rightarrow \text{Ab}(S_{\text{et}})^{\mathbb{N}}$, $R\Gamma_{\text{et}}(S, \{\mathcal{F}_i\})$ etc. for the étale versions.

For $C \in D^+(\text{Ab})$, we have

$$RT_l(C)[1] \cong R\text{Hom}(\mathbb{Q}_l/\mathbb{Z}_l, C)[1].$$

We have a natural morphisms $C^\cdot \rightarrow RT_l(C^\cdot)[1]$ by applying $R\mathrm{Hom}(\cdot, C^\cdot)$ to the morphisms $\mathbb{Q}_l/\mathbb{Z}_l[-1] \rightarrow \mathbb{Q}/\mathbb{Z}[-1] \rightarrow \mathbb{Z}$. This induces a natural morphism

$$C^\cdot \otimes \mathbb{Z}_l \rightarrow RT_l(C^\cdot)[1]$$

since $RT_l(C^\cdot)$ is represented by a complex of \mathbb{Z}_l -modules. (Note that \mathbb{Z}_l is flat and hence the functor $(\cdot) \otimes \mathbb{Z}_l$ is exact inducing a triangulated functor on the derived categories.) The above morphism is an isomorphism if C^\cdot has finitely generated cohomology groups. On the other hand, if C^\cdot has uniquely divisible cohomology groups, then $RT_l(C^\cdot) = 0$.

Let $\varepsilon: S_{\mathrm{fppf}} \rightarrow S_{\mathrm{et}}$ be the morphism of sites defined by the identity functor on the underlying categories. Combining all the above and (6), we have for any $\mathcal{F} \in \mathrm{Ab}(S_{\mathrm{fppf}})$ a natural morphism and isomorphisms

$$\begin{aligned} R\Gamma_W(S, R\varepsilon_*\mathcal{F}) \otimes \mathbb{Z}_l &\rightarrow RT_l(R\Gamma_W(S, R\varepsilon_*\mathcal{F}))[1] \\ &\xrightarrow{\sim} RT_l(R\Gamma_{\mathrm{et}}(S, R\varepsilon_*\mathcal{F}))[1] \\ &\cong RT_l(R\Gamma_{\mathrm{fppf}}(S, \mathcal{F}))[1] \\ &\cong R\Gamma_{\mathrm{fppf}}(S, RT_l(\mathcal{F}))[1]. \end{aligned}$$

The first morphism is an isomorphism if the groups $H_W^*(S, R\varepsilon_*\mathcal{F})$ are finitely generated. If \mathcal{F} is represented by a smooth group scheme, then its fppf cohomology agrees with the étale cohomology ([Mil80, III, Rmk. 3.11 (b)]), so $R\varepsilon_*\mathcal{F} = \mathcal{F}$. With Prop. 6.4, we have the following.

Proposition 9.1. *For any prime number l , we have canonical morphisms*

$$\begin{aligned} R\Gamma_W(S, \mathcal{A}) \otimes \mathbb{Z}_l &\rightarrow R\Gamma_{\mathrm{fppf}}(S, RT_l(\mathcal{A}))[1], \\ R\Gamma_W(S, \mathcal{A}^0) \otimes \mathbb{Z}_l &\rightarrow R\Gamma_{\mathrm{fppf}}(S, RT_l(\mathcal{A}^0))[1]. \end{aligned}$$

The right-hand sides are canonically isomorphic to $R\Gamma_{\mathrm{et}}(S, RT_l(\mathcal{A}))[1]$, $R\Gamma_{\mathrm{et}}(S, RT_l(\mathcal{A}^0))[1]$, respectively, if $l \neq p$. These morphisms are isomorphisms if $\mathrm{III}(A)$ is finite.

We can give a more explicit description of the objects $RT_l(\mathcal{A})$ and $RT_l(\mathcal{A}^0)$ and their fppf cohomology in some cases as follows.

Proposition 9.2. *For each place v , let $i_v: \mathrm{Spec} k(v) \hookrightarrow S$ denote the inclusion morphism. Assume that $l \neq p$ or A has semistable reduction everywhere.*

(a) We have canonical ML-isomorphisms

$$\begin{aligned} \underline{T}_l(\mathcal{A}) &\cong \underline{T}_l(\mathcal{A}^0), \quad R^1 \underline{T}_l(\mathcal{A}^0) = 0, \\ R^1 \underline{T}_l(\mathcal{A}) &\cong \bigoplus_v i_{v*} \pi_0(\mathcal{A}_v) \otimes \mathbb{Z}_l \quad (\text{a constant system}), \end{aligned}$$

(b) The group $H_{\text{fppf}}^n(S, {}_l \mathcal{A}^0)$ is finite for any $n \geq 0$ and $i \geq 1$.

(c) The natural morphism from $H_{\text{fppf}}^n(S, \underline{T}_l(\mathcal{A}^0))$ to $\varprojlim_i H_{\text{fppf}}^n(S, {}_l \mathcal{A}^0)$ is an isomorphism.

(d) We have a long exact sequence

$$\begin{aligned} \cdots \rightarrow H_{\text{fppf}}^n(S, \underline{T}_l(\mathcal{A}^0)) &\rightarrow H_{\text{fppf}}^n(S, R\underline{T}_l(\mathcal{A})) \rightarrow \bigoplus_v H_{\text{et}}^{n-1}(k(v), \pi_0(\mathcal{A}_v)) \otimes \mathbb{Z}_l \\ &\rightarrow H_{\text{fppf}}^{n+1}(S, \underline{T}_l(\mathcal{A}^0)) \rightarrow \cdots \end{aligned}$$

Under the stated assumption, the closed group subschemes ${}_l \mathcal{A}^0$ of \mathcal{A}^0 over S are quasi-finite, flat and separated over S by [Mil06, III, Cor. C.9]. They are étale if $l \neq p$, and finite over the locus of S where \mathcal{A} has good reduction.

Proof. (a) We have an exact sequence

$$0 \rightarrow \mathcal{A}^0 \rightarrow \mathcal{A} \rightarrow \bigoplus_v i_{v*} \pi_0(\mathcal{A}_v) \rightarrow 0$$

in $\text{Ab}(S_{\text{fppf}})$. Under the stated assumption, the multiplication by l^i on \mathcal{A}^0 is faithfully flat for any i by [Mil06, III, Cor. C.9]. Hence $R^n \underline{T}_l(\mathcal{A}^0) = 0$ for $n \geq 1$ by (11). Since the component groups are finite, we know that $\underline{T}_l(i_{v*} \pi_0(\mathcal{A}_v))$ is ML-zero and $R^1 \underline{T}_l(i_{v*} \pi_0(\mathcal{A}_v))$ is ML-isomorphic to $i_{v*} \pi_0(\mathcal{A}_v) \otimes \mathbb{Z}_l$ by (11). This implies the result.

(b) Consider the Hochschild-Serre spectral sequence

$$E_2^{st} = H_{\text{et}}^s(\mathbb{F}_q, H_{\text{et}}^t(\overline{S}, \overline{\mathcal{A}})) \implies H_{\text{et}}^{s+t}(S, \mathcal{A}).$$

Using Prop. 6.2, we know that the kernels and cokernels of multiplication by l^i on the E_2 -terms are finite for all i . Hence the same is true for $H_{\text{et}}^n(S, \mathcal{A})$ and therefore for $H_{\text{et}}^n(S, \mathcal{A}^0)$. The long exact sequence associated with the sequence $0 \rightarrow {}_l \mathcal{A}^0 \rightarrow \mathcal{A}^0 \xrightarrow{l^i} \mathcal{A}^0 \rightarrow 0$ in $\text{Ab}(S_{\text{fppf}})$ gives an exact sequence

$$0 \rightarrow H_{\text{et}}^{n-1}(S, \mathcal{A}^0) \otimes \mathbb{Z}/l^i \mathbb{Z} \rightarrow H_{\text{fppf}}^n(S, {}_l \mathcal{A}^0) \rightarrow {}_l H_{\text{et}}^n(S, \mathcal{A}^0) \rightarrow 0.$$

Thus the middle term is also finite.

(c) The previous assertion implies that the first derived inverse limit $\varprojlim_i^1 H_{\text{fppf}}^n(S, {}_l \mathcal{A}^0)$ is zero. Hence the result follows from [Jan88, (3.1)].

(d) The assertions above and the long exact sequence for H_{fppf}^* give the result. \square

Corollary 9.3. *Assume that $l \neq p$ or A has semistable reduction everywhere. We have a canonical homomorphism*

$$H_W^n(S, \mathcal{A}^0) \otimes \mathbb{Z}_l \rightarrow \varprojlim_i H_{\text{fppf}}^{n+1}(S, {}_l\mathcal{A}^0)$$

for any n . It is an isomorphism if $\text{III}(A)$ is finite.

Note that if $l = p$ and A has non-semistable reduction at some v , then the multiplication by p on \mathcal{A} and \mathcal{A}^0 has a non-flat kernel and a non-representable cokernel.

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